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OPTIMAL CONTROL OF ELECTRORHEOLOGICAL CLUTCH DESCRIBED BY NONLINEAR PARABOLIC EQUATION WITH NONLOCAL BOUNDARY CONDITIONS.

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Abstract. The operation of the electrorheological clutch is simulated by a nonlinear parabolic equation which describes the motion of electrorheological fluid in the gap between the driving and driven rotors. In this case, the velocity of the driving rotor is prescribed on one part of the boundary. Nonlocal nonlinear boundary condition is given on a part of the boundary, which corresponds to the driven rotor [25]. A problem on optimal control of acceleration or braking of the driven rotor is formulated and studied. Functions of time of the angular velocity of the driving rotor and of the voltages are considered to be controls. In the case that the clutch acts as an accelerator, the energy consumed in the acceleration of the driven rotor is minimized under the restriction that at some instant, the angular velocity and the acceleration of the driven rotor are localized within given regions. In the case of braking, the energy production is maximized. The existence of a solution of optimal control problem is proved and necessary optimality conditions are established.

Key words. Electrorheological fluid, parabolic equation, nonlocal boundary condition, existence, optimality condition.

1. Introduction

Electrorheological fluids are smart materials that are composed of small polarizable particles dispersed in nonconducting dielectric liquids. With an applied electric field, the dielectric mismatch creates polarization forces that cause the particles to form chains aligned with the electric field. Because of this, the fluid becomes anisotropic. The apparent viscosity (the resistance to flow) in the direction orthogonal to the direction of electric field abruptly increases. It can increase by several orders of magnitude for electric fields of the order of 1 $kVmm^{-1}$ [27]. The apparent viscosity in the direction of the electric field changes not so drastically [30]. These effects are both rapid and reversible. Due to their remarkable properties, electrorheological fluids have various applications in electromechanical devices such as clutches, shock absorber, valves and others [6]. Electrorheological clutches are used in automotive industry, in robotic devices, industrial forklifts, medicine, etc. They provide smooth controlled acceleration or braking, create needed angular velocity and needed resistance to rotation, and so on.

A constitutive equation of electrorheological fluids which describes the main peculiarities and among them the anisotropy of these fluids was developed in [11]. Different problems on flow of electrorheological fluids and close to them were studied in [7], [8], [11], [12], [20], [23], [24]. Some models of cylindrical and radial electrorheological clutches were developed

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and considered in [9], [33], [34]. In [25] a problem on dynamics of electrorheological clutch is formulated and studied.

Below we formulate and study a problem on optimal control of electrorheological clutch by using the model of clutch developed in [25].

In Section 2, we present governing equations and the model of electrorheological clutch. The direct problem for the clutch reduces to finding the velocity function of the electrorheological fluid, which is the solution of the nonlinear parabolic equation that satisfies nonlocal nonlinear boundary condition on the surface of the driven rotor and the Dirichlet condition on the other part of the boundary.

The nonlocal boundary condition contains an integral over the surface of the driven rotor and time of the tangential component of the surface force that acts on the driven rotor. The surface force is a nonlinear function of the derivatives of the velocity function. In order for the function of surface force to be integrable, the velocity function should be smooth, and hence, only a smooth velocity function can be the solution of the direct problem.

In Section 3, we present special functional spaces that are used in the paper. The theorem on the existence and the uniqueness of the solution of the direct problem is contained in Section 4.

In Section 5, we formulate optimal control problem for the clutch. The functions of time of the voltages applied to electrodes and of the angular velocity of the driving rotor are considered to be controls. In this case, the coefficients of the parabolic equation and the nonlocal operator of boundary condition depend on the control. The energy consumed in the acceleration (braking) of the driven rotor is minimized (maximized) under the restriction that at an instant T the angular velocity and the acceleration of the driven rotor to be in given regions. In addition, restrictions on values of the controls at all instants of time and on norms of the controls are given.

We assume that the admissible controls are smooth, the functions of voltages and angular velocity of the driven rotor are elements of the spaces $H^1(0,T)$ and $H^2(0,T)$, respectively. The reason is that, on the one hand, the control should be sufficiently smooth for the solvability of the direct problem, on the other hand, smooth controls produce smooth acceleration and braking without pushes and strokes. This is in general agreement with the purpose of the electrorheological clutch, and, in general, only smooth controls are employed in actual practice.

In Sections 6 and 7, we prove that the solution of the direct problem and the goal functional and the functional of entry into the given region at the instant T are continuously Fréchet differentiable with respect to the controls. The derivatives of the functional are calculated by the use of the conjugate state which is the solution of the conjugate problem.

The conjugate state is defined by the method of transposition. Since the solution of the direct problem is smooth, the conjugate state belongs to the space of product of negative Banach spaces. The Petrov-Galerkin method is considered for numerical solution of the conjugate problem. We prove that approximate solutions obtained by the Petrov-Galerkin method converge to the exact solution of the conjugate problem.

Existence result for optimal control problem and necessary optimality conditions are established in Section 8. In Section 9, we consider optimal control problem with point-finite restrictions on the values of the controls. The special case that the clutch is cylindrical is briefly reviewed in Section 10.

2. Model of the electrorheological clutch.

2.1. **Outline of the clutch and governing equations for the fluid.** A scheme of the electrorheological clutch of the firm Bayer[4] is presented in Figure 1. The clutch consists of the input and output rotors which are the driving and the driven rotors. The driving rotor is a shaft with disks, and the driven rotor is a shell with disks. An electrorheological fluids is sandwiched between the driving and the driven rotors.

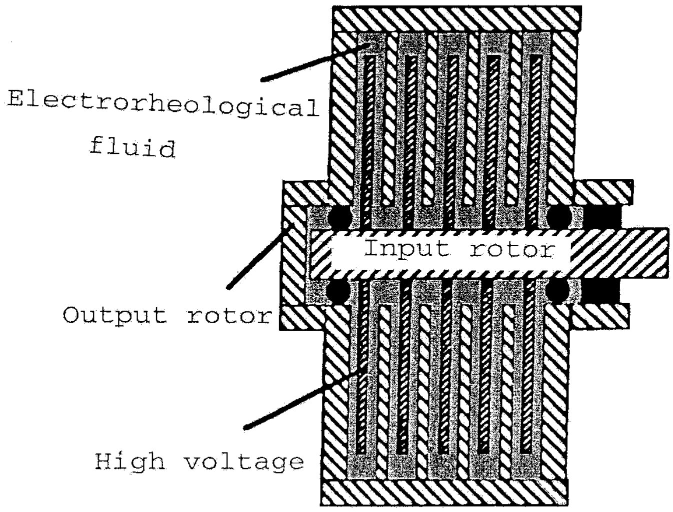

FIGURE 1. Schematic representation of the electrorheological clutch of the firm Bayer.

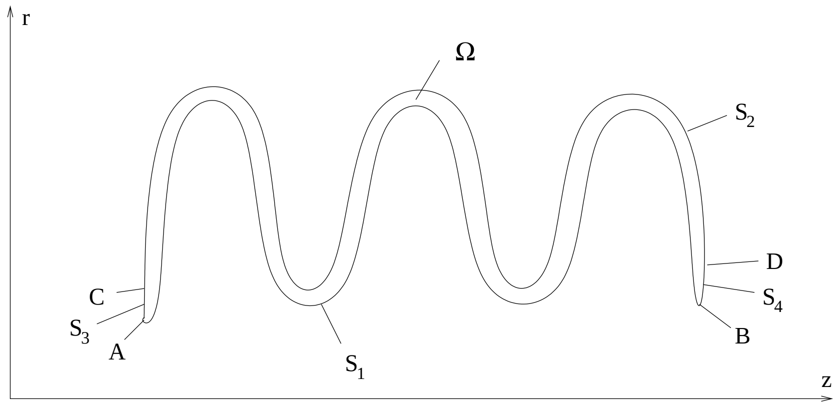

FIGURE 2. Domain of flow of the electrorheological fluid in cylindrical coordinate system.

A voltage is applied to the surfaces of the disks of the driving rotor which serve as electrodes, whereas the surfaces of the disks of the driven rotor act as counter electrodes. By varying the voltage, one varies the viscosity of the electrorheological fluid and the torque acting on the driven rotor.

We use the cylindrical coordinates system r, α, z. It is assumed that the flow of the fluid is axially symmetric, and in line with the scheme of the electrorheological clutch presented in Figure 1, we consider a domain of flow Ω of the form shown in Figure 2 in the cylindrical coordinates r, z. Here the boundary S of the domain Ω consists of four parts: the part S_1 corresponds to the surface of the driving rotor, the part S_2 corresponds to the surface of the driven rotor, the parts S_3 and S_4 unite the boundaries S_1 and S_2. The points A and B are the ends of S_1 and also the points C and D are the ends of S_2.

Since the gap between the driving and driven rotors is small, it is assumed that in the mobile orthonormal basis e_r, e_α, e_z of the cylindrical coordinate system r, α, z, the velocity vector of the fluid u has the following form $u(r,z,t) = (0, u_\alpha(r,z,t), 0)$ (t is a time variable), i.e. only peripheral component of the velocity is nonvanishing. We denote $u = u_\alpha$.

The viscosity function is defined by (see [25])

$$\varphi(I(u), |E|^2) = b_0(|E|^2)(\lambda + I(u))^{-\frac{1}{2}} + \psi_0(I(u), |E|^2). \tag{2.1}$$

Here b_0 and ψ_0 are functions of corresponding arguments, $|E|$ is the module of the vector of electric field strength, λ a small positive parameter, $I(u)$ the second invariant of the rate of strain tensor

$$I(u) = \frac{1}{2}\Big(\frac{\partial u}{\partial r} - \frac{u}{r}\Big)^2 + \frac{1}{2}\Big(\frac{\partial u}{\partial z}\Big)^2. \tag{2.2}$$

The motion equation have the following form (see [25])

$$\frac{\partial}{\partial r}\Big(\varphi\Big(\frac{\partial u}{\partial r} - \frac{u}{r}\Big)\Big) + \frac{\partial}{\partial z}\Big(\varphi\frac{\partial u}{\partial z}\Big) + \frac{2}{r}\varphi\Big(\frac{\partial u}{\partial r} - \frac{u}{r}\Big) = \rho\,\frac{\partial u}{\partial t} \qquad \text{in} \quad Q = \Omega \times (O,T), \tag{2.3}$$

where ρ is the density a positive constant and $T < \infty$.

We denote by $R_1(s)$, $R_2(s)$ the first components of the coordinates (r,z) of the points s of S_1 and S_2, i.e. $R_1(s) = r(s)$, $s \in S_1$, $R_2(s) = r(s)$, $s \in S_2$.

