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Understanding and Basic Mental Models of the concept of derivative

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An empirical study with 501 university students of mathematics, mainly at the beginning of their studies, investigated the relation between the understanding—concentrated in basic mental models (BMMs)—of the derivative and the capability of solving derivative problems. The results show that the BMM of tangent slope is of central importance, it is much less for the local rate of change, while the local linearization plays no role. The study emphasizes the meaning of BMMs for successfully solving problems related to the concept of derivative.

Keywords: Basic Mental Model, concept understanding, concept image, derivative, tangent slope, rate of change, local linearity.

The understanding of the concept of the derivative is a central goal of the teaching of calculus. In the following, we refer to Basic Mental Models (BMMs) as a core part of the "concept image" and the framework of Zandieh (2000) concerning this understanding of this concept.

Concept definition – concept image

The relation *concept image – concept definition* has been studied in many different contexts since the early 1980s (Tall a. Vinner, 1981). "Concept definition (is) a form of words used to specify that concept" and "Concept image describes the total cognitive structure that is associated with the concept […]. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (Tall & Vinner, 1981, p. 152).

This relation has been studied für many different concepts, for example, numbers, functions or limits. A problem concerning this framework is that the *concept definition* is very narrow, mainly concentrated on the definitions(s) of the concept, while the *concept image* is a too extensive concept. This calls for a possibility to have a concentrated core of a concept concerning the *concept image*.

Basic Mental Models of the concept of derivative

The concept of Basic Mental Model (BMM, in German: Grundvorstellung) has been well established in German-speaking didactics of mathematics for many years (vom Hofe & Blum, 2016). A BMM of a mathematical concept is a content-related interpretation that gives meaning to this concept, providing relations to meaningful contexts (see Greefrath et al., 2016). BMMs of mathematical concepts can be considered within the theoretical framework of *Concept Image–Concept Definition* (Tall & Vinner, 1981). They are parts or subsets of the *Concept Image* of a mathematical concept. While *Concept Image* refers to all individual mental images identified with the concept, BMMs are the *core* or *central components* of these images. *Normative BMMs* are derived from didactic analysis of mathematical concepts, *individual BMMs* are specific BMMs in a student's mind.

Based on didactical analyses, Greefrath et al. (2023) identified four BMMs of the first derivative of a function at a point (Table 2) and validated them in a comprehensive empirical study.

Table 2: Basic Mental Models of the first derivative of a function at a point (Greefrath et al., 2016)

BMM of the local rate of change (RC)	A varying quantity is considered. The derivative is the local rate of change of this quantity. If the quantity changes with time, the derivative has the meaning of an instantaneous velocity or speed.
BMM of the tangent slope (TS)	The graph of a function in a coordinate system is considered. The derivative is the slope of the tangent to the graph at a point. The development of this BMM is closely related to the development of the concept of a tangent to a graph.
BMM of the local linearity (LL)	If you "zoom in" more and more on the graph of a function at a point of interest, the graph locally appears more and more as a straight line. The derivative is the slope of this straight line. Software for dynamic mathematics with the possibility of zooming in on the screen is particularly useful for developing this BMM.
BMM of the amplification factor (AF)	A quantity y is dependent on another quantity x. The derivative indicates how strongly small changes in the quantity x affect the dependent quantity y. ($\Delta y \approx m \cdot \Delta x$)

Understanding the concept of derivative

BMMs are an important element for understanding and applying a concept. However, understanding a concept is much more than knowing one or some BMMs of the concept. There are many ways to describe the concept of *understanding*. Skemp (1976) distinguishes between instrumental and relational understanding. Another distinction is that in *conceptual knowledge*—about the properties of a concept, relationships between different properties and relationships to other concepts—and *procedural knowledge* – knowing how to solve equations, how to calculate limits or derivatives of functions (Hurrel, 2021). Zandieh (2000) proposed a comprehensive theoretical framework for analysing students' understanding of the concept of derivative. She distinguishes *multiple representations of the derivative* on the one hand, and on the other she considers the *derivative as a ratio, a limit and a function* (Table 1). The framework is developed for school mathematics, however, it might also be used for the university level (Feudel & Biehler, 2021). There are also some follow-up developments concerning this framework (for example, Radmehr & Turgut, 2024)

