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On Lower Bounds of Second-Order Chord Power Integrals of Convex Discs

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Abstract

For planar convex discs K with positive area $A(K)$, boundary length $L(\partial K)$ and second-order chord power integral $I_2(K)$, we study the ratio $L(\partial K) I_2(K)/A^2(K)$ and give reasons supporting the conjecture that its uniform lower bound is $32/3$ attained exactly for circles. In particular, using the Ambartzumian-Pleijel representation of $I_2(K)$ we derive formulas for $I_2(K)$ in case of general triangles, rectangles, and regular N -gons. In these cases and for ellipses we can prove this inequality. A related conjecture is formulated for the class of convex N -gons which exhibits a strengthening of the isoperimetric inequality as well as of Carleman's inequality for convex N -gons. An extension to higher dimensions is discussed at the end of the paper.

Keywords : MOTION-INVARIANT POISSON LINE PROCESSES, ISOPERIMETRIC INEQUALITY, CARLEMAN'S INEQUALITY, MEAN BREADTH, CHORD LENGTH DISTRIBUTION, REGULAR N -GONS, ELLIPSES

MSC 2000 : PRIMARY 52A40 60G05 SECONDARY 52A07 52A22

1 Introduction - Posing the Problem

We consider a motion-invariant Poisson line tessellation $T_\lambda = \bigcup_{i \in \mathbb{Z}^1} g(P_i, \Phi_i)$ with line density λ in an increasing planar region ϱK with $1 \leq \varrho \uparrow \infty$, where K is some fixed convex disc containing \mathbf{o} as inner point. Here, $g(p, \varphi)$ denotes an unoriented straight line in a Cartesian x_1x_2 -coordinate system with normal unit vector $(\cos \varphi, \sin \varphi)$ directed in the upper half-plane and signed perpendicular distance $p \in \mathbb{R}^1$ from the origin \mathbf{o} , formally written

$$g(p, \varphi) = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \cos \varphi + x_2 \sin \varphi = p \} \quad , \quad \varphi \in [0, \pi) \quad , \quad p \in \mathbb{R}^1 .$$

T_λ is driven by an independently marked stationary Poisson process $\Pi_\lambda = \{[P_i, \Phi_i] : i \in \mathbb{Z}^1\}$ on the real line \mathbb{R}^1 with intensity λ and typical mark Φ_0 uniformly distributed on $[0, \pi)$, see Chapt. 3 in [5] for details.

Put

$$X_\varrho = \sqrt{\varrho} \left(\frac{\Psi_L(\varrho K)}{L(\partial(\varrho K))} - \frac{\lambda}{\pi} \right) \quad \text{and} \quad Y_\varrho = \sqrt{\varrho} \left(\frac{\Psi_V(\varrho K)}{A(\varrho K)} - \frac{\lambda^2}{\pi} \right),$$

where $\Psi_L(\varrho K)$ resp. $\Psi_V(\varrho K)$ denotes the total number of lines of T_λ hitting ϱK resp. the total number of line crossings generated by T_λ in ϱK . In [3] we have proved the following bivariate central limit theorem :

$$\begin{pmatrix} X_\varrho \\ Y_\varrho \end{pmatrix} \xrightarrow{\varrho \rightarrow \infty} \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\lambda}{\pi L(\partial K)} & \frac{2\lambda^2}{\pi L(\partial K)} \\ \frac{2\lambda^2}{\pi L(\partial K)} & \frac{4\lambda^3}{\pi^3} \frac{I_2(K)}{A^2(K)} \end{pmatrix} \right) \quad (1)$$

with a strictly positive-definite covariance matrix in the Gaussian limit (1) implying that

$$\frac{I_2(K)}{A^2(K)} > \frac{\pi^2}{L(\partial K)} \approx \frac{9.8696}{L(\partial K)}. \quad (2)$$

Here, for $n = 0, 1, 2, \dots$,

$$I_n(K) = \int_0^\pi \int_{\mathbb{R}^1} L^n(K \cap g(p, \varphi)) \, dp \, d\varphi$$

denotes the n th-order *chord power integral* (briefly: CPI) of K .