We use the notations: $S_{iT} = S_i \times (0,T)$, $i = 1,2,3,4$, $S_T = S \times (0,T)$, ω and ω_1 are the angular velocities of the driving and the driven rotors, respectively. The angular velocity of the driving rotor ω is assumed to be assigned, that is, the velocity of the fluid on S_{1T} is given by

$$u\big|_{S_{1T}} = \omega R_1. \tag{2.4}$$

The tangential α component of the surface force acting on the driven rotor creates the torque (the rotation moment), that, in its turn, induces the angular acceleration of the driven rotor. The angular velocity is defined by the integration of the angular acceleration over time.

In that way, there arises the following nonlocal boundary condition on the surface of the driven rotor, see [25]:

$$u(s,t) = \omega_1(t)R_2(s) = R_2(s)\{\omega_1(0) - (\rho_0 I_0)^{-1}\int_0^t \Big[\int_{S_2}\varphi\Big(\Big(\frac{\partial u}{\partial r} - \frac{u}{r}\Big)\nu_1 + \frac{\partial u}{\partial z}\nu_3\Big)R_2\,ds + M_{ex}\Big]\,d\tau\}, \quad (s,t) \in S_{2T}. \tag{2.5}$$

Here the parameter t in the integrand is denoted by τ, ν_1 and ν_3 are radial and axial components of the unit outward normal ν to the boundary S of Ω, I_0 and ρ_0 the axial moment of inertia and the density of the driven rotor, and M_{ex} is the moment of an external load. The integrand in (2.5) is the α component of the surface force that acts on the driven rotor.

The boundary conditions on S_{3T} and S_{4T} are defined as follows:

$$u(s,t) = P(\omega(t)R_1, \omega_1(t)R_2)(s), \qquad s \in \overline{S}_3 \bigcup \overline{S}_4, \qquad t \in (0,T), \tag{2.6}$$

where P is an operator of extension from $S_1 \bigcup S_2$ to S.

The initial condition is given by

$$u(r,z,0)=u_0(r,z), \qquad r,z\in\Omega. \tag{2.7}$$

In this case the following conditions of concordance are assumed to be satisfied

$$\begin{aligned} u_0\big|_{S_1} &= \omega(0)R_1, \qquad u_0\big|_{S_2}=\omega_1(0)R_2. \\ u_0\Big|_{S_3\bigcup S_4} &= P(\omega(0)R_1,\omega_1(0)R_2)\Big|_{S_3\bigcup S_4}. \end{aligned} \tag{2.8}$$

By virtue of (2.8), the value $\omega(0)$ and $\omega_1(0)$ are considered to be given.

2.2. **Problem for electric field.** We consider Maxwell's equations in the following form (see e.g. [18]):

$$\begin{aligned} \operatorname{curl} E+\frac{1}{c}\frac{\partial B}{\partial t} &= 0, \qquad \operatorname{div} B=0, \\ \operatorname{curl} H-\frac{1}{c}\frac{\partial D}{\partial t} &= 0, \qquad \operatorname{div} D=0. \end{aligned} \tag{2.9}$$

Here E is the electric field, B the magnetic induction, D the electric displacement, H the magnetic field, c the speed of light. One can assume that

$$D=\epsilon E, \qquad B=\mu H, \tag{2.10}$$

where ϵ is the dielectric permittivity, μ the magnetic permeability.

Since electrorheological fluids are dielectrics the magnetic field H can be neglected. Then (2.9), (2.10) give the following relations:

$$\operatorname{curl} E=0, \tag{2.11}$$

$$\operatorname{div}(\epsilon E)=0. \tag{2.12}$$

It follows from (2.11) that there exists a function of potential θ such that

$$E=-\operatorname{grad}\theta, \tag{2.13}$$

and (2.12) implies

$$\operatorname{div}(\epsilon \operatorname{grad}\theta)=0 \quad \text{in} \quad \Omega. \tag{2.14}$$

In our case $\operatorname{grad}\theta=\left(\frac{\partial\theta}{\partial r},\frac{\partial\theta}{\partial z}\right)$ and equation (2.14) takes the form

$$\frac{\partial}{\partial r}\Big(\epsilon\frac{\partial\theta}{\partial r}\Big)+\frac{\epsilon}{r}\frac{\partial\theta}{\partial r}+\frac{\partial}{\partial z}\Big(\epsilon\frac{\partial\theta}{\partial z}\Big)=0 \quad \text{in} \quad \Omega. \tag{2.15}$$

We consider θ as a function of r, z and t, t is a parameter in the equation (2.15). Put the following boundary conditions.

$$\theta(s,t)=\begin{cases} l(t), \quad s\in S_1, \quad t\in(0,T), \\ 0, \quad s\in S_2, \quad t\in(0,T), \\ P(l(t),0)(s), \quad s\in\overline{S_3}\bigcup\overline{S_4}, \quad t\in(0,T). \end{cases} \tag{2.16}$$

Here $l(t)$ is the voltage applied to the electrode that is housed on the surface of the driving rotor S_1, and P is the operator of extension from $S_1\bigcup S_2$ to S that is used in (2.6).

The dielectric permittivity ϵ is assumed to be a positive constant. Thus, the problem for electric field reduces to the solution of the elliptic problem (2.15), (2.16) at each instant of time t.

3. Some functional spaces.

We assume that the domain of flow Ω satisfies the following condition:

(A0): Ω is a bounded simply connected domain in the plane (r,z) with a boundary S of the class C^2, in addition, $S=\bigcup_{i=1}^4 \overline{S}_i$, where S_i are open nonempty subsets of S such that $S_i \bigcap S_j=\emptyset$ at $i\neq j$. Moreover, the intersection of $\overline{\Omega}$ and the axis z is an empty set.

For $m\in\mathbb{N}$ and $q\geq 1$, we denote by $H_q^m(\Omega)$ the Sobolev space for the cylindrical coordinate r, z with the following norm:

$$\|h\|_{H_q^m(\Omega)}=\Big(\int_\Omega \sum_{k_1+k_2\leq m}\Big|\frac{\partial^{k_1+k_2}}{\partial r^{k_1}\partial z^{k_2}}\,h\Big|^q\,drdz\Big)^{\frac{1}{q}}. \tag{3.1}$$

Let Λ be the transformation of Ω with the change of coordinates r, α, z for the Cartesian coordinates x_1, x_2, x_3.

Since the set $\overline{\Omega}$ and the axis z are separated, the norm (3.1) is equivalent to the norm of Sobolev space $W_q^m(\Lambda)^2$ for axially symmetric in Λ functions, see [22], [25].

The notation $H_q^{2-\frac{1}{q}}(S)$ denotes the space of traces of functions from $H_q^2(\Omega)$ on S. The space $H_q^{2-\frac{1}{q}}(S)$ consists of all functions from $L_q(S)$, whose derivatives of nonintegral order $2-\frac{1}{q}$ belong to $L_q(S)$, see e.g. [1]; $H_q^{2-\frac{1}{q}}(S_i)$ is the space of traces of functions from $H_q^2(\Omega)$ on S_i, $i=1,2,3,4$.

We define the norm in $H_q^{2-\frac{1}{q}}(S)$ as follows:

$$\|w\|_{H_q^{2-\frac{1}{q}}(S)}=\inf_{\substack{h\in H_q^2(\Omega)\\ h|_S=w}}\|h\|_{H_q^2(\Omega)}. \tag{3.2}$$

The norm in $H_q^{2-\frac{1}{q}}(S_i)$ is defined similarly.

Once $r,z,t\to h(r,z,t)$ is a function given in Q, we consider h as a function of t with values in a space of functions of r,z. We denote the space of linear continuous operators mapping a normed space Y into a normed space Z by $\mathcal{L}(Y,Z)$. The dual of Y space is denoted by Y^*, and by (f,h) the duality between Y^* and Y, where $f\in Y^*$ and $h\in Y$.

We will use the anisotropic Sobolev space $H_q^{2,1}(Q)$ with the following norm:

$$\|v\|_{H_q^{2,1}(Q)}=\Big(\int_0^T(\|v(t)\|^q_{H_q^2(\Omega)}\,dt+\int_Q\Big|\frac{dv}{dt}\Big|^q\,dr\,dz\,dt\Big)^{\frac{1}{q}}. \tag{3.3}$$

The next lemma on traces follows from the known results, see e.g. [17], Lemma 3.4, Chapter 2 and [32].

Lemma 3.1. *Suppose that the condition* (A0) *is satisfied. Then the following inequalities hold:*

$$\|v(t)\|_{H_q^{2-\frac{2}{q}}(\Omega)}\leq c_1\|v\|_{H_q^{2,1}(Q)},\qquad v\in H_q^{2,1}(Q),\qquad t\in[0,T], \tag{3.4}$$

$$\Big\|\frac{\partial v}{\partial r}\Big|_{S_T}\Big\|_{H_q^{1-\frac{1}{q},\frac{1}{2}-\frac{1}{2q}}(S_T)}\leq c_2\|v\|_{H_q^{2,1}(Q)},\qquad \Big\|\frac{\partial v}{\partial z}\Big|_{S_T}\Big\|_{H_q^{1-\frac{1}{q},\frac{1}{2}-\frac{1}{2q}}(S_T)}\leq c_3\|v\|_{H_q^{2,1}(Q)}, \tag{3.5}$$

$$\|v|_{S_T}\|_{H_q^{2-\frac{1}{q},1-\frac{1}{2q}}(S_T)}\leq c_4\|v\|_{H_q^{2,1}(Q)},\qquad v\in H_q^{2,1}(Q). \tag{3.6}$$

Notice that the norms presented in the left-hand side of (3.4)–(3.6) are defined by analogy with the norm of (3.2).

We assume that the operator of extension P being used in (2.6) and (2.8) satisfies the conditions

$$\begin{aligned} &P \in \mathcal{L}(H_q^{2-\frac{1}{q}}(S_1) \times H_q^{2-\frac{1}{q}}(S_2), H_q^{2-\frac{1}{q}}(S)),\\ &P(w_1, w_2)\Big|_{S_i} = w_i, \quad i = 1, 2; \quad q > 4. \end{aligned} \tag{3.7}$$

One can define the operator P so that the functions $P(w_1, w_2)\Big|_{S_i}$, $i = 3, 4$, are polynomials of s of the third degree and the function $P(w_1, w_2)$ and its derivatives at points A, C, B, D (see Figure 2) are continuous. One can also define the operator P such that the functions $P(w_1, w_2)\Big|_{S_i}$, $i = 3, 4$ are close to affine functions and the conditions (3.7) are satisfied.

