Universität Augsburg

Institut für Mathematik

Xuejun Xu, Huangxin Chen, Ronald H.W. Hoppe

Local Multilevel Methods for Adaptive Nonconforming Finite Element Methods

Preprint Nr. 21/2009 — 14. August 2009
Institut für Mathematik, Universitätsstraße, D-86135 Augsburg http://www.math.uni-augsburg.de/

Impressum:

Herausgeber:

Institut für Mathematik
Universität Augsburg
86135 Augsburg
`http://www.math.uni-augsburg.de/pages/de/forschung/preprints.shtml`

ViSdP:

Ronald H.W. Hoppe
Institut für Mathematik
Universität Augsburg
86135 Augsburg

LOCAL MULTILEVEL METHODS FOR ADAPTIVE NONCONFORMING FINITE ELEMENT METHODS

XUEJUN XU, HUANGXIN CHEN, AND RONALD H.W. HOPPE

Abstract. In this paper, we propose a local multilevel product algorithm and its additive version for linear systems arising from adaptive nonconforming finite element approximations of second order elliptic boundary value problems. The abstract Schwarz theory is applied to analyze the multilevel methods with Jacobi or Gauss-Seidel smoothers performed on local nodes on coarse meshes and global nodes on the finest mesh. It is shown that the local multilevel methods are optimal, i.e., the convergence rate of the multilevel methods is independent of the mesh sizes and mesh levels. Numerical experiments are given to confirm the theoretical results.

.

Introduction

Multigrid methods and other multilevel preconditioning methods for nonconforming finite elements have been studied by many researchers (cf. [4], [5], [6], [7], [14], [18], [19], [20], [21], [23], [26], [27], [28],[32], [34]). The BPX framework developed in [4] provides a unified convergence analysis for nonnested multigrid methods. Duan *et al.* [14] extended the result to general V-cycle nonnested multigrid methods, but only the case of full elliptic regularity was considered. Besides, Brenner [7] established a framework for the nonconforming V-cycle multigrid method under less restrictive regularity assumptions. For multilevel preconditioning methods, Oswald developed a hierarchical basis multilevel method [19] and a BPX-type multilevel preconditioner [20] for nonconforming finite elements. Vassilevski and Wang [26] presented some multilevel algorithms for nonconforming finite element methods and obtained a uniform convergence result without additional regularity beyond H^1. Furthermore, Hoppe and Wohlmuth [15] considered multilevel preconditioned conjugate gradient methods for nonconforming P_1 finite element approximations with respect to adaptively generated hierarchies of nonuniform meshes based on residual type a posteriori error estimators.

Recent studies (cf., e.g., [2], [10], [11], [17], [24]) indicate optimal convergence properties of adaptive conforming and nonconforming finite element methods. Therefore, in order to achieve an optimal numerical solution, it is imperative to study efficient iterative algorithms for the solution of linear systems arising from adaptive

2000 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.

Key words and phrases. local multilevel methods, adaptive nonconforming finite element methods, convergence analysis, optimality.

The work of the first two authors was supported in parts by the National Basic Research Program of China (Grant No. 2005CB321701) and the National Natural Science Foundation of China (Grant No. 10731060).

The work of the third author was supported in parts by NSF grants DMS-0707602, DMS-0810156, DMS-0811153, and DMS-0914788.

finite element methods (**AFEM**). Since the number of degrees of freedom N per level may not grow exponentially with mesh levels, as Mitchell has pointed out in [16] for adaptive conforming finite element methods, the number of operations used for multigrid methods with smoothers performed on all nodes can be as bad as $O(N^2)$, and a similar situation may also occur in the nonconforming case.

For adaptive conforming finite element methods, Wu and Chen [29] have obtained uniform convergence for the multigrid V-cycle algorithm which performs Gauss-Seidel smoothing on newly generated nodes and those old nodes where the support of the associated nodal basis function has changed. To our knowledge, so far there does not exist an optimal multilevel method for nonconforming finite element methods on locally refined meshes. The reason is that the theoretical analysis for the local multilevel methods is rather difficult. Indeed, there are two difficulties which need to be overcome. First, the Xu and Zikatanov identity [31], on which the proof in [29] depends, can not be applied directly, because the multilevel spaces are nonnested in this situation. The second difficulty is how to establish the strengthened Cauchy-Schwarz inequality on nonnested multilevel spaces. In this paper, we will construct a special prolongation operator from the coarse space to the finest space, and obtain the key global strengthened Cauchy-Schwarz inequality. Two multilevel methods, the product and additive version, are proposed. Applying the well-known Schwarz theory (cf. [25]), we show that local multilevel methods for adaptive nonconforming finite element methods are optimal, i.e., the convergence rate of the multilevel algorithms is independent of mesh sizes and mesh levels.

The remainder of this paper is organized as follows: In section 2, we introduce some notations and briefly review nonconforming P1 finite element methods. Section 3 is concerned with the study of condition number estimates of linear systems arising from adaptive nonconforming finite element methods by applying the techniques presented by Bank and Scott in [1]. The following section 4 is devoted to the derivation of a local multilevel product algorithm and its additive version. In section 5, we develop an abstract Schwarz theory based on three assumptions whose verification is carried out for local Jacobi and local Gauss-Seidel smoothers, respectively. Finally, in the last section we give some numerical experiments to confirm the theoretical analysis.

1. Notations and Preliminaries

Throughout this paper, we adopt standard notation from Lebesgue and Sobolev space theory (cf., e.g., [13]). In particular, we refer to $(\cdot,\cdot)$ as the inner product in $L^2(\Omega)$ and to $\|\cdot\|_{1,\Omega}$ as the norm in the Sobolev space $H^1(\Omega)$. We further use $A \lesssim B$, if $A \leq CB$ with a positive constant C depending only on the shape regularity of the meshes. $A \approx B$ stands for $A \lesssim B \lesssim A$. We consider elliptic boundary value problems in polyhedral domains $\Omega \subset R^n$, $n \geq 2$. However, for the sake of simplicity the analysis of local multilevel methods will be restricted to the 2D case.

Given a bounded, polygonal domain $\Omega \subset R^2$, we consider the following second order elliptic boundary value problem

$$\mathcal{L}u := -\mathrm{div}(a(x)\nabla u) = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

The choice of a homogeneous Dirichlet boundary condition is made for ease of presentation only. Similar results are valid for other types of boundary conditions and equation (1.1) with a lower order term as well. We further assume that the coefficient functions in (1.1) satisfy the following properties:
(**a**) $a(\cdot)$ is a measurable function and there exist constants $\beta_1 \geq \beta_0 > 0$ such that

$$\beta_0 \leq a(x) \leq \beta_1 \quad \text{f.a.a. } x \in \Omega; \tag{1.3}$$

(**b**) $f \in L^2(\Omega)$.

The weak formulation of (1.1) and (1.2) is to find $u \in V := H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad , \quad v \in V, \tag{1.4}$$

where the bilinear form $a : V \times V \to \mathbb{R}$ is given by

$$a(u,v) = (a\nabla u, \nabla v) \quad , \quad u,v \in V. \tag{1.5}$$

Since the bilinear form (1.5) is bounded and V-elliptic, the existence and uniqueness of the solution of (1.4) follows from the Lax-Milgram theorem.

Throughout this paper, we work with families of shape regular meshes $\{\mathcal{T}_i, i = 0,1,...,J\}$, where $\mathcal{T}_0$ is an intentionally chosen coarse initial triangulation, the others are obtained by adaptive procedures, refined by the newest vertex bisection algorithm. It has been proved that there exists a constant $\theta > 0$ such that

$$\theta_T \geq \theta \quad , \quad T \in \mathcal{T}_i, \; i = 1,2,..., \tag{1.6}$$

where θ_T is the minimum angle of the element T. The set of edges on $\mathcal{T}_i$ is denoted by $\mathcal{E}_i$, and the set of interior and boundary edges by $\mathcal{E}_i^0$ and $\mathcal{E}_i^{\partial\Omega}$, respectively. Correspondingly, let $\mathcal{M}_i$ denote all the middle points of $\mathcal{E}_i$ and $\mathcal{M}_i^0$ be the middle points of $\mathcal{E}_i^0$. We refer to $\mathcal{N}_i$ as the set of interior nodes of $\mathcal{T}_i$. For any $E \in \mathcal{E}_i$, $h_{i,E}$ and $m_{i,E}$ denote the length and the midpoint of E. The patch $\omega_{i,E}, E \in \mathcal{E}_i^0$, is the union of two elements in $\mathcal{T}_i$ sharing E. For any $T \in \mathcal{T}_i$, $h_{i,T}$ and x_T stand for the diameter and the barycenter of T.

We denote by V_J the lowest order nonconforming Crouzeix-Raviart finite element space with respect to $\mathcal{T}_J$, i.e.,

$$V_J = \{v_J \in L^2(\Omega) \mid v_J|_T \in P_1(T), T \in \mathcal{T}_J \;,\; \int_E [v_J] ds = 0, E \in \mathcal{E}_J\}.$$

Here, $[v_J]|_E$ refers to the jump of v_J across $E \in \mathcal{E}_J^0$ and is set to zero for $E \in \mathcal{E}_J^{\partial\Omega}$. Moreover, we define the conforming P_1 finite element space by

$$V_J^c = \{v_J^c \in V \mid v_J^c|_T \in P_1(T), T \in \mathcal{T}_J\}.$$

The nonconforming finite element approximation of (1.4) is to find $u_J \in V_J$ such that

$$a_J(u_J, v_J) = (f, v_J) \quad , \quad v_J \in V_J, \tag{1.7}$$

where $a_J(\cdot,\cdot)$ stands for the mesh-dependent bilinear form

$$a_J(u_J, v_J) = \sum_{T \in \mathcal{T}_J} (a\nabla u_J, \nabla v_J)_{0,T}. \tag{1.8}$$

Existence and uniqueness of the solution u_J again follows from the Lax-Milgram theorem. In the sequel, we refer to $\|\cdot\|_{1,J}$ as the mesh-dependent energy norm

$$\|v_J\|_{1,J}^2 = \sum_{T\in\mathcal{T}_J} \|v_J\|_{1,T}^2 .$$

For brevity, we will drop the subscript J from some of the above quantities, if no confusion is possible, e.g., we will write h_T instead of $h_{J,T}$ and $a(\cdot,\cdot)$ instead of $a_J(\cdot,\cdot)$.

2. Condition number estimate

The computation of the solution u_J of (1.7) always requires to solve a matrix equation using a particular basis for V_J. Suppose that $\{\phi_i,\ i=1,...,N\}$ is a given basis for V_J, where N is the dimension of V_J, and define the matrix A and the vector F according to

$$A_{ij} := a(\phi_i,\phi_j) \quad \text{and} \quad F_i := (f,\phi_i) \quad , \quad i,j=1,...,N.$$

Then, equation (1.7) is equivalent to the linear algebraic system

$$AX = F, \tag{2.1}$$

where $u_J = \sum_{i=1}^N u_i\phi_i$ and $X=(u_i)$.

In this section, we will not restrict ourselves to the two-dimensional case, but consider domains $\Omega \subset R^n, n\geq 2$. We will specify conditions on V_J and the basis $\{\phi_i,\ i=1,...,N\}$ that will allow us to establish upper bounds for the condition number of A.

We assume that $\mathcal{T}_J$ contains at most $\alpha_1^{n/2}N$ elements, with α_1 denoting a fixed constant. The following estimates hold true (cf., e.g., [13]):

$$\|v\|_{1,T}^2 \lesssim h_T^{n-2}\|v\|_{\mathcal{L}^\infty(T)}^2 \lesssim \|v\|_{\mathcal{L}^{2n/(n-2)}(T)}^2 \ ,\ T\in\mathcal{T}_J \ ,\ v\in V_J \ ,\ n\geq 3. \tag{2.2}$$

In the special case of two dimensions ($n=2$), we supplement the following inequality to the latter one in (2.2),

$$\|v\|_{\mathcal{L}^\infty(T)} \lesssim h_T^{-2/p}\|v\|_{\mathcal{L}^p(T)} \quad , \quad T\in\mathcal{T}_J \ ,\ v\in V_J \ ,\ 1\leq p\leq\infty. \tag{2.3}$$

Under the assumptions on the domain Ω, there exists a continuous embedding $H^1(\Omega)\hookrightarrow\mathcal{L}^p(\Omega)$. For $n\geq 3$, Sobolev's inequality

$$\|v\|_{\mathcal{L}^{2n/(n-2)}(\Omega)} \leq C\|v\|_{1,\Omega} \quad , \quad v\in H^1(\Omega). \tag{2.4}$$

holds true. In two dimensions, we have a more explicit estimate (cf., e.g., [1])

$$\|v\|_{\mathcal{L}^p(\Omega)} \leq C\sqrt{p}\|v\|_{1,\Omega} \quad , \quad v\in H^1(\Omega) \ ,\ p<\infty. \tag{2.5}$$

As far as the basis $\{\phi_i,\ i=1,...,N\}$ of V_J is concerned, we assume that it is a local basis:

$$\max_{1\leq i\leq N} \text{cardinality}\{T\in\mathcal{T}_J : \text{supp}(\phi_i)\cap T\neq\varnothing\} \leq \alpha_2. \tag{2.6}$$

Finally, we impose a more important assumption with regard to the scaling of the basis:

$$h_T^{n-2}\|v\|_{\mathcal{L}^\infty(T)}^2 \lesssim \sum_{\text{supp}(\phi_i)\cap T\neq\varnothing} v_i^2 \lesssim h_T^{n-2}\|v\|_{\mathcal{L}^\infty(T)}^2 \quad , \quad T\in\mathcal{T}_J, \tag{2.7}$$

where $v = \sum_{i=1}^{N} v_i\phi_i$ and (v_i) is arbitrary. For instance, if $\{\psi_i,\, i = 1, ..., N\}$ denote the Crouzeix-Raviart P1 nonconforming basis functions, we define a new scaled basis $\{\phi_i,\, i = 1, ..., N\}$ by

$$\phi_i := h_i^{(2-n)/2}\psi_i,$$

where h_i is the diameter of the support of ψ_i. Then, the new basis satisfies assumption (2.7). We also impose the same assumption (2.7) for the conforming finite element basis, when utilized in the sequel.