From integral geometry it well-known, see [4] or [5], that $I_0(K) = L(\partial K)$,

$$I_1(K) = \pi A(K) \quad , \quad I_2(K) = \int_K \int_K \frac{dx \, dy}{\|y - x\|} \quad , \quad I_3(K) = 3 A^2(K).$$

Note that the CPIs $I_n(K)$, $n \geq 0$, are connected with some unique *chord length distribution function* $C_K(x)$, $x \geq 0$, such that the n th moment of C_K equals the ratio $I_n(K)/I_0(K)$. However, C_K does not always determine completely the shape of the convex disc K , as shown first by C.L. Mallows and J.M.C. Clark in 1970, see [4].

Nevertheless, in image analysis and other fields of application C_K and even only the first four CPIs $I_0(K), \dots, I_4(K)$ are used to ‘characterize’ the shape of convex particles.

Further, we mention that the ratio $I_2(K)/A^2(K)$ can be interpreted as mean reciprocal Euclidean distance $\mathbb{E}\|P_K - Q_K\|^{-1}$ of two independent and uniformly distributed points P_K and Q_K in K . This could be useful for computer-based simulations of the second-order CPI $I_2(K)$.

From the view point of *optimal experimental design* it is the aim to minimize the limiting variance of Y_ϱ in (1) given e.g. the perimeter $L(\partial K)$ or other side condition on K . In other words, we have to find the smallest possible ratio $I_2(K)/A^2(K)$ and the minimizing disc K , when $L(\partial K)$ is fixed.

In the rest of this paper we discuss several special cases which motivate in particular the lower estimate in

Conjecture I For any convex discs K with inner point the inclusion

$$\frac{10.6667}{L(\partial K)} \approx \frac{32}{3L(\partial K)} \leq \frac{I_2(K)}{A^2(K)} \leq \frac{16}{3\sqrt{\pi}A(K)} \quad (3)$$

holds which establishes a slight strengthening of the well-known isoperimetric inequality, see e.g. [4], with equality on both sides if and only if K is a circle with diameter $L(\partial K)/\pi$.

Remark 1 The upper bound in (3) is a celebrated result by T. Carleman published already in 1919, see Thm. 8.6.6 in [5] for generalizations to higher-order CPIs in any dimension. A hint to the lower bound is given in [1].

2 Second-Order CPIs for Some Convex Discs

For any circular disc $B_r = \{x \in \mathbb{R}^2 : \|x\| \leq r\}$ with radius $r > 0$, it is a simple exercise to confirm that $I_2(B_r) = 16\pi r^3/3$ and together with $L(\partial B_r) = 2\pi r$, $A(B_r) = \pi r^2$ we see that lower and upper bound in (3) coincide.

For an ellipse $E_{ab} = \{(x_1, x_2) \in \mathbb{R}^2 : b^2 x_1^2 + a^2 x_2^2 \leq a^2 b^2\}$ with semi-axes a, b satisfying $a \geq b > 0$ and numerical excentricity $\varepsilon = \sqrt{a^2 - b^2}/a$, we arrive after lengthy computations at, see [3]:

$$I_2(E_{ab}) = \frac{32ab^2}{3} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - \varepsilon^2 \sin^2 \varphi}} = \frac{32ab^2}{3} F\left(\frac{\pi}{2}, \varepsilon\right),$$

where $F(\frac{\pi}{2}, \varepsilon)$ is Legendre's notation for a complete elliptic integral of first kind. On the other hand, using the complete elliptic integral of second kind $E(\frac{\pi}{2}, \varepsilon)$ we can express the perimeter $L(\partial E_{ab})$ by

$$L(\partial E_{ab}) = 4a E\left(\frac{\pi}{2}, \varepsilon\right) = 4a \int_0^{\frac{\pi}{2}} \sqrt{1 - \varepsilon^2 \sin^2 \varphi} \, d\varphi.$$

By applying the Cauchy-Schwarz inequality and the well-known area formula $A(E_{ab}) = \pi a b$ we obtain that

$$\begin{aligned} \frac{32}{3} &= \frac{32}{3} \left(\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{(1 - \varepsilon^2 \sin^2 \varphi)^{1/4}}{(1 - \varepsilon^2 \sin^2 \varphi)^{1/4}} \, d\varphi \right)^2 \leq \frac{128}{3\pi^2} E\left(\frac{\pi}{2}, \varepsilon\right) F\left(\frac{\pi}{2}, \varepsilon\right) \\ &= \frac{128}{\pi^2} \frac{L(\partial E_{ab})}{4a} \frac{3I_2(E_{ab})}{32ab^2} = \frac{L(\partial E_{ab}) I_2(E_{ab})}{A^2(E_{ab})}, \end{aligned}$$

where '=' holds iff $\varepsilon = 0$. Thus, we have proved

Theorem 1 *For any ellipse E_{ab} both inequalities in (3) hold and equality on each side is attained iff $a = b$.*