We introduce the following vector space

$$\begin{aligned} U =& \{u | u \in H_q^{2,1}(Q), q > 4, u(s,t) = \mu_1(t)R_1(s), (s,t) \in S_{1T},\\ & u(s,t) = \mu_2(t)R_2(s), (s,t) \in S_{2T},\ u(s,t) = P(\mu_1(t)R_1, \mu_2(t)R_2)(s),\\ & s \in (\overline{S}_3 \bigcup \overline{S}_4), t \in (0,T),\ \mu_1 \in H_q^{1-\frac{1}{2q}}(0,T),\ \mu_2 \in H_q^{\frac{3}{2}-\frac{1}{2q}}(0,T)\}. \end{aligned} \tag{3.8}$$

The space U is equipped with the norm

$$\|u\|_U = \|u\|_{H_q^{2,1}(Q)} + \|R_1^{-1}u\Big|_{S_{1T}}\|_{H_q^{1-\frac{1}{2q}}(0,T)} + \|R_2^{-1}u\Big|_{S_{2T}}\|_{H_q^{\frac{3}{2}-\frac{1}{2q}}(0,T)}. \tag{3.9}$$

It follows from $(A0)$ that there exists positive constants $k_1 - k_4$ such that

$$k_1 \le R_1(s) \le k_2,\ s \in S_1, \quad k_3 \le R_2(s) \le k_4,\ s \in S_2, \quad k_2 < k_3. \tag{3.10}$$

Therefore, the norm (3.9) is correctly defined.

Theorem 3.1. *Suppose that the conditions* $(A0)$ *and* (3.7) *are satisfied. Then the space* U *being equipped with the norm* (3.9) *is a Banach space.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in U, i.e. for an arbitrary $\varepsilon > 0$, there exists N_ε whereby

$$\|u_m - u_n\|_U < \varepsilon \qquad \text{at } m, n > N_\varepsilon. \tag{3.11}$$

Taking into account that $H_q^{2,1}(Q)$ is a Banach space, we obtain from (3.6), (3.9) and (3.11) that there exists a function $u \in H_q^{2,1}(Q)$ such that

$$u_n \to u \quad \text{in } H_q^{2,1}(Q), \tag{3.12}$$

$$u_n\Big|_{S_T} \to u\Big|_{S_T} \quad \text{in } H^{2-\frac{1}{q}, 1-\frac{1}{2q}}(S_T). \tag{3.13}$$

Since $H_q^{1-\frac{1}{2q}}(0,T)$ and $H_q^{\frac{3}{2}-\frac{1}{2q}}(0,T)$ are Banach spaces, (3.9) and (3.11) yield

$$\begin{aligned} &R_1^{-1}u_n\Big|_{S_{1T}} \to \alpha_1 \quad \text{in } H_q^{1-\frac{1}{2q}}(0,T),\\ &R_2^{-1}u_n\Big|_{S_{2T}} \to \alpha_2 \quad \text{in } H_q^{\frac{3}{2}-\frac{1}{2q}}(0,T). \end{aligned} \tag{3.14}$$

Considering that $S \in C^2$, we obtain by (3.10) and (3.14) that

$$u_n\Big|_{S_{1T}} \to \alpha_1 R_1 \quad \text{in } H_q^{1-\frac{1}{2q}}(0,T;C^2(\overline{S}_1)),$$
$$u_n\Big|_{S_{2T}} \to \alpha_2 R_2 \quad \text{in } H_q^{\frac{3}{2}-\frac{1}{2q}}(0,T;C^2(\overline{S}_2)). \tag{3.15}$$

Relations (3.13) and (3.15) imply

$$u\Big|_{S_{1T}} = \alpha_1 R_1, \quad u\Big|_{S_{2T}} = \alpha_2 R_2. \tag{3.16}$$

It follows from (3.8) that

$$u_n\Big|_{S_{1T}} = \mu_1^n R_1, \quad u_n\Big|_{S_{2T}} = \mu_2^n R_2, \quad \mu_1^n \in H_q^{1-\frac{1}{2q}}(0,T),\ \mu_2^n \in H_q^{\frac{3}{2}-\frac{1}{2q}}(0,T). \tag{3.17}$$

Because of this (3.15) gives

$$\mu_1^n \to \alpha_1 \quad \text{in } H_q^{1-\frac{1}{2q}}(0,T), \quad \mu_2^n \to \alpha_2 \quad \text{in } H_q^{\frac{3}{2}-\frac{1}{2q}}(0,T). \tag{3.18}$$

(3.8) and (3.17) yield

$$u_n(s,t) = P(\mu_1^n(t)R_1,\, \mu_2^n(t)R_2)(s), \qquad s \in (\overline{S}_3 \bigcup \overline{S}_4), \qquad t \in (0,T). \tag{3.19}$$

By (3.13), (3.18) and (3.19), we receive

$$u(s,t) = P(\alpha_1(t)R_1,\, \alpha_2(t)R_2)(s), \qquad s \in (\overline{S}_3 \bigcup \overline{S}_4), \qquad t \in (0,T). \tag{3.20}$$

From (3.12), (3.16) and (3.20) it is apparent that $u \in U$, and our lemma is proved.

We set the following space

$$V = \{(f,y,e) | (f,y,e) \in L_q(Q) \times H_q^{2-\frac{1}{q},1-\frac{1}{2q}}(S_T) \times H_q^{2-\frac{2}{q}}(\Omega),$$
$$y\Big|_{S_{1T}} = \mu_1 R_1,\ y\Big|_{S_{2T}} = \mu_2 R_2,\ y\Big|_{(\overline{S}_3 \bigcup \overline{S}_4)\times(0,T)} = P(\mu_1 R_1, \mu_2 R_2),$$
$$\mu_1 \in H_q^{1-\frac{1}{2q}}(0,T),\ \mu_2 \in H_q^{\frac{3}{2}-\frac{1}{2q}}(0,T),\ y\Big|_{t=0} = e\Big|_S\}. \tag{3.21}$$

The space V is provided with the norm

$$\|(f,y,e)\|_V = \|f\|_{L_q(Q)} + \|R_1^{-1} y\Big|_{S_{1T}}\|_{H_q^{1-\frac{1}{2q}}(0,T)}$$
$$+ \|R_2^{-1} y\Big|_{S_{2T}}\|_{H_q^{\frac{3}{2}-\frac{1}{2q}}(0,T)} + \|e\|_{H_q^{2-\frac{2}{q}}(\Omega)}. \tag{3.22}$$

Theorem 3.2. *Suppose that the conditions (AO) and (3.7) are fulfilled. Then V is a Banach space.*

Proof. Let $\{q_n = (f_n, y_n, e_n)\}$ be a Cauchy sequence in V, i.e.

$$\|(f_n - f_k, y_n - y_k, e_n - e_k)\|_V \to 0 \quad \text{as } n,k \to \infty.$$

Since $L_q(Q)$, $H_q^{1-\frac{1}{2q}}(0,T)$, $H_q^{\frac{3}{2}-\frac{1}{2q}}(0,T)$, and $H_q^{2-\frac{2}{q}}(\Omega)$ are Banach spaces, relations (3.21) and (3.22) yield

$$f_n \to f \qquad \text{in } L_q(Q), \tag{3.23}$$

$$R_1^{-1} y_n\Big|_{S_{1T}} = \mu_{1n} \to \mu_1 \qquad \text{in } H_q^{1-\frac{1}{2q}}(0,T), \tag{3.24}$$

$$R_2^{-1} y_n\Big|_{S_{2T}} = \mu_{2n} \to \mu_2 \qquad \text{in } H_q^{\frac{3}{2}-\frac{1}{2q}}(0,T), \tag{3.25}$$

$$y_n\Big|_{(\overline{S}_3 \bigcup \overline{S}_4)\times(0,T)} = P(\mu_{1n}R_1, \mu_{2n}R_2), \tag{3.26}$$

$$e_n \to e \quad \text{in } H_q^{2-\frac{2}{q}}(\Omega), \tag{3.27}$$

in addition

$$R_1\mu_{1n}(0) = e_n\Big|_{S_1}, \quad R_2\mu_{2n}(0) = e_n\Big|_{S_2}, \tag{3.28}$$

$$P(\mu_{1n}(0)R_1,\ \mu_{2n}(0)R_2) = e_n\Big|_{(\overline{S}_3 \bigcup \overline{S}_4)}. \tag{3.29}$$

(3.27) and the embedding results (see e.g. [5], Chapter 5, Section 24) yield

$$e_n\Big|_S \to e\Big|_S \quad \text{in } H_q^{2-\frac{3}{q}}(S). \tag{3.30}$$

Taking into account (3.24)–(3.26) and (3.28)–(3.30), we obtain

$$e\Big|_S = \begin{cases} R_1\mu_1(0) & \text{on } S_1, \\ R_2\mu_2(0) & \text{on } S_2, \\ P(\mu_1(0)R,\ \mu_2(0)R_2) & \text{on } \overline{S}_3 \bigcup \overline{S}_4. \end{cases} \tag{3.31}$$

Define a function y on the set S_T as follows:

$$y = \begin{cases} \mu_1 R_1 & \text{on } S_{1T}, \\ \mu_2 R_2 & \text{on } S_{2T}, \\ P(\mu_1 R_1,\ \mu_2 R_2) & \text{on } (\overline{S}_3 \bigcup \overline{S}_4) \times (0,T). \end{cases} \tag{3.32}$$

(3.31) and (3.32) yield $y|_{t=0} = e|_S$, and by (3.7), we have $y \in H_q^{2-\frac{1}{q},1-\frac{1}{2q}}(S_T)$. Therefore, the function $q = (f, y, e)$, that is defined by (3.23), (3.32) and (3.27), belongs to the space V and $q_n = (f_n, y_n, e_n)$ converges to $q = (f, y, e)$ in V. ■

4. Existence results and second formulation of the direct problem.

The next theorem follows from known results, see e.g. [29], Section 4.1.

Theorem 4.1. *Suppose that the conditions* (*A*0) *and* (3.7) *are satisfied. Let also ϵ be a positive constant and $l \in C([0,T])$. Then for an arbitrary $t \in [0,T]$ there exists a unique solution $\theta(t)$ of the problem* (2.15), (2.16) *such that $\theta(t) \in H_q^2(\Omega)$, moreover, $\theta \in C([0,T]; H_q^2(\Omega))$ and*

$$E = -\operatorname{grad}\theta = -\Big(\frac{\partial\theta}{\partial r}, \frac{\partial\theta}{\partial z}\Big) \in C([0,T]; H_q^1(\Omega)^2), \qquad q > 4. \tag{4.1}$$

Remark. By virtue of the Theorem 4.1, an operator $F \in \mathcal{L}(C([0,T]), C([0,T]; H_q^1(\Omega)^2))$ is defined such that

$$C([0,T]) \ni l \to Fl = E \in C([0,T]; H_q^1(\Omega)^2), \qquad q > 4. \tag{4.2}$$

We assume that the viscosity function φ is defined by (2.1), where b_0 and ψ_0 satisfy the following conditions:

(A1): b_0 is a function twice continuously differentiable in $\mathbb{R}_+$ and, in addition,

$$0 \le b_0(y) \le a_0, \qquad y \in \mathbb{R}_+, \tag{4.3}$$

where a_0 is a positive constant, $\mathbb{R}_+ = \{y | y \in \mathbb{R},\ y \ge 0\}$.

