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OPTIMALITY OF LOCAL MULTILEVEL METHODS ON ADAPTIVELY REFINED MESHES FOR ELLIPTIC BOUNDARY VALUE PROBLEMS

XUEJUN XU* , HUANGXIN CHEN*, AND R.H.W. HOPPE†

Abstract. A local multilevel product algorithm and its additive version are analyzed for linear systems arising from the application of adaptive finite element methods to second order elliptic boundary value problems. The abstract Schwarz theory is applied to verify uniform convergence of local multilevel methods featuring Jacobi and Gauss-Seidel smoothing only on local nodes. By this abstract theory, convergence estimates can be further derived for the hierarchical basis multigrid method and the hierarchical basis preconditioning method on locally refined meshes, where local smoothing is performed only on new nodes. Numerical experiments confirm the optimality of the suggested algorithms.

1. Introduction. Multigrid or multilevel methods belong to the most efficient methods to solve large linear systems arising from the discretization of elliptic boundary value problems by finite element methods. The convergence properties of multigrid methods for conforming finite elements have been studied by many authors (cf., e.g., [9], [10], [11], [8], [14], [19], [24], [32]). The hierarchical basis multilevel method ([34], [35]) and the hierarchical basis multigrid method [6] have been developed by H. Yserentant *et al.* for finite element methods on quasi-uniform meshes. In particular, using the notions of space decomposition and subspace correction, a unified theory has been established in [32] for a general class of iterative algorithms such as multigrid methods, overlapping domain decomposition methods, and hierarchical basis methods.

In this paper, we study local multilevel methods for adaptive finite element methods (**AFEM**) applied to second order elliptic boundary value problems. Mesh adaptivity based on a posteriori error estimators has become a powerful tool for solving partial differential equations. It is known that the convergence property of **AFEM** with the newest vertex bisection algorithm is optimal in the sense that the finite element discretization error is proportional to $N^{-1/2}$ in the energy norm, where N is the number of degrees of freedom on the underlying mesh (cf., e.g., [7], [16], [22], [28]). Since the number of nodes per level may not grow exponentially with the mesh levels, as has been pointed out in [23], the number of operations used for multigrid methods with smoothers performed on all nodes can be as large as $O(N^2)$. Therefore, it is interesting to study efficient iterative algorithms to solve the linear systems arising from **AFEM** procedures. Numerical experiments in [23] indicate the optimality of local multigrid methods performing smoothing only on newly created nodes and their neighbors.

In recent years, some techniques have been developed to handle problems on locally refined meshes. One approach in [20], [21] and [31] is the fast adaptive composite grid (**FAC**) method, using global and local uniform grids both to define the composite grid problem and to interact for fast solution, which is very suitable for parallel computation. Other approaches have been developed as well such as multilevel adaptive techniques (**MLAT**) studied in [3], [12], [13], [24], and multigrid methods for locally refined finite element meshes [1], [2], [18], [25], [26]. We emphasize that these locally refined meshes obey restrictive conditions which are not satisfied by the newest vertex bisection algorithm which will be used for adaptivity in this work. As far as **AFEM** procedures featuring the newest vertex bisection algorithm are concerned, Wu and Chen [30] have been the first to show that the multigrid V-cycle algorithm performng Gauss-Seidel smoothing on new nodes and those old nodes where the support of the associated nodal basis function has changed can guarantee uniform convergence of the algorithm.

The objective of this paper is to utilize the well-known Schwarz theory [29] to study local multilevel methods with local Jacobi or local Gauss-Seidel smoothing. Within this framework

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we can also derive convergence estimates for the hierarchical basis multigrid method and the hierarchical basis preconditioning method on locally refined meshes, where the local smoothers are performed only on new nodes. In this paper, the main difficulty is how to obtain a global strengthened Cauchy-Schwarz inequality which is a key assumption in the Schwarz theory. We will prove that the global strengthened Cauchy-Schwarz inequality holds true not only for the local Gauss-Seidel iteration, but also for the local Jacobi iteration. Moreover, we point out that the Xu and Zikatanov identity [33], on which the proof in [30] depends, can not be directly used in this paper. The convergence estimate in [30] can only be deduced for multiplicative smoothers and thus is not available for additive smoothers such as the Jacobi iteration. Finally, for the hierarchical basis multigrid method, a nontrivial stability splitting property on locally refined meshes is obtained.

The remainder of this paper is organized as follows: In section 2, we introduce basic notations and briefly review the conforming P1 finite element method on locally refined meshes. In section 3, we propose a local multilevel product algorithm (or a local multigrid method) and its additive version. In section 4, we present the abstract theory based on three assumptions whose verification is carried out for local Jacobi and local Gauss-Seidel smoothers, respectively. We further derive and analyze the hierarchical basis multigrid method and the hierarchical basis preconditioning method on locally refined meshes in section 5. Finally, in the last section we present numerical results for some representative test examples that confirm our theoretical analysis.

2. Notations and Preliminaries. Throughout this paper, we adopt standard notations from Lebesgue and Sobolev space theory (cf. e.g. [17]). In particular, we refer to $(\cdot,\cdot)$ as the inner product and to $\|\cdot\|_{1,\Omega}$ as the norm on the Sobolev space $H^1(\Omega)$. We further use $A \lesssim B$, if $A \le CB$ with a positive constant C depending only on the shape regularity of the meshes. $A \approx B$ stands for $A \lesssim B \lesssim A$. For simplicity, we restrict ourselves to the 2D case.

Given a bounded, polygonal domain $\Omega \subset R^2$, we consider the following second order elliptic boundary value problem

$$\mathcal{L}u := -\mathrm{div}(a(x)\nabla u) = f \quad \text{in } \Omega, \tag{2.1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

The choice of a homogeneous Dirichlet boundary condition is made for ease of presentation only. Similar results are valid for other types of boundary conditions and equation (2.1) with a lower order term as well. We further assume that the coefficient function a and the right-hand side f in (2.1) satisfy the following properties:

(**a**) a is a measurable function and there exist constants $\beta_1 \ge \beta_0 > 0$ such that

$$\beta_0 \le a(x) \le \beta_1 \quad \text{for almost all } x \in \Omega, \tag{2.3}$$

(**b**) $f \in L^2(\Omega)$.

The weak formulation of (2.1) and (2.2) is to find $u \in V := H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad , \quad v \in V, \tag{2.4}$$

where the bilinear form $a : V \times V \to \mathbb{R}$ is given by

$$a(u,v) = (a\nabla u, \nabla v) \quad , \quad u,v \in V. \tag{2.5}$$

Since the bilinear form (2.5) is bounded and V-elliptic, the existence and uniqueness of the solution of (2.4) follows from the Lax-Milgram theorem.

Throughout this paper, we work with families of shape regular meshes $\{\mathcal{T}_i,\ i = 0, 1, ..., J\}$, where $\mathcal{T}_0$ is an intentionally chosen coarse initial triangulation, the others are obtained by the adaptive procedures using the newest vertex bisection algorithm. It has been proved in [4] that there exists a constant $\theta > 0$ such that

$$\theta_T \ge \theta \quad , \quad T \in \mathcal{T}_i\ ,\ i = 1, 2, ..., \tag{2.6}$$

where θ_T is the minimum angle of the element T. The set of edges on $\mathcal{T}_i$ is denoted by $\mathcal{E}_i$, and the set of interior and boundary edges by $\mathcal{E}_i^0$ and $\mathcal{E}_i^{\partial\Omega}$, respectively. We refer to $\mathcal{N}_i$ as the set of interior nodes of $\mathcal{T}_i$. The domain Ω_i^z is the union of elements containing $z \in \mathcal{N}_i$ and h_i^z refers to the shortest edge of $\mathcal{E}_i(\Omega_i^z)$. For any $T \in \mathcal{T}_i$, $h_{i,T}$ stands for the diameter of T.

We refer to V_J as the conforming P_1 finite element space

$$V_J = \{v_J \in V \mid v_J|_T \in P_1(T), T \in \mathcal{T}_J\}.$$

The conforming finite element approximation of (2.4) is to find $u_J \in V_J$ such that

$$a(u_J, v_J) = (f, v_J) \quad , \quad v_J \in V_J. \tag{2.7}$$

The existence and uniqueness of the solution u_J follows again from the Lax-Milgram theorem.

The computation of the solution u_J of (2.7) always requires a constructive approach involving the conversion of the variational equation into a matrix equation using a particular basis for V_J. Suppose that $\{\phi_i,\, i = 1, ..., N\}$ is a given basis for V_J, where N is the dimension of V_J, and define the matrix A and the vector F via

$$A_{ij} := a(\phi_i, \phi_j) \quad \text{and} \quad F_i := (f, \phi_i) \quad , \quad i, j = 1, ..., N.$$

Then equation (2.7) is equivalent to

$$AX = F, \tag{2.8}$$

where $u_J = \sum_{i=1}^N u_i\phi_i$ and $X = (u_i)$.

Bank and Scott [5] have shown that the ℓ_2-condition number of the linear system (2.8) does not necessarily degrade as the mesh is refined locally and can be bounded by

$$\mathcal{K}_2(A) \lesssim N(1 + |\log(Nh^2_{\min}(\mathcal{T}_J))|), \tag{2.9}$$

where $h_{\min}(\mathcal{T}_J) = \min\{h_{J,T} : T \in \mathcal{T}_J\}$. Moreover, the upper bound is sharp.

Based on the estimate (2.9), we know that standard iterative methods for solving the large linear system (2.8) will converge very slowly. The objective of this paper is to design efficient multilevel solvers of optimal computational complexity.