The computation of CPIs for a rectangle R_{ab} with sides a and b is an immediate consequence of a more general formula for convex N -gons P_N which will be derived in the next Sect. 3. We get the formula

$$I_2(R_{ab}) = \frac{2(a^3 + b^3) - 8(\sqrt{a^2 + b^2})^3}{3} + 4ab^2 I\left(\frac{a}{b}\right) + 4a^2 b I\left(\frac{b}{a}\right),$$

where

$$I(x) = \int_0^x \sqrt{t^2 + 1} \, dt = \frac{1}{2} \left[x \sqrt{x^2 + 1} + \log\left(x + \sqrt{x^2 + 1}\right) \right]. \quad (4)$$

Note that $I(-x) = I(x)$, $I'(x) > 0$, $I''(x) > 0$ and $I'''(x) > 0$ for $x > 0$.

Theorem 2 *Given the semiperimeter $s = a + b$, the ratio $I_2(R_{ab})/a^2 b^2$ takes its minimum for the square with side length $s/2$, i.e.*

$$\frac{I_2(R_{ab})}{A^2(R_{ab})} \geq \frac{8}{3s} \left[3 \log(1 + \sqrt{2}) + 1 - \sqrt{2} \right] \approx \frac{11.8928}{L(\partial R_{ab})}. \quad (5)$$

Proof After tedious calculations we find that the relation (5) is equivalent to

$$\frac{3(a+b)}{2a^2b^2} I_2(R_{ab}) = f\left(\frac{a}{a+b}\right) + f\left(\frac{b}{a+b}\right) \geq 2f\left(\frac{1}{2}\right),$$

where

$$f(x) = \frac{3}{x} \log \frac{x + \sqrt{1 - 2x(1-x)}}{1-x} - \frac{1}{1-x + \sqrt{1 - 2x(1-x)}} \quad \text{for } 0 < x < 1.$$

In fact, it can be shown that $f''(x) > 0$ for $0 < x < 1$, i.e., $f(\cdot)$ is strictly convex so that $(f(x) + f(1-x))/2 \geq f(1/2) = 2(3 \log(1 + \sqrt{2}) + 1 - \sqrt{2})$ exhibits a special case of Jensen's inequality. \square

Finally, we consider an arbitrary triangle Δ_{abc} with side-lengths a, b , and c and semi-perimeter $s = (a + b + c)/2$ such that each side is shorter than the sum of the other two sides. We use the below formula (6) to rewrite $I_2(K_3)$ in terms of the function (4) and express the trigonometric functions of the angles by the sides a, b, c , and s . Summarizing all expressions leads directly to the comparatively simple formula

$$\frac{I_2(\Delta_{abc})}{A^2(\Delta_{abc})} = \frac{4}{3} \left[\frac{1}{a} \log\left(\frac{s}{s-a}\right) + \frac{1}{b} \log\left(\frac{s}{s-b}\right) + \frac{1}{c} \log\left(\frac{s}{s-c}\right) \right], \quad (6)$$

where Heron's formula gives $A(\Delta_{abc}) = \sqrt{s(s-a)(s-b)(s-c)}$.

Theorem 3 *Among all planar triangles with perimeter $2s = a + b + c$, the ratio $I_2(\Delta_{abc})/A^2(\Delta_{abc})$ attains its minimum for the equilateral triangle, i.e. for $a = b = c = 2s/3$.*

Proof Using the (strict) convexity of the function

$$g(x) = \frac{1}{x} \log\left(\frac{s}{s-x}\right) = \frac{1}{s} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x}{s}\right)^{k-1} \quad \text{for } 0 < x < s,$$

we may apply Jensen's inequality yielding

$$\frac{I_2(\Delta_{abc})}{A^2(\Delta_{abc})} = \frac{4}{3} \left(g(a) + g(b) + g(c) \right) \geq 4g\left(\frac{a+b+c}{3}\right) = \frac{12 \log 3}{L(\partial\Delta_{abc})}$$

