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On Lower Bounds of Second-Order Chord Power Integrals of Convex Discs

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Abstract

For planar convex discs K with positive area $A(K)$, boundary length $L(\partial K)$ and second-order chord power integral $I_2(K)$, we study the ratio $L(\partial K)\, I_2(K)/A^2(K)$ and give reasons supporting the conjecture that its uniform lower bound is $32/3$ attained exactly for circles. In particular, using the Ambartzumian-Pleijel representation of $I_2(K)$ we derive formulas for $I_2(K)$ in case of general triangles, rectangles, and regular N-gons. In these cases and for ellipses we can prove this inequality. A related conjecture is formulated for the class of convex N-gons which exhibits a strengthening of the isoperimetric inequality as well as of Carleman's inequality for convex N-gons. An extension to higher dimensions is discussed at the end of the paper.

Keywords : motion-invariant Poisson line processes, isoperimetric inequality, Carleman's inequality, mean breadth, chord length distribution, regular N-gons, ellipses

MSC 2000 : Primary 52A40 60G05 Secondary 52A07 52A22

1 Introduction - Posing the Problem

We consider a motion-invariant Poisson line tessellation $T_\lambda = \bigcup_{i\in\mathbb{Z}^1} g(P_i,\Phi_i)$ with line density λ in an increasing planar region ϱK with $1 \le \varrho \uparrow \infty$, where K is some fixed convex disc containing $\mathbf{o}$ as inner point. Here, $g(p,\varphi)$ denotes an unoriented straight line in a Cartesian x_1x_2-coordinate system with normal unit vector $(\cos\varphi, \sin\varphi)$ directed in the upper half-plane and signed perpendicular distance $p \in \mathbb{R}^1$ from the origin $\mathbf{o}$, formally written

$$g(p,\varphi) = \{\, (x_1,x_2) \in \mathbb{R}^2 : x_1\, \cos\varphi + x_2\, \sin\varphi = p \,\} \ , \ \ \varphi \in [0,\pi) \ , \ \ p \in \mathbb{R}^1 \, .$$

T_λ is driven by an independently marked stationary Poisson process $\Pi_\lambda = \{[P_i,\Phi_i] : i \in \mathbb{Z}^1\}$ on the real line $\mathbb{R}^1$ with intensity λ and typical mark Φ_0 uniformly distributed on $[0,\pi)$, see Chapt. 3 in [5] for details.

Put

$$X_\varrho = \sqrt{\varrho}\,\Big(\frac{\Psi_L(\varrho\, K)}{L(\partial(\varrho\, K))} - \frac{\lambda}{\pi}\Big) \text{ and } Y_\varrho = \sqrt{\varrho}\,\Big(\frac{\Psi_V(\varrho\, K)}{A(\varrho\, K)} - \frac{\lambda^2}{\pi}\Big),$$

where $\Psi_L(\varrho\, K)$ resp. $\Psi_V(\varrho\, K)$ denotes the total number of lines of T_λ hitting $\varrho\, K$ resp. the total number of line crossings generated by T_λ in $\varrho\, K$. In [3] we have proved the following bivariate central limit theorem :

$$\begin{pmatrix} X_\varrho \\ Y_\varrho \end{pmatrix} \underset{\varrho\to\infty}{\Longrightarrow} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\lambda}{\pi\, L(\partial K)} & \frac{2\,\lambda^2}{\pi\, L(\partial K)} \\ \frac{2\,\lambda^2}{\pi\, L(\partial K)} & \frac{4\,\lambda^3}{\pi^3}\,\frac{I_2(K)}{A^2(K)} \end{pmatrix}\right) \tag{1}$$

with a strictly positive-definite covariance matrix in the Gaussian limit (1) implying that

$$\frac{I_2(K)}{A^2(K)} > \frac{\pi^2}{L(\partial K)} \approx \frac{9.8696}{L(\partial K)}\,. \tag{2}$$

Here, for $n = 0, 1, 2, \ldots$,

$$I_n(K) = \int_0^\pi \int_{\mathbb{R}^1} L^n(K \cap g(p,\varphi))\,\mathrm{d}p\,\mathrm{d}\varphi$$

denotes the nth-order *chord power integral* (briefly: CPI) of K.