Table 1: Zandieh's framework for understanding the derivative

	Graphical	**Verbal**	**Symbolic**	**Physical**
Ratio	Slope of the secant line	Average rate of change	Difference quotient	Average velocity
Limit	Slope of the tangent line or slope of the curve under magnification	Instantaneous rate of change	Limit of the difference quotient	Instantaneous velocity
Function	Graph of the derivative function	Instantaneous rate of change as a function	Derivative as a function	Velocity as a function of time

Each column of this framework can be seen as an element of the concept image or/and of a BMM of the concept of derivative. We are interested in the understanding of the concept of derivative in a point of a function and in the relation between the function and its derivative function. This means, we refer to the limit and the function row of Table 1. The BMMs of the derivative are a part of the concept image and specify it in the sense of a core image or core perception.

Research questions

Our hypothesis is that BMMs are helpful–and maybe essential–for solving problems. In the following, firstly, we are interested how students solve mathematical derivative problems. Secondly, we want to quantify the relation between students' capability to solving problems and their BMMs. We are interested in the following research questions:

1. To what extend are students able to solve derivative problems?
2. What is the relation between students' capability to solve derivative problems and their BMMs?

Methods

A test with two groups of problems was developed. In the capability–test, mathematics students had to answer questions concerning the concept of derivative. In the following BMM–test, the students were classified with regard to their individual BMMs. The test is designed for university students of mathematics at the beginning of their teacher training courses or undergraduate courses in scientific disciplines like computer science, physics, chemistry or biology. A total of 501 students took part in the study. They were recruited in mathematics preparatory courses of teacher training programs (all school types) at four different German universities.

Problem Group 1: Capability–Test

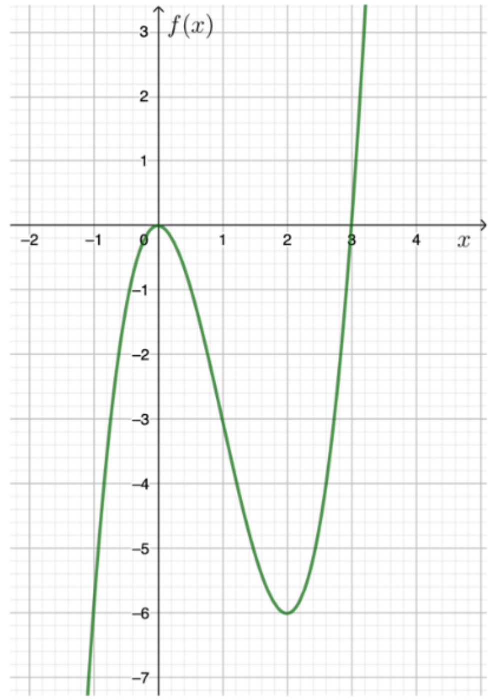
Seven tasks were developed that relate to the first derivative of a real function at a specific point or in an interval of the domain of definition or to the entire derivative function. According to the classification scheme by Zandieh (2000) (cf. Table 1), the tasks refer to different representations – *Graphical*: relationship between graph and first derivative; *Verbal*: verbal descriptions of properties of the derivative; *Symbolic*: interpretation of a statement on the symbolic level; *Physical*: interpretation of an environmental situation.

Two types of tasks were used. For type 1 (tasks 1 to 4), three suggested solutions were given for each task, for which the student had to decide whether the statement given in each case was correct or incorrect.

Problem 1.1

The adjacent figure shows the graph of a real polynomial function f of third degree. Mark whether the following statements regarding the derivative function f' are true or false.

	true	false
All the values of the function f' are negative in the interval]0; 2[.		
The function f' is strictly monotonically increasing in the interval]0; 2[.		
The function f' has a local minimum at the position $x = 0$.		