with equality iff $a = b = c = 2s/3$. \square

3 Higher-Order CPIs for Convex Polygons

Following the ideas of A. Pleijel (1956) and R.V. Ambartzumian (1970-74), see references in [4], we can express $I_n(K)$ as double integral over the boundary ∂K . Combining two formulas for $I_n(K)$ given in [4], see Chapt. I.3.5, p. 37, we obtain

$$I_n(K) = -\frac{n}{2(n-1)} \int_{\partial K} \int_{\partial K} \|s_1 - s_2\|^{n-1} \cos(\theta_1 + \theta_2) ds_1 \wedge ds_2$$

for $n \geq 1$, where $\theta_1 = \theta_1(s_1)$ (resp. $\theta_2 = \theta_2(s_2)$) denotes the angle between the tangent at s_1 (resp. s_2) on ∂K and the chord joining s_1 and s_2 .

Now, let $K = K_N$ be a convex polygon having N vertices A_1, \dots, A_N (anti-clockwise) with inner angle α_i at A_i and edges $L_i = \overline{A_i A_{i+1}}$, where $A_{N+i} = A_i$, $L_{N+i} = L_i$ and $\alpha_{N+i} = \alpha_i$ for $i = 1, \dots, N$. Obviously, $\partial K_N = L_1 \cup \dots \cup L_N$. If $s_1 \in L_i \setminus \{A_{i+1}\}$ and $s_2 \in L_{i+j} \setminus \{A_{i+j}\}$, then we get a $(j+2)$ -gon with vertices $s_1, A_{i+1}, \dots, A_{i+j}, s_2$ such that $\theta_1 + \theta_2 + \alpha_{i+1} + \dots + \alpha_{i+j} = j\pi$ implying $\cos(\theta_1 + \theta_2) = (-1)^j \cos(\alpha_{i+1} + \dots + \alpha_{i+j})$. After some simple rearrangement we find that n th-order CPI $I_n(K_N)$ equals

$$\begin{aligned} & \sum_{i=1}^N \sum_{j=1}^{N-1} (-1)^{j+1} \cos\left(\sum_{k=1}^j \alpha_{i+k}\right) I_n(L_i, L_{i+j}) - \frac{1}{n^2-1} \sum_{i=1}^N |L_i|^{n+1} \\ &= 2 \sum_{j=1}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^{j+1} \sum_{i=1}^N \cos\left(\sum_{k=1}^j \alpha_{i+k}\right) I_n(L_i, L_{i+j}) - \frac{1}{n^2-1} \sum_{i=1}^N |L_i|^{n+1} \\ & \quad - (1 + (-1)^N) (-1)^{N/2} \sum_{i=1}^{N/2} \cos\left(\sum_{k=1}^{N/2} \alpha_{i+k}\right) I_n(L_i, L_{i+\frac{N}{2}}), \end{aligned} \quad (7)$$

where $|L_i| = \|A_i - A_{i+1}\|$ and

$$I_n(L_i, L_{i+j}) = \frac{n}{2(n-1)} \int_{L_i} \int_{L_{i+j}} \|s_1 - s_2\|^{n-1} ds_1 \wedge ds_2.$$

We will apply the previous formula only for $n = 2$ so that it remains to compute the integral $I_2(\overline{AB}, \overline{CD})$ for a convex quadrangle $\square ABCD$:

$$\begin{aligned} J_{\varepsilon_1, \varepsilon_2}(a, c, e) &= \int_0^a \int_0^c \left\| A - C + \frac{s}{a}(B - A) - \frac{t}{c}(D - C) \right\| dt ds \\ &= \int_0^a \int_0^c \sqrt{s^2 + t^2 + e^2 - 2se \cos \varepsilon_1 - 2te \cos \varepsilon_2 + 2st \cos(\varepsilon_2 - \varepsilon_1)} dt ds, \end{aligned}$$

where $a = \|A - B\|$, $c = \|C - D\|$, $e = \|A - C\|$, $\varepsilon_1 = \angle(CAB)$, and $\varepsilon_2 = \angle(ACD)$ determine $\square ABCD$. Remarkably, the double integral $J_{\varepsilon_1, \varepsilon_2}(a, c, e)$ can be expressed as closed formula in terms of the function (4). The details of the rather lengthy computation are omitted. We obtain that, for $\varepsilon_1 \neq \varepsilon_2$,