For the analysis of the condition number, we propose a prolongation operator from V_J to $\tilde{V}^c_{J+1}$, where $\tilde{V}^c_{J+1}$ is the conforming finite element space based on $\tilde{\mathcal{T}}_{J+1}$. $\tilde{\mathcal{T}}_{J+1}$ is an auxiliary triangulation, only used in the analysis, which is obtained from $\mathcal{T}_J$ by subdividing each $T \in \mathcal{T}_J$ into 2^n simplices by joining the midpoints of the edges. We refer to T as an element in $\mathcal{T}_J$ with vertices $x_k, k = 1, ..., n+1$, and denote the midpoints of its edges by $m_1, ..., m_s$, where s is the number of edges of T, e.g., $s = 3$ if $n = 2$.

In case $n = 2$, the prolongation operator $I_J^{J+1} : V_J \to \tilde{V}^c_{J+1}$ is defined by

$$I_J^{J+1}v(m_l) = v(m_l)\ ,\ l = 1, ..., 3 \quad , \quad I_J^{J+1}v(x_k) = \beta_k\ ,\ k = 1, ..., 3,$$

where β_k is the average of v in x_k. Moreover, $I_J^{J+1}(x_k) = 0$, if x_k is located on the Dirichlet boundary. The stability analysis of I_J^{J+1} has been derived when $\tilde{\mathcal{T}}_{J+1}$ is obtained from $\mathcal{T}_J$ by the above bisection algorithm. In the **AFEM** procedures we use the newest vertex bisection algorithm. The associated stability analysis of $I_{i-1}^{i}, i = 1, ..., J$, will be given in the appendix of this paper.

As in the case $n \geq 3$, we define $I_J^{J+1} : V_J \to \tilde{V}^c_{J+1}$ according to

$$I_J^{J+1}v(m_l) = \alpha_l\ ,\ l = 1, ..., s \quad , \quad I_J^{J+1}v(x_k) = \beta_k\ ,\ k = 1, ..., n+1,$$

where α_l and β_k are the averages of v at m_l and x_k respectively. $I_J^{J+1}(x_k) = 0$ or $I_J^{J+1}(m_l) = 0$, if x_k or m_l is situated on the Dirichlet boundary. The associated stability analysis of I_J^{J+1} can be obtained analogously.

We now give bounds on the condition number of the matrix $A := (a(\phi_i, \phi_j))$, where $\{\phi_i,\, i = 1, ..., N\}$ is the scaled basis for V_J satisfying the above assumptions.

In the general case $n \geq 3$, we have the following result.

Theorem 2.1. Suppose that the nonconforming finite element space V_J satisfies (2.2) and the basis $\{\phi_i,\, i = 1, ..., N\}$ satisfies (2.6) and (2.7). Then, the ℓ_2-condition number $\mathcal{K}_2(A)$ of A is bounded by

$$\mathcal{K}_2(A) \lesssim N^{2/n}. \tag{2.8}$$

Proof. We set $v = \sum_{i=1}^{N} v_i\phi_i$, then

$$a(v, v) = X^t A X,$$

where $X = (v_i)$. By a similar technique as in the proof of Theorem 4.1 in [1], we have

$$a(v, v) \lesssim X^t X.$$

On the other hand, we apply the prolongation operator I_J^{J+1} to v, and set

$$I_J^{J+1}v = \sum_{x_i \in \mathcal{N}_{J+1}(\tilde{\mathcal{T}}_{J+1})} I_J^{J+1}v(x_i)\tilde{\phi}_{i,J+1},$$

where $\{\tilde{\phi}_{i,J+1}\}$ is the conforming finite element basis of $\tilde{V}^c_{J+1}$. By Hölder's inequality, Sobolev's inequality, and the stability of I_J^{J+1}, we derive a complementary inequality according to

$$
\begin{aligned}
X^t X &\leq \sum_{T\in\tilde{\mathcal{T}}_{J+1}} \sum_{\mathrm{supp}(\phi_{i,J+1})\cap T\neq\varnothing} I_J^{J+1} v^2(x_i) \lesssim \sum_{T\in\tilde{\mathcal{T}}_{J+1}} h_T^{n-2} \|I_J^{J+1} v\|^2_{\mathcal{L}^\infty(T)} \\
&\lesssim \sum_{T\in\tilde{\mathcal{T}}_{J+1}} \|I_J^{J+1} v\|^2_{\mathcal{L}^{2n/(n-2)}(T)} \lesssim N^{2/n} \|I_J^{J+1} v\|^2_{\mathcal{L}^{2n/(n-2)}(\Omega)} \\
&\lesssim N^{2/n} \|I_J^{J+1} v\|^2_{1,\Omega} \lesssim N^{2/n} \|v\|^2_{1,J} \lesssim N^{2/n} a(v,v).
\end{aligned}
$$

Using the above estimates, we obtain

$$N^{-2/n} X^t X \lesssim X^t A X \lesssim X^t X,$$

which implies that

$$N^{-2/n} \lesssim \lambda_{\min}(A) \quad \text{and} \quad \lambda_{\max}(A) \lesssim 1.$$

Recalling

$$\mathcal{K}_2(A) = \lambda_{\max}(A)/\lambda_{\min}(A),$$

the above two estimates yield (2.8). □

In the special case $n = 2$, a similar result can be deduced as follows.

Theorem 2.2. Suppose that the nonconforming finite element space V_J satisfies (2.2) and (2.3), and that the basis $\{\phi_i,\, i = 1, ..., N\}$ satisfies (2.6) and (2.7). Then, the ℓ_2-condition number $\mathcal{K}_2(A)$ of A is bounded by

$$\mathcal{K}_2(A) \lesssim N(1 + |\log(N h^2_{\min}(\mathcal{E}_J))|). \tag{2.9}$$

Proof. As in the proof of the above theorem, it suffices to show that

$$N(1 + |\log(N h^2_{\min}(\mathcal{E}_J))|)^{-1} X^t X \lesssim X^t A X \lesssim X^t X. \tag{2.10}$$

We set $v = \sum_{i=1}^N v_i \phi_i$, $X = (v_i)$ and $a(v,v) = X^t A X$. Then, $a(v,v) \lesssim X^t X$ holds true as in Theorem 5.1 in [1].

As far as the lower bound in (2.10) is concerned, as in the proof of Theorem 2.1 we have ($p > 2$)

$$
\begin{aligned}
X^t X &\leq \sum_{T\in\tilde{\mathcal{T}}_{J+1}} \sum_{\mathrm{supp}(\phi_{i,J+1})\cap T\neq\varnothing} I_J^{J+1} v^2(x_i) \lesssim \sum_{T\in\tilde{\mathcal{T}}_{J+1}} \|I_J^{J+1} v\|^2_{\mathcal{L}^\infty(T)} \\
&\lesssim \sum_{T\in\tilde{\mathcal{T}}_{J+1}} h_T^{-4/p} \|I_J^{J+1} v\|^2_{\mathcal{L}^p(T)} \lesssim (\sum_{T\in\tilde{\mathcal{T}}_{J+1}} h_T^{-4/(p-2)})^{(p-2)/p} \|I_J^{J+1} v\|^2_{\mathcal{L}^p(T)} \\
&\lesssim (\sum_{T\in\tilde{\mathcal{T}}_{J+1}} h_T^{-4/(p-2)})^{(p-2)/p} p \, \|I_J^{J+1} v\|^2_{1,T} \lesssim (\sum_{T\in\tilde{\mathcal{T}}_{J+1}} h_T^{-4/(p-2)})^{(p-2)/p} p \, a(v,v) \\
&\lesssim N(N h^2_{\min}(\mathcal{E}_J))^{-2/p} p \, a(v,v).
\end{aligned}
$$

The special choice $p = \max\{2, |\log(N h^2_{\min}(\mathcal{E}_J))|\}$ allows to conclude. □

For a fixed triangulation, the conforming P1 finite element space is contained in the nonconforming P1 finite element space. Hence, the sharpness of the bounds in Theorem 2.2 can be verified by the same example as in [1].

3. Local multilevel methods

The above section clearly shows that for the solution of a large scale problem the convergence of standard iterations such as Gauss-Seidel or CG will become very slow. This motivates the construction of more efficient iterative algorithms for those algebraic systems resulting from adaptive nonconforming finite element approximations.

We will derive our local multilevel methods for adaptive nonconforming finite element discretizations based on the Crouzeix-Raviart elements. As a prerequisite, we again use the prolongation operator $I_{i-1}^i : V_{i-1} \to V_i$ defined as in section 2. Now, $\mathcal{T}_i$ represents a refinement of $\mathcal{T}_{i-1}$ by the newest vertex bisection algorithm, I_{i-1}^i defines the values of $I_{i-1}^i v$ at the vertices of elements of level i, yielding a continuous piecewise linear function on $\mathcal{T}_i$. $I_{i-1}^i v$ being a function in V_i, it naturally represents a function in the finest space V_J. Hence, the operator I_{i-1} given by

$$I_{i-1}v := I_{i-1}^i v \quad , \quad v \in V_{i-1},$$

defines an intergrid operator from V_{i-1} to V_J.

For $0 \le i \le J$, we define $A_i : V_i \to V_i$ by means of

$$(A_i v, w) = a_i(v, w) \quad , \quad w \in V_i.$$

We also define projections P_i, $P_i^0 : V_J \to V_i$ according to

$$a_i(P_i v, w) = a(v, I_i w) \quad , \quad (P_i^0 v, w) = (v, I_i w) \quad , \quad v \in V_J,\, w \in V_i.$$

For any node $z \in \mathcal{N}_i$, we use the notation φ_i^z to represent the associated nodal conforming basis function of V_i^c. Let $\tilde{\mathcal{N}}_i^c$ be the set of new nodes and those old nodes where the support of the associated basis function has changed, i.e.,

$$\tilde{\mathcal{N}}_i^c = \{z \in \mathcal{N}_i : z \in \mathcal{N}_i \setminus \mathcal{N}_{i-1} \text{ or } z \in \mathcal{N}_{i-1} \text{ but } \varphi_i^z \neq \varphi_{i-1}^z\}.$$

Let $\tilde{\mathcal{M}}_i$ represent the set of midpoints on which local smoothers are performed:

$$\tilde{\mathcal{M}}_i := \{m_{i,E} \in \mathcal{M}_i : m_{i,E} \in \mathcal{M}_i^0(\hat{\mathcal{T}}_i)\},$$

where $\hat{\mathcal{T}}_i = \bigcup_{x_i^z \in \tilde{\mathcal{N}}_i^c} \{\text{supp}(\varphi_i^z)\}$.