(A2): ψ is a function twice continuously differentiable in $\mathbb{R}^2_+$ and the following inequalities hold:

$$a_2 \ge \psi_0(y_1, y_2) \ge a_1, \tag{4.4}$$

$$\psi_0(y_1, y_2) + 2\,\frac{\partial \psi_0}{\partial y_1}(y_1, y_2) y_1 \ge a_3, \tag{4.5}$$

$$\Big|\frac{\partial \psi_0}{\partial y_1}(y_1, y_2)\Big| y_1 \le a_4, \qquad (y_1, y_2) \in \mathbb{R}^2_+, \tag{4.6}$$

where $a_1 - a_4$ are positive constants.

Let us dwell on the physical sense of the inequalities (4.3)-(4.6). Relations (4.3) and (4.4) indicates that the viscosity function is bounded from below and above by positive constants. In this case, the viscosity at small values of $I(u)$ is large, because λ is a small positive constant in (2.2). The inequality (4.5) implies that for fixed value of $|E|$, the derivative of the function $I(u) \to D(I(u))$ is positive, where D(I(u)) is the second invariant of the stress deviator

$$D(I(u)) = 4[\varphi(I(u), |E|^2)]^2 I(u).$$

This means that in the case of simple shear flow, the stress increases with increasing shear rate. The inequality (4.6) is a restriction on $|\frac{\partial \psi_0}{\partial y_1}|$ for large values of y_1.

The inequalities (4.3)–(4.6) as well as the assumption that λ is a small positive constant are natural from the physical point of view. The viscosity function is identified by approximation of a set of flow curves which are obtained experimentally by viscometric testing for different electric fields. The inequalities (4.3)–(4.6) and the assumption that $\lambda > 0$ and small are consisted with the shapes of the flow curves and enable one to approximate a set of flow curves over a wide range of shear rates with a high degree of accuracy, (see [11], [12], [20]).

We also suppose that

$$\omega \in H_q^{1-\frac{1}{2q}}(0, T), \tag{4.7}$$

$$u_0 \in H_q^{2-\frac{2}{q}}(\Omega), \qquad q > 4, \tag{4.8}$$

$$M_{ex} \in C([0, T]). \tag{4.9}$$

Theorem 4.2. *Assume that the conditions* (A0), (A1) *and* (A2) *are satisfied. Let also the terms* (2.8), (3.7), (4.7)–(4.9) *hold, and a function* $E \in C([0, T]; H^1_q(\Omega)^2)$ *is given. Then there exists a unique function* $u \in U$ *which is the solution of the problem* (2.3), (2.1), (2.4)–(2.7).

A result that is very close to Theorem 4.2 is proved in [25]. Theorem 4.2 is argued just as it is done in [25]. Because of this, we only adduce the main steps and concepts of the proof.
1. The problem under consideration is approximated by a problem with a delay (in this connection see also proof of Lemma 6.2 below). This enables to treat nonlocal boundary conditions as inhomogeneous Dirichlet boundary conditions.
2. By using the implicit function theorem, the results on smoothness of solutions of linear parabolic problems, and the method of extension by parameter, the existence and the uniqueness of the solution of our nonlinear problem with the Dirichlet boundary conditions, that is the problem with a delay, is proved. By virtue of (2.5), (3.5) and (3.7), the solution of this

problem belongs to the space U.
3. Applying the results of compact embedding for anisotropic Besov spaces, a priory estimates for solutions of the problem with a delay are obtained. These estimates are independent of the parameter of delay and permit to apply the contraction mapping principle with parameter on each small subinterval of time. By passing to the limit as the parameter of delay tends to zero, we obtain that there exists a unique function $u \in U$ which is the solution of the problem (2.3), (2.1), (2.4)–(2.7).

Below the problem (2.3), (2.1), (2.4)–(2.7) in which $E = Fl$ will be called the direct problem.

It follows from the Remark and Theorem 4.2 that at given function of voltages $l \in H^{\beta}(0,T)$,

$\beta \in (1/2,1)$ and angular velocity of the driving rotor $\omega \in H_q^{1-\frac{1}{2q}}(0,T)$, $\omega(0) = R_1^{-1}\, u_0|_{S_1}$ (see (2.8)), there exists a unique solution of the direct problem.

We consider the functions l and ω as controls. Define a set of controls as follows:

$$G = \{g | g = (l,\omega),\ l \in H^{\beta}(0,T),\ \beta \in (1/2,1),\ \omega \in H_q^{1-\frac{1}{2q}}(0,T),\ \omega(0) = R_1^{-1}\, u_0|_{S_1}\}. \tag{4.10}$$

Here and below, the space $H_{\xi}^{p}(0,T)$ is denoted by $H^p(0,T)$ at $\xi = 2$ and $p \in \mathbb{R}$. The set G is equipped with the topology generated by the topology of $H^{\beta}(0,T) \times H_q^{1-\frac{1}{2q}}(0,T)$.

By virtue of the Remark and Theorem 4.2, it is defined an operator $N : G \to U$ such that

$$G \ni g = (l,\omega) \to N(g) = u \in U, \tag{4.11}$$

where u is the solution of the direct problem.

We set an operator $L : H^{\beta}(0,T) \times U \to L_q(Q)$ as follows:

$$\begin{aligned} L(l,u) &= \rho \frac{\partial u}{\partial t} - 2\,\frac{\partial}{\partial r}[\varphi(I(u),|Fl|^2)\varepsilon_1(u)] \\ &-2\,\frac{\partial}{\partial z}[\varphi(I(u),|Fl|^2)\varepsilon_2(u)] - \frac{4}{r}\varphi(I(u),|Fl|^2)\varepsilon_1(u) \quad \text{in } Q. \end{aligned} \tag{4.12}$$

Here $\varepsilon_1(u)$ and $\varepsilon_2(u)$ are the components of the rate of strain tensor

$$\varepsilon_1(u) = \frac{1}{2}\Big(\frac{\partial u}{\partial r} - \frac{u}{r}\Big), \qquad \varepsilon_2(u) = \frac{1}{2}\,\frac{\partial u}{\partial z}, \qquad I(u) = 2(\varepsilon_1(u))^2 + 2\,(\varepsilon_2(u))^2. \tag{4.13}$$

Define also an operator $B_1 : G \times U \to H_q^{2-\frac{1}{q},1-\frac{1}{2q}}(S_T)$ in the form

$$B_1(g,u)(s,t) = \begin{cases} u(s,t) - \omega(t)R_1(s), \quad (s,t) \in S_{1T}, \\ u(s,t) - X(g,u)(t)R_2(s), \quad (s,t) \in S_{2T}, \\ u(s,t) - P(\omega(t)R_1, X(g,u)(t)R_2)(s), \\ (s,t) \in (\overline{S}_3 \bigcup \overline{S}_4) \times (0,T), \end{cases} \tag{4.14}$$

where

$$X(g,u)(t) = [\omega_1(0) - (\rho_0 I_0)^{-1} \int_0^t [M_{ex} + 2\int_{S_2} \varphi(I(u),|Fl|^2)(\varepsilon_1(u)\nu_1 + \varepsilon_2(u)\nu_3)R_2\, ds]d\tau, \tag{4.15}$$

and $\omega_1(0) = R_2^{-1}\, u_0\Big|_{S_2}$, see (2.8). The parameter t in the integrand in (4.15) is denoted by τ.

We assign an operator $B_2 : U \to H_q^{2-\frac{2}{q}}(\Omega)$ by

$$B_2(u) = u(0) - u_0. \tag{4.16}$$

We introduce a mapping $J : G \times U \to V$ as follows:

$$(G \times U) \ni (g, u) \to J(g, u) = \{L(l, u), B_1(g, u), B_2(u)\}. \tag{4.17}$$

We consider the problem: For given g in G, find a function $u \in U$ such that

$$J(g, u) = 0. \tag{4.18}$$

It is easy to check that the function u that meets the condition (4.18) is the solution of the direct problem. By virtue of Theorem 4.2, there exists a unique solution of the problem (4.18) and $u = N(g)$, see (4.11).

Therefore, N is an implicit function defined by equation (4.18), i. e.

$$J(g, N(g)) = 0, \qquad g \in G. \tag{4.19}$$

The next result follows from Theorem 4.2.

Corollary. Suppose that the conditions $(A0)$, $(A1)$ and $(A2)$ are satisfied. Let also the terms (2.8), (3.7), (4.7)–(4.9) are fulfilled. Then for an arbitrary $g \in G$, there exists a unique function $N(g) \in U$ such that (4.19) is met.

5. Optimal control problem.

For given $g \in G$, the energy that is expended in the acceleration or in the braking of the driven rotor is defined as follows:

$$\Psi_0(g) = \int_0^T \int_{S_1} \varphi(I(N(g)), |Fl|^2) \Big[\Big(\frac{\partial N(g)}{\partial r} - \frac{N(g)}{r} \Big) \nu_1 + \frac{\partial N(g)}{\partial z} \nu_3 \Big] R_1 \omega \, ds \, dt. \tag{5.1}$$

The right-hand side of (5.1) is the integral over S_{1T} of the scalar product of the surface forces acting of the fluid and the velocity of the fluid.

In the case that the clutch works as an accelerator $\Psi_0(g) > 0$, i.e. clutch consumes energy. Once the clutch functions as a brake $\Psi_0(g) < 0$; in this case the clutch gives out energy. Therefore, the functional Ψ_0 should be minimized in both cases.

The angular velocity of the driven rotor ω_1 as a function of g is defined by

$$\omega_1(t) = (R_2^{-1} N(g)\Big|_{S_{2T}})(t), \qquad t \in [0, T]. \tag{5.2}$$

Define a functional

$$\Psi_1(g) = \Big[(R_2^{-1} N(g)\Big|_{S_{2T}})(T) - k_0 \Big]^2 + k_1 \Big[\Big(\frac{d}{dt} \Big(R_2^{-1} N(g)\Big|_{S_{2T}} \Big) \Big)(T) - k_2 \Big]^2, \tag{5.3}$$

where k_0, k_1, k_2 are constants, $k_1 > 0$.

We assign the following set of admissible control:

$$G_a = \{g | g = (l, \omega) \in H^1(0, T) \times H^2(0, T), \ \|l\|^2_{H^1(0,T)} \le e_1,$$
$$0 \le l(t) \le e_2, \ t \in [0, T], \ \|\omega\|^2_{H^2(0,T)} \le e_3, \ e_4 \le \omega(t) \le e_5, \ t \in [0, T],$$
$$\omega(0) = R_1^{-1} u_0\Big|_{S_1}, \ \Psi_1(g) \le e_6\}. \tag{5.4}$$

Here $e_1 - e_6$ are constants. Under a reasonable (from the engineering point of view) choice of the constants $e_1 - e_6$, the set G_a is nonempty.