$$J_{\varepsilon_1, \varepsilon_2}(a, c, e) =$$

$$\frac{e^3 \sin^3 \varepsilon_1}{3 \sin(\varepsilon_2 - \varepsilon_1)} \left[I\left(\frac{e \cos \varepsilon_1 - a}{e \sin \varepsilon_1}\right) - I(\cot \varepsilon_1) \right] - \frac{e^3 \sin^3 \varepsilon_2}{3 \sin(\varepsilon_2 - \varepsilon_1)} \left[I\left(\frac{e \cos \varepsilon_2 - c}{e \sin \varepsilon_2}\right) - I(\cot \varepsilon_2) \right]$$

$$+ \frac{(e_1 \sin \varepsilon_1 + c \sin(\varepsilon_2 - \varepsilon_1))^3}{3 \sin(\varepsilon_2 - \varepsilon_1)} \left[I\left(\frac{e \cos \varepsilon_1 - c \cos(\varepsilon_2 - \varepsilon_1)}{e \sin \varepsilon_1 + c \sin(\varepsilon_2 - \varepsilon_1)}\right) - I\left(\frac{e \cos \varepsilon_1 - c \cos(\varepsilon_2 - \varepsilon_1) - a}{e \sin \varepsilon_1 + c \sin(\varepsilon_2 - \varepsilon_1)}\right) \right]$$

$$- \frac{(e_2 \sin \varepsilon_2 - a \sin(\varepsilon_2 - \varepsilon_1))^3}{3 \sin(\varepsilon_2 - \varepsilon_1)} \left[I\left(\frac{e \cos \varepsilon_2 - a \cos(\varepsilon_2 - \varepsilon_1)}{e \sin \varepsilon_2 - a \sin(\varepsilon_2 - \varepsilon_1)}\right) - I\left(\frac{e \cos \varepsilon_2 - a \cos(\varepsilon_2 - \varepsilon_1) - c}{e \sin \varepsilon_2 - a \sin(\varepsilon_2 - \varepsilon_1)}\right) \right]$$

and

$$J_{\varepsilon, \varepsilon}(a, c, e) = \frac{e^3 \sin^3 \varepsilon}{3} \left[J\left(\frac{a + c - e \cos \varepsilon}{e \sin \varepsilon}\right) - J\left(\frac{c - e \cos \varepsilon}{e \sin \varepsilon}\right) - J\left(\frac{a - e \cos \varepsilon}{e \sin \varepsilon}\right) + J(\cot \varepsilon) \right]$$

for $\varepsilon = \varepsilon_1 = \varepsilon_2$ with $J(x) = 3xI(x) - (\sqrt{x^2 + 1})^3$.

If $\varepsilon_1 = 0$, i.e. $e = a$ and $C \equiv B$, then

$$J_{0, \varepsilon}(a, c, a) = \frac{a^3 \sin^2 \varepsilon}{3} \left[I\left(\frac{c - a \cos \varepsilon}{a \sin \varepsilon}\right) + I(\cot \varepsilon) \right] + \frac{c^3 \sin^2 \varepsilon}{3} \left[I\left(\frac{a - c \cos \varepsilon}{c \sin \varepsilon}\right) + I(\cot \varepsilon) \right].$$

Inserting $J_{\varepsilon, \varepsilon}(a, a, e)$ resp. $J_{0, \varepsilon}(a, c, a)$ in (7) for $N = 4$ resp. $N = 3$ with adequately chosen angles expressed in terms of the sides leads to the formulas for $I_2(R_{ab})$ resp. $I_2(\Delta_{abc})$ given in Sect. 2.

In particular for a triangle Δ_{abc} with inner angles α, β, γ and semi-perimeter $s = (a + b + c)/2$ we use the relations

$$\tan \frac{\alpha}{2} = \frac{A(\Delta_{abc})}{s(s-a)}, \quad \tan \frac{\beta}{2} = \frac{A(\Delta_{abc})}{s(s-b)}, \quad \tan \frac{\gamma}{2} = \frac{A(\Delta_{abc})}{s(s-c)},$$

and Heron's formula $A(\Delta_{abc}) = \sqrt{s(s-a)(s-b)(s-c)}$ to obtain (6).