For convenience, we set $\tilde{\mathcal{M}}_i = \{m_i^k, k = 1, ..., \tilde{n}_i\}$,where $\tilde{n}_i$ is the cardinality of $\tilde{\mathcal{M}}_i$, and refer to $\phi_i^k = \phi_i^{m_i^k}$ as the Crouzeix-Raviart nonconforming finite element basis function associated with m_i^k. Then, for $k = 1, ..., \tilde{n}_i$ let $P_i^k, Q_i^k : V_i \to V_i^k = \text{span}\{\phi_i^k\}$ be defined by

$$a_i(P_i^k v, \phi_i^k) = a_i(v, \phi_i^k) \quad , \quad (Q_i^k v, \phi_i^k) = (v, \phi_i^k) \quad , \quad v \in V_i\,,$$

and let $A_i^k : V_i^k \to V_i^k$ be defined by

$$(A_i^k v, \phi_i^k) = a_i(v, \phi_i^k) \quad , \quad v \in V_i^k.$$

It is easy to see that the following relationship holds true:

$$A_i^k P_i^k = Q_i^k A_i. \tag{3.1}$$

We assume that the local smoothing operator $R_i : V_i \to V_i$ is nonnegative, symmetric or nonsymmetric with respect to the inner product $(\cdot, \cdot)$. It will be precisely defined and further studied in section 4. For $i = 1, ..., J-1$, R_i is only performed on local midpoints $\tilde{\mathcal{M}}_i$ (we refer to Figure 1 for an illustration). R_0 is solved directly, i.e., $R_0 = A_0^{-1}$. On the finest level, R_J is carried out on all

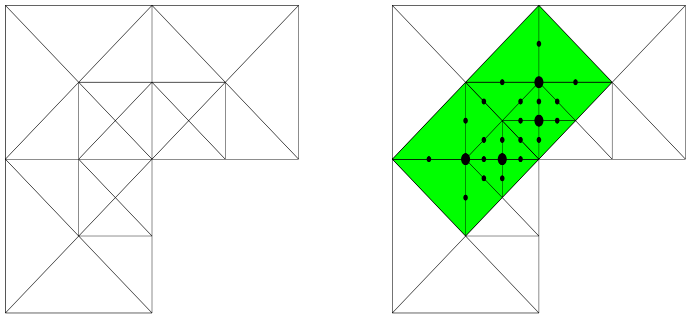

FIGURE 1. *Coarse mesh (left), fine mesh (right) and illustration of* $\tilde{\mathcal{M}}_i$*: the big nodes on the right refer to* $\tilde{\mathcal{N}}_i$*, the small nodes refer to* $\tilde{\mathcal{M}}_i, i = 1, ..., J-1$.

midpoints $\mathcal{M}_J^0$, i.e., $\tilde{n}_J = \#\mathcal{M}_J^0$. For simplicity, we set $A = A_J$ and denote by I_J and P_J the identity operator on the finest space V_J. We set

$$S_i := I_i R_i A_i P_i \quad , \quad i = 0, 1, ..., J.$$

Now, we scale S_i as follows:

$$T_i := \mu_{J,i} S_i \quad , \quad i = 0, 1, ..., J. \tag{3.2}$$

where $\mu_{J,i} > 0$ is a parameter, independent of mesh sizes and mesh levels, chosen to satisfy

$$a(T_i v, T_i v) \leq \omega_i\, a(T_i v, v) \quad , \quad v \in V_J \ , \ w_i < 2.$$

We will also drop the subscript J from $\mu_{J,i}$ since no confusion is possible in the convergence analysis.

With the sequences of operators $\{T_i, i = 0, 1, ..., J\}$, we can now state the local multilevel algorithm for adaptive nonconforming finite element methods as follows.

Algorithm 3.1. **Local multilevel product algorithm (LMPA)**

Given an arbitrarily chosen initial iterate $u^0 \in V_J$, we seek $u^n \in V_J$ as follows:
(i) Let $v_0 = u^{n-1}$. For $i = 0, 1, ..., J$, compute v_{i+1} by

$$v_{i+1} = v_i + T_i(u_J - v_i). \tag{3.3}$$

(ii) Set $u^n = v_{J+1}$.

Algorithm 3.2. **Local multilevel additive algorithm (LMAA)**

Let $T = \sum_{i=0}^{J} T_i$ and let u_J be the exact solution of (1.7). Find $\tilde{u}_J \in V_J$ such that

$$T\tilde{u}_J = \tilde{f}, \tag{3.4}$$

where $\tilde{f} = \sum_{i=0}^{J} T_i u_J$.

In view of the operator equation

$$A_i P_i = P_i^0 A,$$

the function $\tilde{f}$ in (3.4) is formally defined by the exact finite element solution u_J which can be computed directly, and so does the iteration (3.3).

Obviously, there exists a unique solution $\tilde{u}_J$ of (3.4) coinciding with u_J for (1.7). The conjugate-gradient method can be used to solve the new problem, if T is symmetric. We can also apply the conjugate-gradient method to the symmetric version of **LMAA** (**SLMAA**) by solving

$$\frac{(T+T^*)}{2}\tilde{u}_J = \hat{f}$$

instead of (3.4), where $\hat{f} = \sum_{i=0}^{J} \frac{(T_i+T_i^*)}{2} u_J$ and T^* denotes the adjoint operator of T with respect to the inner product $a(\cdot,\cdot)$.

4. Convergence theory

In this section, we provide an abstract theory concerned with the convergence of local multilevel methods for linear systems arising from adaptive nonconforming finite element methods. We will use the well-known Schwarz theory developed in [25], [30] and [35] to analyze the algorithms.

Let $\{T_i, i = 0, 1, ..., J\}$ be a sequence of operators from the finest space V_J to itself. The abstract theory provides an estimate for the norm of the error operator

$$E = (I - T_J)\cdots(I - T_1)(I - T_0) = \prod_{i=0}^{J}(I - T_i),$$

where I is the identity operator in V_J. The convergence estimate for the algorithm **LMPA** is then obtained by the norm estimate for E. The abstract theory can be invoked due to the following assumptions.

(**A1**). Each operator T_i is nonnegative with respect to the inner product $a(\cdot,\cdot)$, and there exists a positive constant $\omega_i < 2$, which depends on μ_i, such that

$$a(T_i v, T_i v) \leq \omega_i \, a(T_i v, v) \quad , \quad v \in V_J.$$

(**A2**). **Stability:** There exists a constant K_0 such that

$$a(v, v) \leq \frac{K_0}{\mu} a(Tv, v) \quad , \quad v \in V_J,$$

where $\mu = \min_{0\leq i\leq J}\{\mu_i\}$.

(**A3**). **Global strengthened Cauchy-Schwarz inequality:** There exists a constant K_1 such that

$$\sum_{i=0}^{J}\sum_{j=0}^{i-1} a(T_i v, T_j u) \leq K_1 (\sum_{i=0}^{J} a(T_i v, v))^{1/2} (\sum_{j=0}^{J} a(T_j u, u))^{1/2} \ , \ v, u \in V_J.$$

As in the proof of (4.1) in [33], it is easy to show that the following inequality holds true for the algorithms **LMPA** and **LMAA** with local smoothers chosen as Jacobi or Gauss-Seidel iterations (especially $K_2 = 1$ in the Jacobi case):

$$\sum_{i=0}^{J} a(T_i v, u) \leq K_2 (\sum_{i=0}^{J} a(T_i v, v))^{1/2} (\sum_{i=0}^{J} a(T_i u, u))^{1/2} \quad , \quad v, u \in V_J. \tag{4.1}$$

Theorem 4.1. Let the assumptions **A1-A3** be satisfied. Then, for the algorithm 3.1 the norm of the error operator E can be bounded as follows (*cf.* [25], [30], [35])

$$a(Ev, Ev) \leq \delta \, a(v, v) \quad , \quad v \in V_J,$$

where $\delta = 1 - \frac{\mu(2-\omega)}{K_0(K_1+K_2)^2}$, $\omega = \max_{0\leq i\leq J}\{\omega_i\}$.

For the additive multilevel algorithm 3.2, the following theorem provides a spectral estimate for the operator $T = \sum_{i=0}^{J} T_i$ when T is symmetric with respect to the inner product $a(\cdot,\cdot)$.

Theorem 4.2. If T is symmetric with respect to $a(\cdot,\cdot)$ and assumptions **A1-A3** hold true, then we have (*cf.* [25], [30], [35])

$$\frac{\mu}{K_0}\, a(v,v) \le a(Tv,v) \le (2K_1+\omega)\, a(v,v) \quad , \quad v \in V_J.$$

When T is nonsymmetric with respect to $a(\cdot,\cdot)$, similar analysis can be done for the spectral estimate of the symmetric part $\frac{T+T^*}{2}$.

Remark 5.1. It should be pointed out that the convergence result for **LMPA** or for the preconditioned conjugate gradient method by **LMAA** depends on the parameter μ, which will be observed in our numerical experiments. The convergence rate deteriorates for decreasing μ.

Next, we will apply the above convergence theory to **LMPA** and **LMAA** by verifying assumptions **A1-A3** for the adaptive nonconforming finite element method. There are two classes of smoothers R_i, Jacobi and Gauss-Seidel iterations, which will be considered separately.

4.1. **Local Jacobi smoother.** First, for $v \in V_J$ we consider the decomposition

$$v = \sum_{i=0}^{J} v_i \ , \ v_J = v - \tilde{v} \ , \ v_i = (\Pi_i - \Pi_{i-1})\tilde{v} \ , \ i = 0,1,...,J-1, \tag{4.2}$$

where $\tilde{v} = \tilde{\Pi}_{J-1} v$ and $\tilde{\Pi}_{J-1} v$ represents a local regularization of v in V_{J-1}^c (c.f. [9]), e.g., by a Clément-type interpolation. $\Pi_i : V_{J-1}^c \to V_i^c$ stands for the Scott-Zhang interpolation operator [22].

The local Jacobi smoother is defined as an additive smoother (cf. [3]):

$$R_i := \gamma \sum_{k=1}^{\tilde{n}_i} (A_i^k)^{-1} Q_i^k, \tag{4.3}$$

where γ is a suitably chosen positive scaling factor. Due to (3.1), we have

$$T_0 = \mu_0 I_0 P_0 \ , \ T_i = \mu_i I_i R_i A_i P_i = \mu_i \gamma I_i \sum_{k=1}^{\tilde{n}_i} P_i^k P_i \ , \ i = 1,...,J. \tag{4.4}$$

4.1.1. *Verification of assumption A1.*

Lemma 4.1. Let T_i, $i \ge 0$, be defined by (4.4). Then, we have

$$a(T_i v, T_i v) \le \omega_i\, a(T_i v, v) \quad , \quad v \in V_J \ , \ \omega_i < 2.$$

Moreover, T_i is symmetric and nonnegative in V_J. Therefore, assumption **A1** is satisfied.

Proof. Following (4.4), for $v, w \in V_J$ we deduce

$$a(T_i v, w) = a(\mu_i I_i R_i A_i P_i v, w) = a_i(\mu_i R_i A_i P_i v, P_i w) = (\mu_i R_i A_i P_i v, A_i P_i w).$$

In view of the definition of R_i in (4.3), we can easily see that R_i is symmetric and nonnegative in V_i. Hence, T_i is symmetric and nonnegative in V_J.

It is easy to show that the stated result holds true for T_0. Actually, we have

$$a(T_0 v, T_0 v) \le \mu_0^2 C_0 a_0(P_0 v, P_0 v) = \mu_0 C_0 a(T_0 v, v).$$

Let $\omega_0 = \mu_0 C_0$. We choose $\mu_0 < 2/C_0$ such that $\omega_0 < 2$.

For $T_i, i \geq 1$, we set

$$K_i^k = \{P_i^m : \mathrm{supp}(I_i P_i^k v) \cap \mathrm{supp}(I_i P_i^m v) \neq \varnothing \ , \ v \in V_i\}$$

and

$$\gamma_{k,m} = \begin{cases} 1 & \text{if } \mathrm{supp}(I_i P_i^k v) \cap \mathrm{supp}(I_i P_i^m v) \neq \varnothing \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

The cardinality of K_i^k is bounded by a constant depending only on the minimum angle θ in (1.6). For $v \in V_i$, $i = 1, \ldots, J$, Hölder's inequality implies

$$\begin{aligned} \sum_{k,m=1}^{\tilde{n}_i} |a(I_i P_i^k v, I_i P_i^m v)| &= \sum_{k,m=1}^{\tilde{n}_i} \gamma_{k,m} |a(I_i P_i^k v, I_i P_i^m v)| \\ \leq \sum_{k,m=1}^{\tilde{n}_i} \gamma_{k,m} |a(I_i P_i^k v, I_i P_i^k v)| &\leq C_i \sum_{k=1}^{\tilde{n}_i} a(I_i P_i^k v, I_i P_i^k v). \end{aligned} \tag{4.5}$$

Taking advantage of the definition of T_i in (4.4), (4.5), and the stability of I_i, for $v \in V_J$ we have

$$\begin{aligned} a(T_i v, T_i v) &= \mu_i^2 \gamma^2 a(\sum_{k=1}^{\tilde{n}_i} I_i P_i^k P_i v, \sum_{k=1}^{\tilde{n}_i} I_i P_i^k P_i v) \\ &\leq \mu_i^2 \gamma^2 \sum_{k,m=1}^{\tilde{n}_i} |a(I_i P_i^k P_i v, I_i P_i^m P_i v)| \\ &\leq \mu_i^2 \gamma^2 C_i \sum_{k=1}^{\tilde{n}_i} a(I_i P_i^k P_i v, I_i P_i^k P_i v) \\ &\leq \mu_i^2 \gamma^2 C_0 C_i \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_i v) \\ &= \mu_i^2 \gamma^2 C_0 C_i \sum_{k=1}^{\tilde{n}_i} a(I_i P_i^k P_i v, v) = \mu_i \gamma C_0 C_i a(T_i v, v). \end{aligned}$$

The proof is completed by setting $\omega_i = \mu_i \gamma C_0 C_i$ and choosing

$$0 < \gamma < 1 \quad \text{and} \quad 0 < \mu_i < \frac{2}{\gamma C_0 C_i} \tag{4.6}$$

such that $\omega_i < 2$.