Problem 1.2

The function values $f(t)$ give the outside temperature measured at time t. Mark which interpretation of the expression

$$\lim_{t \to t_o} \frac{f(t) - f(t_o)}{t - t_o}$$

is correct or incorrect.

	true	false
The expression indicates how the temperature changes between t_o and t.		
The expression indicates how the temperature changes at the time t_o.		
The expression indicates how high the temperature is at the time t_o.		

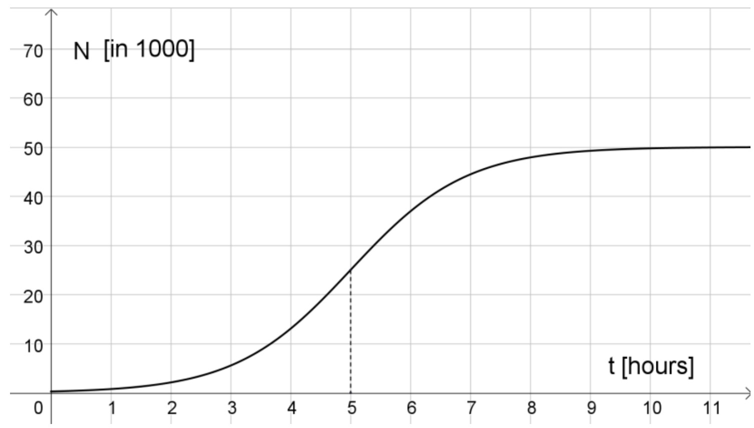
To reduce the guessing probability, a problem is only considered as correct if all subtasks are correct, otherwise, it is wrong (guessing probability 12.5 %).

Problem Group 2: BMM-Test

In Greefrath et al. (2021, 2023), a test was developed and tested to assess BMMs for university students of mathematics. The basic idea of this test is to present the students problems with argumentations that suggest certain BMMs. For each item, students are asked to mark on a five-point Likert scale from (– –) to (+ +) to indicate the extent to which the answer corresponds to their own thinking. The complete test consists of 13 tasks, eight items were selected for the test used here. Moreover, we skipped the BMM AF because it is not yet emphasized in high school and is useful only for quite limited kinds of problems. Students should be able to work on tasks with different BMMs and looking at problems from different perspectives. Therefore, test contexts were chosen that are conceivable with all BMMs. An example of the BMM–test is the following item:

Problem 2.1

The figure below shows the number of bacteria N in a nutrient solution as a function of the time t (in hours).



The special meaning of the time t = 5 can be explained in different ways.

	– –	–	o	+	+ +
If you zoom in on the graph, it appears like a straight line at every point. This has the greatest gradient for t = 5.					
The slope of the tangent to the graph is greatest at t = 5.					
The number of bacteria changes most rapidly at the time t = 5.					

Evaluation method

Homogeneity was checked with Cronbach's alpha. Correlations of sum scores were measured with Pearson's correlation coefficient. Linear regression was used in the analysis of dependencies. All calculations were carried out with the statistical program R.

Results

Capability of the Derivative

The mean values and standard deviations of the capability-test are shown in Table 3. Cronbach's α for problem 2 to 4 indicate good homogeneity, whereas these for problem 1 are more heterogeneous. For problems 5 to 7 exactly one answer is correct, determining Cronbach's alpha is not possible.

Tab. 3: Means and standard deviations of the capability–test

	Mean value	**Standard deviation**	**Cronbach α**
Task 1	0.24	0.43	0.26
Task 2	0.61	0.49	0.62
Task 3	0.28	0.45	0.60
Task 4	0.61	0.49	0.69
Task 5	0.62	0.49	–

Task 6	0.66	0.48	–
Task 7	0.84	0.37	–
Average	0.55	0.24	0.62

Overall, the capability to solve problems is not unidimensional. The value for Cronbach's alpha is 0.62. Especially the problems with a given graph (2, 4, 5, 7) are solved better on average than the others.