From integral geometry it well-known, see [4] or [5], that $I_0(K) = L(\partial K)$,

$$I_1(K) = \pi\, A(K) \quad , \quad I_2(K) = \int_K \int_K \frac{\mathrm{d}x\,\mathrm{d}y}{\|y - x\|} \quad , \quad I_3(K) = 3\, A^2(K)\,.$$

Note that the CPIs $I_n(K)$, $n \geq 0$, are connected with some unique *chord length distribution function* $C_K(x)$, $x \geq 0$, such that the nth moment of C_K equals the ratio $I_n(K)/I_0(K)$. However, C_K does not always determine completely the shape of the convex disc K, as shown first by C.L. Mallows and J.M.C. Clark in 1970, see [4].

Nevertheless, in image analysis and other fields of application C_K and even only the first four CPIs $I_0(K), ..., I_4(K)$ are used to 'characterize' the shape of convex particles.

Further, we mention that the ratio $I_2(K)/A^2(K)$ can be interpreted as mean reciprocal Euclidean distance $\mathsf{E}\|P_K - Q_K\|^{-1}$ of two independent and uniformly distributed points P_K and Q_K in K. This could be useful for computer-based simulations of the second-order CPI $I_2(K)$.

From the view point of *optimal experimental design* it is the aim to minimize the limiting variance of Y_ϱ in (1) given e.g. the perimeter $L(\partial K)$ or other side condition on K. In other words, we have to find the smallest possible ratio $I_2(K)/A^2(K)$ and the minimizing disc K, when $L(\partial K)$ is fixed.

In the rest of this paper we discuss several special cases which motivate in particular the lower estimate in

Conjecture I For any convex discs K with inner point the inclusion

$$\frac{10.6667}{L(\partial K)} \approx \frac{32}{3\,L(\partial K)} \leq \frac{I_2(K)}{A^2(K)} \leq \frac{16}{3\,\sqrt{\pi\,A(K)}} \tag{3}$$

holds which establishes a slight strengthening of the well-known isoperimetric inequality, see e.g. [4], with equality on both sides if and only if K is a circle with diameter $L(\partial K)/\pi$.

Remark 1 The upper bound in (3) is a celebrated result by T. Carleman published already in 1919, see Thm. 8.6.6 in [5] for generalizations to higher-order CPIs in any dimension. A hint to the lower bound is given in [1].

2 Second-Order CPIs for Some Convex Discs

For any circular disc $B_r = \{\, x \in \mathbb{R}^2 : \|x\| \leq r \,\}$ with radius $r > 0$, it is a simple exercise to confirm that $I_2(B_r) = 16\,\pi\,r^3/3$ and together with $L(\partial B_r) = 2\,\pi\,r$, $A(B_r) = \pi\,r^2$ we see that lower and upper bound in (3) coincide.

For an ellipse $E_{ab} = \{\, (x_1, x_2) \in \mathbb{R}^2 : b^2\,x_1^2 + a^2\,x_2^2 \leq a^2\,b^2 \,\}$ with semi-axes a, b satisfying $a \geq b > 0$ and numerical excentricity $\varepsilon = \sqrt{a^2 - b^2}/a$, we arrive after lengthy computations at, see [3] :

$$I_2(E_{ab}) = \frac{32\,a\,b^2}{3} \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\varphi}{\sqrt{1 - \varepsilon^2\,\sin^2\varphi}} = \frac{32\,a\,b^2}{3}\,\mathsf{F}(\frac{\pi}{2}, \varepsilon)\,,$$

where $\mathsf{F}(\frac{\pi}{2},\varepsilon)$ is Legendre's notation for a complete elliptic integral of first kind. On the other hand, using the complete elliptic integral of second kind $\mathsf{E}(\frac{\pi}{2},\varepsilon)$ we can express the perimeter $L(\partial E_{ab})$ by

$$L(\partial E_{ab}) = 4\,a\,\mathsf{E}(\frac{\pi}{2},\varepsilon) = 4\,a \int\limits_0^{\frac{\pi}{2}} \sqrt{1-\varepsilon^2\,\sin^2\varphi}\,\mathrm{d}\varphi\,.$$