4 Second-Order CPIs for Regular N -gons

We consider a regular N -gon P_N with vertices A_1, \dots, A_N , edge length a giving the circumradius $r_N = a/2 \sin \varphi_N$ with $\varphi_N = \pi/N$. The vertices $A_1, \dots, A_N (= A_0)$ can be identified with multiples of the unit roots $A_k = r_N (\cos(2k\varphi_N), \sin(2k\varphi_N))$. Hence, for $k = 0, 1, \dots, N-1$, we have

$$\|\overline{A_0 A_k}\| = 2 r_N \sin(k\varphi_N) = a \frac{\sin(k\varphi_N)}{\sin(\varphi_N)}.$$

Applying formula (6) for $I_2(K_N)$ in case of a regular N -gon P_N combined with the integral expressions in Sect. 3 and $A(P_N) = N a^2/4 \tan \varphi_N$ and $L(\partial P_N) = N a$ leads to the following closed-term expression of $I_2(P_N)$:

Theorem 4 *For any $N = 3, 4, 5, \dots$ we have*

$$I_2(P_N) = \frac{A^2(P_N)}{L(\partial P_N)} c_N = \frac{a^3 N c_N}{16 \tan^2 \varphi_N} \quad \text{with} \quad c_N = b_N - a_N,$$

where

$$a_N = \frac{16}{3 \cos \varphi_N} \sum_{k=1}^{\lfloor \frac{N-3}{2} \rfloor} \frac{\sin^2(k\varphi_N) \sin^2((k+1)\varphi_N)}{\cos(k\varphi_N) \cos((k+1)\varphi_N)} \log\left(\frac{\tan \frac{(k+1)\varphi_N}{2}}{\tan \frac{k\varphi_N}{2}}\right)$$

and

$$b_N = \frac{16 (1 + \cos \varphi_N)^3}{3 \sin \varphi_N \sin(2\varphi_N)} \log\left(\frac{1 + \sin \frac{\varphi_N}{2}}{\cos \frac{\varphi_N}{2}}\right) \quad \text{for odd } N \geq 3,$$

$$b_N = \frac{16}{3} \left[\frac{1 + \sin^2 \varphi_N}{\sin^2 \varphi_N} \log\left(\frac{1 + \sin \varphi_N}{\cos \varphi_N}\right) - \tan \frac{\varphi_N}{2} \right] \quad \text{for even } N \geq 4.$$

For $N \in \{3, 4, 5, 6\}$ with $\cos \varphi_N$ expressible by square root terms we get :

$$\begin{aligned} c_3 &= 12 \log 3, \quad c_5 = \frac{10(2 + \sqrt{5})}{3} \left[2 \log(2 + \sqrt{5}) - (8 - 3\sqrt{5}) \log 5 \right] \\ c_4 &= \frac{16}{3} \left[3 \log(1 + \sqrt{2}) + 1 - \sqrt{2} \right], \quad c_6 = \frac{4}{3} \left[11 \log 3 - 2 \log(2 + \sqrt{3}) - 4(2 - \sqrt{3}) \right] \\ c_8 &= \frac{16}{3} \left[(5 + 2\sqrt{2}) \log\left(1 + \sqrt{4 + 2\sqrt{2}}\right) - (8 + \sqrt{2}) \log(1 + \sqrt{2}) - (1 + \sqrt{2})(\sqrt{4 - 2\sqrt{2}} - 1) \right] \\ &\quad + \frac{8}{3} \left[(2 + \sqrt{2}) \log\left(\sqrt{2} - 1 + \sqrt{4 - 2\sqrt{2}}\right) - (3\sqrt{2} - 4) \log\left(1 + \sqrt{2} + \sqrt{4 + 2\sqrt{2}}\right) \right] \end{aligned}$$

Numerical values of the ratio $c_N = L(\partial P_N) I_2(P_N)/A^2(P_N)$

N	c_N	N	c_N	N	c_N	N	c_N	N	c_N
3	13.183347	11	10.812517	19	10.715294	27	10.690730	40	10.677629
4	11.892838	12	10.789036	20	10.710545	28	10.689041	50	10.673683
5	11.412043	13	10.770821	21	10.706460	29	10.687524	60	10.671539
6	11.172030	14	10.756401	22	10.702921	30	10.686156	70	10.670247
7	11.033211	15	10.744788	23	10.699834	31	10.684919	80	10.669408
8	10.945188	16	10.735297	24	10.697126	32	10.683796	90	10.668832
9	10.885692	17	10.727439	25	10.694736	33	10.682773	100	10.668421
10	10.843523	18	10.720859	26	10.692617	34	10.681840	1000	10.666684

We can prove that $c_N > 32/3$ for $N \geq 3$ and $c_N \downarrow 32/3$ as $N \rightarrow \infty$.