3. Local multilevel methods. In this section, we develop local multilevel methods for solving linear systems arising from **AFEM** procedures. For any $0 \le i \le J$, define $A_i : V_i \to V_i$, the discretization operator on level i, by

$$(A_iv, w) = a(v, w) \quad , \quad v, w \in V_i.$$

Then the finite element discretization of (2.4) is to find $u_i \in V_i$ such that

$$A_iu_i = f_i, \tag{3.1}$$

where $f_i \in V_i$ satisfies $(f_i, v) = (f, v)$, $v \in V_i$. We also define projections P_i, $Q_i : V_J \to V_i$,

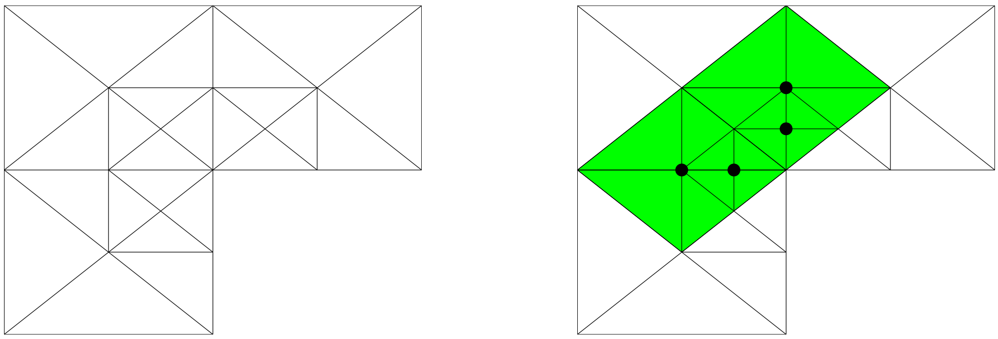
$$a(P_iv, w) = a(v, w) \quad , \quad (Q_iv, w) = (v, w) \quad , \quad v \in V_J,\, w \in V_i.$$

For any node $z \in \mathcal{N}_i$, we use the notation φ_i^z to represent the associated nodal conforming finite element basis function of V_i. Let $\tilde{\mathcal{N}}_i$ be the set of new nodes and those old nodes where the support of the associated basis function has changed (see Figure 3.1), i.e.,

$$\tilde{\mathcal{N}}_i = \{z \in \mathcal{N}_i : z \in \mathcal{N}_i \setminus \mathcal{N}_{i-1} \text{ or } z \in \mathcal{N}_{i-1} \text{ but } \varphi_i^z \neq \varphi_{i-1}^z\}.$$

For convenience, we set $\tilde{\mathcal{N}}_i = \{x_i^k, k = 1, ..., \tilde{n}_i\}$, where $\tilde{n}_i$ is the cardinality of $\tilde{\mathcal{N}}_i$, and we denote by $\phi_i^k = \phi_i^{x_i^k}$ the conforming P_1 finite element basis function associated with x_i^k. We define local projections $P_i^k, Q_i^k : V_J \to V_i^k := \text{span}\{\phi_i^k\}$ by

$$a(P_i^kv, \phi_i^k) = a(v, \phi_i^k) \quad , \quad (Q_i^kv, \phi_i^k) = (v, \phi_i^k) \quad , \quad v \in V_J,$$



The coarse level $\mathcal{T}_{i-1}$ *The fine level $\mathcal{T}_i$*

FIG 3.1. *Illustration of $\tilde{\mathcal{N}}_i$: the big dots in the right figure refer to $\tilde{\mathcal{N}}_i$.*

and $A_i^k : V_i^k \to V_i^k$ by

$$(A_i^k v, \phi_i^k) = a(v, \phi_i^k) \quad , \quad v \in V_i^k.$$

We further refer to $R_i : V_i \to V_i$ as a local smoothing operator which is assumed to be nonnegative, symmetric or nonsymmetric with respect to the inner product $(\cdot,\cdot)$. For $i = 1, ..., J$, R_i is only performed on local nodes $\tilde{\mathcal{N}}_i$. The linear system on the coarsest mesh is solved directly, i.e., $R_0 = A_0^{-1}$.

We now state the local multilevel algorithms for **AFEM** as follows.

ALGORITHM 3.1. **Local Multigrid algorithm(LMG)**
The standard Multigrid V-cycle algorithm solves (3.1) by the following iterative method:

$$u_i^{n+1} = u_i^n + B_i(f_i - A_i u_i^n).$$

The operators $B_i : V_i \to V_i, 0 \le i \le J$ are recursively defined as follows:
(V-cycle algorithm). Let $B_0 = A_0^{-1}$. For $i \ge 1$ and $g \in V_i$, we define $B_i g = x_3$.
(i). Pre-smoothing: $x_1 = R_i^t f$,
(ii). Correction: $x_2 = x_1 + B_{i-1} Q_{i-1}(g - A_i x_1)$,
(iii). Post-smoothing: $x_3 = x_2 + R_i(g - A_i x_2)$.

ALGORITHM 3.2. **Local multilevel additive algorithm(LMAA)**
For $B_J = \sum_{i=0}^{J} R_i Q_i$, find $u_J \in V_J$ such that

$$B_J A_J u_J = B_J f_J. \tag{3.2}$$

REMARK 3.1. *The CG method can be used to solve the new problem* (3.2), *if $B_J A_J$ is symmetric with respect to the inner product $a(\cdot,\cdot)$.*

4. The abstract theory. In this section, we present the abstract theory concerned with the convergence of local multilevel methods for linear systems arising from **AFEM** procedures. We will use the well-known Schwarz theory developed in [29], [32] and [36] to analyze the algorithms **LMG** and **LMAA**. To this end, we set

$$T_i := R_i A_i P_i,\ i = 0, 1, ..., J, \quad \text{and} \quad T := \sum_{i=0}^{J} T_i.$$

The abstract theory provides an estimate for the norm of the error operator

$$E = (I - T_J) \cdots (I - T_1)(I - T_0) = \prod_{i=0}^{J} (I - T_i),$$

where I is the identity operator in V_J. Convergence estimates for algorithm 3.1 are obtained by upper bounds for E in the energy norm $\|\cdot\|_a := a(\cdot,\cdot)^{\frac{1}{2}}$. To this end, we impose the following assumptions:

(**A1**). Each operator T_i is nonnegative with respect to the inner product $a(\cdot,\cdot)$, and there exists a positive constant $\omega_i < 2$ such that

$$a(T_i v, T_i v) \le \omega_i \; a(T_i v, v) \quad , \quad v \in V_J \; , \; i = 0,1,...,J.$$

(**A2**). **Stability:** There exists a constant K_0 such that

$$a(v,v) \le K_0 \; a(Tv,v) \quad , \quad v \in V_J.$$

(**A3**). **Global strengthened Cauchy-Schwarz inequality:** There exists a constant K_1 such that

$$\sum_{i=0}^{J}\sum_{j=0}^{i-1} a(T_i v, T_j u) \le K_1 \; (\sum_{i=0}^{J} a(T_i v, v))^{1/2} (\sum_{j=0}^{J} a(T_j u, u))^{1/2} \quad , \quad v, u \in V_J.$$

We remark that the following inequality should also be verified for local Jacobi and local Gauss-Seidel smoothers, which can not be deduced by the Cauchy-Schwarz inequality for T_i directly.

$$\sum_{i=0}^{J} a(T_i v, u) \le K_2 \; (\sum_{i=0}^{J} a(T_i v, v))^{1/2} (\sum_{i=0}^{J} a(T_i u, u))^{1/2} \quad , \quad v, u \in V_J. \tag{4.1}$$

THEOREM 4.1. *If assumptions* **A1-A3** *are satisfied, then the norm of the error operator E can be bounded as follows* (*cf.* [29], [32], [36]):

$$a(Ev, Ev) \le \delta \, a(v,v) \quad , \quad v \in V_J,$$

where $\delta = 1 - (2-\omega)/(K_0(K_1+K_2)^2)$, $\omega = \max_i\{\omega_i\}$. Hence, for algorithm 3.1 *there holds*

$$\|I - B_J A_J\|_a = \|EE^*\|_a \le \delta.$$

For the additive multilevel algorithm 3.2, the following theorem provides a spectral estimate for the operator $T = \sum_{i=0}^{J} T_i$ when T is symmetric with respect to $a(\cdot,\cdot)$.

THEOREM 4.2. *If T is symmetric with respect to $a(\cdot,\cdot)$ and assumptions* **A1-A3** *hold true, then we have* (*cf.* [29], [32], [36])

$$\frac{1}{K_0}\, a(v,v) \le a(Tv,v) \le (2K_1 + \omega)\, a(v,v) \quad , \quad v \in V_J.$$

We begin to apply the abstract theory to algorithm 3.1 and algorithm 3.2 by verifying assumptions **A1-A3** for adaptive finite element methods. There are two classes of smoothers for R_i, Jacobi and Gauss-Seidel iterations, which will be investigated separately.

4.1. Local Jacobi smoother. First, we consider the decomposition of $v \in V_J$ according to

$$v = \sum_{i=0}^{J} v_i \quad , \quad v_0 = \Pi_0 v \quad , \quad v_i = (\Pi_i - \Pi_{i-1})v \quad , \quad i = 1,...,J, \tag{4.2}$$

where $\Pi_i : V_J \to V_i$ is the Scott-Zhang interpolation operator [27].