We remark that due to the fact that I_J is the identity we may choose $\mu_J = 1$ and $0 < \gamma < 1$ such that $\omega_J = \gamma C_J < 2$. □

4.1.2. *Verification of assumption A2.*

Lemma 4.2. Let $\{T_i, i = 0, 1, ..., J\}$ be defined by (4.4). Then, there exists a constant K_0 such that

$$a(v, v) \leq \frac{K_0}{\mu} a(Tv, v) \quad , \quad v \in V_J \quad , \quad \mu = \min_{0 \leq i \leq J} \{\mu_i\}.$$

Proof. Due to the decomposition of v in (4.2) and $I_i v_i = v_i$, $i = 0, 1, ..., J$, where v_i is defined by (4.2), there holds

$$a(v,v) = \sum_{i=0}^{J} a(v_i, v) = \sum_{i=0}^{J} a(I_i v_i, v) = \sum_{i=0}^{J} a_i(v_i, P_i v). \tag{4.7}$$

For $i = 1, ..., J$, we have

$$\begin{aligned} a_i(v_i, P_i v) &= \sum_{k=1}^{\tilde{n}_i} a_i(v_i(m_i^k)\phi_i^k, P_i v) = \sum_{k=1}^{\tilde{n}_i} a_i(v_i(m_i^k)\phi_i^k, P_i^k P_i v) \\ &\leq \sum_{k=1}^{\tilde{n}_i} a_i^{1/2}(v_i(m_i^k)\phi_i^k, v_i(m_i^k)\phi_i^k v) \cdot a_i^{1/2}(P_i^k P_i v, P_i^k P_i v) \\ &\leq (\sum_{k=1}^{\tilde{n}_i} a_i(v_i(m_i^k)\phi_i^k, v_i(m_i^k)\phi_i^k v))^{1/2} (\sum_{k=1}^{\tilde{n}_i} a(I_i P_i^k P_i v, v))^{1/2}. \end{aligned} \tag{4.8}$$

Following (4.7), we deduce

$$\begin{aligned} a(v,v) &= \sum_{i=0}^{J} a_i(v_i, P_i v) \\ &\leq (a_0(v_0, v_0) + \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} a_i(v_i(m_i^k)\phi_i^k, v_i(m_i^k)\phi_i^k))^{1/2} \\ &\cdot (a(I_0 P_0 v, v) + \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} a(I_i P_i^k P_i v, v))^{1/2}. \end{aligned} \tag{4.9}$$

Since $a_i(\phi_i^k, \phi_i^k) \approx 1$, we have

$$a_i(v_i(m_i^k)\phi_i^k, v_i(m_i^k)\phi_i^k) \approx v_i^2(m_i^k).$$

We note that the following inequality can be derived similarly as Lemma 3.3 in [29]

$$\sum_{i=1}^{J-1} \sum_{k=1}^{\tilde{n}_i} v_i^2(m_i^k) \lesssim a(\tilde{v}, \tilde{v}) = a(\tilde{\Pi}_{J-1} v, \tilde{\Pi}_{J-1} v) \lesssim a(v,v).$$

For the initial level, we have

$$a_0(v_0, v_0) = a_0(\Pi_0 \tilde{v}, \Pi_0 \tilde{v}) \lesssim a(\tilde{v}, \tilde{v}) \lesssim a(v,v).$$

For the finest level, there holds

$$\sum_{k=1}^{\tilde{n}_J} v_J^2(m_J^k) \lesssim \sum_{k=1}^{\tilde{n}_J} (h_J^k)^{-2} \|v - \tilde{\Pi}_{J-1} v\|^2_{\mathcal{L}^2(\omega_J^k)} \lesssim a(v,v),$$

where $h_J^k = h_{J,E}, m_J^k \in E, E \in \mathcal{E}_J^0$. Hence, we have

$$a_0(v_0, v_0) + \sum_{i=1}^{J} \sum_{k=1}^{\tilde{n}_i} v_i^2(m_i^k) \lesssim a(v,v). \tag{4.10}$$

Combining the above inequalities, we conclude that there exists a constant $\tilde{K}_0$ independent of mesh sizes and mesh levels such that

$$\begin{aligned} a(v,v) &\le \frac{\tilde{K}_0}{\min_{0\le i\le J}\{\mu_i\}}(\mu_0 a(I_0P_0v,v) + \sum_{i=1}^{J}\sum_{k=1}^{\tilde{n}_i} a(\mu_i I_i P_i^k P_i v, v)) \\ &\le \frac{\tilde{K}_0}{\mu\gamma}\sum_{i=0}^{J} a(T_i v, v) = \frac{\tilde{K}_0}{\mu\gamma} a(Tv,v). \end{aligned}$$

We thus obtain the stated result by setting $K_0 = \tilde{K}_0/\gamma$. □

4.1.3. *Verification of assumption A3.* As a prerequisite to verify assumption **A3**, we provide the following key lemma which will be proved in the appendix.

Lemma 4.3. For $i = 1, ..., J$, let $\mathcal{T}_i$ be a refinement of $\mathcal{T}_{i-1}$ by the newest vertex bisection algorithm and denote by $\tilde{\Omega}_j^k$ the support of $I_j\phi_j^k$. Then, for $m_j^k \in \tilde{\mathcal{M}}_j$ we have

$$\sum_{i=j+1}^{J}\sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} (\frac{h_i^l}{h_j^k})^{3/2} \lesssim 1\ ,\ \sum_{i=j+1}^{J}\sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \tilde{\Omega}_j^k}} (\frac{h_i^l}{h_j^k})^{3} \lesssim 1, \tag{4.11}$$

where $\mathcal{E}_{j+1}^k = \mathcal{E}_{j+1}(\tilde{\Omega}_j^k)$. Likewise, for $m_i^l \in \tilde{\mathcal{M}}_i$,

$$\sum_{j=1}^{i-1}\sum_{\substack{m_j^k\in\tilde{\mathcal{M}}_j,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} (\frac{h_i^l}{h_j^k})^{1/2} \lesssim 1\ ,\ \sum_{j=1}^{i-1}\sum_{\substack{m_j^k\in\tilde{\mathcal{M}}_j,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \tilde{\Omega}_j^k}} (\frac{h_i^l}{h_j^k})^{1/2} \lesssim 1. \tag{4.12}$$

We are now in a position to verify assumption **A3**.

Lemma 4.4. There exists a constant K_1 independent of mesh sizes and mesh levels such that assumption **A3** holds true.

Proof. In view of (4.4), we have

$$\begin{aligned} \sum_{i=1}^{J}\sum_{j=1}^{i-1} a(T_i v, T_j u) &= \gamma^2 \sum_{j=1}^{J}\sum_{i=j+1}^{J}\sum_{k=1}^{\tilde{n}_j} a(\mu_j I_j P_j^k P_j u, \sum_{l=1}^{\tilde{n}_i}\mu_i I_i P_i^l P_i v) \\ &= \gamma^2 \sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} a(\mu_j I_j P_j^k P_j u, \sum_{i=j+1}^{J}\sum_{l=1}^{\tilde{n}_i}\mu_i I_i P_i^l P_i v). \end{aligned}$$

Setting $\omega = \sum_{i=j+1}^{J}\sum_{l=1}^{\tilde{n}_i}\mu_i I_i P_i^l P_i v$, we have

$$\begin{aligned} a(\mu_j I_j P_j^k P_j u, \omega) &= a_j(\mu_j P_j^k P_j u, P_j^k P_j \omega) \\ &\le a_j^{1/2}(\mu_j P_j^k P_j u, \mu_j P_j^k P_j u) a_j^{1/2}(P_j^k P_j\omega, P_j^k P_j\omega), \end{aligned}$$

whence

$$\begin{aligned} \sum_{i=1}^{J}\sum_{j=1}^{i-1} a(T_i v, T_j u) \le \gamma^2 (\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} a_j(\mu_j P_j^k P_j u, \mu_j P_j^k P_j u))^{1/2} \\ \cdot (\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} a_j(P_j^k P_j\omega, P_j^k P_j\omega))^{1/2}. \end{aligned} \tag{4.13}$$

In view of (4.6), it is obvious that

$$\gamma\mu_j < \frac{2}{C_0 C_j} \lesssim 1 \quad , \quad 1 \le j \le J, \tag{4.14}$$

and there also holds $\gamma\mu_J = \gamma < \frac{2}{C_J} \lesssim 1$. If we choose $\mu_J = 1$, then

$$\begin{aligned}\gamma\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} a_j(\mu_j P_j^k P_j u, \mu_j P_j^k P_j u) \\ = \sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} \gamma\mu_j a(\mu_j I_j P_j^k P_j u, u) \lesssim \sum_{j=1}^{J} a(T_j u, u).\end{aligned} \tag{4.15}$$

Next, it suffices to show that

$$\gamma\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} a_j(P_j^k P_j\omega, P_j^k P_j\omega) \lesssim \sum_{i=2}^{J} a(T_i v, v). \tag{4.16}$$

Clearly, $a_j(\phi_j^k, \phi_j^k) \approx 1$. We note that

$$P_j^k P_j I_i P_i^l P_i v = \frac{a_j(P_j I_i P_i^l P_i v, \phi_j^k)}{a_j(\phi_j^k, \phi_j^k)}\phi_j^k \approx a_j(P_j I_i P_i^l P_i v, \phi_j^k)\phi_j^k,$$

which leads us to

$$a_j(P_j^k P_j\omega, P_j^k P_j\omega) \approx (\sum_{i=j+1}^{J}\sum_{l=1}^{\tilde{n}_i} a_j(P_j\mu_i I_i P_i^l P_i v, \phi_j^k))^2.$$

Similarly, $P_i^l P_i v \approx a_i(P_i v, \phi_i^l)\phi_i^l$. It follows that

$$\begin{aligned}&a_j(P_j\mu_i I_i P_i^l P_i v, \phi_j^k) = a(\mu_i I_i P_i^l P_i v, I_j\phi_j^k) = a_i(\mu_i P_i^l P_i v, P_i I_j \phi_j^k) \\ &\approx a_i(a_i(\mu_i P_i v, \phi_i^l)\phi_i^l, P_i I_j\phi_j^k) = a(I_i\phi_i^l, I_j\phi_j^k)a_i(\mu_i P_i v, \phi_i^l).\end{aligned}$$

Since $I_j\phi_j^k$ is conforming and piecewise linear on $\mathcal{T}_{j+1}|_{\tilde{\Omega}_j^k}$, we obtain

$$\begin{aligned}a(I_i\phi_i^l, I_j\phi_j^k) &= \sum_{\substack{T\subset\tilde{\Omega}_j^k,\\ T\in\mathcal{T}_{j+1}}} \int_T a(x)\nabla I_i\phi_i^l\cdot\nabla I_j\phi_j^k \\ &= \sum_{\substack{T\subset\tilde{\Omega}_j^k,\\ T\in\mathcal{T}_{j+1}}} \int_{\partial T} a(x)\frac{\partial I_j\phi_j^k}{\partial n} I_i\phi_i^l - \sum_{\substack{T\subset\tilde{\Omega}_j^k,\\ T\in\mathcal{T}_{j+1}}} \int_T (\nabla a(x)\cdot\nabla I_j\phi_j^k) I_i\phi_i^l.\end{aligned}$$

We set $d_j^k = \max\{h_{j+1,T} : T \in \tilde{\Omega}_j^k, T \in \mathcal{T}_{j+1}\}$. By the minimum angle property in (1.6) we have $d_j^k \approx h_j^k$. Similarly, $d_i^l \approx h_i^l$. Observing (1.3), $|\frac{\partial I_j\phi_j^k}{\partial n}| \lesssim (d_j^k)^{-1} \approx (h_j^k)^{-1}$ and

$$\begin{aligned}a_i(\mu_i P_i v, \phi_i^l) &= a_i(\mu_i P_i^l P_i v, \phi_i^l) \\ &\le a_i^{1/2}(\mu_i P_i^l P_i v, \mu_i P_i^l P_i v) a_i^{1/2}(\phi_i^l, \phi_i^l) \lesssim a^{1/2}(\mu_i^2 I_i P_i^l P_i v, v),\end{aligned} \tag{4.17}$$

we deduce

$$
\begin{aligned}
(4.18)\qquad & \sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \tilde{\Omega}_j^k}} a(I_i\phi_i^l, I_j\phi_j^k)a_i(\mu_i P_i v,\phi_i^l)\\
\lesssim & \sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} \frac{h_i^l}{h_j^k} a^{1/2}(\mu_i^2 I_i P_i^l P_i v, v)\\
+ & \sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \tilde{\Omega}_j^k}} \frac{(h_i^l)^2}{h_j^k} a^{1/2}(\mu_i^2 I_i P_i^l P_i v, v).
\end{aligned}
$$