Since the viscosity function φ depends on the module of the vector of electric field E, we reckon that $l(t) \ge 0$ for all $t \in [0, T]$.

We consider the following optimal control problem: Find g_0 satisfying

$$g_0 = (l_0, \omega_0) \in G_a, \quad \Psi_0(g_0) = \inf \Psi_0(g), \quad g \in G_a. \tag{5.5}$$

In the case that the clutch acts as an accelerator $\Psi_0(g) > 0$, and (5.5) denotes that the energy, consumed in the acceleration of the driven rotor, is minimized under the restriction that an instant T, the angular velocity and the acceleration of the driven rotor are localized within given regions.

In the case that $\Phi_0(g) < 0$, the energy production is maximized under the above restriction.

For the integrand in (5.5) to be integrable and the direct problem to be solvable, the functions u, l and ω must be sufficiently smooth. The restriction $\omega \in H_q^{1-\frac{1}{2q}}(0,T)$ in the set of controls G in (4.10) is necessary for the solvability of the direct problem in the space $H_q^{2,1}(Q)$ and for the integrability of the integrand in (2.5); the restriction $l \in H^\beta(0,T)$ in (4.10) is very close to necessary one, see [25].

However, we suppose that $(l,\omega) \in H^1(0,T) \times H^2(0,T)$ and impose restrictions on the values of the norms of l and ω in $H^1(0,T)$ and in $H^2(0,T)$ in the set of admissible controls G_a. The optimal control problem can certainly be considered under a somewhat weaker restrictions on the smoothness of the functions l and ω. But our restrictions are the weakest, which concur with the requirement for the clutch to provide smooth acceleration or braking without pushes and shocks.

It is well-known that nonsmooth and especially discontinuous controls result in shocks, vibrations, and sometimes in destructions. Because of this, only smooth controls are used in actual practice.

6. AUXILIARY RESULTS.

Lemma 6.1. *Suppose that the conditions (A_0), $(A1)$, $(A2)$, (3.7) and (4.8) are satisfied. Then the function J defined by (4.17), (4.12)–(4.16) is a continuously Fréchet differentiable mapping of $G \times U$ into V, and at any point $(g,u) \in G \times U$, $g = (l,\omega)$, the Fréchet derivative $J'(g,u)$ of the mapping J is defined as follows:*

$$J'(g,u)((h,e),v) = (L'(l,u)(h,v),\ B_1'(g,u)((h,e),v),\ B_2'(u)v). \tag{6.1}$$

Here $v \in U$, $(h,e) \in G_1$, where G_1 is vector space joined to the affine space G,

$$G_1 = \{(h,e) | h \in H^\beta(0,T),\ e \in H_q^{1-\frac{1}{2q}}(0,T),\ e(0) = 0\}, \tag{6.2}$$

and

$$L'(l,u)(h,v) = \frac{\partial L}{\partial l}(l,u)h + \frac{\partial L}{\partial u}(l,u)v, \tag{6.3}$$

$$B_1'(g,u)((h,e),v) = \frac{\partial B_1}{\partial l}(g,u)h + \frac{\partial B_1}{\partial \omega}(g,u)e + \frac{\partial B_1}{\partial u}(g,u)v, \tag{6.4}$$

$$B_2'(u)v = v(0). \tag{6.5}$$

The partial derivatives of the operators L and B_1 have the following forms:

$$\frac{\partial L}{\partial l}(l,u)h = -4\frac{\partial}{\partial r}\Big[\frac{\partial \varphi}{\partial y_2}(I(u), |Fl|^2)(Fl, Fh)\varepsilon_1(u)\Big]$$
$$-4\frac{\partial}{\partial z}\Big[\frac{\partial \varphi}{\partial y_2}(I(u), |Fl|^2)(Fl, Fh)\varepsilon_2(u)\Big] - \frac{8}{r}\frac{\partial \varphi}{\partial y_2}(I(u), |Fl|^2)(Fl, Fh)\varepsilon_1(u) \quad \text{in } Q, \tag{6.6}$$

where (Fl, Fh) *is the scalar product in* $\mathbb{R}^2$ *of the vectors* $(Fl)(r,z,t)$ *and* $(Fh)(r,z,t)$*, and*

$$\begin{aligned}
&\frac{\partial L}{\partial u}(l,u)v = \rho\frac{\partial v}{\partial t} - \\
&-2\,\frac{\partial}{\partial r}\Big[\varphi(I(u),|Fl|^2)\varepsilon_1(v) + 4\,\frac{\partial\varphi}{\partial y_1}(I(u),|Fl|^2)(\varepsilon_1(u)\varepsilon_1(v)+\varepsilon_2(u)\varepsilon_2(v))\varepsilon_1(u)\Big] \\
&-2\,\frac{\partial}{\partial z}\Big[\varphi(I(u),|Fl|^2)\varepsilon_2(v) + 4\,\frac{\partial\varphi}{\partial y_1}(I(u),|Fl|^2)(\varepsilon_1(u)\varepsilon_1(v)+\varepsilon_2(u)\varepsilon_2(v))\varepsilon_2(u)\Big] \\
&-\frac{4}{r}\Big[\varphi(I(u),|Fl|^2)\varepsilon_1(v) + 4\,\frac{\partial\varphi}{\partial y_1}(I(u),|Fl|^2)(\varepsilon_1(u)\varepsilon_1(v)+\varepsilon_2(u)\varepsilon_2(v))\varepsilon_1(u)\Big] \quad \text{in } Q,
\end{aligned} \tag{6.7}$$

where (see (2.1)*)*

$$\begin{aligned}
\frac{\partial\varphi}{\partial y_1}\,(I(u),|Fl|^2) &= -\frac{1}{2}\,b_0\,(|Fl|^2)(\lambda+I(u))^{-\frac{3}{2}} + \frac{\partial\psi_0}{\partial y_1}\,(I(u),|Fl|^2), \\
\frac{\partial\varphi}{\partial y_2}\,(I(u),|Fl|^2) &= \frac{\partial b_0}{\partial y}\,(|Fl|^2)(\lambda+I(u))^{-\frac{1}{2}} + \frac{\partial\psi_0}{\partial y_2}\,(I(u),|Fl|^2),
\end{aligned} \tag{6.8}$$

$$\Big(\frac{\partial B_1}{\partial l}(g,u)h\Big)(s,t) = \begin{cases} 0 \quad \text{on } S_{1T}, \\ -\Big(\frac{\partial X}{\partial l}(g,u)h\Big)(t)R_2(s), \quad (s,t)\in S_{2T}, \\ -\Big(\frac{\partial X}{\partial l}(g,u)h\Big)(t)P(0,R_2)(s), \\ (s,t)\in(\overline{S}_3\bigcup\overline{S}_4)\times(0,T), \end{cases} \tag{6.9}$$

$$\Big(\frac{\partial B_1}{\partial \omega}(g,u)e\Big)(s,t) = \begin{cases} -e(t)R_1(s), \quad (s,t)\in S_{1T}, \\ 0 \quad \text{on } S_{2T}, \\ -e(t)P(R_1,0)(s), \quad (s,t)\in(\overline{S}_3\bigcup\overline{S}_4)\times(0,T), \end{cases} \tag{6.10}$$

$$\Big(\frac{\partial B_1}{\partial u}(g,u)v\Big)(s,t) = \begin{cases} v(s,t), \quad (s,t)\in S_{1T}, \\ v(s,t) - \Big(\frac{\partial X}{\partial u}(g,u)v\Big)(t)R_2(s), \quad (s,t)\in S_{2T}, \\ v(s,t) - \Big(\frac{\partial X}{\partial u}(g,u)v\Big)(t)P(0,R_2)(s), \\ (s,t)\in(\overline{S}_3\bigcup\overline{S}_4)\times(0,T). \end{cases} \tag{6.11}$$

Here

$$\begin{aligned}
&\Big(\frac{\partial X}{\partial u}(g,u)v\Big)(t) = -2\,(\rho_0 I_0)^{-1}\int_0^t\int_{S_2}[\varphi(I(u),|Fl|^2)(\varepsilon_1(v)\nu_1+\varepsilon_2(v)\nu_3) \\
&+4\,\frac{\partial\varphi}{\partial y_1}(I(u),|Fl|^2)(\varepsilon_1(u)\varepsilon_1(v)+\varepsilon_2(u)\varepsilon_2(v))(\varepsilon_1(u)\nu_1+\varepsilon_2(u)\nu_3)]\,R_2\,ds\,d\tau,
\end{aligned} \tag{6.12}$$

$$\begin{aligned}
\Big(\frac{\partial X}{\partial l}(g,u)h\Big)(t) = -4\,(\rho_0 I_0)^{-1}\int_0^t\int_{S_2}\Big[\frac{\partial\varphi}{\partial y_2}(I(u),|Fl|^2)(Fl,Fh)(\varepsilon_1(u)\nu_1 \\
+\varepsilon_2(u)\nu_3)\Big]R_2\,ds\,d\tau.
\end{aligned} \tag{6.13}$$

In this case

$$J'(g,u)) \in \mathcal{L}(G_1\times U, V), \qquad (g,u)\in G\times U. \tag{6.14}$$

Lemma 6.1 is proved by analogy with Lemma 5.1 from [25]. At first it is proved that at any point $(g,u) \in G \times U$ the operator J is Gâteaux differentiable and its Gâteaux derivative is defined by (6.1) and subsequent relations. The partial derivatives of the operator B_1, that are defined by (6.9), (6.10) and (6.11), are obtained using the following relations:

$$P(\omega(t)R_1,\ X(g,u)(t)R_2)(s) = P((\omega(t)R_1,0) + (0, X(g,u)(t)R_2))(s) = \\ \omega(t)P(R_1,0)(s) + X(g,u)(t)P(0,R_2)(s), \tag{6.15}$$

which follows from (3.7).

It is next proved that the function $(g,u) \to J'(g,u)$ is a continuous mapping of $G \times U$ into $\mathcal{L}(G_1 \times U, V)$; in this case we take into account that the embedding of $H_q^{2,1}(Q)$ into $C([0,T]; C^1(\overline{\Omega}))$ is continuous at $q > 4$ (see [5], Theorem 10.4, Chapter 3). Therefore, the Gâteaux derivative of J is the Fréchet derivative.

Lemma 6.2. *Suppose that the conditions* (A0), (A1), (A2), (3.7), (4.8), *and* (4.9) *are satisfied. Then for an arbitrary pair* $(g,u) \in G \times U$, $g = (l,\omega)$, *the operator* $\frac{\partial J}{\partial u}(g,u) = \Big(\frac{\partial L}{\partial u}(l,u),\ \frac{\partial B_1}{\partial u}(g,u),\ B_2'(u)\Big)$ *is an isomorphism of* U *onto* V, *that is the inverse operator* $\Big(\frac{\partial J}{\partial u}(g,u)\Big)^{-1}$ *of* $\frac{\partial J}{\partial u}(g,u)$ *is a linear continuous mapping of* V *onto* U.