By applying the Cauchy-Schwarz inequality and the well-known area formula $A(E_{ab}) = \pi\,a\,b$ we obtain that

$$\begin{aligned}\frac{32}{3} &= \frac{32}{3}\left(\frac{2}{\pi}\int\limits_0^{\frac{\pi}{2}} \frac{(\,1-\varepsilon^2\,\sin^2\varphi\,)^{1/4}}{(\,1-\varepsilon^2\,\sin^2\varphi\,)^{1/4}}\,\mathrm{d}\varphi\right)^2 \le \frac{128}{3\,\pi^2}\,\mathsf{E}(\frac{\pi}{2},\varepsilon)\,\mathsf{F}(\frac{\pi}{2},\varepsilon)\\ &= \frac{128}{\pi^2}\,\frac{L(\partial E_{ab})}{4\,a}\,\frac{3\,I_2(E_{ab})}{32\,a\,b^2} = \frac{L(\partial E_{ab})\,I_2(E_{ab})}{A^2(E_{ab})}\,,\end{aligned}$$

where '=' holds iff $\varepsilon = 0$. Thus, we have proved

Theorem 1 *For any ellipse E_{ab} both inequalities in (3) hold and equality on each side is attained iff $a = b$.*

The computation of CPIs for a rectangle R_{ab} with sides a and b is an immediate consequence of a more general formula for convex $N-$gons P_N which will be derived in the next Sect. 3. We get the formula

$$I_2(R_{ab}) = \frac{2\,(\,a^3+b^3\,) - 8\,(\sqrt{a^2+b^2}\,)^3}{3} + 4\,a\,b^2\,I\Big(\,\frac{a}{b}\,\Big) + 4\,a^2\,b\,I\Big(\,\frac{b}{a}\,\Big)\,,$$

where

$$I(x) = \int\limits_0^x \sqrt{t^2+1}\,\mathrm{d}t = \frac{1}{2}\Big[\,x\,\sqrt{x^2+1} + \log\Big(\,x+\sqrt{x^2+1}\,\Big)\Big]\,. \tag{4}$$

Note that $I(-x) = I(x)$, $I'(x) > 0$, $I''(x) > 0$ and $I'''(x) > 0$ for $x > 0$.

Theorem 2 *Given the semiperimeter $s = a + b$, the ratio $I_2(R_{ab})/a^2\,b^2$ takes its minimum for the square with side length $s/2$, i.e.*

$$\frac{I_2(R_{ab})}{A^2(R_{ab})} \ge \frac{8}{3\,s}\Big[\,3\,\log(\,1+\sqrt{2}\,) + 1 - \sqrt{2}\,\Big] \approx \frac{11.8928}{L(\partial R_{ab})}\,. \tag{5}$$

Proof After tedious calculations we find that the relation (5) is equivalent to

$$\frac{3(a+b)}{2\,a^2\,b^2}\;I_2(R_{ab}) = f(\frac{a}{a+b}) + f(\frac{b}{a+b}) \geq 2\,f(\frac{1}{2})\;,$$

where

$$f(x) = \frac{3}{x}\,\log\frac{x+\sqrt{1-2\,x\,(1-x)}}{1-x} - \frac{1}{1-x+\sqrt{1-2\,x\,(1-x)}} \quad \text{for} \quad 0 < x < 1\,.$$

In fact, it can be shown that $f''(x) > 0$ for $0 < x < 1$, i.e., $f(\cdot)$ is strictly convex so that $(\,f(x) + f(1-x)\,)/2 \geq f(1/2) = 2\,(\,3\,\log(\,1+\sqrt{2}\,) + 1 - \sqrt{2}\,)$ exhibits a special case of Jensen's inequality. □

Finally, we consider an arbitrary triangle Δ_{abc} with side-lengths a, b, and c and semi-perimeter $s = (a+b+c)/2$ such that each side is shorter than the sum of the other two sides. We use the below formula (6) to rewrite $I_2(K_3)$ in terms of the function (4) and express the trigonometric functions of the angles by the sides a, b, c, and s. Summarizing all expessions leads directly to the comparatively simple formula

$$\frac{I_2(\Delta_{abc})}{A^2(\Delta_{abc})} = \frac{4}{3}\,\left[\frac{1}{a}\,\log\Big(\frac{s}{s-a}\Big) + \frac{1}{b}\,\log\Big(\frac{s}{s-b}\Big) + \frac{1}{c}\,\log\Big(\frac{s}{s-c}\Big)\right]\,, \tag{6}$$

where Heron's formula gives $A(\Delta_{abc}) = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}\,.$

Theorem 3 *Among all planar triangles with perimeter* $2\,s = a+b+c$, *the ratio* $I_2(\Delta_{abc})/A^2(\Delta_{abc})$ *attains its minimum for the equilateral triangle, i.e.* for $a = b = c = 2\,s/3$.