Hence, combining (4.11), (4.17) and (4.18), we have

$$
\begin{aligned}
(4.19)\qquad a_j(P_j^kP_j\omega, P_j^kP_j\omega) \lesssim & \Big(\sum_{i=j+1}^{J}\sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} \frac{h_i^l}{h_j^k} a^{1/2}(\mu_i^2 I_i P_i^l P_i v, v)\Big)^2\\
& + \Big(\sum_{i=j+1}^{J}\sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \tilde{\Omega}_j^k}} \frac{(h_i^l)^2}{h_j^k} a^{1/2}(\mu_i^2 I_i P_i^l P_i v, v)\Big)^2\\
\lesssim & \Big(\sum_{i=j+1}^{J}\sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} \Big(\frac{h_i^l}{h_j^k}\Big)^{1/2} a(\mu_i^2 I_i P_i^l P_i v, v)\Big)\Big(\sum_{i=j+1}^{J}\sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} \Big(\frac{h_i^l}{h_j^k}\Big)^{3/2}\Big)\\
& + \Big(\sum_{i=j+1}^{J}\sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \tilde{\Omega}_j^k}} \frac{h_i^l}{\sqrt{h_j^k}} a(\mu_i^2 I_i P_i^l P_i v, v)\Big)\Big(\sum_{i=j+1}^{J}\sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \tilde{\Omega}_j^k}} \Big(\frac{h_i^l}{\sqrt{h_j^k}}\Big)^3\Big)\\
\lesssim & \sum_{i=j+1}^{J}\sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} \Big(\frac{h_i^l}{h_j^k}\Big)^{1/2} a(\mu_i^2 I_i P_i^l P_i v, v)\\
& + \sum_{i=j+1}^{J}\sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \tilde{\Omega}_j^k}} \frac{h_i^l}{\sqrt{h_j^k}} a(\mu_i^2 I_i P_i^l P_i v, v).
\end{aligned}
$$

We set $\delta(m_i^l, m_j^k) = 1$, if $I_i\phi_i^l \not\equiv 0$ on $\mathcal{E}_{j+1}^k$, and $\delta(m_i^l, m_j^k) = 0$, otherwise, $\tilde{\delta}(m_i^l, m_j^k) = 1$, if $I_i\phi_i^l \not\equiv 0$ on $\tilde{\Omega}_j^k$, and $\tilde{\delta}(m_i^l, m_j^k) = 0$, otherwise. By (4.12) and (4.14), we obtain

$$
\begin{aligned}
(4.20)\qquad &\gamma \sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} a_j(P_j^k P_j\omega, P_j^k P_j\omega) \\
&\lesssim \gamma \sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j}\sum_{i=j+1}^{J} \sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} (\frac{h_i^l}{h_j^k})^{1/2} a(\mu_i^2 I_i P_i^l P_i v, v) \\
&+ \gamma \sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j}\sum_{i=j+1}^{J} \sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \tilde{\Omega}_j^k}} \frac{h_i^l}{\sqrt{h_j^k}} a(\mu_i^2 I_i P_i^l P_i v, v) \\
&= \gamma \sum_{i=2}^{J}\sum_{m_i^l\in\tilde{\mathcal{M}}_i} (\sum_{j=1}^{i-1}\sum_{m_j^k\in\tilde{\mathcal{M}}_j} (\frac{h_i^l}{h_j^k})^{1/2}\delta(m_i^l, m_j^k)) a(\mu_i^2 I_i P_i^l P_i v, v) \\
&+ \gamma \sum_{i=2}^{J}\sum_{m_i^l\in\tilde{\mathcal{M}}_i} (\sum_{j=1}^{i-1}\sum_{m_j^k\in\tilde{\mathcal{M}}_j} \frac{h_i^l}{\sqrt{h_j^k}}\tilde{\delta}(m_i^l, m_j^k)) a(\mu_i^2 I_i P_i^l P_i v, v) \\
&\lesssim \sum_{i=2}^{J}\sum_{m_i^l\in\tilde{\mathcal{M}}_i} \gamma\mu_i a(\mu_i I_i P_i^l P_i v, v)(1+\sqrt{h_i^l}) \\
&\lesssim \sum_{i=2}^{J}\sum_{m_i^l\in\tilde{\mathcal{M}}_i} a(\mu_i I_i P_i^l P_i v, v).
\end{aligned}
$$

Hence, (4.16) is verified. Combining (4.13-4.16), we obtain

$$
(4.21)\qquad \sum_{i=1}^{J}\sum_{j=1}^{i-1} a(T_i v, T_j u) \lesssim (\sum_{i=2}^{J} a(T_i v, v))^{1/2}(\sum_{j=1}^{J} a(T_j u, u))^{1/2}.
$$

A similar analysis can be used to derive

$$
(4.22)\qquad \sum_{i=1}^{J} a(T_i v, T_0 u) \lesssim (\sum_{i=1}^{J} a(T_i v, v))^{1/2} a(T_0 u, u)^{1/2},
$$

which, together with (4.21), completes the proof of the lemma. □

4.2. **Local Gauss-Seidel smoother.** In this subsection, we will verify assumptions **A1-A3** for the multilevel methods with a local Gauss-Seidel smoother R_i which is defined by

$$R_i := (I - E_i^{\tilde{n}_i})A_i^{-1},$$

where $E_i^{\tilde{n}_i} = (I - P_i^{\tilde{n}_i})\cdots(I - P_i^1) = \prod_{k=1}^{\tilde{n}_i}(I - P_i^k)$. For brevity, we set $E_i := E_i^{\tilde{n}_i}$, since no confusion is possible. We have

$$
(4.23)\qquad T_0 = \mu_0 I_0 P_0\ ,\ T_i = \mu_i I_i R_i A_i P_i = \mu_i I_i (I - E_i) P_i\ ,\ i = 1, ..., J.
$$

The decomposition of v is the same as (4.2).

For $i = 1, ..., J$, let $E_i^0 = I, E_i^{k-1} := (I - P_i^{k-1}) \cdots (I - P_i^1), k = 2, \ldots, \tilde{n}_i$. It is easy to see that

$$I - E_i = \sum_{k=1}^{\tilde{n}_i} P_i^k E_i^{k-1}. \tag{4.24}$$

As in Lemma 4.5 in [33], there also holds

$$\begin{aligned} a_i(P_i v, P_i u) - a_i(E_i P_i v, E_i P_i u) \\ = \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k E_i^{k-1} P_i v, E_i^{k-1} P_i u) \ , \ v, u \in V_J. \end{aligned} \tag{4.25}$$

4.2.1. *Verification of assumption A1.* We consider the case $i \geq 1$, since for T_0 assumption **A1** has been verified in Lemma 4.1.

Lemma 4.5. Let T_i, $i \geq 1$, be defined by (4.23). Then, T_i is nonnegative in V_J and there holds

$$a(T_i v, T_i v) \leq \omega_i \, a(T_i v, v) \quad , \quad v \in V_J \ , \ \omega_i < 2.$$

Proof. Due to (4.23) and (4.24) we have

$$\begin{aligned} a(T_i v, T_i v) &= \mu_i^2 a(I_i(I - E_i)P_i v, I_i(I - E_i)P_i v) \\ &= \mu_i^2 \sum_{k,m=1}^{\tilde{n}_i} a(I_i P_i^k E_i^{k-1} P_i v, I_i P_i^m E_i^{m-1} P_i v). \end{aligned}$$

Using (4.25), the same techniques as in (4.5), and the stability of I_i, we obtain

(4.26)

$$\begin{aligned} a(T_i v, T_i v) \leq & \mu_i^2 C_i \sum_{k=1}^{\tilde{n}_i} a(I_i P_i^k E_i^{k-1} P_i v, I_i P_i^k E_i^{k-1} P_i v) \\ \leq & \mu_i^2 C_0 C_i \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k E_i^{k-1} P_i v, P_i^k E_i^{k-1} P_i v) \\ = & \mu_i^2 C_0 C_i (a_i(P_i v, P_i v) - a_i(E_i P_i v, E_i P_i v)) \\ = & \mu_i^2 C_0 C_i (a_i(P_i v, P_i v) - a_i((I - (I - E_i))P_i v, (I - (I - E_i))P_i v)) \\ = & \mu_i^2 C_0 C_i (2a_i((I - E_i)P_i v, P_i v) - a_i((I - E_i)P_i v, (I - E_i)P_i v)) \\ \leq & \mu_i^2 C_0 C_i (2a_i((I - E_i)P_i v, P_i v) - \frac{1}{C_0} a(I_i(I - E_i)P_i v, I_i(I - E_i)P_i v)) \\ = & 2\mu_i C_0 C_i a(T_i v, v) - C_i a(T_i v, T_i v), \end{aligned}$$

whence

$$a(T_i v, T_i v) \leq \frac{2\mu_i C_0 C_i}{1 + C_i} a(T_i v, v).$$

Obviously, the nonnegativeness of T_i follows from the above inequality. Setting $\omega_i = \frac{2\mu_i C_i C_0}{1+C_i}$, and choosing $0 < \mu_i < \frac{1+C_i}{2C_0 C_i}$ such that $\omega_i < 2$, the lemma is proved. We remark that we can choose $\mu_J = 1$, since I_J is the identity. □

4.2.2. *Verification of assumption A2.*

Lemma 4.6. Let $\{T_i, i = 0, 1, ..., J\}$ be defined as in (4.23). There exists a constant K_0 such that

$$a(v, v) \leq \frac{K_0}{\mu} a(Tv, v) \quad , \quad v \in V_J \ , \ \mu = \min_{0 \leq i \leq J}\{\mu_i\}.$$

Proof. In view of the decomposition of v in (4.2), we have $a(v, v) = \sum_{i=0}^{J} a_i(v_i, P_i v)$. For $i = 1, ..., J$, we also have (cf. (4.8))

$$a_i(v_i, P_i v) \leq (\sum_{k=1}^{\tilde{n}_i} a_i(v_i(m_i^k)\phi_i^k, v_i(m_i^k)\phi_i^k))^{1/2} \cdot (\sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_i v))^{1/2}.$$

Since $I - E_i^{k-1} = \sum_{m=1}^{k-1} P_i^m E_i^{m-1}$, we deduce

$$\begin{aligned}
&\sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_i v) \\
&= \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k E_i^{k-1} P_i v) + \sum_{k=1}^{\tilde{n}_i} \sum_{m=1}^{k-1} a_i(P_i^k P_i v, P_i^k P_i^m E_i^{m-1} P_i v) \\
&\leq (\sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_i v))^{1/2} (\sum_{k=1}^{\tilde{n}_i} a_i(P_i^k E_i^{k-1} P_i v, E_i^{k-1} P_i v))^{1/2} \\
&+ \sum_{k,m=1}^{\tilde{n}_i} |a_i(P_i^k P_i v, P_i^m E_i^{m-1} P_i v)|.
\end{aligned}$$

Furthermore, using the same technique as in (4.5), we have

$$\begin{aligned}
&\sum_{k,m=1}^{\tilde{n}_i} |a_i(P_i^k P_i v, P_i^m E_i^{m-1} P_i v)| \\
&\lesssim (\sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_i v))^{1/2} (\sum_{k=1}^{\tilde{n}_i} a_i(P_i^k E_i^{k-1} P_i v, E_i^{k-1} P_i v))^{1/2},
\end{aligned}$$

Then, it follows from (4.26) that

$$\sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_i v) \lesssim \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k E_i^{k-1} P_i v, E_i^{k-1} P_i v) \lesssim \frac{1}{\mu_i} a(T_i v, v).$$

Hence,

$$a_0(P_0 v, P_0 v) + \sum_{k=1}^{\tilde{n}_i} a_i(P_i^k P_i v, P_i^k P_i v) \lesssim \sum_{i=0}^{J} \frac{1}{\mu_i} a(T_i v, v).$$

Finally, similar to the analysis of (4.9) and (4.10), we deduce that assumption **A2** holds true. □

4.2.3. *Verification of assumption A3.*

Lemma 4.7. There exists a constant K_1 independent of mesh sizes and mesh levels such that assumption **A3** holds true for $\{T_i, i = 0, 1, ..., J\}$ defined by (4.23).