The local Jacobi smoother is defined as an additive smoother (cf. [9]):

$$R_i := \gamma \sum_{k=1}^{\tilde{n}_i} (A_i^k)^{-1} Q_i^k, \tag{4.3}$$

where γ is an appropriately chosen positive scaling factor. Due to the definition of R_i, we have

$$T_0 = P_0 \quad , \quad T_i = R_i A_i P_i = \gamma \sum_{k=1}^{\tilde{n}_i} P_i^k \quad , \quad i = 1, ..., J. \tag{4.4}$$

It is easy to deduce that $K_2 = 1$ in (4.1) in the Jacobi case. Therefore, we only need to verify assumptions **A1-A3**. Actually,

$$\begin{aligned}
a(Tv, u) &= a(P_0 v, P_0 u) + \gamma \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} a(P_i^k v, P_i^k u) \\
&\leq a(P_0 v, v)^{1/2} a(P_0 u, u)^{1/2} + \gamma \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} a(P_i^k v, v)^{1/2} a(P_i^k u, u)^{1/2} \\
&\leq (a(P_0 v, v) + \gamma \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} a(P_i^k v, v))^{1/2} (a(P_0 u, u) + \gamma \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} a(P_i^k u, u))^{1/2} \\
&= (\sum_{i=0}^{J} a(T_i v, v))^{1/2} (\sum_{i=0}^{J} a(T_i u, u))^{1/2}.
\end{aligned}$$

4.1.1. Verification of assumption A1. Assumption **A1** is easily obtained for T_0. We analyze the case $i \geq 1$.

LEMMA 4.1. *Let T_i, $i \geq 1$, be defined by* (4.4). *Then, there holds*

$$a(T_i v, T_i v) \leq \omega_i \; a(T_i v, v) \quad , \quad v \in V_J \quad , \quad \omega_i < 2.$$

Moreover, T_i is symmetric and nonnegative on V_J. Therefore, assumption **A1** *is satisfied.*

Proof. Following (4.4), for $v, w \in V_J$ we deduce

$$a(T_i v, w) = a(R_i A_i P_i v, w) = a(R_i A_i P_i v, P_i w) = (R_i A_i P_i v, A_i P_i w).$$

In view of the definition of R_i by (4.3), it follows that R_i is symmetric and nonnegative in V_i. Hence, T_i is symmetric and nonnegative in V_J. We set

$$K_i^k = \{P_i^m : \operatorname{supp}(P_i^k v) \cap \operatorname{supp}(P_i^m v) \neq \varnothing,\ v \in V_i, m = 1, \ldots, \tilde{n}_i\}. \tag{4.5}$$

Then, the cardinality of K_i^k is bounded by a constant depending only on the minimum angle θ in (2.6). Based on this fact and Hölder's inequality, there exists a constant C_i such that

$$\sum_{k,m=1}^{\tilde{n}_i} |a(P_i^k v, P_i^m v)| \leq C_i \sum_{k=1}^{\tilde{n}_i} a(P_i^k v, P_i^k v) \quad , \quad v \in V_i. \tag{4.6}$$

By the definition of T_i in (4.4) and observing (4.6), for $v \in V_J$ we obtain

$$\begin{aligned}
a(T_i v, T_i v) &= \gamma^2 a(\sum_{k=1}^{\tilde{n}_i} P_i^k v, \sum_{k=1}^{\tilde{n}_i} P_i^k v) \leq \gamma^2 \sum_{k,m=1}^{\tilde{n}_i} |a(P_i^k v, P_i^m v)| \\
&\leq \gamma^2 C_i \sum_{k=1}^{\tilde{n}_i} a(P_i^k v, P_i^k v) = \gamma^2 C_i \sum_{k=1}^{\tilde{n}_i} a(P_i^k v, v) = \gamma C_i a(T_i v, v).
\end{aligned}$$

The proof is completed by setting $\omega_i = \gamma C_i$ and choosing $0 < \gamma < 1$ such that $\omega_i < 2$. □

4.1.2. Verification of assumption A2. We will rely on the decomposition (4.2).

LEMMA 4.2. *Let $\{T_i, i = 0, 1, ..., J\}$ be defined by* (4.4)*. Then, there exists a constant K_0 such that*

$$a(v, v) \leq K_0 a(Tv, v) \quad , \quad v \in V_J.$$

Proof. Due to (4.2), we have

$$a(v, v) = \sum_{i=0}^{J} a(v_i, v), \tag{4.7}$$

and for $i = 1, ..., J$, we obtain

$$\begin{aligned} a(v_i, v) &= \sum_{k=1}^{\tilde{n}_i} a(v_i(x_i^k)\phi_i^k, v) \leq \sum_{k=1}^{\tilde{n}_i} a^{1/2}(v_i(x_i^k)\phi_i^k, v_i(x_i^k)\phi_i^k v) a^{1/2}(P_i^k v, P_i^k v) \\ &\leq (\sum_{k=1}^{\tilde{n}_i} a(v_i(x_i^k)\phi_i^k, v_i(x_i^k)\phi_i^k v))^{1/2} (\sum_{k=1}^{\tilde{n}_i} a(P_i^k v, v))^{1/2}. \end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8) yields

$$\begin{aligned} & a(v, v) = \sum_{i=0}^{J} a(v_i, v) \\ & \leq \Big(a(v_0, v_0) + \sum_{i=1}^{J}\sum_{k=1}^{\tilde{n}_i} a(v_i(x_i^k)\phi_i^k, v_i(x_i^k)\phi_i^k)\Big)^{1/2} \Big(a(P_0 v, v) + \sum_{i=1}^{J}\sum_{k=1}^{\tilde{n}_i} a(P_i^k v, v)\Big)^{1/2}. \end{aligned} \tag{4.9}$$

Since $a(\phi_i^k, \phi_i^k) \approx 1$, there holds

$$a(v_i(x_i^k)\phi_i^k, v_i(x_i^k)\phi_i^k) \approx v_i^2(x_i^k).$$

The following inequality has been proved in Lemma 3.3 of [30],

$$\sum_{i=1}^{J}\sum_{k=1}^{\tilde{n}_i} v_i^2(x_i^k) \lesssim a(v, v).$$

For the initial level, we have

$$a(v_0, v_0) = a(\Pi_0 v, \Pi_0 v) \lesssim a(v, v).$$

Thus, we obtain

$$a(v_0, v_0) + \sum_{i=1}^{J}\sum_{k=1}^{\tilde{n}_i} v_i^2(x_i^k) \lesssim a(v, v). \tag{4.10}$$

Combining the above inequalities, we conclude that there exists a constant $\tilde{K}_0$ independent of mesh sizes and mesh levels such that

$$a(v, v) \leq \tilde{K}_0 \Big(a(P_0 v, v) + \sum_{i=1}^{J}\sum_{k=1}^{\tilde{n}_i} a(P_i^k v, v)\Big) \leq \frac{\tilde{K}_0}{\gamma} \sum_{i=0}^{J} a(T_i v, v) = \frac{\tilde{K}_0}{\gamma}\, a(Tv, v).$$

We obtain the desired result by setting $K_0 = \tilde{K}_0/\gamma$. □

4.1.3. Verification of assumption A3. As a prerequisite to verify assumption **A3**, we need the following key lemma which has been derived in [30].

LEMMA 4.3. *For $i = 1, ..., J$ let $\mathcal{T}_i$ be a refinement of $\mathcal{T}_{i-1}$ by the newest vertex bisection algorithm and let Ω_j^k be the support of ϕ_j^k. Then, for $x_j^k \in \tilde{\mathcal{N}}_j$ we have*

$$\sum_{i=j+1}^{J} \sum_{\substack{x_i^l \in \tilde{\mathcal{N}}_i, \\ x_i^l \in \mathcal{E}_j^k}} (\frac{h_i^l}{h_j^k})^{3/2} \lesssim 1 \quad , \quad \sum_{i=j+1}^{J} \sum_{\substack{x_i^l \in \tilde{\mathcal{N}}_i, \\ x_i^l \in \Omega_j^k}} (\frac{h_i^l}{h_j^k})^{3} \lesssim 1, \tag{4.11}$$

where $\mathcal{E}_j^k = \mathcal{E}_j(\Omega_j^k)$. Moreover, for $x_i^l \in \tilde{\mathcal{N}}_i$ there holds

$$\sum_{j=1}^{i-1} \sum_{\substack{x_j^k \in \tilde{\mathcal{N}}_j, \\ x_i^l \in \mathcal{E}_j^k}} (\frac{h_i^l}{h_j^k})^{1/2} \lesssim 1 \quad , \quad \sum_{j=1}^{i-1} \sum_{\substack{x_j^k \in \tilde{\mathcal{N}}_j, \\ x_i^l \in \Omega_j^k}} (\frac{h_i^l}{h_j^k})^{1/2} \lesssim 1. \tag{4.12}$$

Now we are in a position to verify assumption **A3**.

LEMMA 4.4. *There exists a constant K_1 independent of mesh sizes and mesh levels such that assumption* **A3** *holds true.*

Proof. In view of (4.4), we have

$$\begin{aligned}\sum_{i=1}^{J} \sum_{j=1}^{i-1} a(T_i v, T_j u) &= \gamma^2 \sum_{j=1}^{J} \sum_{i=j+1}^{J} \sum_{k=1}^{\tilde{n}_j} a(P_j^k u, \sum_{l=1}^{\tilde{n}_i} P_i^l v) \\ &= \gamma^2 \sum_{j=1}^{J} \sum_{k=1}^{\tilde{n}_j} a(P_j^k u, \sum_{i=j+1}^{J} \sum_{l=1}^{\tilde{n}_i} P_i^l v).\end{aligned}$$

Setting $\omega = \sum_{i=j+1}^{J} \sum_{l=1}^{\tilde{n}_i} P_i^l v$, there holds

$$a(P_j^k u, \omega) = a(P_j^k u, P_j^k \omega) \leq a^{1/2}(P_j^k u, P_j^k u)\ a^{1/2}(P_j^k \omega, P_j^k \omega),$$

whence

$$\sum_{i=1}^{J} \sum_{j=1}^{i-1} a(T_i v, T_j u) \leq \gamma^2 \Big(\sum_{j=1}^{J} \sum_{k=1}^{\tilde{n}_j} a(P_j^k u, P_j^k u) \Big)^{1/2} \Big(\sum_{j=1}^{J} \sum_{k=1}^{\tilde{n}_j} a(P_j^k \omega, P_j^k \omega) \Big)^{1/2}. \tag{4.13}$$

It is obvious that

$$\gamma \sum_{j=1}^{J} \sum_{k=1}^{\tilde{n}_j} a(P_j^k u, P_j^k u) = \gamma \sum_{j=1}^{J} \sum_{k=1}^{\tilde{n}_j} a(P_j^k u, u) = \sum_{j=1}^{J} a(T_j u, u). \tag{4.14}$$

We also have

$$\gamma \sum_{j=1}^{J} \sum_{k=1}^{\tilde{n}_j} a(P_j^k \omega, P_j^k \omega) \lesssim \sum_{i=2}^{J} a(T_i v, v). \tag{4.15}$$