Proof Using the (strict) convexity of the function

$$g(x) = \frac{1}{x}\,\log\Big(\frac{s}{s-x}\Big) = \frac{1}{s}\,\sum_{k=1}^{\infty}\frac{1}{k}\,\Big(\frac{x}{s}\Big)^{k-1} \quad \text{for} \quad 0 < x < s\;,$$

we may apply Jensen's inequality yielding

$$\frac{I_2(\Delta_{abc})}{A^2(\Delta_{abc})} = \frac{4}{3}\,\Big(\,g(a) + g(b) + g(c)\,\Big) \geq 4\,g\Big(\frac{a+b+c}{3}\Big) = \frac{12\,\log 3}{L(\partial\Delta_{abc})}$$

with equality iff $a = b = c = 2s/3\,.$ □

3 Higher-Order CPIs for Convex Polygons

Following the ideas of A. Pleijel (1956) and R.V. Ambartzumian (1970-74), see references in [4], we can express $I_n(K)$ as double integral over the boundary ∂K. Combining two formulas for $I_n(K)$ given in [4] , see Chapt. I.3.5 , p. 37 , we obtain

$$I_n(K) = -\frac{n}{2\,(n-1)} \int\limits_{\partial K}\int\limits_{\partial K} \|s_1 - s_2\|^{n-1} \cos(\theta_1 + \theta_2)\, \mathrm{d}s_1 \wedge \mathrm{d}s_2$$

for $n \geq 1$, where $\theta_1 = \theta_1(s_1)$ (resp. $\theta_2 = \theta_2(s_2)$) denotes the angle between the tangent at s_1 (resp. s_2) on ∂K and the chord joining s_1 and s_2.

Now, let $K = K_N$ be a convex polygon having N vertices $A_1, \ldots, A_N$ (anti-clockwise) with inner angle α_i at A_i and edges $L_i = \overline{A_i\,A_{i+1}}$, where $A_{N+i} = A_i$, $L_{N+i} = L_i$ and $\alpha_{N+i} = \alpha_i$ for $i = 1, \ldots, N$. Obviously, $\partial K_N = L_1 \cup \cdots \cup L_N$. If $s_1 \in L_i \setminus \{A_{i+1}\}$ and $s_2 \in L_{i+j} \setminus \{A_{i+j}\}$, then we get a $(j+2)-$gon with vertices $s_1, A_{i+1}, \ldots, A_{i+j}, s_2$ such that $\theta_1 + \theta_2 + \alpha_{i+1} + \cdots + \alpha_{i+j} = j\,\pi$ implying $\cos(\theta_1 + \theta_2) = (-1)^j \cos(\alpha_{i+1} + \cdots + \alpha_{i+j})$. After some simple rearrangement we find that nth-order CPI $I_n(K_N)$ equals

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N-1}(-1)^{j+1} \cos\Big(\sum_{k=1}^{j} \alpha_{i+k}\Big)\, I_n(L_i, L_{i+j}) - \frac{1}{n^2-1}\sum_{i=1}^{N}|L_i|^{n+1}\\
&= 2 \sum_{j=1}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^{j+1} \sum_{i=1}^{N} \cos\Big(\sum_{k=1}^{j} \alpha_{i+k}\Big)\, I_n(L_i, L_{i+j}) - \frac{1}{n^2-1}\sum_{i=1}^{N}|L_i|^{n+1}\\
&\quad -\big(1 + (-1)^N\big)\,(-1)^{N/2} \sum_{i=1}^{N/2} \cos\Big(\sum_{k=1}^{N/2} \alpha_{i+k}\Big)\, I_n(L_i, L_{i+\frac{N}{2}})\,,
\end{aligned} \tag{7}$$

where $|L_i| = \|A_i - A_{i+1}\|$ and

$$I_n(L_i, L_{i+j}) = \frac{n}{2\,(n-1)} \int\limits_{L_i}\int\limits_{L_{i+j}} \|s_1 - s_2\|^{n-1}\, \mathrm{d}s_1 \wedge \mathrm{d}s_2\,.$$