Proof. We consider the problem: Given $(g,u) \in G \times U$ and $(f,y,e) \in V$, find $v \in U$ such that

$$\frac{\partial J}{\partial u}(g,u)v = (f,y,e), \tag{6.16}$$

that is

$$\frac{\partial L}{\partial u}(l,u)v = f \quad \text{in } Q, \tag{6.17}$$

$$\frac{\partial B_1}{\partial u}(g,u)v = y \quad \text{on } S_T, \tag{6.18}$$

$$v(0) = e \quad \text{in } \Omega. \tag{6.19}$$

The problem (6.17)–(6.19) is approximated by a problem with a delay δ, where δ is a small positive constant. In this case, the operator $\frac{\partial B_1}{\partial u}(g,u)$ is approximated by the operator $Z_\delta(g,u)$ that is defined as follows:

$$\Big(Z_\delta(g,u)v\Big)(s,t) = \begin{cases} v(s,t), \quad (s,t) \in S_{1T}, \\ v(s,t) - \Big(Y_\delta(g,u)v\Big)(t)R_2(s), \quad (s,t) \in S_{2T}, \\ v(s,t) - \Big(Y_\delta(g,u)v\Big)(t)P(0,R_2)(s), \\ (s,t) \in (\overline{S}_3 \bigcup \overline{S}_4) \times (0,T), \end{cases} \tag{6.20}$$

where

$$\Big(Y_\delta(g,u)v\Big)(t) = -2\,(\rho_0 I_0)^{-1} \int_{-\delta}^{t-\delta} \int_{S_2} [\varphi(I(u),|Fl|^2)(\varepsilon_1(v)\nu_1 + \varepsilon_2(v)\nu_3) \\ +4\,\frac{\partial \varphi}{\partial y_1}(I(u),|Fl|^2)(\varepsilon_1(u)\varepsilon_1(v) + \varepsilon_2(u)\varepsilon_2(v))(\varepsilon_1(u)\nu_1 + \varepsilon_2(u)\nu_3)]\,R_2\,ds\,d\tau. \tag{6.21}$$

Here we take

$$v(t) = e, \quad u(t) = u(0) = u_0, \quad l(t) = l(0) \quad \text{at } t \in [-\delta, 0]. \tag{6.22}$$

We consider the following problem with a delay: Find $v_\delta \in U$ such that

$$\frac{\partial L}{\partial u}(l,u)v_\delta = f \quad \text{in } Q, \tag{6.23}$$

$$Z_\delta(g,u)v_\delta = y \quad \text{on } S_T, \tag{6.24}$$

$$v_\delta(t) = e \quad \text{at } t \in [-\delta, 0] \quad \text{in } \Omega. \tag{6.25}$$

By virtue of (6.22) and (6.25), the value of the function $t \to (Y_\delta(g,u)v_\delta)(t)$ are known on the segment $[0,\delta]$. Because of this, the relation (6.24) is reduced to the following Dirichlet boundary condition at $t \in (0,\delta]$.

$$v_\delta(s,t) = \begin{cases} y(s,t), \quad (s,t) \in S_1 \times (0,\delta], \\ \Big(Y_\delta(g,u)v_\delta\Big)(t)R_2(s) + y(s,t), \quad (s,t) \in S_2 \times (0,\delta], \\ \Big(Y_\delta(g,u)v_\delta\Big)(t)P(0,R_2)(s) + y(s,t), \\ (s,t) \in (\overline{S}_3 \bigcup \overline{S}_4) \times (0,\delta]. \end{cases} \tag{6.26}$$

The results of [25] and [17], Theorem 9.1, Chapter 4, imply that there exists a unique solution of the problem (6.23)–(6.25) on the time segment $[0,\delta]$. Analogously to the above, using the solution on $[0,\delta]$, we reduce the condition (6.24) to the Dirichlet boundary condition at $t \in (\delta, 2\,\delta]$ and prolong the solution of our problem on the segment $[0,2\delta]$. Since $H_q^{2,1}(Q) \in C([0,T];C^1(\overline{\Omega}))$, we obtain by (6.21), (6.26), and (3.6) that $v_\delta \in H_q^{2,1}(\Omega \times (0,2\delta))$. In this way, we prove that there exists the unique solution of the problem (6.23)–(6.25) on the whole interval $(0,T)$.

By repeating the arguments of the proof of Theorem 4.2 from [25], we establish that there exists the unique solution v of the problem (6.16) and $v_\delta \to v$ in U as δ tends to zero. Since the right-hand side (f,y,e) in (6.16) is an arbitrary triple from V, the Banach theorem on inverse operator (see e.g. [15], Chapter II, section 5) implies that the inverse operator $\Big(\frac{\partial J}{\partial u}(g,u)\Big)^{-1}$ of $\frac{\partial J}{\partial u}(g,u)$ is a linear continuous mapping of V onto U, and the Lemma is proved.

The Corollary of Theorem 4.2, Lemmas 6.1, and 6.2, and the results on implicit function (see [31], Chapter 3, Section 8) lead to the following theorem:

Theorem 6.1. *Suppose that the conditions* (A0), (A1) *and* (A2) *are satisfied. Let also* (2.8), (3.7), (4.8), *and* (4.9) *hold. Then the function $N : g \to N(g)$ defined by* (4.19) *is a continuously Fréchet differentiable mapping of G into U and its derivative is defined as follows:*

$$N'(g) = -\Big(\frac{\partial J}{\partial u}(g,N(g))\Big)^{-1} \circ \frac{\partial J}{\partial g}(g,N(g)) \tag{6.27}$$

where $\frac{\partial J}{\partial g}(g,N(g)) \in \mathcal{L}(G_1,V)$,

$$\frac{\partial J}{\partial g}(g,N(g)) = \Big\{\frac{\partial L}{\partial l}(l,N(g)), \Big(\frac{\partial B_1}{\partial l}(g,N(g)), \frac{\partial B_1}{\partial \omega}(g,N(g))\Big), 0\Big\}. \tag{6.28}$$

7. Differentiation of the functionals Ψ_0 and Ψ_1.

7.1. **Calculation of derivatives.** . We introduce the following functional:

$$\Phi_0(u,g) = \int_0^T \int_{S_1} \varphi(I(u),|Fl|^2)\Big[\Big(\frac{\partial u}{\partial r} - \frac{u}{r}\Big)\nu_1 + \frac{\partial u}{\partial z}\nu_3\Big] R_1 \omega \, ds\, dt,$$
$$u \in U, \quad g = (l,\omega) \in G. \tag{7.1}$$

Then

$$\Psi_0(g) = \Phi_0(N(g), g), \qquad g \in G, \tag{7.2}$$

and the Fréchet derivative of Ψ_0 is defined as follows:

$$\Psi_0'(g)(h,e) = \Big(\frac{\partial \Phi_0}{\partial u}(N(g), g) \circ N'(g)\Big)(h,e) + \frac{\partial \Phi_0}{\partial g}(N(g), g)(h,e), \qquad (h,e) \in G_1. \tag{7.3}$$

Taking (6.27) into account, we obtain

$$\begin{aligned}
&\Big(\frac{\partial \Phi_0}{\partial u}(N(g), g) \circ N'(g)\Big)(h,e) \\
&\qquad = -\Big(\frac{\partial \Phi_0}{\partial u}(N(g), g) \circ \Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^{-1} \circ \frac{\partial J}{\partial g}(g, N(g))\Big)(h,e) \\
&\qquad = -\Big(\frac{\partial \Phi_0}{\partial u}(N(g), g), \Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^{-1} \Big(\frac{\partial J}{\partial g}(g, N(g))\Big)(h,e)\Big) \\
&\qquad = -\Big(\Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^{*-1} \circ \frac{\partial \Phi_0}{\partial u}(N(g), g), \Big(\frac{\partial J}{\partial g}(g, N(g))\Big)(h,e)\Big).
\end{aligned} \tag{7.4}$$

Here in the second equality, we denote by $(.,.)$ the scalar product of the elements of U^* and U, in the third inequality $(.,.)$ denotes the scalar product of the elements of V^* and V, where U^* and V^* are dual spaces to U and V. The third inequality is obtained by using the equality

$$\Big(\Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^{-1}\Big)^* = \Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^{*-1},$$

that follows from known results, see e.g. [16], Section 2, Chapter 12 or [13] Theorem 6.5.11.

Let

$$p_0 = -\Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^{*-1} \circ \frac{\partial \Phi_0}{\partial u}(N(g), g). \tag{7.5}$$

Since $\frac{\partial \Phi_0}{\partial u}(N(g), g) \in U^*$ and $\Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^{*-1} \in \mathcal{L}(U^*, V^*)$, we have $p_0 \in V^*$, in addition, (7.5) implies

$$\Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^* p_0 = -\frac{\partial \Phi_0}{\partial u}(N(g), g) \in U^*,$$

that is

$$\Big(\Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^* p_0, v\Big) = -\Big(\frac{\partial \Phi_0}{\partial u}(N(g), g), v\Big), \qquad v \in U. \tag{7.6}$$

Taking Lemma 6.2 and the relation (7.6) into account, we obtain that p_0 is the unique solution of the following problem:

$$\begin{aligned}
&p_0 \in V^*, \\
&\Big(p_0, \frac{\partial J}{\partial u}(g, N(g))v\Big) = -\Big(\frac{\partial \Phi_0}{\partial u}(N(g), g), v\Big), \qquad v \in U.
\end{aligned} \tag{7.7}$$

It follows from (7.3)–(7.5) that the Fréchet derivative of the functional Ψ_0 in G is defined as follows:

$$\begin{aligned}
\Psi_0'(g)(h,e) &= \Big(p_0, \Big(\frac{\partial J}{\partial g}(g, N(g))\Big)(h,e)\Big) + \frac{\partial \Phi_0}{\partial g}(N(g), g)(h,e), \\
&\qquad g = (l, \omega) \in G, \quad (h,e) \in G_1.
\end{aligned} \tag{7.8}$$

Here $\frac{\partial J}{\partial g}(g, N(g))$ is given by (6.28) and

$$\begin{aligned}\frac{\partial \Phi_0}{\partial g}(N(g), g)(h, e) = \int_0^T \int_{S_1} \Big[2\, \frac{\partial \varphi}{\partial y_2}(I(N(g)), |Fl|^2)(Fl, Fh) \\ \times\Big(\Big(\frac{\partial N(g)}{\partial r} - \frac{N(g)}{r}\Big)\nu_1 + \frac{\partial N(g)}{\partial z}\nu_3\Big)R_1\omega \\ +\varphi(I(N(g)), |Fl|^2)\Big(\Big(\frac{\partial N(g)}{\partial r} - \frac{N(g)}{r}\Big)\nu_1 + \frac{\partial N(g)}{\partial z}\nu_3\Big)R_1 e\Big]\, ds\, dt. \qquad (7.9)\end{aligned}$$

The Fréchet derivative of the functional Ψ_1 is calculated by analogy with above.