We note that

$$a(\phi_j^k, \phi_j^k) \approx 1 \quad \text{and} \quad P_j^k P_i^l v = \frac{a(P_i^l v, \phi_j^k)}{a(\phi_j^k, \phi_j^k)} \phi_j^k \approx a(P_i^l v, \phi_j^k) \phi_j^k,$$

and hence,

$$a(P_j^k \omega, P_j^k \omega) \approx (\sum_{i=j+1}^{J} \sum_{l=1}^{\tilde{n}_i} a(P_i^l v, \phi_j^k))^2.$$

Furthermore, due to

$$P_i^l v = \frac{a(v,\phi_i^l)}{a(\phi_i^l,\phi_i^l)}\phi_i^l \approx a(v,\phi_i^l)\phi_i^l,$$

it follows that

$$a(P_i^l v,\phi_j^k) \approx a(a(v,\phi_i^l)\phi_i^l,\phi_j^k) = a(\phi_i^l,\phi_j^k)a(v,\phi_i^l).$$

Since ϕ_j^k is conforming and piecewise linear on $\mathcal{T}_j|_{\Omega_j^k}$, we obtain

$$a(\phi_i^l,\phi_j^k) = \sum_{\substack{T\subset\Omega_j^k,\\ T\in\mathcal{T}_j}}\int_T a(x)\nabla\phi_i^l\cdot\nabla\phi_j^k = \sum_{\substack{T\subset\Omega_j^k,\\ T\in\mathcal{T}_j}}\int_{\partial T} a(x)\frac{\partial\phi_j^k}{\partial n}\phi_i^l - \sum_{\substack{T\subset\Omega_j^k,\\ T\in\mathcal{T}_j}}\int_T(\nabla a(x)\cdot\nabla\phi_j^k)\phi_i^l.$$

Note that

$$a(v,\phi_i^l) = a(P_i^l v,\phi_i^l) \le a^{1/2}(P_i^l v,P_i^l v)a^{1/2}(\phi_i^l,\phi_i^l) \lesssim a^{1/2}(P_i^l v,v).$$

Moreover, observing $|\frac{\partial\phi_j^k}{\partial n}| \lesssim (h_j^k)^{-1}$ and (2.3), it follows that

$$\sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\Omega_j^k}} a(\phi_i^l,\phi_j^k)a(v,\phi_i^l) \lesssim \sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\mathcal{E}_j^k}}\frac{h_i^l}{h_j^k}a^{1/2}(P_i^l v,v) + \sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\Omega_j^k}}\frac{(h_i^l)^2}{h_j^k}a^{1/2}(P_i^l v,v).$$

Consequently, in view of (4.11) we get

$$\begin{aligned}
a(P_j^k\omega,P_j^k\omega) &\lesssim (\sum_{i=j+1}^J \sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\Omega_j^k}} a(\phi_i^l,\phi_j^k)a(v,\phi_i^l))^2 \\
&\lesssim (\sum_{i=j+1}^J \sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\mathcal{E}_j^k}} \frac{h_i^l}{h_j^k}a^{1/2}(P_i^l v,v))^2 + (\sum_{i=j+1}^J \sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\Omega_j^k}} \frac{(h_i^l)^2}{h_j^k}a^{1/2}(P_i^l v,v))^2 \\
&\lesssim (\sum_{i=j+1}^J \sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\mathcal{E}_j^k}} (\frac{h_i^l}{h_j^k})^{1/2}a(P_i^l v,v))\cdot(\sum_{i=j+1}^J \sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\mathcal{E}_j^k}} (\frac{h_i^l}{h_j^k})^{3/2}) \\
&\quad +(\sum_{i=j+1}^J \sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\Omega_j^k}} \frac{h_i^l}{\sqrt{h_j^k}}a(P_i^l v,v))\cdot(\sum_{i=j+1}^J \sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\Omega_j^k}} (\frac{h_i^l}{\sqrt{h_j^k}})^3) \\
&\lesssim (\sum_{i=j+1}^J \sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\mathcal{E}_j^k}} (\frac{h_i^l}{h_j^k})^{1/2}a(P_i^l v,v) + \sum_{i=j+1}^J \sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\Omega_j^k}} \frac{h_i^l}{\sqrt{h_j^k}}a(P_i^l v,v))(1+(h_j^k)^{3/2}) \\
&\lesssim \sum_{i=j+1}^J \sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\mathcal{E}_j^k}} (\frac{h_i^l}{h_j^k})^{1/2}a(P_i^l v,v) + \sum_{i=j+1}^J \sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\Omega_j^k}} \frac{h_i^l}{\sqrt{h_j^k}}a(P_i^l v,v).
\end{aligned}$$

We set $\delta(x_i^l,x_j^k)=1$, if $x_i^l\in\mathcal{E}_j^k$, and $\delta(x_i^l,x_j^k)=0$, otherwise, as well as $\tilde{\delta}(x_i^l,x_j^k)=1$, if $x_i^l\in\Omega_j^k$, and $\tilde{\delta}(x_i^l, x_j^k) = 0$, otherwise. In view of (4.12), we obtain

$$\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} a(P_j^k\omega, P_j^k\omega) \lesssim \sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j}\sum_{i=j+1}^{J}\sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\mathcal{E}_j^k}} (\frac{h_i^l}{h_j^k})^{1/2} a(P_i^l v, v) + \sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j}\sum_{i=j+1}^{J}\sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\Omega_j^k}} \frac{h_i^l}{\sqrt{h_j^k}} a(P_i^l v, v)$$

$$= \sum_{i=2}^{J}\sum_{x_i^l\in\tilde{\mathcal{N}}_i}(\sum_{j=1}^{i-1}\sum_{x_j^k\in\tilde{\mathcal{N}}_j}(\frac{h_i^l}{h_j^k})^{1/2}\delta(x_i^l, x_j^k))a(P_i^l v, v) + \sum_{i=2}^{J}\sum_{x_i^l\in\tilde{\mathcal{N}}_i}(\sum_{j=1}^{i-1}\sum_{x_j^k\in\tilde{\mathcal{N}}_j}\frac{h_i^l}{\sqrt{h_j^k}}\tilde{\delta}(x_i^l, x_j^k))a(P_i^l v, v)$$

$$\lesssim \sum_{i=2}^{J}\sum_{x_i^l\in\tilde{\mathcal{N}}_i} a(P_i^l v, v)(1+\sqrt{h_i^l}) \lesssim \sum_{i=2}^{J}\sum_{x_i^l\in\tilde{\mathcal{N}}_i} a(P_i^l v, v).$$

This completes the proof of (4.15). Combining (4.13)-(4.15), we deduce

$$\sum_{i=1}^{J}\sum_{j=1}^{i-1} a(T_i v, T_j u) \lesssim (\sum_{i=2}^{J} a(T_i v, v))^{1/2}(\sum_{j=1}^{J} a(T_j u, u))^{1/2}. \tag{4.16}$$

A similar analysis can be done to derive

$$\sum_{i=1}^{J} a(T_i v, T_0 u) \lesssim (\sum_{i=1}^{J} a(T_i v, v))^{1/2} a(T_0 u, u)^{1/2}. \tag{4.17}$$

Together with (4.16), this inequality provides the assertion. □

4.2. Local Gauss-Seidel smoother. We will now apply the abstract theory to the local Gauss-Seidel smoother R_i which is defined by

$$R_i := (I - E_i^{\tilde{n}_i})A_i^{-1},$$

where $E_i^{\tilde{n}_i} = (I - P_i^{\tilde{n}_i})\cdots(I - P_i^1) = \prod_{k=1}^{\tilde{n}_i}(I - P_i^k)$. For simplicity, we set $E_i := E_i^{\tilde{n}_i}$, since no confusion is possible. It is easy to see that

$$T_0 = P_0 \quad , \quad T_i = R_i A_i P_i = (I - E_i)P_i = I - E_i \quad , \quad i = 1, ..., J. \tag{4.18}$$

The decomposition of v is the same as in (4.2). The following identity plays a key role in the subsequent analysis.

Lemma 4.5. *For $i = 1, ..., J$, there holds*

$$a(v, u) - a(E_i v, E_i u) = \sum_{k=1}^{\tilde{n}_i} a(P_i^k E_i^{k-1} v, E_i^{k-1} u) \quad , \quad v, u \in V_J, \tag{4.19}$$

where $E_i^0 = I$ and E_i^{k-1} is defined by

$$E_i^{k-1} := (I - P_i^{k-1})\cdots(I - P_i^1) \quad , \quad k = 2, \ldots, \tilde{n}_i.$$

Proof. Obviously, there holds

$$E_i^{k-1} - E_i^k = P_i^k E_i^{k-1},$$

and hence,

$$I - E_i = \sum_{k=1}^{\tilde{n}_i} P_i^k E_i^{k-1}. \tag{4.20}$$

Note that

$$a(E_i^{k-1}v, E_i^{k-1}u) = a(E_i^k v, E_i^k u) + a(P_i^k E_i^{k-1}v, P_i^k E_i^{k-1}u),$$

which implies

$$a(v,u) - a(E_i v, E_i u) = \sum_{k=1}^{\tilde{n}_i} a(P_i^k E_i^{k-1}v, E_i^{k-1}u).$$

Hence, (4.19) is verified. □

4.2.1. Verification of assumption A1. We consider the case $i \geq 1$, since assumption **A1** is obviously true for T_0.