We will apply the previous formula only for $n = 2$ so that it remains to compute the integral $I_2(\overline{AB}, \overline{CD})$ for a convex quadrangle $\square ABCD$:

$$J_{\varepsilon_1,\varepsilon_2}(a, c, e) = \int_0^a\int_0^c \Big\|\, A - C + \frac{s}{a}\,(B - A) - \frac{t}{c}\,(D - C)\,\Big\|\; \mathrm{d}t\,\mathrm{d}s$$

$$= \int_0^a\int_0^c \sqrt{s^2 + t^2 + e^2 - 2\,s\,e\,\cos\varepsilon_1 - 2\,t\,e\,\cos\varepsilon_2 + 2\,s\,t\,\cos(\varepsilon_2 - \varepsilon_1)}\,\mathrm{d}t\,\mathrm{d}s\,,$$

where $a = \|A - B\|$, $c = \|C - D\|$, $e = \|A - C\|$, $\varepsilon_1 = \angle(CAB)$, and $\varepsilon_2 = \angle(ACD)$ determine $\square ABCD$. Remarkably, the double integral $J_{\varepsilon_1,\varepsilon_2}(a,c,e)$ can be expressed as closed formula in terms of the function (4). The details of the rather lengthy computation are omitted. We obtain that, for $\varepsilon_1 \neq \varepsilon_2$,

$$J_{\varepsilon_1,\varepsilon_2}(a,c,e) =$$

$$\frac{e^3 \sin^3 \varepsilon_1}{3 \sin(\varepsilon_2 - \varepsilon_1)} \Big[I\Big(\frac{e \cos \varepsilon_1 - a}{e \sin \varepsilon_1} \Big) - I(\cot \varepsilon_1) \Big] - \frac{e^3 \sin^3 \varepsilon_2}{3 \sin(\varepsilon_2 - \varepsilon_1)} \Big[I\Big(\frac{e \cos \varepsilon_2 - c}{e \sin \varepsilon_2} \Big) - I(\cot \varepsilon_2) \Big]$$

$$+ \frac{(e_1 \sin \varepsilon_1 + c \sin(\varepsilon_2 - \varepsilon_1))^3}{3 \sin(\varepsilon_2 - \varepsilon_1)} \Big[I\Big(\frac{e \cos \varepsilon_1 - c \cos(\varepsilon_2 - \varepsilon_1)}{e \sin \varepsilon_1 + c \sin(\varepsilon_2 - \varepsilon_1)} \Big) - I\Big(\frac{e \cos \varepsilon_1 - c \cos(\varepsilon_2 - \varepsilon_1) - a}{e \sin \varepsilon_1 + c \sin(\varepsilon_2 - \varepsilon_1)} \Big) \Big]$$

$$- \frac{(e_2 \sin \varepsilon_2 - a \sin(\varepsilon_2 - \varepsilon_1))^3}{3 \sin(\varepsilon_2 - \varepsilon_1)} \Big[I\Big(\frac{e \cos \varepsilon_2 - a \cos(\varepsilon_2 - \varepsilon_1)}{e \sin \varepsilon_2 - a \sin(\varepsilon_2 - \varepsilon_1)} \Big) - I\Big(\frac{e \cos \varepsilon_2 - a \cos(\varepsilon_2 - \varepsilon_1) - c}{e \sin \varepsilon_2 - a \sin(\varepsilon_2 - \varepsilon_1)} \Big) \Big]$$

and

$$J_{\varepsilon,\varepsilon}(a,c,e) = \frac{e^3 \sin^3 \varepsilon}{3} \Big[J\Big(\frac{a + c - e \cos \varepsilon}{e \sin \varepsilon} \Big) - J\Big(\frac{c - e \cos \varepsilon}{e \sin \varepsilon} \Big) - J\Big(\frac{a - e \cos \varepsilon}{e \sin \varepsilon} \Big) + J(\cot \varepsilon) \Big]$$

for $\varepsilon = \varepsilon_1 = \varepsilon_2$ with $J(x) = 3\,x\,I(x) - (\sqrt{x^2+1})^3$.