Proof. We set $\xi_i = T_i v$. It follows from (4.23) that

$$\begin{aligned}
&\sum_{i=1}^{J}\sum_{j=1}^{i-1} a(T_i v, T_j u) = \sum_{j=1}^{J}\sum_{i=j+1}^{J} a(\xi_i, \mu_j I_j (I - E_j) P_j u) \\
&= \sum_{j=1}^{J}\sum_{i=j+1}^{J} \mu_j a_j (P_j \xi_i, (I - E_j) P_j u) = \sum_{j=1}^{J}\sum_{i=j+1}^{J} \mu_j \sum_{k=1}^{\tilde{n}_j} a_j (P_j \xi_i, P_j^k E_j^{k-1} P_j u) \\
&= \sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} \mu_j a_j (P_j^k P_j \sum_{i=j+1}^{J} \xi_i, P_j^k E_j^{k-1} P_j u).
\end{aligned}$$

Further, Hölder's inequality yields

$$\begin{aligned}
\sum_{i=1}^{J}\sum_{j=1}^{i-1} a(T_i v, T_j u) \le & (\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} \mu_j^2 a_j (P_j^k E_j^{k-1} P_j u, E_j^{k-1} P_j u))^{1/2} \\
& \cdot (\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} a_j (\sum_{i=j+1}^{J} P_j^k P_j \xi_i, \sum_{i=j+1}^{J} P_j^k P_j \xi_i))^{1/2}.
\end{aligned} \tag{4.27}$$

In view of the estimate of (4.26) in Lemma 4.5 and $\mu_j < \frac{1+C_j}{2C_0C_j} \lesssim 1$, for $j = 1, \dots, J$, we find

$$\begin{aligned}
\sum_{k=1}^{\tilde{n}_j} \mu_j^2 a_j (P_j^k E_j^{k-1} P_j u, E_j^{k-1} P_j u) &\le (2\mu_j a(T_j u, u) - \frac{1}{C_0} a(T_j u, T_j u)) \\
&\le 2 \sum_{j=1}^{J} \mu_j a(T_j u, u) \lesssim a(T_j u, u),
\end{aligned} \tag{4.28}$$

whence

$$\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} \mu_j^2 a_j (P_j^k E_j^{k-1} P_j u, E_j^{k-1} P_j u) \lesssim \sum_{j=1}^{J} a(T_j u, u).$$

Next, we show that

$$\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j} a_j (\sum_{i=j+1}^{J} P_j^k P_j \xi_i, \sum_{i=j+1}^{J} P_j^k P_j \xi_i) \lesssim \sum_{i=2}^{J} a(T_i v, v). \tag{4.29}$$

We note that $P_j^k P_j \xi_i = \frac{a_j(P_j\xi_i, \phi_j^k)}{a_j(\phi_j^k, \phi_j^k)} \phi_j^k \approx a_j(P_j \xi_i, \phi_j^k)\phi_j^k$, and similarly $P_i^l E_i^{l-1} P_i v \approx a_i(E_i^{l-1} P_i v, \phi_i^l)\phi_i^l$. Then, there holds

$$a_j(\sum_{i=j+1}^{J} P_j^k P_j \xi_i, \sum_{i=j+1}^{J} P_j^k P_j \xi_i) \lesssim (\sum_{i=j+1}^{J} a_j(P_j \xi_i, \phi_j^k))^2.$$

Moreover,

$$\begin{aligned} &a_j(P_j\xi_i,\phi_j^k) = a_j(P_j\mu_i I_i(I-E_i)P_iv,\phi_j^k) = \mu_i a(I_i(I-E_i)P_iv, I_j\phi_j^k) \\ &= \mu_i \sum_{l=1}^{\tilde{n}_i} a_i(P_i^l E_i^{l-1}P_iv, P_iI_j\phi_j^k) \approx \mu_i \sum_{l=1}^{\tilde{n}_i} a_i(\phi_i^l, P_iI_j\phi_j^k) a_i(E_i^{l-1}P_iv,\phi_i^l) \\ &= \mu_i \sum_{l=1}^{\tilde{n}_i} a(I_i\phi_i^l, I_j\phi_j^k) a_i(E_i^{l-1}P_iv,\phi_i^l). \end{aligned}$$

Similar to the analysis of (4.20) in Lemma 4.4 in the Jacobi case, and due to Lemma 4.3, we have

$$\begin{aligned} &\sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j}\Big(\sum_{i=j+1}^{J} a_j(\xi,\phi_j^k)\Big)^2 \\ &\lesssim \sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j}\sum_{i=j+1}^{J} \sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} \Big(\frac{h_i^l}{h_j^k}\Big)^{1/2}\mu_i^2 a_i(P_i^lE_i^{l-1}P_iv, E_i^{l-1}P_iv) \\ &\quad + \sum_{j=1}^{J}\sum_{k=1}^{\tilde{n}_j}\sum_{i=j+1}^{J} \sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \tilde{\Omega}_j^k}} \frac{h_i^l}{\sqrt{h_j^k}}\mu_i^2 a_i(P_i^lE_i^{l-1}P_iv, E_i^{l-1}P_iv) \\ &\lesssim \sum_{i=2}^{J}\sum_{m_i^l\in\tilde{M}_i} \mu_i^2 a_i(P_i^lE_i^{l-1}P_iv, E_i^{l-1}P_iv) \sum_{j=1}^{i-1}\sum_{m_j^k\in\tilde{M}_j}\Big(\frac{h_i^l}{h_j^k}\Big)^{1/2}\delta(m_i^l,m_j^k) \\ &\quad + \sum_{i=2}^{J}\sum_{m_i^l\in\tilde{M}_i} \mu_i^2 a_i(P_i^lE_i^{l-1}P_iv, E_i^{l-1}P_iv) \sum_{j=1}^{i-1}\sum_{m_j^k\in\tilde{M}_j}\frac{h_i^l}{\sqrt{h_j^k}}\tilde{\delta}(m_i^l,m_j^k) \\ &\lesssim \sum_{i=2}^{J}\sum_{m_i^l\in\tilde{M}_i} \mu_i^2 a_i(P_i^lE_i^{l-1}P_iv, E_i^{l-1}P_iv)(1+\sqrt{h_i^l}) \lesssim \sum_{i=2}^{J} a(T_iv,v). \end{aligned}$$

Hence, (4.29) is verified. In view of (4.26), (4.27) and (4.29), it follows that

$$\sum_{i=1}^{J}\sum_{j=1}^{i-1} a(T_iv,T_ju) \lesssim \Big(\sum_{i=2}^{J} a(T_iv,v)\Big)^{1/2}\Big(\sum_{j=1}^{J} a(T_ju,u)\Big)^{1/2}. \tag{4.30}$$

We further deduce

$$\sum_{i=1}^{J} a(T_iv,T_0u) \lesssim \Big(\sum_{i=1}^{J} a(T_iv,v)\Big)^{1/2} a(T_0u,u)^{1/2},$$

which, together with (4.30), implies Lemma 4.7. □

Numerical results

In this section, for selected test examples we present numerical results that illustrate the optimality of algorithm 4.1 and algorithm 4.2. The implementation is based on the FFW toolbox [8]. The local error estimators and the strategy **MARK** for the selection of elements and edges for refinement have been realized as in the algorithm ANFEM II in [12]. In the following examples, both **LMPA** and **LMAA** are considered as preconditioners for the conjugate gradient method, i.e., a symmetric version of **LMPA (SLMPA)** has been used in the computations. Likewise, a symmetric version of **LMAA (SLMAA)** is employed when the smoother is nonsymmetric, otherwise, **LMAA** is directly applied. The algorithms **LMPA** and **LMAA** require $O(N\log N)$ and $O(N)$ operations respectively, where N is the number of degrees of freedom (DOFs) (cf. [26]).

The estimate (A.1) in the appendix indicates that the prolongation operator I_i from V_i to V_J would increase the energy by a constant C_0 at worst, which is essential in the convergence analysis of the local multilevel methods. We can weaken the influence by a well chosen scaling number $\mu_{J,i}$ in (3.2). As seen from Theorem 4.1 and Theorem 4.2, the uniform convergence rate of **LMPA** or the preconditioned conjugate gradient method by **LMAA** will deteriorate for decreasing scaling number $\mu = \min_{0\le i\le J}\{\mu_{J,i}\}$. This property will be observed in the following Example 6.1. We always choose $\mu_{J,J} = 1$ in the computations.

For the preconditioned conjugate gradient method, the iteration stops when it satisfies

$$\|r_i^0 - A_i r_i^n\|_{0,\Omega} \le \epsilon \|r_i^0\|_{0,\Omega} \quad , \quad \epsilon = 10^{-6},$$

where $\{r_i^k : k = 1, 2, ...\}$ stands for the set of iterative solutions of the residual equation $A_i x = r_i^0$.

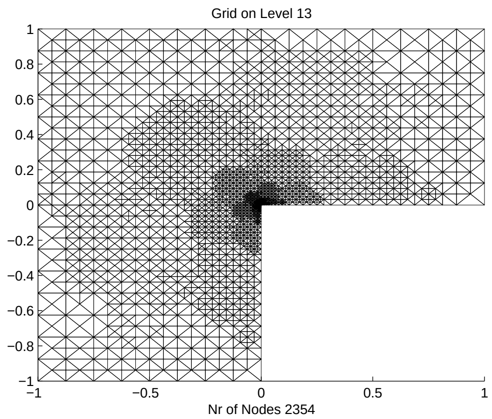

FIGURE 2. *Locally refined mesh at the* 13*-th refinement level (Example* 6.1*).*

At the i-th level, let $u_i^0 = u_{i-1}, r_i^n = f_i - A_i u_i^n$, and set

$$\epsilon_0 = (r_i^0)^t B_i r_i^0, \qquad \epsilon_n = (r_i^n)^t B_i r_i^n,$$

where B_i is the local multilevel iteration. The number of iteration steps required to achieve the desired accuracy is denoted by **iter**. We further denote by $\rho = (\sqrt{\epsilon_n}/\sqrt{\epsilon_0})^{1/\mathbf{iter}}$ the average reduction factor.

Example 4.1. On the L-shaped domain $\Omega = [-1,1] \times [-1,1] \backslash (0,1] \times [-1,0)$, we consider the following elliptic boundary value problem

$$\begin{aligned} -\Delta(0.5u) + u &= f(x,y) \quad \text{in } \Omega, \\ u &= g(x,y) \quad \text{on } \partial\Omega, \end{aligned}$$

where f and g are chosen such that $u(r,\theta) = r^{\frac{2}{3}} \sin(\frac{2}{3}\theta)$ is the exact solution (in polar coordinates).

TABLE 1. Number of iterations and average reduction factor ρ on each level for the respective algorithms with scaling number $\mu_{J,i} = 0.8$ and $\mu_{J,J} = 1$, $0 \le i \le J-1$, $J \ge 1$. For the conjugate gradient method without preconditioning, only the number of iterations is given (Example 6.1).

Level	DOFs	CG	SLMPA-GS		SLMPA-Jacobi		SLMAA-GS		LMAA-Jacobi	
		iter	iter	ρ	iter	ρ	iter	ρ	iter	ρ
13	6831	206	9	0.2203	12	0.3184	34	0.6732	46	0.7475
14	11293	242	10	0.2408	12	0.3185	35	0.6783	47	0.7526
15	18121	310	10	0.2395	12	0.3179	35	0.6807	48	0.7567
16	30385	369	10	0.2412	12	0.3156	35	0.6833	49	0.7594
17	49825	458	10	0.2430	12	0.3141	36	0.6853	49	0.7614
18	80893	560	10	0.2400	12	0.3115	35	0.6852	49	0.7623
19	135060	700	10	0.2391	12	0.3079	35	0.6847	49	0.7624
20	219441	858	10	0.2405	12	0.3052	35	0.6838	50	0.7640
21	359337	1053	10	0.2375	12	0.3020	35	0.6844	50	0.7641
22	598091	1331	10	0.2353	12	0.2988	35	0.6845	49	0.7629
23	964580	1491	10	0.2356	12	0.2970	35	0.6848	50	0.7645
24	1592958	1715	10	0.2315	11	0.2873	35	0.6840	49	0.7631

TABLE 2. Average reduction factors ρ (**SLPMA-GS**) for different scaling numbers (Example 6.1).

Level	$\mu_{J,0} = \cdots = \mu_{J,J-1} = \alpha$, $\mu_{J,J} = 1$					
	$\alpha = 1.8$	$\alpha = 1.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.2$	$\alpha = 0.1$
13	0.2448	0.2340	0.2196	0.2737	0.4100	0.5125
14	0.2462	0.2410	0.2394	0.2740	0.4162	0.5189
15	0.2479	0.2410	0.2393	0.2738	0.4234	0.5292
16	0.2507	0.2426	0.2410	0.2729	0.4228	0.5337
17	0.2508	0.2444	0.2426	0.2722	0.4194	0.5274
18	0.2488	0.2414	0.2397	0.2697	0.4163	0.5239
19	0.2484	0.2408	0.2387	0.2668	0.4148	0.5225
20	0.2482	0.2419	0.2400	0.2567	0.4088	0.5215
21	0.2490	0.2390	0.2370	0.2532	0.4067	0.5207
22	0.2666	0.2368	0.2347	0.2488	0.4023	0.5182
23	0.2666	0.2371	0.2351	0.2451	0.3979	0.5088
24	0.2678	0.2331	0.2310	0.2423	0.3949	0.5058

For ease of notation, we refer to **SLMPA-GS**,**SLMAA-GS** and **SLMPA-Jacobi**,**LMAA-Jacobi** as the preconditioned conjugate gradient method by **SLMPA** and **SLMAA** with local Gauss-Seidel smoothing and local Jacobi smoothing, respectively. For the Jacobi iteration, the scaling factor is chosen according to $\gamma = 0.8$.