LEMMA 4.6. *Let T_i, $i \geq 1$, be defined by* (4.18). *Then, T_i is nonnegative on V_J and there holds*

$$a(T_i v, T_i v) \leq \omega_i a(T_i v, v) \quad , \quad v \in V_J \quad , \quad \omega_i < 2.$$

Proof. Recalling $T_i = I - E_i, i \geq 1$ and (4.20), we obtain

$$a(T_i v, T_i v) = a((I - E_i)v, (I - E_i)v) = \sum_{k,m=1}^{\tilde{n}_i} a(P_i^k E_i^{k-1}v, P_i^m E_i^{m-1}v).$$

Using Lemma 4.5 and the same techniques as in (4.6), we deduce

$$\begin{aligned} a(T_i v, T_i v) &\leq C_i \sum_{k=1}^{\tilde{n}_i} a(P_i^k E_i^{k-1}v, E_i^{k-1}v) \\ &= C_i(a(v,v) - a(E_i v, E_i v)) = C_i(2a(T_i v, v) - a(T_i v, T_i v)), \end{aligned} \tag{4.21}$$

whence

$$a(T_i v, T_i v) \leq \frac{2C_i}{C_i + 1} a(T_i v, v).$$

This implies nonnegativeness of T_i. Setting $\omega_i = (2C_i)/(C_i + 1) < 2$ completes the proof. □

4.2.2. Verification of assumption A2.

LEMMA 4.7. *Let $\{T_i, i = 0, 1, ..., J\}$ be given as in* (4.18). *Then, there exists a constant K_0 such that*

$$a(v,v) \leq K_0 a(Tv, v) \quad , \quad v \in V_J.$$

Proof. From the decomposition of v in (4.2), it follows that $a(v,v) = \sum_{i=0}^J a(v_i, v)$. Similar to (4.8), for $i = 1, ..., J$ we also have

$$a(v_i, v) \leq (\sum_{k=1}^{\tilde{n}_i} a(v_i(x_i^k)\phi_i^k, v_i(x_i^k)\phi_i^k))^{1/2} (\sum_{k=1}^{\tilde{n}_i} a(P_i^k v, P_i^k v))^{1/2},$$

Due to the identity $I - E_i^{k-1} = \sum_{m=1}^{k-1} P_i^m E_i^{m-1}$, we deduce

$$\begin{aligned} &\sum_{k=1}^{\tilde{n}_i} a(P_i^k v, P_i^k v) = \sum_{k=1}^{\tilde{n}_i} a(P_i^k v, P_i^k E_i^{k-1}v) + \sum_{k=1}^{\tilde{n}_i} \sum_{m=1}^{k-1} a(P_i^k v, P_i^k P_i^m E_i^{m-1}v) \\ &\leq \Big(\sum_{k=1}^{\tilde{n}_i} a(P_i^k v, P_i^k v)\Big)^{1/2} \Big(\sum_{k=1}^{\tilde{n}_i} a(P_i^k E_i^{k-1}v, E_i^{k-1}v)\Big)^{1/2} + \sum_{k,m=1}^{\tilde{n}_i} |a(P_i^k v, P_i^m E_i^{m-1}v)|. \end{aligned}$$

Furthermore, by Hölder's inequality and (4.6)

$$\sum_{k,m=1}^{\tilde{n}_i} |a(P_i^k v, P_i^m E_i^{m-1} v)| \lesssim \Big(\sum_{k=1}^{\tilde{n}_i} a(P_i^k v, P_i^k v)\Big)^{1/2} \Big(\sum_{k=1}^{\tilde{n}_i} a(P_i^k E_i^{k-1} v, E_i^{k-1} v)\Big)^{1/2}.$$

Then, it follows from (4.21) that

$$\sum_{k=1}^{\tilde{n}_i} a(P_i^k v, P_i^k v) \lesssim \sum_{k=1}^{\tilde{n}_i} a(P_i^k E_i^{k-1} v, E_i^{k-1} v) \lesssim a(T_i v, v), \tag{4.22}$$

whence

$$a(P_0 v, P_0 v) + \sum_{k=1}^{\tilde{n}_i} a(P_i^k v, P_i^k v) \lesssim \sum_{i=0}^{J} a(T_i v, v).$$

Similar to the analysis of (4.9) and (4.10), we deduce that assumption **A2** holds true. □

4.2.3. Verification of assumption A3.

LEMMA 4.8. *There exists a constant K_1 independent of mesh sizes and mesh levels such that assumption* **A3** *holds true for $\{T_i, i = 0, 1, ..., J\}$ defined by* (4.18).

Proof. For $\xi_i = T_i v$, it follows from (4.18) that

$$\begin{aligned}\sum_{i=1}^{J}\sum_{j=1}^{i-1} a(T_i v, T_j u) &= \sum_{j=1}^{J}\sum_{i=j+1}^{J} a(\xi_i, (I - E_j)u) \\ &= \sum_{j=1}^{J}\sum_{i=j+1}^{J}\sum_{k=1}^{\tilde{n}_j} a(P_j^k \xi_i, P_j^k E_j^{k-1} u) = \sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} a(P_j^k \sum_{i=j+1}^{J} \xi_i, P_j^k E_j^{k-1} u).\end{aligned}$$

By Hölder's inequality there holds

$$\begin{aligned}&\sum_{i=1}^{J}\sum_{j=1}^{i-1} a(T_i v, T_j u) \\ &\le \Big(\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} a(P_j^k E_j^{k-1} u, E_j^{k-1} u)\Big)^{1/2} \Big(\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} a(\sum_{i=j+1}^{J} P_j^k \xi_i, \sum_{i=j+1}^{J} P_j^k \xi_i)\Big)^{1/2}.\end{aligned} \tag{4.23}$$

Next, we show that

$$\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} a(\sum_{i=j+1}^{J} P_j^k \xi_i, \sum_{i=j+1}^{J} P_j^k \xi_i) \lesssim \sum_{i=2}^{J} a(T_i v, v). \tag{4.24}$$

Due to

$$P_j^k \xi_i = \frac{a(\xi_i, \phi_j^k)}{a(\phi_j^k, \phi_j^k)} \phi_j^k \approx a(\xi_i, \phi_j^k)\phi_j^k \quad , \quad P_i^l E_i^{l-1} v = \frac{a(E_i^{l-1} v, \phi_i^l)}{a(\phi_i^l, \phi_i^l)} \phi_i^l \approx a(E_i^{l-1} v, \phi_i^l)\phi_i^l,$$

we have

$$a(\sum_{i=j+1}^{J} P_j^k \xi_i, \sum_{i=j+1}^{J} P_j^k \xi_i) \approx (\sum_{i=j+1}^{J} a(\xi_i, \phi_j^k))^2,$$

and

$$a(\xi_i, \phi_j^k) = a((I - E_i)v, \phi_j^k) = \sum_{l=1}^{\tilde{n}_i} a(P_i^l E_i^{l-1} v, \phi_j^k) \approx \sum_{l=1}^{\tilde{n}_i} a(\phi_i^l, \phi_j^k) a(E_i^{l-1} v, \phi_i^l).$$

By Lemma 4.3 and a similar technique as in the previous subsection, we obtain

$$\begin{aligned}
&\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j}(\sum_{i=j+1}^{J} a(\xi_i,\phi_j^k))^2 \\
\lesssim &\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j}\sum_{i=j+1}^{J}\sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\mathcal{E}_j^k}}(\frac{h_i^l}{h_j^k})^{1/2}a(P_i^lE_i^{l-1}v,E_i^{l-1}v) \\
&+\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j}\sum_{i=j+1}^{J}\sum_{\substack{x_i^l\in\tilde{\mathcal{N}}_i,\\ x_i^l\in\Omega_j^k}}\frac{h_i^l}{\sqrt{h_j^k}}a(P_i^lE_i^{l-1}v,E_i^{l-1}v) \\
\lesssim &\sum_{i=2}^{J}\sum_{x_i^l\in\tilde{N}_i}a(P_i^lE_i^{l-1}v,E_i^{l-1}v)\sum_{j=1}^{i-1}\sum_{x_j^k\in\tilde{N}_j}(\frac{h_i^l}{h_j^k})^{1/2}\delta(x_i^l,x_j^k) \\
&+\sum_{i=2}^{J}\sum_{x_i^l\in\tilde{N}_i}a(P_i^lE_i^{l-1}v,E_i^{l-1}v)\sum_{j=1}^{i-1}\sum_{x_j^k\in\tilde{N}_j}\frac{h_i^l}{\sqrt{h_j^k}}\tilde{\delta}(x_i^l,x_j^k) \\
\lesssim &\sum_{i=2}^{J}\sum_{x_i^l\in\tilde{N}_i}a(P_i^lE_i^{l-1}v,E_i^{l-1}v)(1+\sqrt{h_i^l})\lesssim\sum_{i=2}^{J}a(T_iv,v).
\end{aligned}$$

Hence, (4.24) is verified. In view of (4.19), (4.21), (4.23) and (4.24), it follows that

$$\sum_{i=1}^{J}\sum_{j=1}^{i-1}a(T_iv,T_ju)\lesssim(\sum_{i=2}^{J}a(T_iv,v))^{1/2}(\sum_{j=1}^{J}a(T_ju,u))^{1/2}. \tag{4.25}$$

Moreover, we deduce

$$\sum_{i=1}^{J}a(T_iv,T_0u)\lesssim(\sum_{i=1}^{J}a(T_iv,v))^{1/2}a(T_0u,u)^{1/2},$$

which, together with (4.25), allows to conclude. □

Next, we show that (4.1) holds true for the Gauss-Seidel case. Actually, similar to the Jacobi case, by (4.22) we have

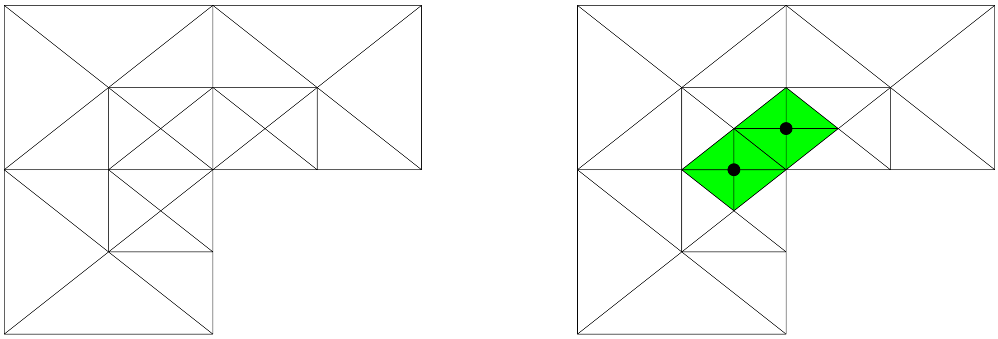
$$\begin{aligned}
a(Tv,u)&=a(P_0v,P_0u)+\sum_{i=1}^{J}\sum_{k=1}^{\tilde{n}_i}a(P_i^kE_i^{k-1}v,P_i^ku) \\
&\le a(P_0v,v)^{1/2}a(P_0u,u)^{1/2}+\sum_{i=1}^{J}\sum_{k=1}^{\tilde{n}_i}a(P_i^kE_i^{k-1}v,E_i^{k-1}v)^{1/2}a(P_i^ku,P_i^ku)^{1/2} \\
&\le (a(P_0v,v)+\sum_{i=1}^{J}\sum_{k=1}^{\tilde{n}_i}a(P_i^kE_i^{k-1}v,E_i^{k-1}v))^{1/2}(a(P_0u,u)+\sum_{i=1}^{J}\sum_{k=1}^{\tilde{n}_i}a(P_i^ku,P_i^ku))^{1/2} \\
&\lesssim(\sum_{i=0}^{J}a(T_iv,v))^{1/2}(\sum_{i=0}^{J}a(T_iu,u))^{1/2}.
\end{aligned}$$