We consider the following functional on the space U.

$$\Phi_1(u) = \Big[\Big(R_2^{-1}u\Big|_{S_{2T}}\Big)(T) - k_0\Big]^2 + k_1\Big[\Big(\frac{d}{dt}\Big(R_2^{-1}u\Big|_{S_{2T}}\Big)\Big)(T) - k_2\Big]^2. \qquad (7.10)$$

(5.3) yields

$$\Psi_1(g) = \Phi_1(N(g)). \qquad (7.11)$$

There exists a unique function p_1 such that

$$\begin{aligned}&p_1 \in V^*, \\ &\Big(p_1, \frac{\partial J}{\partial u}(g, N(g))v\Big) = -\Big(\frac{\partial \Phi_1}{\partial u}(N(g)), v\Big), \qquad v \in U. \qquad (7.12)\end{aligned}$$

The Fréchet derivative of Ψ_1 is defined by the following formula:

$$\Psi_1'(g)(h, e) = \Big(p_1, \Big(\frac{\partial J}{\partial g}(g, N(g))\Big)(h, e)\Big), \quad (h, e) \in G_1. \qquad (7.13)$$

The solution of the problem (7.7) depends on $g \in G$. Because of this, we denote it by $p_0(g)$. Taking into account (7.5), Lemma 6.2, and Theorem 6.1, we obtain that

$$g \to p_0(g) \text{ is a continuous mapping of } G \text{ into } V^*. \qquad (7.14)$$

By analogy, denoting the solution of the problem (7.12) by $p_1(g)$, we get

$$g \to p_1(g) \text{ is a continuous mapping of } G \text{ into } V^*. \qquad (7.15)$$

Lemma 6.1, Theorem 6.1, and (7.8), (7.13), (7.14), (7.15) lead to the following result:

Theorem 7.1. *Suppose that the conditions (A0), (A1), and (A2) are satisfied. Let also* (2.8), (3.7), (4.8), *and* (4.9) *hold, and the functionals $\Psi_i, i = 1, 2$ be defined by* (5.1) *and* (5.3). *Then the functionals Ψ_i are continuously Fréchet differentiable in G, and their derivatives are defined by* (7.8) *and* (7.13), *where p_0 and p_1 are the solutions of the problems* (7.7) *and* (7.12), *respectively.*

7.2. **Approximate solution of problems (7.7) and (7.12).** We consider the Petrov-Galerkin method for numerical solution of the problems (7.7) and (7.12). Let $\{V_k^*\}_{k=1}^\infty$ and $\{U_k\}_{k=1}^\infty$ be sequences of finite dimensional subspaces in V^* and U, respectively, such that

$$\lim_{k\to\infty} \inf_{h \in V_k^*} \|p - h\|_{V^*} = 0, \quad p \in V^*, \qquad (7.16)$$

$$\lim_{k\to\infty} \inf_{q \in U_k} \|w - q\|_U = 0, \quad w \in U, \qquad (7.17)$$

and the dimensions of the spaces V_k^* and U_k are equal for each k.

We seek an approximate solution of the problem (7.7) in the form

$$p_k \in V_k^*,$$
$$\Big(p_k, \frac{\partial J}{\partial u}(g, N(g))\, q\Big) = -\Big(\frac{\partial \Phi_0}{\partial u}(N(g), g), q\Big), \quad q \in U_k. \tag{7.18}$$

Denote the operator of restriction of U^* onto U_k^* by s_k, if $\gamma \in U^*$ then

$$\| s_k \gamma\|_{U^*} = \sup |(\gamma, q)|,\ q \in U_k,\ \| q\|_U = 1. \tag{7.19}$$

We assume that there exists a constant b such that

$$\| p \|_{V^*} \leq b \| s_k\Big(\Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^* p\Big) \|_{U^*}, \quad p \in V_k^*, \quad k \in \mathbb{N}. \tag{7.20}$$

By Lemma 6.2 there exists a constant b_1 such that

$$\| p \|_{V^*} \leq b_1 \| \Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^* p \|_{U^*}, \quad p \in V^*. \tag{7.21}$$

Because of this, the assumption (7.20) is quite natural.

Theorem 7.2. *Suppose that the conditions (A0), (A1), and (A2) are satisfied. Let* (2.8), (3.7), (4.8), *and* (4.9) *hold, and* $g \in G$. *Assume also that the terms* (7.16), (7.17), *and* (7.20) *are fulfilled and the dimensions of the spaces* V_k^* *and* U_k *are equal for each* k. *Then for any* k *there exists a unique solution of the problem* (7.18) *and* $p_k \to p_0$ *in* V^*, *where* p_0 *is the solution of problem* (7.7).

Proof. Since the dimensions of the spaces V_k^* and U_k are equal and $\frac{\partial J}{\partial u}(g, N(g))$ is an isomorphism of U onto V, there exists a unique solution of the problem (7.18) for any k.

It follows from (7.18) and (7.6) that

$$s_k\Big(\Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^* p_k\Big) = -s_k \frac{\partial \Phi_0}{\partial u}(N(g), g) = s_k\Big(\Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^* p_0\Big). \tag{7.22}$$

By (7.16) there exists a sequence $\{\tilde{p}_k\}_{k=1}^\infty$ satisfying

$$\tilde{p}_k \in {V_k}^*, \quad \tilde{p}_k \to p_0 \text{ in } V^*. \tag{7.23}$$

Taking (7.20) and (7.22) into account, we obtain

$$\| p_k - \tilde{p}_k \|_{V^*} \leq b \| s_k\Big(\Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^* (p_k - \tilde{p}_k)\Big) \|_{U^*}$$
$$= b \| s_k\Big(\Big(\frac{\partial J}{\partial u}(g, N(g))\Big)^* (p_0 - \tilde{p}_k)\Big) \|_{U^*} \leq c \| p_0 - \tilde{p}_k \|_{V^*} . \tag{7.24}$$

(7.23) and (7.24) imply that $p_k \to p_0$ in V^*. ■

Approximate solutions of the problem (7.12) are also defined as solutions of the problem (7.18), where $\frac{\partial \Phi_0}{\partial u}(N(g), g)$ is replaced by $\frac{\partial \Phi_1}{\partial u}(N(g))$.

For solving the problem (7.18) it is convenient to transform the domain Ω onto a domain Ω_0 which is a rectangle with rounded of angles. To accomplish this, the domain Ω is slightly extended near the points A and B (see Figure 2) to a domain Ω_1 such that $\overline{\Omega}$ is inscribed into $\overline{\Omega}_1$, and Ω_1 is transformed by a C^2 diffeomorphism F onto a rectangular domain Ω_2, i. e. $\overline{\Omega}_2 = F(\overline{\Omega}_1)$.

We take $\Omega_0 = F(\Omega)$, then $\overline{\Omega}_0$ is inscribed into $\overline{\Omega}_2$. Finite dimensional spaces $\check{U}_k$ in the form of tensor product of splines are constructed in $\overline{\Omega}_2 \times [0, T]$. The restrictions of the functions from $\check{U}_k$ to $\overline{\Omega}_0 \times [0, T]$ are used for numerical solution of the transformed problem (7.18) in the domain $\Omega_0 \times (0, T)$. In this case, U_k are the restrictions of the functions from $F^{-1}(\check{U}_k)$ to $\overline{\Omega} \times [0, T]$, where F^{-1} is the inverse diffeomorphism of F.

An information relative to such approach may be found in [21], Sections 1.14, 6.7, 6.8, [26] Chapter 2, Sections 7, 8.

Consider the issue of construction of the subspaces V_k^* or $F(V_k^*)$. The dual space to $L_q(Q)$ for the function f, see (3.21), is $L_l(Q)$ with $1/l + 1/q = 1$. The function μ_1 belongs to the space $H_q^{1-\frac{1}{2q}}(0,T)$. It follows from the embedding theorems that $H^2(0,T) \subset H_q^{1-\frac{1}{2q}}(0,T) \subset L_2(0,T)$. Therefore, $L_2(0,T) \subset (H_q^{1-\frac{1}{2q}}(0,T))^* \subset (H^2(0,T))^*$. Since the space $L_2(0,T)$ is dense in $(H^2(0,T))^*$, it is also dense in $(H_q^{1-\frac{1}{2q}}(0,T))^*$.

By analogy it is established that $L_2(0,T)$ is dense in $(H_q^{\frac{3}{2}-\frac{1}{2q}}(0,T))^*$ and $L_2(\Omega)$ is dense in $(H_q^{2-\frac{2}{q}}(\Omega))^*$.

It follows from the above, that the subspaces V_k^* can be constructed in the form of step functions given in $Q, (0,T)$ and Ω. The conditions (7.16), (7.17) and (7.20) are fulfilled for the spaces that are constructed by the above plan.

It should be mentioned that the commonly used finite elements methods, which are based on the Galerkin and Faedo-Galerkin schemes, do not ensure the converges of approximate solutions to the exact one with respect to the norms of $H_q^{2,1}(Q)$ and $H_q^2(\Omega)$ for the direct problem and the problem (2.15), (2.16) respectively. The Petrov-Galerkin method with the above approximation of the spaces U, V^*, and $H_q^2(\Omega)$ can be used for numerical solution of the direct problem and the problem (2.15), (2.16).

For the cylindrical clutch, R_1 and R_2 are positive constants, and Ω is a one-dimensional domain $\Omega = (R_1, R_2)$ (see Section 10). In that case, the problem for electric field, the direct and optimal control problems are significantly simplified.

8. Existence result for problem (5.5) and necessary optimality conditions.

Theorem 8.1. *Suppose that the conditions* (A0), (A1) *and* (A2) *are satisfied. Let also* (2.8), (3.7), (4.8), (4.9) *hold and the set* G_a *be nonempty. Then there exists a solution of the optimal control problem* (5.5).