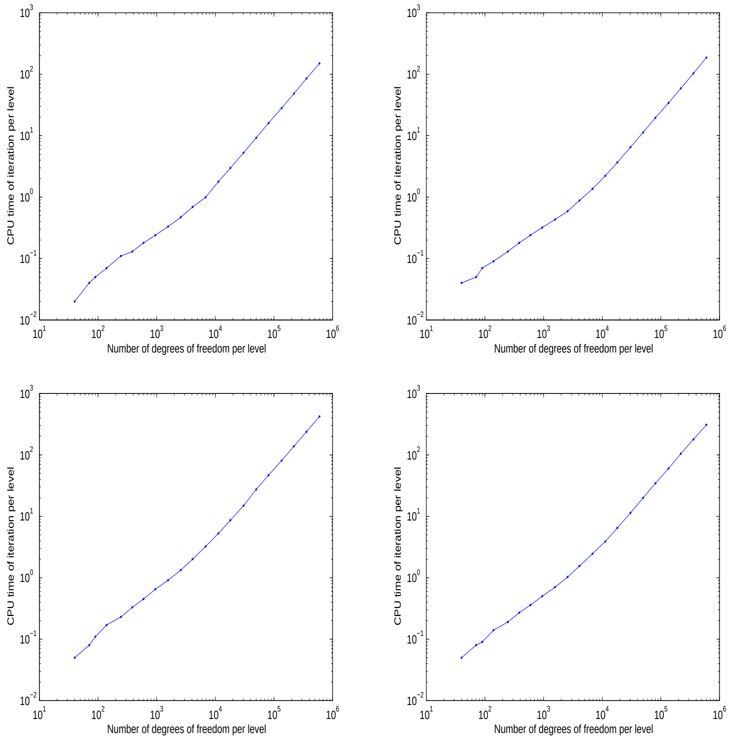

FIGURE 3. *CPU times for* **SLMPA-GS**, **SLMPA-Jacobi**, **SLMAA-GS**, *and* **LMAA-Jacobi** *(from left to right and top to bottom)*

At first, we choose $\mu_{J,i} = 0.8$ $(0 \leq i < J)$ to illustrate the optimality of our algorithms. Figure 2 displays the locally refined mesh at the 13-th refinement level. As seen from Table 1, the number of iterative steps of the conjugate gradient method without preconditioning (**CG**) increases quickly with the mesh levels. However, for the algorithms **SLMPA-GS**, **SLMPA-Jacobi**, **SLMAA-GS** and **LMAA-Jacobi** we observe that the number of iteration steps and the average reduction factors are all bounded independently of the mesh sizes and the mesh levels. These results and Figure 3, displaying the CPU times (in seconds) for the respective algorithms, demonstrate the optimality of the algorithms and thus confirm the theoretical analysis.

Next, we choose different scaling numbers to illustrate how they influence the convergence behavior of the local multilevel methods. We only list the results for **SLMPA-GS**. A similar behavior can be observed for the other algorithms. We choose $\mu_{J,0} = \cdots = \mu_{J,J-1} = \alpha$ and $\mu_{J,J} = 1$, and thus $\mu = \min\{\alpha, 1\}$. Table 2 shows that for a fixed α, **SLMPA-GS** converges almost uniformly. The last four numbers of each row in Table 2 show that for a fixed level the average reduction factor of **SLMPA-GS** deteriorates for decreasing μ. If $\alpha \geq 1$, then $\mu = \min\{\alpha, 1\} = 1$, and the convergence rate will also deteriorate as α increases. This is also observed for the first numbers of each row in Table 2. In particular, for $\mu = 1$ the convergence rate of **SLMPA-GS** deteriorates only with respect to ω_i (the spectral bound of T_i), which increases linearly with $\mu_{J,i}$.

Example 4.2. We consider Poisson's equation

$$-\Delta u = 1 \qquad \text{in}\,\Omega,$$

with Dirichlet boundary conditions on a domain with a crack, namely $\Omega = \{(x, y) : |x| + |y| \leq 1\}\backslash\{(x, y) : 0 \leq x \leq 1, y = 0\}$. The exact solution is $r^{1/2}\sin(\theta/2) - \frac{1}{4}r^2$ (in polar coordinates).

In this example, we choose $\mu_{J,J} = 1$, $\mu_{J,i} = 1$ and $\mu_{J,i} = 0.8$ $(0 \leq i < J, J \geq 1)$, respectively, for the local multilevel methods with local Gauss-Seidel smoothing and local Jacobi smoothing.

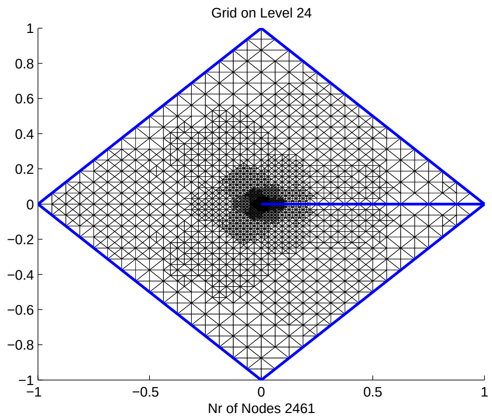

FIGURE 4. *Locally refined mesh at the* 24*-th refinement level (Example* 6.2*).*

TABLE 3. Number of iterations and average reduction factors ρ on each level for the respective algorithms with scaling numbers $\mu_{J,J} = 1, \mu_{J,i} = 1, \mu_{J,i} = 0.8$,$0 \leq i \leq J-1, J \geq 1$. For the conjugate gradient method without preconditioning, only the number of iterations is given (Example 6.2).

Level	DOFs	CG	SLMPA-GS		SLMAA-GS		SLMPA-Jacobi		LMAA-Jacobi	
		iter	iter	ρ	iter	ρ	iter	ρ	iter	ρ
28	18206	287	11	0.2610	51	0.7720	14	0.3756	65	0.8154
30	29108	341	10	0.2555	50	0.7662	13	0.3507	65	0.8151
32	46105	417	10	0.2403	52	0.7745	14	0.3615	65	0.8161
34	73571	523	11	0.2628	55	0.7854	14	0.3773	70	0.8271
36	116866	634	10	0.2511	52	0.7768	13	0.3309	63	0.8105
38	184155	768	10	0.2340	52	0.7764	14	0.3601	67	0.8212
40	292148	942	10	0.2513	55	0.7880	14	0.3708	70	0.8286
42	462599	1168	10	0.2395	52	0.7765	12	0.3181	64	0.8141
44	727564	1404	10	0.2337	53	0.7808	13	0.3511	68	0.8243
46	1150917	1536	10	0.2435	54	0.7852	14	0.3615	70	0.8275

Figure 4 displays the locally refined mesh at the 24-th refinement level. The numbers in Table 3 and the CPU times (in seconds) displayed in Figure 5 show a similar behavior as in the previous example and thus also support the theoretical findings.

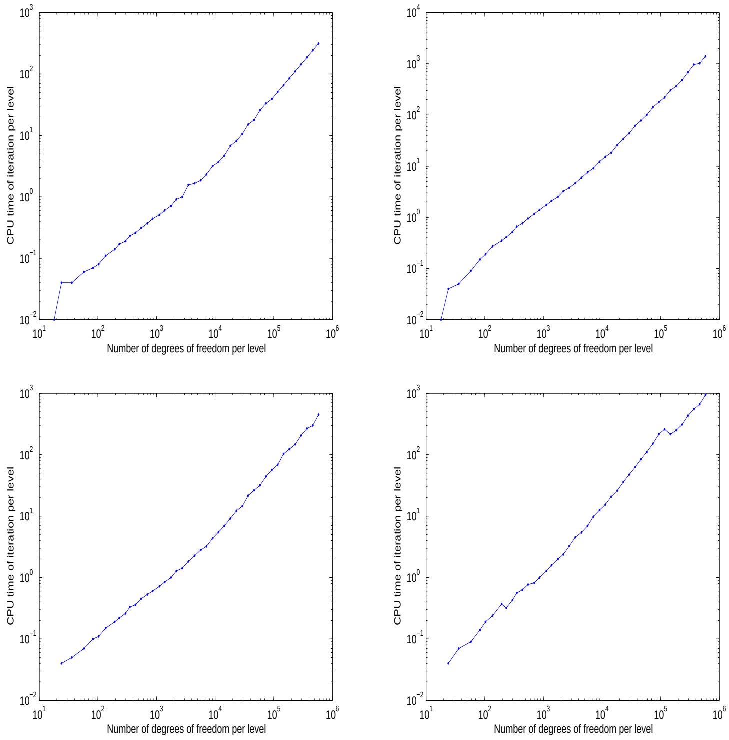

FIGURE 5. *CPU times for* **SLMPA-GS**, **SLMAA-GS**, **SLMPA-Jacobi**, *and* **LMAA-Jacobi** *(from left to right and top to bottom)*

APPENDIX

In this appendix, we analyze the stability of I_i and provide the proof of Lemma 4.3.

Proof of the stability result of the prolongation operator I_i. Let $\mathcal{T}_{i+1}$ be the refined triangulation obtained from $\mathcal{T}_i$ by the algorithm stated in section 2 or by the newest vertex bisection. Then, there exist constants C_0 and $\tilde{C}_0$ such that

$$(I_i v, I_i v) \leq \tilde{C}_0(v,v) \quad , \quad a(I_i v, I_i v) \leq C_0 a_i(v,v) \quad , \quad v \in V_i. \tag{A.1}$$

Since the analysis for the first bisection algorithm has been done in [26], we only give the proof for the refinement by the newest vertex bisection.

The first inequality in (A.1) is trivial for $I_i v$ being defined by local averaging. It suffices to derive the second one.

The origin of vertices of $T \in \mathcal{T}_{i+1}$ includes four cases depending whether the vertex of T is the midpoint of an edge or a node in $\mathcal{T}_i$. In particular, let m, n denote the number of vertices of $T \in \mathcal{T}_{i+1}$ representing midpoints or nodes in $\mathcal{T}_i$, respectively. Setting $S = \{(m,n) : m+n = 3,\ m,n = 0,1,2,3\}$, we have $\#S = 4$. We only consider one of the possible cases: the vertices of $T \in \mathcal{T}_{i+1}$ are all nodes in $\mathcal{T}_i$, i.e., T is not refined in the transition from $\mathcal{T}_i$ to $\mathcal{T}_{i+1}$, e.g., $T_2 \in \mathcal{T}_{i+1}$ is also $K_2 \in \mathcal{T}_i$ in Figure 6. A similar analysis can be carried out in all other cases.

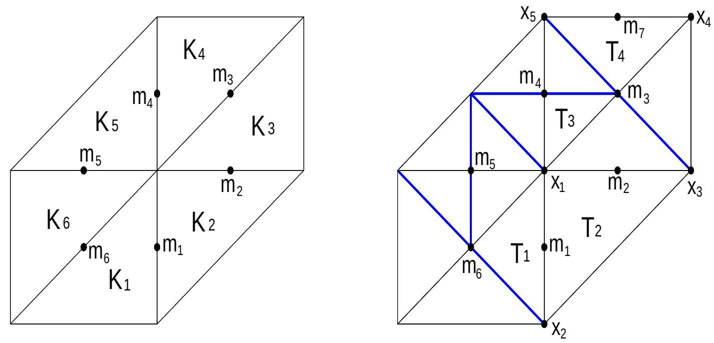

FIGURE 6. *The left figure illustrates a local grid from $\mathcal{T}_i$, the right one displays its refinement as part of $\mathcal{T}_{i+1}$.*

Note that $a(I_iv, I_iv)|_{T_2}$ can be bounded by

$$C((I_iv(x_1) - I_iv(x_2))^2 + (I_iv(x_1) - I_iv(x_3))^2) \tag{A.2}$$

for some constant C. We recall that $I_iv(x_i)$ is the average of v at x_i over the triangles $K_l, l = 1, ..., M_{x_i}$, where M_{x_i} is the number of triangles containing x_i. Hence, the first term of (A.2) can be written as

$$\frac{1}{M_{x_1}} \sum_{l=1}^{M_{x_1}} (v|_{K_l}(x_1) - v(m_1)) + \frac{1}{M_{x_2}} \sum_{s=1}^{M_{x_2}} (v(m_1) - v|_{K_s}(x_2)). \tag{A.3}$$

A similar result can be obtained for the second term of (A.2). Since

$$v|_{K_l}(x_1) - v(m_1) = v|_{K_l}(x_1) - v(m_l) + \sum_{j=1}^{l-1} (v(m_{j+1}) - v(m_j)),$$

it suffices to find a constant C such that the first term of (A.3) can be bounded by

$$\sum_{l=1}^{M_{x_1}} (v|_{K_l}(x_1) - v(m_1) \leq C a_i(v, v)|_{\tilde{K}}, \tag{A.4}$$

where $\tilde{K} = \cup_{l=1}^{M_{x_1}} K_l$. The same analysis can be carried out for the second term of (A.3). Following (A.2-A.4), we get

$$a(I_iv, I_iv)|_{T_2} \leq C a_i(v, v)|_{\tilde{T_2}} \tag{A.5}$$

with some constant C, where $\tilde{T_2}$ is a patch of triangles in $\mathcal{T}_i$ also containing the vertices of T_2.