Basic Mental Models

Table 4 shows the mean values and standard deviations as well as Cronbach's alpha of the sum scores for TS, RC and LL. We confirmed the results in Greefrath et al. (2023), that a three-dimensional model for the BMMs fits quite well.

Table 4: Mean and standard deviations for the TS, RC and LL

	TS	RC	LL
Mean value	0.81	0.62	0.57
Standard deviation	0.20	0.21	0.22
Cronbach's alpha	0.86	0.79	0.82

Relation between Problem-solving capability and BMMs

All correlations – except the correlation of capability and LL – are (even highly) significantly different from 0. This shows that TS and RC are associated with capability of solving derivative problems, whereas it is weak for LL (see Table 5).

Table 5: Correlation matrix

	Problem-solving capability	TS	RC	LL
Problem-solving capability	1.00	0.32	0.17	0.05
TS	0.32**	1.00	0.29**	0.30**
RC	0.17**	0.29**	1.00	0.31**
LL	0.05	0.30**	0.31**	1.00

Discussion

The Capability-test covers many aspects of the two higher levels Limit and Function of Zandieh's (2000) framework, as well as all graphical, verbal, symbolic and physical representations. Furthermore, the test shows different levels of difficulty – mean values of 0.24 to 0.84 – with the items in the first column of the framework having higher solution rates than the items without graphical representation. In the test, more procedural aspects of knowledge such as understanding the derivation rules were not taken into account.

In the students' BMMs, the tangent slope is most prominent, while rate of change and local linearization occur much less frequently. Particularly in view of the stronger emphasis on modelling skills in school analysis (in curricula and textbooks, less in actual teaching), one might have expected a stronger emphasis on the local rate of change. The basic idea of local linearity plays a rather subordinate role. Presumably, typical learning processes in mathematics lessons play a central role here, in which the tangent concept is supported above all by the predominance of intra-mathematical problems. The repeatedly demanded view of the derivative as a local rate of change is probably not yet sufficiently developed due to too few modelling examples.

The ranking of the basic concepts tangent slope, local rate of change and local linearization does not change if only students with high or low problem-solving capabilities are considered. In the sub-group of students with lower problem solving abilities, the preference for the BMM tangent slope is much lower than in the high achieving group. but the other BMMs are even lower. Hence, even in this subgroup the BMM tangent slope is the best predictor of at least limited success. However, it can be assumed that BMMs are useful for increasing the capability to solve derivative problems and consequently the understanding of the concept of derivative (according to vom Hofe and Blum, 2016).

Furthermore, the results of a linear regression show that the tangent slope can predict the ability to solve derivative problems with a relatively high weight (again, especially for the items where a graph is given), this is the case for the rate of change to a lesser but also significant extent, while the concept of local linearization does not play any role.

Limitations

In terms of subject selection, this test clearly selects students by limiting them to mathematics students. However, this at least allows the assumption that the capability test would be worse with a non-subject-specific test group, for example, students from social studies.

Furthermore, it must be taken into account that the capability-test only consists of 7 questions and the BMM-test is based on self-reporting by the students. Developing a test which determines the "actual BMMs", is a research problem for the future.

Conclusion

The result of this study shows that the BMM of tangent slope is of highest priority for the solving of derivative problems. This justifies the special importance of the graphical representation, which is also prominently emphasized in Zandieh's framework. The rather low correlations between BMMs and the problem-solving capabilities shows that these are different categories, but they are not independent, as highly significant correlations exist. It still needs to be clarified what significance the different BMMs have for certain content areas and problems. This can provide indications how and when BMMs should be developed in school mathematics. Bednorz et al. (2023) have proposed a learning path for this purpose. Based on the results of this study, it can be assumed that the BMMs RC and LL are more difficult to teach and require well thought-out special strategies. However, the local rate of change is important in the school context, especially in relation to application tasks, and local linearization is important on university mathematics level. This means that the development of all BMMs is important for the broad and global understanding of a concept, as they show different perspectives and offer different ways of using and working with the concept.

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