5. Hierarchical basis multilevel method. In this section, we will discuss the hierarchical basis multigrid method (**HBMG**) and the hierarchical basis preconditioning method (**HBP**) on locally refined meshes. The hierarchical basis method is based on the decomposition of V_J into subspaces given by

$$V_J = \sum_{i=0}^{J} \bar{V}_i,\ \bar{V}_0 = I_0 V_J \quad , \quad \bar{V}_i = (I_i - I_{i-1})V_J \quad , \quad i = 1, ..., J. \tag{5.1}$$

Here, $I_i : V_J \to V_i$ is the nodal value interpolation.

By means of the above decomposition, we can derive the convergence result for **HBMG** by verifying assumptions **A1-A3** as in section 4. Compared with the above local multigrid method, the smoothing operator $R_i (1 \le i \le J)$ in **HBMG** is carried out only on the set of new nodes, e.g., $\bar{\mathcal{N}}_i = \mathcal{N}_i \setminus \mathcal{N}_{i-1}$. We set $\bar{n}_i = \#\bar{\mathcal{N}}_i$. The operators A_i, P_i, Q_i are all well defined by the subspaces $\{\bar{V}_i : i = 0, 1, ..., J\}$.



The coarse level $\mathcal{T}_{i-1}$ *The fine level $\mathcal{T}_i$*

FIG 5.1. *Illustration of $\bar{\mathcal{N}}_i$: the big dots in the right figure refer to $\bar{\mathcal{N}}_i$.*

For brevity, we only provide the convergence analysis of **HBMG** for local Jacobi smoothing. A similar analysis can be carried out for **HBMG** in case of local Gauss-Seidel smoothing. We note that the assumptions **A1** and **A3** can be verified as in Lemma 4.1 and Lemma 4.4. For the stability assumption **A2**, we have the following result.

LEMMA 5.1. *There exists a constant K_0 such that*

$$a(v, v) \le K_0 (1 + |\log h_{\min}|)^2 a(Tv, v) \quad , \quad v \in V_J, \tag{5.2}$$

where $h_{\min} = \min\{h_T, T \in \mathcal{T}_J\}$.

Proof. In view of the decomposition (5.1), it follows that

$$v_i = (I_i - I_{i-1})v = \sum_{k=1}^{\bar{n}_i} v(x_i^k)\phi_i^k \quad , \quad i = 1, ..., J.$$

Similar to (4.9), we deduce

$$\begin{aligned} a(v, v) &= \sum_{i=0}^{J} a(v_i, v) \\ &\le \Big(a(v_0, v_0) + \sum_{i=1}^{J} \sum_{k=1}^{\bar{n}_i} a(v(x_i^k)\phi_i^k, v(x_i^k)\phi_i^k) \Big)^{1/2} \Big(a(P_0 v, v) + \sum_{i=1}^{J} \sum_{k=1}^{\bar{n}_i} a(P_i^k v, v) \Big)^{1/2}. \end{aligned} \tag{5.3}$$

Since $a(v_0, v_0) = a(I_0 v, I_0 v) \lesssim a(v, v)$, (5.2) is proved, if we can show that

$$\sum_{i=1}^{J} \sum_{k=1}^{\bar{n}_i} a(v(x_i^k)\phi_i^k, v(x_i^k)\phi_i^k) \lesssim (1 + |\log h_{\min}|)^2 a(v, v). \tag{5.4}$$

For any triangle $K \subset \Omega_i^k \cap \mathcal{T}_i$, let C be a constant representing the L^2 projection of v onto K. We recall the following inequality [34]

$$\|v - C\|_{0,\infty,K} \lesssim (\log \frac{h_i^k}{h_{\min}} + 1)^{1/2} |v|_{1,K}. \tag{5.5}$$

Then, we have

$$\begin{aligned} &|(I_i - I_{i-1})v|_{1,K} = |(I_i - I_{i-1})(v - C)|_{1,K} = |(I - I_{i-1})I_i(v - C)|_{1,K} \\ &\lesssim |I_i(v - C)|_{1,K} \lesssim \|v - C\|_{0,\infty,K} \lesssim (\log \frac{h_i^k}{h_{\min}} + 1)^{1/2} |v|_{1,K}. \end{aligned}$$

Hence, it follows from (5.5) that

$$\sum_{i=1}^{J} \sum_{k=1}^{\bar{n}_i} a(v(x_i^k)\phi_i^k, v(x_i^k)\phi_i^k) \lesssim \sum_{i=1}^{J} \sum_{k=1}^{\bar{n}_i} |(I_i - I_{i-1})v|^2_{1,\Omega_i^k} \lesssim \sum_{i=1}^{J} \sum_{k=1}^{\bar{n}_i} (\log \frac{h_i^k}{h_{\min}} + 1) |v|^2_{1,\Omega_i^k}. \tag{5.6}$$

For the sequence $\{\hat{\mathcal{T}}_i, i = 0, 1, 2, ...\}$, obtained by uniform bisection from the initial mesh $\hat{\mathcal{T}}_0 = \mathcal{T}_0$, we denote by $\hat{\mathcal{N}}_i$ the set of interior nodes of $\hat{\mathcal{T}}_i$, and we set $\hat{h}_i = (\frac{1}{2})^i h_0$. It has been shown in [30] that

$$h_i(z) \approx \hat{h}_{\rho_i(z)} \quad , \quad z \in \hat{\mathcal{N}}_{\rho_i(z)}, \tag{5.7}$$

where

$$\rho_i(z) = [\frac{\log(h_i(z)/h_0)}{\log(1/2)}].$$

We define

$$\sigma(m, z) = \{i : z \in \bar{\mathcal{N}}_i, \rho_i(z) = m, 0 \le i \le J\}.$$

It has been shown in [30] that $\#\sigma(m, z) \lesssim 1$. Let $B(z, c\hat{h}_m) = \{x \in \Omega : |x - z| < c\hat{h}_m\}$ and $M := \max_{z \in \mathcal{N}_J} \rho_J(z)$. Then, $h_{\min} \approx \hat{h}_M$. Combining (5.6) and (5.7), we have

$$\begin{aligned} &\sum_{i=1}^{J} \sum_{k=1}^{\bar{n}_i} a(v(x_i^k)\phi_i^k, v(x_i^k)\phi_i^k) \lesssim \sum_{i=1}^{J} \sum_{k=1}^{\bar{n}_i} (\log \frac{\hat{h}_{\rho_i(x_i^k)}}{h_{\min}} + 1) |v|^2_{1,\Omega_i^k} \\ &\lesssim \sum_{m=0}^{M} \sum_{\substack{z \in \tilde{\mathcal{N}}_i, \rho_i(z) = m \\ 1 \le i \le J}} (\log \frac{\hat{h}_m}{h_{\min}} + 1) |v|^2_{1,B(z,c\hat{h}_m)} = \sum_{m=0}^{M} (\log \frac{\hat{h}_m}{h_{\min}} + 1) \sum_{\substack{z \in \hat{\mathcal{N}}_m \cap \mathcal{N}_J \\ i \in \sigma(m,z)}} |v|^2_{1,B(z,c\hat{h}_m)} \\ &\lesssim \sum_{m=0}^{M} (\log \hat{h}_m - \log h_{\min} + 1) |v|^2_{1,\Omega} \lesssim (1 + |\log h_{\min}|)^2 |v|^2_{1,\Omega}, \end{aligned}$$

which completes the proof of the lemma. □

Finally, we have the following convergence result for the algorithm **HBMG**.

THEOREM 5.1. *For the algorithm* **HBMG** *with local Jacobi or local Gauss-Seidel smoothing, the norm of the error operator E can be bounded as follows*

$$a(Ev, Ev) \le \delta\, a(v, v) \quad , \quad v \in V_J,$$

where

$$\delta = 1 - \frac{2 - \omega}{K_0(1 + |\log h_{\min}|)^2 (1 + K_1)^2} \quad , \quad \omega = \max_i \{\omega_i\}.$$

Hence,

$$\|I - B_J A_J\|_a = \|EE^*\|_a \le \delta.$$

Next, we provide a condition number estimate for the hierarchical basis preconditioning method. Let

$$|||v|||_*^2 = \sum_{z\in\mathcal{N}_0} (I_0 v)^2(z) + \sum_{i=1}^{J} \sum_{z\in\mathcal{N}_i\setminus\mathcal{N}_{i-1}} |(I_i - I_{i-1})v(z)|^2.$$

Similar to (5.4), the following upper bound holds true

$$|||v|||_*^2 \lesssim (1 + |\log h_{\min}|)^2 |v|_{1,\Omega}^2.$$

A lower bound can be derived as in the verification of assumption **A3**:

$$|v|_{1,\Omega}^2 \lesssim |||v|||_*^2.$$

THEOREM 5.2. *Let $\hat{A}_J$ be the stiffness matrix based on the hierarchical basis on locally refined meshes $\mathcal{T}_J$. Then, there holds*

$$\mathrm{cond}(\hat{A}_J) \le C(1 + |\log h_{\min}|)^2,$$

where the constant C is independent of mesh sizes and mesh levels.