Taking into account the ‘discrete isoperimetric inequality’ for N -gons K_N , namely $4N \tan \varphi_N A(K_N) \leq L^2(\partial K_N)$, see [2] for a survey, and the above formulas for $I_2(R_{ab})$, $I_2(\Delta_{abc})$, $I_2(P_N)$ (and also as consequence of numerical experiments) we have enough reasons for

Conjecture II Among all convex N -gons K_N with fixed perimeter $L(\partial K_N)$ the ratio $I_2(K_N)/A^2(K_N)$ attains its minimum for the regular N -gon P_N with edge length $L(\partial K_N)/N$. More precisely, the ‘discrete’ variant of (3)

$$\frac{c_N}{L(\partial K_N)} \leq \frac{I_2(K_N)}{A^2(K_N)} \leq \frac{c_N}{\sqrt{4N \tan \varphi_N A(K_N)}} \quad (8)$$

should be possible to prove with the constants c_N defined in Theorem 4 . Equality on each side holds iff K_N is regular N -gons P_N .

Remark 2 The right-hand inequality is the ‘discrete’ counterpart of Carleman’s inequality. To the best of the author’s knowledge a rigorous proof of it seems to be unknown.

5 An Extension to Higher Dimensions

Let there be given a stationary and isotropic Poisson hyperplane tessellation $T_\lambda^{(d)} = \bigcup_{i \in \mathbb{Z}^1} H(P_i, V_i)$ in \mathbb{R}^d driven by an independently marked stationary Poisson process $\Pi_\lambda = \{[P_i, V_i] : i \in \mathbb{Z}^1\}$ on the real line with intensity λ and typical mark V_0 uniformly distributed on the upper hemisphere \mathbb{S}_+^{d-1} of the unit ball $B_1^{(d)}$, see [3] for details. An unoriented hyperplane in \mathbb{R}^d is defined by $H(p, v) = \{x \in \mathbb{R}^d : \langle v, x \rangle = p\}$ for any $p \in \mathbb{R}^1$ and $v \in \mathbb{S}_+^{d-1}$. Further, by \mathcal{H}_d resp. \mathcal{H}_{d-1} we denote the d - resp. $(d-1)$ -dimensional Hausdorff measure in \mathbb{R}^d and put $\kappa_d = \mathcal{H}_d(B_1^{(d)})$.

The random number of vertices $\Psi_V(\varrho K)$ generated by $T_\lambda^{(d)}$ in ϱK is approximately normally distributed for large ϱ , see Thm. 3.1 in [3]:

$$\sqrt{\varrho} \left(\frac{\Psi_V(\varrho K)}{\mathcal{H}_d(\varrho K)} - \kappa_d \left(\lambda \frac{\kappa_{d-1}}{d \kappa_d} \right)^d \right) \xrightarrow{\varrho \rightarrow \infty} \mathcal{N} \left(0, \frac{\kappa_{d-1}}{d-1} \left(\lambda \frac{\kappa_{d-1}}{d \kappa_d} \right)^{2d-1} \frac{J_d(K)}{\mathcal{H}_d^2(K)} \right)$$

with

$$J_d(K) = \frac{2}{(d-1)\kappa_{d-1}} \int_{\mathbb{R}^1} \int_{\mathbb{S}_+^{d-1}} \mathcal{H}_{d-1}^2(K \cap H(p, v)) \mathcal{H}_{d-1}(dv) dp = \int_K \int_K \frac{dx dy}{\|x - y\|}.$$

The right-hand equality can be shown by combining the affine Blaschke- Petkantschin formula of Thm. 7.2.7 with Thm. 8.6.4 in [5]. As a by-product it turns out that $J_d(K)$ coincides (up to a constant) with the d th-order CPI of the convex body $K \subset \mathbb{R}^d$. Taking into account that $L(\partial K) = \pi b_2(K)$ for any convex disc $K \in \mathbb{R}^2$, where $b_d(K)$ denotes the *mean breadth* of a convex body $K \in \mathbb{R}^d$, we formulate as a natural extension of (3) the following

Conjecture III For any convex body $K \subset \mathbb{R}^d$ with inner points, it holds

$$\frac{(2^d d!)^2 \kappa_{d-1}}{(d-1)(2d)! \kappa_d} \frac{2}{b_d(K)} \leq \frac{J_d(K)}{\mathcal{H}_d^2(K)} \leq \frac{(2^d d!)^2 \kappa_{d-1}}{(d-1)(2d)! \kappa_d} \left(\frac{\kappa_d}{\mathcal{H}_d(K)} \right)^{\frac{1}{d}} \quad (9)$$

with equality on each side iff K is a d -ball $B_r^{(d)}$ with some radius $r > 0$.