Proof. Let $\{g_n = (l_n, \omega_n)\}$ be a minimizing sequence, i.e.

$$\{g_n\} \subset G_a, \quad \lim \Psi_0(g_n) = \inf \Psi_0(g), \quad g \in G_a. \tag{8.1}$$

It follows from (8.1) and (5.4) that a subsequence $\{g_m = (l_m, \omega_m)\}$ can be extracted from the sequence $\{g_n\}$ such that

$$l_m \rightharpoonup l_0 \quad \text{in } H^1(0,T), \quad \omega_m \rightharpoonup \omega_0 \quad \text{in } H^2(0,T). \tag{8.2}$$

The embedding theorem and (8.2) imply

$$l_m \to l_0 \quad \text{in } \ H^\beta(0,T) \text{ and in } C([0,T]), \quad \omega_m \to \omega_0 \quad \text{in } H_q^{1-\frac{1}{2q}}(0,T) \text{ and in } \ C^1([0,T]). \tag{8.3}$$

Theorem 7.1 and (8.3) yield

$$\Psi_1(g_0) \le e_6, \quad g_0 = (l_0, \omega_0). \tag{8.4}$$

(5.4), (8.2), and (8.3) imply

$$\begin{aligned} &\|l_0\|^2_{H^1(0,T)} \le e_1, \quad 0 \le l_0(t) \le e_2 \quad \text{at } t \in [0,T], \\ &\|\omega_0\|^2_{H^2(0,T)} \le e_3, \quad e_4 \le \omega_0(t) \le e_5 \quad \text{at } t \in [0,T]. \end{aligned} \tag{8.5}$$

By (5.4) and (8.3), we get

$$\omega_0(0) = R_1^{-1} u_0\Big|_{S_1}. \tag{8.6}$$

The relations (5.4), (8.4)–(8.6) mean that

$$g_0 = (l_0, \omega_0) \in G_a. \tag{8.7}$$

By Theorem 7.1, the functional Ψ_0 is continuous in G. Thus relations (8.1), (8.3), and (8.7) imply that $g_0 = (l_0, \omega_0)$ is a solution of the problem (5.5). ■

Let

$$G_0 = \{g | g = (l, \omega) \in H^1(0,T) \times H^2(0,T),\ 0 \le l(t) \le e_2,\ e_4 \le \omega(t) \le e_5,\ t \in [0,T],$$
$$\omega(0) = R_1^{-1} u_0\Big|_{S_1}\}. \tag{8.8}$$

The set G_0 is convex and the functionals

$$l \to \|l\|^2_{H^1(0,T)} - e_1,\ \omega \to \|\omega\|^2_{H^2(0,T)} - e_3$$

are continuously Fréchet differentiable in $H^1(0,T)$ and $H^2(0,T)$, respectively. The functionals Ψ_0 and Ψ_1 are also continuously Fréchet differentiable in G. Thus, by applying the known results (see e.g. [10], Chapter 2, Section 1, [28], Theorem 4.1, [31], Chapter 3, Section 10), we obtain that the optimal control $g_0 = (l_0, \omega_0)$ satisfies the following necessary optimality conditions:

Theorem 8.2. *Suppose that the conditions (A0), (A1), and (A2) are satisfied. Let* (2.8), (3.7), (4.8), (4.9) *hold and assume that the set G_a is nonempty. Then the following conditions are fulfilled:*

There exist Lagrange multipliers $\lambda_0 \ge 0$, $\lambda_1 \ge 0$, $\lambda_2 \ge 0$, $\lambda_3 \ge 0$ such that

$$\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \ne 0, \tag{8.9}$$

$$\lambda_0 \Psi_0'(g_0)(g - g_0) + \lambda_1 \Psi_1'(g_0)(g - g_0) + \lambda_2 (l_0, l - l_0)_{H^1(0,T)}$$
$$+ \lambda_3 (\omega_0, \omega - \omega_0)_{H^2(0,T)} \ge 0, \quad g = (\lambda, \omega) \in G_0, \tag{8.10}$$

$$\lambda_1(\Psi_1(g_0) - e_6) = 0, \quad \lambda_2(\| l_0 \|^2_{H^1(0,T)} - e_1) = 0,$$
$$\lambda_3(\| \omega_0 \|^2_{H^2(0,T)} - e_3) = 0, \tag{8.11}$$

where $(.,.)_{H^1(0,T)}$ and $(.,.)_{H^2(0,T)}$ are scalar products in $H^1(0,T)$ and $H^2(0,T)$.

If the functionals $\Psi_1'(g_0)$, $(l_0, .)_{H^1(0,T)}$, and $(\omega_0, .)_{H^2(0,T)}$ are linearly independent, then $\lambda_0 \ne 0$ and one can take $\lambda_0 = 1$.

It is easy to see that the condition of the linear independence of the functionals $\Psi_1'(g_0)$, $(l_0, .)_{H^1(0,T)}$, and $(\omega_0, .)_{H^2(0,T)}$ is practically always satisfied

9. Optimal control problem with point-finite restrictions.

The continuum of restrictions that is contained in G_0 is inconvenient for numerical solution of the problem (5.5). Because of this, we consider optimal control problem with restrictions on values of l and ω at discrete points.

Let m be a positive integer, and let k denotes the corresponding time-step: $k = T/m$, and t_n be the subdivisions of $[0, T]$, $t_n = nk,\ n = 0, 1, 2, ..., m$. Let

$$\mathcal{G}_m = \{g | g = (l, \omega) \in H^1(0, T) \times H^2(0, T),$$
$$0 \leq l(t_n) \leq e_2,\ e_4 \leq \omega(t_n) \leq e_5,\ n = 0, 1, 2, ..., m,\ \omega(0) = R_1^{-1} u_0\Big|_{S_1}\}, \tag{9.1}$$

$$\mathcal{G}_{am} = \{g | g = (l, \omega) \in \mathcal{G}_m,\ \|l\|^2_{H^1(0,T)} \leq e_1,\ \|\omega\|^2_{H^2(0,T)} \leq e_3,\ \Psi_1(g) \leq e_6\}. \tag{9.2}$$

We consider the following optimal control problem: Find g_m satisfying

$$g_m = (l_m, \omega_m) \in \mathcal{G}_{am},\ \Psi_0(g_m) = \inf \Psi_0(g),\ g \in \mathcal{G}_{am}. \tag{9.3}$$

Theorem 9.1. *Suppose that the conditions (A0), (A1), and (A2) are satisfied. Let* (2.8), (3.7), (4.8), (4.9) *hold, and assume that the set G_a is nonempty. Then for any m there exists a solution of the problem* (9.3)*, and the function g_m satisfies the conditions* (8.9), (8.10), (8.11) *in which g_0 and G_0 are replaced by g_m and $\mathcal{G}_m$. A subsequence $\{g_i\}$ can be extracted from the sequence $\{g_m\}$ such that*

$$g_i = (l_i, \omega_i) \to g_0 = (l_0, \omega_0) \ \textit{in}\ C([0, T]) \times C^1([0, T]), \tag{9.4}$$

$$N(g_i) \to N(g_0) \ \textit{in}\ H^{2,1}_q(Q), \tag{9.5}$$

where $g_0 = (l_0, \omega_0)$ is a solution of the problem (5.5)*.*

Proof. It follows from the proof of theorems 8.1 and 8.2 that there exists a solution of the problem (9.3) for any m, and the function g_m satisfies the conditions (8.9), (8.10), (8.11), wherein g_0 and G_0 are replaced by g_m and $\mathcal{G}_m$

By (9.2) the sequence $\{g_m = (l_m, \omega_m)\}$ is bounded in $H^1(0, T) \times H^2(0, T)$. Therefore, a subsequence $\{g_i = (l_i, \omega_i)\}$ can be extracted such that

$$l_i \rightharpoonup \hat{l} \quad \text{in } H^1(0, T), \quad \omega_i \rightharpoonup \hat{\omega} \quad \text{in } H^2(0, T), \tag{9.6}$$

$$g_i = (l_i, \omega_i) \to \hat{g} = (\hat{l}, \hat{\omega}) \text{ in } H^\beta(0, T) \times H_q^{1-\frac{1}{2q}}(0, T)$$
$$\text{and in } C^\alpha([0, T]) \times C^{1+\alpha}([0, T]),\ \alpha \in (0, 1/2]. \tag{9.7}$$

(9.2), (9.3), and (9.7) yield

$$\lim \Psi_0(g_i) = \Psi_0(\hat{g}) \leq \inf \Psi_0(g),\ g \in G_a. \tag{9.8}$$

By analogy with the proof of Theorem 8.1, we obtain with the use of (9.6) and (9.7) that $\hat{g} \in G_a$. Because of this, (9.8) implies

$$\Psi_0(\hat{g}) = \inf \Psi_0(g), \quad g \in G_a. \tag{9.9}$$

Therefore, the function $g_0 = \hat{g}$ is a solution of the problem (5.5). ■

We emphasize that in practice, the restrictions on the values of the functions l and ω are non strictly (not exactly) specified. Because of this, for practical purposes, one can solve the problem (9.3) at moderate values of m.

10. Cylindrical clutch

In the case of cylindrical clutch, R_1 and R_2 are positive constants, $R_2 > R_1$ and $\delta = R_2 - R_1$ is small as compared to R_1 and the length of the clutch. Because of this, one can consider that $\Omega = (R_1, R_2)$. Then the motion equation (2.3) takes the following form:

$$\frac{\partial}{\partial r}\Big(\varphi\Big(\frac{\partial u}{\partial r} - \frac{u}{r}\Big)\Big) + \frac{2}{r}\varphi\Big(\frac{\partial u}{\partial r} - \frac{u}{r}\Big) = \rho\,\frac{\partial u}{\partial t} \qquad \text{in} \quad Q = (R_1, R_2) \times (O, T). \tag{10.1}$$

The viscosity function φ is defined by (2.2) with $I(u) = \frac{1}{2}\Big(\frac{\partial u}{\partial r} - \frac{u}{r}\Big)^2$. The boundary conditions take the form

$$u(R_1, t) = \omega(t) R_1. \tag{10.2}$$

$$u(R_2, t) = R_2\{\omega_1(0) - (\rho_0 I_0)^{-1} \int_0^t \Big[\Big(\varphi\Big(\frac{\partial u}{\partial r} - \frac{u}{r}\Big)\Big)\Big|_{R_2} R_2 l_1 + M_{ex}\Big]\, d\tau\} \quad t \in (0, T), \tag{10.3}$$

where l_1 is the length of the clutch. The initial condition is given by

$$u(r, 0) = u_0(r), \quad r \in (R_1, R_2). \tag{10.4}$$

We suppose that the following conditions of concordance are satisfied:

$$u_0(R_1) = \omega(0) R_1, \quad u_0(R_2) = \omega_1(0) R_2. \tag{10.5}$$

By analogy with the above, it is proved the following result:

Theorem 10.1. *Let the conditions (A1) and (A2) be satisfied. Suppose that the terms (10.5) hold and*

$$\omega \in H_q^{1-\frac{1}{2q}}(0, T), \tag{10.6}$$

$$u_0 \in H_q^{2-\frac{2}{q}}(R_1, R_2), \quad q > 3, \tag{10.7}$$

$$M_{ex} \in C([0, T]). \tag{10.8}$$

Let also a function $E \in C([0, T]; H_q^1(R_1, R_2))$ is given. Then there exists a unique function $u \in H_q^{2,1}((R_1, R_2) \times (0, T))$ which is the solution of the problem (10.1)-(10.4)

All the above results with relevant simplifications remain valid for the cylindrical clutch.

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