Let S be the matrix which transforms the representations of the finite element functions of V_J with respect to the hierarchical basis into the representations with respect to the usual nodal basis. We then have the representation $\hat{A}_J = S^T A_J S$. Since

$$\mathrm{cond}(S^T A_J S) = \mathrm{cond}(SS^T A_J),$$

the hierarchical basis method can be interpreted as a preconditioning method for A_J with the preconditioner SS^T.

6. Numerical results. In this section, we present several examples to illustrate the optimality of algorithm 3.1 and algorithm 3.2. For algorithm 3.2, we test the PCG method for **LMAA** with local Jabobi smoothing. Furthermore, in order to compare the two methods, we present examples for **HBMG** and **HBP** on locally refined meshes. We remark that **LMG** and **HBMG** are implemented with $O(N)$ operations each iteration, where N is the number of degrees of freedom (DOFs, i.e., interior nodes or free nodes). As has been pointed out in [6] and [34], the overall complexity of **HBMG** (the symmetric case, e.g., with local Jacobi smoothing) and the hierarchical basis method used as a preconditioner for CG is $O(N\log(h_{\min})|\log\epsilon|)$ operations, required to reduce the initial error by a given factor ϵ. On the other hand, for **LMAA** with local Jacobi smoothing as a preconditioner for CG, $O(N|\log\epsilon|)$ operations are required. The following implementation is based on the FFW toolbox from [15].

The local error estimator for each element is defined as in [22]. The stopping rule for algorithm 3.1 is as follows: At the i-th level, let $u_i^0 = u_{i-1}, r_i^n = f_i - A_i u_i^n$. Then, the multigrid iteration stops when the following relation is satisfied

$$\frac{\|r_i^n\|_{0,\Omega}}{\|r_i^0\|_{0,\Omega}} \le 10^{-8}.$$

For the PCG method, the stopping criterion is as follows

$$\|r_i^0 - A_i r_i^n\|_{0,\Omega} \le \mathrm{tol}\ \|r_i^0\|_{0,\Omega} \quad , \quad \mathrm{tol} = 10^{-8},$$

where $\{r_i^k : k = 1, 2, ...\}$ denotes the set of iterative solutions of the residual equation $A_i x = r_i^0$.

Example 6.1. Consider the following elliptic boundary value problem with Dirichlet boundary conditions on the L-shaped domain $\Omega = [-1, 1] \times [-1, 1] \backslash (0, 1] \times [-1, 0)$.

$$\begin{aligned} -\Delta u + 0.5u &= f(x, y) \quad \text{in } \Omega, \\ u &= g(x, y) \quad \text{on } \partial\Omega, \end{aligned}$$

where f and g are chosen such that the exact solution in polar coordinates is given by $u(r, \theta) = r^{\frac{2}{3}} \sin(\frac{2}{3}\theta)$.

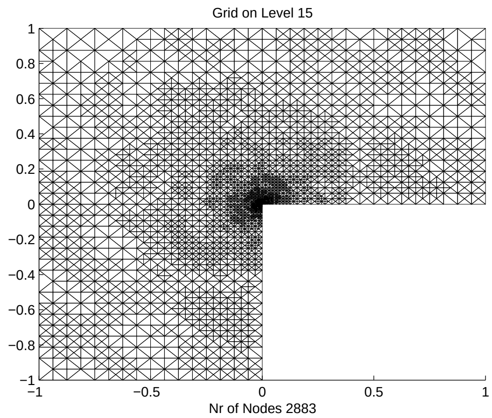


FIG 6.1. *The locally refined mesh after 15 refinement steps (Example 6.1)*

Level	DOFs	LMG-Jacobi	LMG-GS	HBMG-GS
15	2718	25	13	51
17	5255	23	11	59
21	18242	22	11	75
25	58758	23	12	88
27	102397	22	11	88
29	173698	20	10	90
31	290479	20	10	101
33	487869	19	10	113
35	799086	17	9	118

Table 6.1. Iteration steps on each level for the algorithms under comparison

We first present the numerical results for algorithm 3.1 and **HBMG**. We refer to **LMG-Jacobi** as algorithm 3.1 with local Jacobi smoothing ($\gamma = 0.8$), to **LMG-GS** as algorithm 3.1 with local Gauss-Seidel smoothing and to **HBMG-GS** as the hierarchical basis multigrid method with local Gauss-Seidel smoothing. Table 6.1, Figure 6.2, and Figure 6.3 show that the number of iterations and the convergence rate, i.e., the reduction factor $\|I - B_J A_J\|_a$, of algorithm 3.1 with local Jacobi or local Gauss-Seidel smoothing, per level are all bounded independently of mesh sizes and mesh levels, which confirms our theoretical results. For **HBMG-GS**, we observe that the number of iterations depends on the mesh levels. Figures 6.2-6.4 also show that the CPU time (in seconds) of each iteration of **LMG** and **HBMG** is linear with respect to the DOFs.

Next, we study the performance of algorithm 3.2 and the hierarchical basis preconditioning method **(HBP)**. As can be seen from Table 6.2, the number of iterations by CG without preconditioning increases fast with the mesh levels. However, for PCG by LMAA with local Jacobi smoothing **(LMAA-Jacobi)**, the iteration steps per level are both independent of mesh sizes and mesh levels. Similar to **HBMG-GS**, for PCG by **HBP**, the iteration steps depend on the mesh levels. Figures 6.5 and 6.6 show that for these two algorithms the CPU time (in seconds) of each iteration also is in accordance with the theoretical analysis.

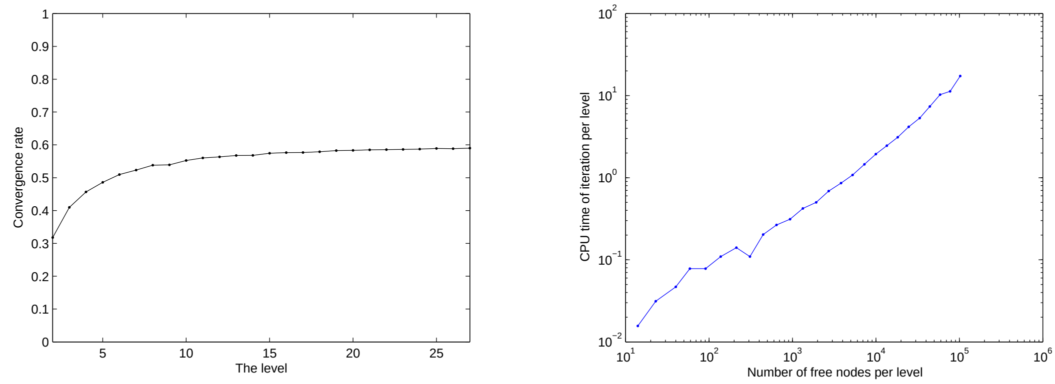


FIG 6.2. *Reduction factor (left) and CPU time (right) per level for* **LMG-Jacobi**.

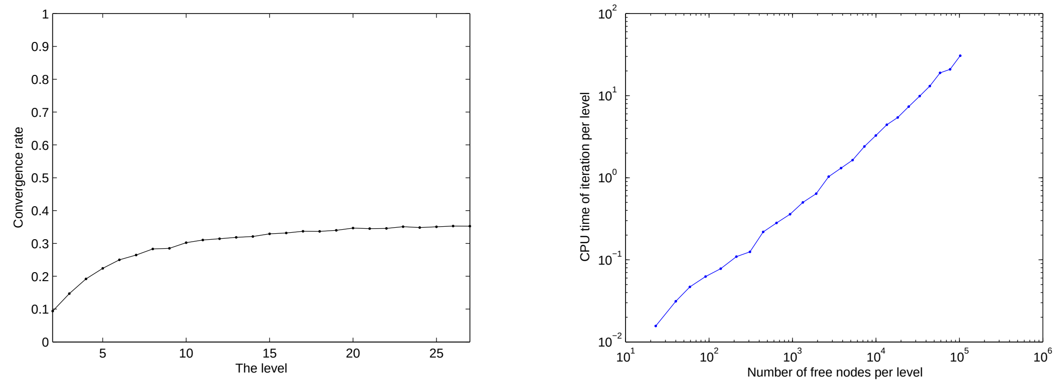


FIG 6.3. *Reduction factor (left) and CPU time (right) per level for* **LMG-GS**.

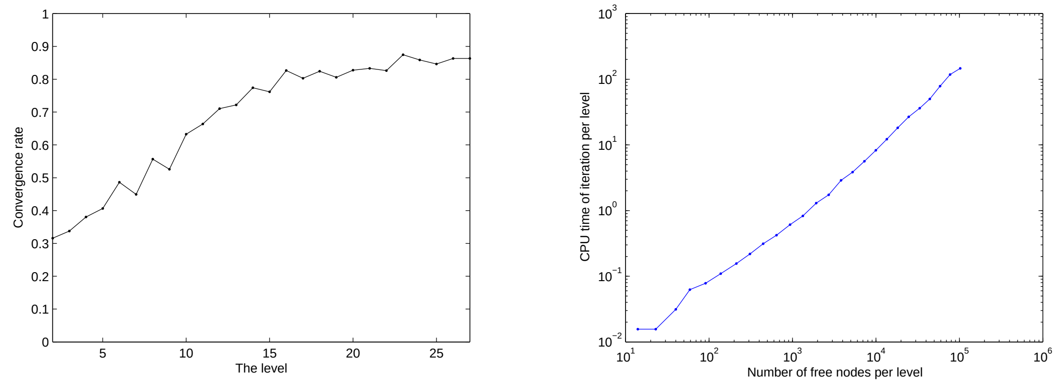


FIG 6.4. *Reduction factor $\|I - B_J A_J\|_a$ (left) and CPU time (right) per level for* **HBMG** *with local Gauss-Seidel smoothing. Here, B_J and A_J are derived based on the hierarchical basis.*

EXAMPLE 6.2. Consider Poisson's equation

$$-\Delta u = 1 \qquad \text{in } \Omega,$$

with Dirichlet boundary conditions on the slit domain $\Omega = \{(x,y) : |x|+|y| \leq 1\}\backslash\{(x,y) : 0 \leq x \leq 1, y = 0\}$. The exact solution (in polar coordinates) is $r^{1/2}\sin(\theta/2) - \frac{1}{4}r^2$.