For $T \in \mathcal{T}_{i+1}$, $\partial T \cap \partial\Omega \neq \varnothing$, let us assume $\partial T_4 \cap \partial\Omega \neq \varnothing$. Then, $a(I_iv, I_iv)|_{T_4}$ can be bounded by

$$C((I_iv(m_3) - I_iv(x_4))^2 + (I_iv(m_3) - I_iv(x_5))^2) = 2C(v(m_3) - v(m_7))^2. \tag{A.6}$$

Combining (A.5) and (A.6) and summing up all $T \in \mathcal{T}_{i+1}$ completes the proof. □

Proof of Lemma 4.3. The proof is similar to Lemma 3.2 in [29]. We only prove the first estimate in (4.11) and (4.12). The second estimate in (4.11) and (4.12) can be obtained similarly.

For the proof of the first estimate in (4.11), we set

$$\rho(m_i^l) = [\frac{\ln(h_i^l/h_0)}{\ln(1/2)}], \qquad h_0 = \max_{E\in\mathcal{E}_0} h_E, \tag{A.7}$$

which characterizes the actual number of refinements of edges in $\mathcal{E}_i$. It is obvious that

$$(\frac{1}{2})^{\rho(m_i^l)+1}h_0 < \rho(m_i^l) \le (\frac{1}{2})^{\rho(m_i^l)}h_0.$$

Denoting by $d(\tilde{\Omega}_j^k)$ the diameter of $\tilde{\Omega}_j^k$, there exists a constant $\beta > 1$ depending only on the minimum angle θ in (1.6) such that $d(\tilde{\Omega}_j^k) \le h_j^k$, whence

$$h_i^l \le d(\tilde{\Omega}_i^l) \le d(\tilde{\Omega}_j^k) \le \beta h_j^k.$$

Due to the definition of (A.7), we have $\rho(m_i^l) \ge \rho(m_j^k) - n_0$, where $n_0 = [\ln\beta/\ln 2]+1$. Thus, for the left term in (4.11) we get

$$\begin{aligned}\sum_{i=j+1}^{J} \sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} (h_i^l)^{3/2} \le &(h_0)^{3/2} \sum_{i=j+1}^{J} \sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} (\frac{1}{\sqrt{2}})^{3\rho(m_i^l)} \\ \le &(h_0)^{3/2} \sum_{m=\rho(m_j^k)-n_0}^{\infty} \sum_{(i,l)\in\sigma_1(m,m_j^k)} (\frac{1}{\sqrt{2}})^{3m},\end{aligned} \tag{A.8}$$

where $\sigma_1(m, m_j^k) = \{(i,l) \mid \rho(m_i^l) = m, j+1 \le i \le J, I_i\phi_i^l \not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k\}$. We define a new set according to

$$\sigma_0(m,y) = \{i : y \in \tilde{\mathcal{M}}_i, \rho(y) = m, 0 \le i \le J\}.$$

In view of the minimum angle condition in the newest vertex bisection, we easily see that

$$\#\sigma_0(m,y) \lesssim 1.$$

Moreover,

$$\#\sigma_1(m, m_j^k) \lesssim \max_{(i,l)\in\sigma_1(m,m_j^k)} \#\sigma_0(m, m_i^l) \cdot \frac{h_j^k}{(\frac{1}{2})^{m+1}h_0} \lesssim \frac{h_j^k}{(\frac{1}{2})^{m+1}h_0}. \tag{A.9}$$

Therefore, (A.8) and (A.9) imply that

$$\sum_{i=j+1}^{J} \sum_{\substack{m_i^l\in\tilde{\mathcal{M}}_i,\\ I_i\phi_i^l\not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} (h_i^l)^{3/2} \lesssim (h_0)^{3/2} \sum_{m=\rho(m_j^k)-n_0}^{\infty} (\frac{1}{\sqrt{2}})^{3m} \frac{h_j^k}{(\frac{1}{2})^{m+1}h_0} \lesssim (h_j^k)^{3/2}.$$

This proves the first estimate in (4.11).

For the second estimate in (4.12), we need to show

$$\#\sigma_2(m, m_i^l) \lesssim 1, \tag{A.10}$$

where $\sigma_2(m, m_i^l) = \{(k,j) \mid m_j^k \in \tilde{\mathcal{M}}_j, \rho(m_j^k) = m, I_i\phi_i^l \not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k, 0 \le j \le i-1\}$. Let

$$\begin{aligned}\mathcal{N}(m, m_i^l) =&\{y : y \in \tilde{\mathcal{M}}_j, \rho(y) = m, |y - m_i^l| \le d(\bar{\bar{\Omega}}_y), \\ &m_i^l \in \bar{\bar{\Omega}}_y, I_i\phi_i^l \not\equiv 0 \text{ on } \mathcal{E}_{j+1}(\tilde{\Omega}_y), 0 \le j \le i-1\},\end{aligned}$$

where for $y \in \tilde{\mathcal{M}}_j$, $\bar{\bar{\Omega}}_y$ is the patch of triangles in $\mathcal{T}_j$ containing the vertices of $\tilde{\Omega}_y$.

For each $y \in \mathcal{N}(m, m_i^l)$, there exists a constant $\tilde{\beta}$ depending only on the minimum angle in (1.6) such that

$$|y - m_i^l| \leq d(\bar{\bar{\Omega}}_y) \leq \tilde{\beta}(\frac{1}{2})^m h_0.$$

On the other hand, for any $y_1, y_2 \in \mathcal{N}(m, m_i^l)$, we have $|y_1 - y_2| \gtrsim (\frac{1}{2})^{m+1} h_0$ and $\#\mathcal{N}(m, m_i^l) \lesssim 1$. Hence,

$$\#\sigma_2(m, m_i^l) \lesssim \#\mathcal{N}(m, m_i^l) \cdot \max_{y \in \mathcal{N}(m, m_i^l)} \#\sigma_0(m, y) \lesssim 1,$$

which proves (A.10). Taking advantage of the preceding estimates, we conclude the proof as follows:

$$\sum_{j=1}^{i-1} \sum_{\substack{m_j^k \in \tilde{\mathcal{M}}_j, \\ I_i\phi_i^l \not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} (h_j^k)^{-1/2} \lesssim h_0^{-1/2} \sum_{j=1}^{i-1} \sum_{\substack{m_j^k \in \tilde{\mathcal{M}}_j, \\ I_i\phi_i^l \not\equiv 0 \text{ on } \mathcal{E}_{j+1}^k}} (\frac{1}{\sqrt{2}})^{-\rho(m_j^k)}$$

$$\lesssim h_0^{-1/2} \sum_{m=0}^{\rho(m_i^l)+n_0} \sum_{(k,j) \in \sigma_2(m, m_i^l)} (\frac{1}{\sqrt{2}})^{-m} \lesssim h_0^{-1/2} (\sqrt{2})^{\rho(m_i^l)+n_0} \lesssim (h_i^l)^{-1/2}.$$

□

References

[1] Randolph E. Bank and L. Ridgway Scott, On the conditioning of finite element equations with highly refined meshes, SIAM J. Numer. Anal., **26**(1989), 1383-1394.

[2] P. Binev, W. Dahmen, and R. DeVore, Adaptive finite element methods with convergence rates, Numer. Math., **97**(2004), 219-268.

[3] J.H. Bramble, Multigird Methods, Pitman, 1993.

[4] J.H.Bramble, J.E.Pasciak, and J.Xu, The Analysis of Multigrid algorithms with nonnested spaces or non-inherited quadratic forms, Math. Comp., **56**(1991), 1-34.

[5] S.C. Brenner, An optimal multigrid method for P1 nonconforming finite elments, Math. Comp., **52**(1989), 1-16.

[6] S.C. Brenner, Convergence of nonconforming multigrid methods without full elliptic regularity, Math. Comp., **68**(1999), 25-53.

[7] S.C. Brenner, Convergence of nonconforming V-cycle and F-cyle multigrid algorithms for second order elliptic boundary value problems, Math. Comp., **73**(2004), 1041-1066.

[8] A. Byfut, J. Gedicke, D. Günther, J. Reininghaus, S. Wiedemann, et al., FFW Documentation, Humboldt University of Berlin, Germany, 2007.

[9] C. Carstensen, Quasi-interpolation and a posteriori error analysis in finite elements methods, RAIRO Model. Math. Anal. Numer., **33**(1999), 1187-1202.

[10] C. Carstensen and R.H.W. Hoppe, Convergence analysis of an adaptive nonconforming finite element emthod, Numer. Math., **103**(2006), 251-266.

[11] J.M. Cascon, C. Kreuzer, R.H. Nochetto, and K.G. Siebert, Quasi-optimal convergence rate for an adaptive finite element method, SIAM J. Numer. Anal., **46**(2008), 2524-2550.

[12] H. Chen, X. Xu, and R.H.W. Hoppe, Convergence and optimality of adaptive nonconforming finite element methods for nonsymmetric and indefinite problems, Preprint No.11/2008, Instituts für Mathematik, Universität Augsburg.

[13] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[14] H. Duan, S. Gao, R. Tan, and S. Zhang, A generalized BPX framework covering the V-cycle nonested multigrid methods, Math. Comp., **76**(2007), 137-152.
[15] R.H.W. Hoppe and B. Wohlmuth, Adaptive multilevel iterative techniques for nonconforming finite element discretizations, East-West J. Numer. Math., **3**(1995), 179-197.
[16] W.F. Mitchell, Optimal multilevel iterative methods for adaptive grids, SIAM J.Sci.Stat.Comput., **13**(1992), 146-167.
[17] P. Morin, R.H. Nochetto, and K.G. Siebert, Convergence of adaptive finite element methods, SIAM Review, **44**(2002), 631-658.
[18] P. Oswald, Integrid transfer operators and multilevel preconditioners for nonconforming discretizations, Appl. Numer. Math., **23**(1996), 139-158.
[19] P. Oswald, On a hierarchical basis multilevel method with nonconforming P_1 elements, Numer. Math., **62**(1992), 189-212.
[20] P. Oswald, On a BPX-preconditioner for P_1 elements, Computing, **51**(1993), 125-133.
[21] P. Oswald, Optimality of multilevel preconditioning for nonconforming P1 finite elements, Numer. Math., **111**(2008), 267-291.
[22] R. Scott and S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions, Math. Comp., **54**(1990), 483-493.
[23] Z. Shi and X. Xu, On a note for nonconforming multigrid for nonsymmetric and indefinite problems, Chinese J. Numer. Math. Appl., **22**(2000), 102-108.
[24] R. Stevenson, Optimality of a standard adaptive finite element method, Foundations of Computional Mathematics, **2**(2007), 245-269.
[25] A. Toselli and O. Widlund, Domain Decomposition Methods - Algorithms and Theory, Springer-Verlag, Berlin, 2005.
[26] P.S. Vassilevski and J. Wang, An application of the abstract multilevel theory to nonconforming finite element methods, SIAM J. Numer. Anal., **32**(1995), 235-248.
[27] S.P. Wong, Preconditioning nonconforming finite element methods for treating Dirichlet boundary conditions, II, Numer. Math., **62**(1992), 413-437.
[28] S.P. Wong, Preconditioning P1 nonconforming finite elements: condition numbers and singular value distributions, Numer. Math., **65**(1993), 253-271.
[29] H.J. Wu and Z.M. Chen, Uniform convergence of multigrid V-cycle on adaptively refined finite element meshes for second order elliptic problems, Science in China, **39**(2006), 1405-1429.
[30] J. Xu, Iterative methods by space decomposition and subspace correction, SIAM Review, **34**(1992), 581-613.
[31] J. Xu and L. Zikatanov, The method of alternating projections and the method of subspace corrections in Hilbert space, J.Amer.Math.Soc., **15**(2002), 573-597.
[32] X. Xu and J. Chen, Multigrid for the Mortar element method for P1 nonconforming element, Numer. Math., **88**(2001), 381-398.
[33] X. Xu, H. Chen, and R.H.W. Hoppe, Optimality of local multilevel methods on adaptively refined meshes for elliptic boundary value problems, submitted to J. Numer. Math., 2009.
[34] X. Xu, L. Li, and W. Chen, A multigrid method for the Mortar-type Morley element approximation of a plate bending problem, SIAM J. Numer. Anal., **39**(2002), 1712-1731.
[35] H. Yserentant, Old and new convergence proofs for multigrid methods, Acta Numerica, **2**(1993), 285-326.

LSEC, Institute of Computational Mathematics, Chinese Academy of Sciences, P.O.Box 2719, Beijing, 100190, People's Republic of China
E-mail address: xxj@lsec.cc.ac.cn

LSEC, Institute of Computational Mathematics, Chinese Academy of Sciences, P.O.Box 2719, Beijing, 100190, People's Republic of China
E-mail address: chx@lsec.cc.ac.cn

Institute of Mathematics, University of Augsburg, D-86159, Augsburg,Germany, and Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA
E-mail address: rohop@math.uh.edu