Due to the generalized Carleman inequality for CPIs, see Thm. 8.6.6 in [5], the upper bound is known for long time, whereas the lower bound of (9) seems to be still unproved. In view of the well-known inequality $2/b_d(K) \leq (d \kappa_d / \mathcal{H}_{d-1}(\partial K))^{1/(d-1)}$, see e.g. [5], one could strenghten the lower bound of (9) in an obvious way. However, replacing $2/b_d(K)$ by $(d \kappa_d / \mathcal{H}_{d-1}(\partial K))^{1/(d-1)}$ in (9) is impossible even in case of ellipsoids in \mathbb{R}^3 . This is seen from the below formulae derived for a three-dimensional ellipsoid

$$E_{abc} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} \leq 1 \right\}$$

with semi-axes $a, b, c > 0$. For the unit vectors $v_{\alpha, \beta} = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \beta)$, $0 \leq \alpha \leq \pi$, $0 \leq \beta \leq 2\pi$, the distances of the support planes $H(p_{\alpha, \beta}, v_{\alpha, \beta})$ of E_{abc} from the origin \mathbf{o} (which just gives the support function $h(v_{\alpha, \beta})$ of E_{abc}) are equal to

$$|p_{\alpha, \beta}| = \sqrt{a^2 \sin^2 \alpha \cos^2 \beta + b^2 \sin^2 \alpha \sin^2 \beta + c^2 \cos^2 \alpha}$$

providing the mean breadth

$$b_3(E_{abc}) = \frac{4}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} |p_{\alpha, \beta}| \sin \alpha \, d\alpha \, d\beta = \frac{1}{2\pi} \int_{\mathbb{S}_+^2} (h(-v) + h(v)) \mathcal{H}_2(dv).$$

Further, after a lengthy calculation including the determination of the semi-axes of the ellipse $E_{abc} \cap H(p, v_{\alpha, \beta})$ for $0 \leq p \leq |p_{\alpha, \beta}|$, it turns out that the ovoid functional $J_3(E_{abc})$ can be expressed as follows ¹:

$$J_3(E_{abc}) = \frac{64}{15} \pi a^2 b^2 c^2 \int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin \alpha}{|p_{\alpha, \beta}|} d\alpha d\beta = \frac{6}{5\pi} \mathcal{H}_3^2(E_{abc}) \int_{\mathbb{S}_+^2} \frac{\mathcal{H}_2(dv)}{h(-v) + h(v)}.$$

Hence, by the Cauchy-Schwarz inequality, it is immediately seen that

$$\frac{b_3(E_{abc}) J_3(E_{abc})}{\mathcal{H}_3^2(E_{abc})} = \frac{3}{5\pi^2} \int_{\mathbb{S}_+^2} \frac{\mathcal{H}_2(dv)}{h(-v) + h(v)} \int_{\mathbb{S}_+^2} (h(-v) + h(v)) \mathcal{H}_2(dv) \geq \frac{3}{5\pi^2} \mathcal{H}_2^2(\mathbb{S}_+^2) = \frac{12}{5},$$

which is nothing else, but the left-hand part of (9) for $d = 3$ and $K = E_{abc}$. In the particular case of a prolate spheroid, e.g. $a \geq b = c$, we get the closed-term expressions

$$\frac{J_3(E_{abb})}{\mathcal{H}_3^2(E_{abb})} = \frac{6}{5a\varepsilon} \log \frac{a}{b} (1 + \varepsilon) \quad \text{and} \quad \mathcal{H}_2(\partial E_{abb}) = 2\pi b^2 + 2\pi ab \frac{\arcsin \varepsilon}{\varepsilon}$$

with numerical excentricity $\varepsilon = \sqrt{a^2 - b^2}/a$ so that we have

$$\sqrt{\mathcal{H}_2(\partial E_{abb})} \frac{J_3(E_{abb})}{\mathcal{H}_3^2(E_{abb})} \sim \frac{6\pi\sqrt{b} \log a}{5\sqrt{a}} \xrightarrow{a \rightarrow \infty} 0 \quad \text{as} \quad a \rightarrow \infty.$$

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