Figure 6.7 displays the locally refined mesh with 1635 nodes after 15 refinement steps. Table 6.3 and Figures 6.8, 6.9 show that the linear increase in CPU time and the convergence rate, i.e.,

Level	DOFs	CG	LMAA-Jacobi	HBP
16	3819	185	38	61
18	7285	257	39	66
20	13524	337	40	71
22	24765	446	41	78
24	44284	576	42	79
26	77431	691	40	85
28	134344	965	41	90
30	224771	1220	41	94
32	376628	1415	41	95
34	625557	1739	41	98

Table 6.2. Iteration steps on each level for the algorithms under comparison

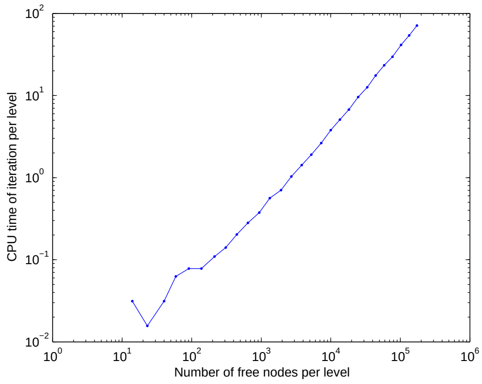


FIG 6.5. *CPU time for LMAA-Jacobi*

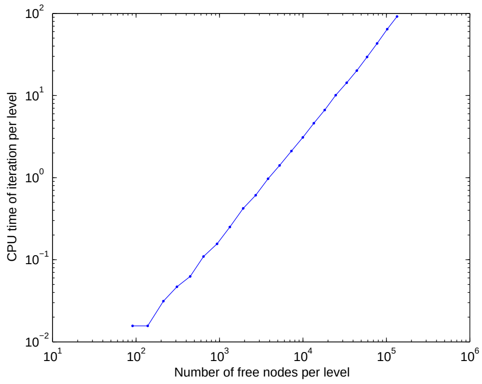


FIG 6.6. *CPU time for HBP*

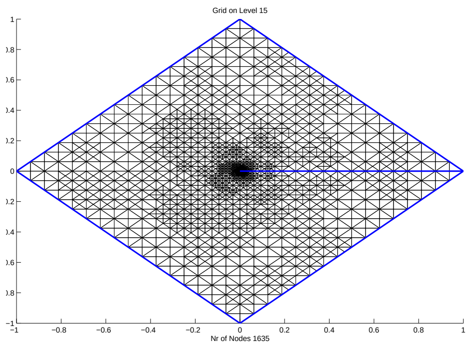


FIG 6.7. *The locally refined mesh after 15 refinement steps (Example 6.2)*

the reduction factor $\|I - B_J A_J\|_a$, are bounded independently of mesh sizes and mesh levels, which indicates the optimality of algorithm **LMG-Jacobi** ($\gamma = 0.8$) and **LMG-GS**.

Level	DOFs	LMG-Jacobi	LMG-GS	HBMG-GS
19	6115	40	20	57
21	11713	39	20	63
25	41460	38	19	76
27	74727	36	18	84
29	144648	36	17	93
31	250576	35	17	99
34	619187	34	17	107
35	795755	34	17	107
36	1075195	32	16	115

Table 6.3. Iteration steps on each level for the algorithms under comparison

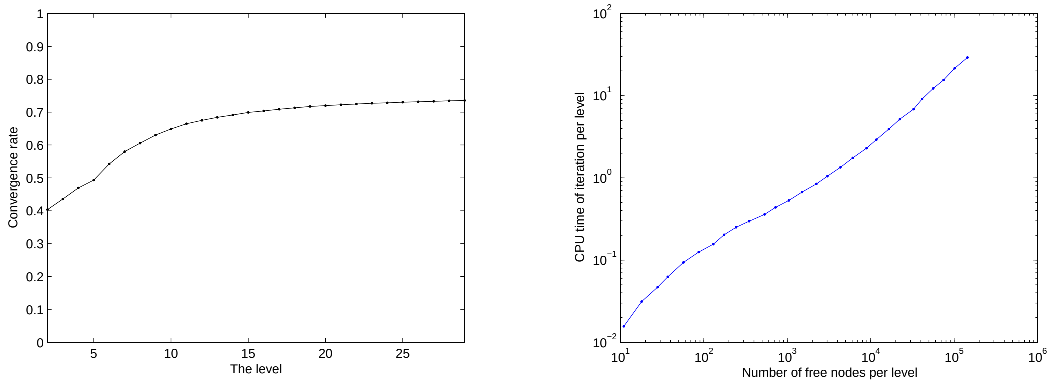


FIG 6.8. *Reduction factor and CPU time per level for* **LMG-Jacobi**.

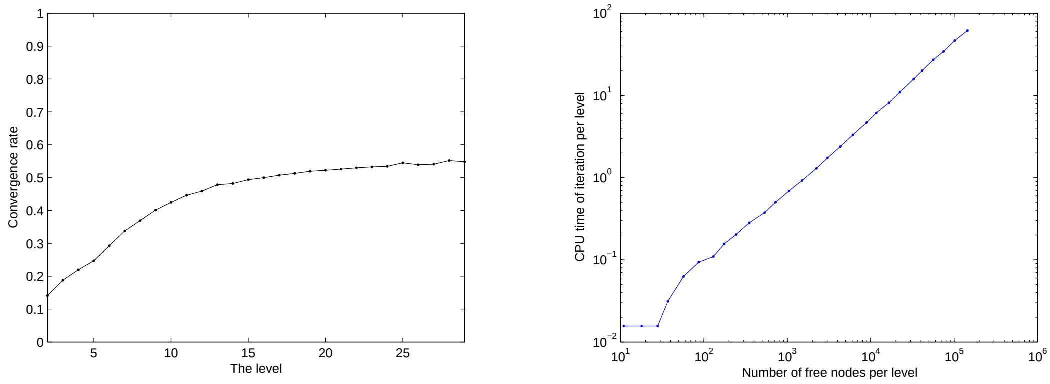


FIG 6.9. *Reduction factor and CPU time per level for* **LMG-GS**.

For algorithm 3.2, Table 6.4 and Figure 6.11 show the optimality of algorithm **LMAA-Jacobi** ($\gamma = 0.8$), whereas Table 6.3, Figure 6.10 and Table 6.4, Figure 6.12 show that the convergence of **HBMG-GS** and PCG by **HBP** also depends on the mesh levels. For these two algorithms, the CPU time (in seconds) of each iteration is almost linear with respect to the DOFs.

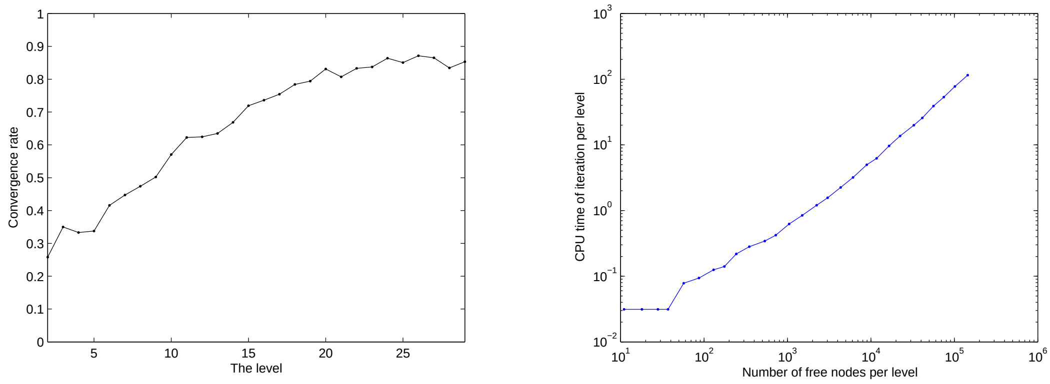


FIG 6.10. *Reduction factor and CPU time per level for* **HBMG-GS**.

Level	DOFs	CG	LMAA-Jacobi	HBP
16	2240	117	39	55
18	4338	195	43	61
20	8936	269	45	67
22	16500	325	47	72
24	32733	483	49	76
26	56161	618	50	82
28	101565	827	51	85
30	184049	1068	50	90
32	327094	1333	52	95
34	619187	1689	53	98

Table 6.4. Iteration steps on each level for the algorithms under comparison

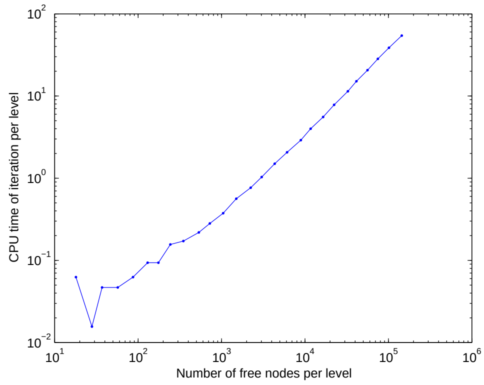


FIG 6.11. *CPU time for LMAA-Jacobi*

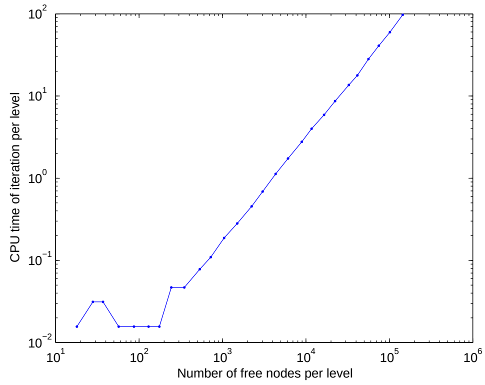


FIG 6.12. *CPU time for HBP*

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