If $\varepsilon_1 = 0$, i.e. $e = a$ and $C \equiv B$, then

$$J_{0,\varepsilon}(a,c,a) = \frac{a^3 \sin^2 \varepsilon}{3} \Big[I\Big(\frac{c - a \cos \varepsilon}{a \sin \varepsilon} \Big) + I(\cot \varepsilon) \Big] + \frac{c^3 \sin^2 \varepsilon}{3} \Big[I\Big(\frac{a - c \cos \varepsilon}{c \sin \varepsilon} \Big) + I(\cot \varepsilon) \Big].$$

Inserting $J_{\varepsilon,\varepsilon}(a,a,e)$ resp. $J_{0,\varepsilon}(a,c,a)$ in (7) for $N = 4$ resp. $N = 3$ with adequately chosen angles expressed in terms of the sides leads to the formulas for $I_2(R_{ab})$ resp. $I_2(\Delta_{abc})$ given in Sect. 2.

In particular for a triangle Δ_{abc} with inner angles α, β, γ and semi-perimeter $s = (a+b+c)/2$ we use the relations

$$\tan\frac{\alpha}{2} = \frac{A(\Delta_{abc})}{s\,(s-a)} \ , \ \tan\frac{\beta}{2} = \frac{A(\Delta_{abc})}{s\,(s-b)} \ , \ \tan\frac{\gamma}{2} = \frac{A(\Delta_{abc})}{s\,(s-c)} \,,$$

and Heron's formula $A(\Delta_{abc}) = \sqrt{s\,(s-a)\,(s-b),(s-c)}$ to obtain (6).

4 Second-Order CPIs for Regular $N-$gons

We consider a regular $N-$gon P_N with vertices $A_1, ..., A_N$, edge length a giving the circumradius $r_N = a/2 \sin\varphi_N$ with $\varphi_N = \pi/N$. The vertices $A_1, ..., A_N (= A_0)$ can be identified with multiples of the unit roots $A_k = r_N\,(\cos(2k\varphi_N), \sin(2k\varphi_N))$. Hence, for $k = 0, 1, ..., N-1$, we have

$$\|\overline{A_0\,A_k}\| = 2\,r_N\,\sin(k\varphi_N) = a\,\frac{\sin(k\,\varphi_N)}{\sin(\varphi_N)}\,.$$

Applying formula (6) for $I_2(K_N)$ in case of a regular $N-$gon P_N combined with the integral expressions in Sect. 3 and $A(P_N) = N\,a^2/4\,\tan\varphi_N$ and $L(\partial P_N) = N\,a$ leads to the following closed-term expression of $I_2(P_N)$:

Theorem 4 *For any $N = 3, 4, 5, ...$ we have*

$$I_2(P_N) = \frac{A^2(P_N)}{L(\partial P_N)}\,c_N = \frac{a^3\,N\,c_N}{16\,\tan^2\varphi_N} \quad \textit{with} \quad c_N = b_N - a_N\,,$$

where

$$a_N = \frac{16}{3\,\cos\varphi_N}\sum_{k=1}^{\lfloor\frac{N-3}{2}\rfloor}\frac{\sin^2(k\,\varphi_N)\,\sin^2((k+1)\,\varphi_N)}{\cos(k\,\varphi_N)\,\cos((k+1)\,\varphi_N)}\,\log\Big(\frac{\tan\frac{(k+1)\,\varphi_N}{2}}{\tan\frac{k\,\varphi_N}{2}}\Big)$$

and

$$b_N = \frac{16\,\big(1+\cos\varphi_N\big)^3}{3\,\sin\varphi_N\,\sin(2\,\varphi_N)}\,\log\Big(\frac{1+\sin\frac{\varphi_N}{2}}{\cos\frac{\varphi_N}{2}}\Big) \qquad \textit{for odd } N \geq 3\,,$$

$$b_N = \frac{16}{3}\Big[\frac{1+\sin^2\varphi_N}{\sin^2\varphi_N}\,\log\Big(\frac{1+\sin\varphi_N}{\cos\varphi_N}\Big) - \tan\frac{\varphi_N}{2}\Big] \quad \textit{for even } N \geq 4\,.$$

For $N \in \{3, 4, 5, 6\}$ with $\cos\varphi_N$ expressible by square root terms we get :

$$c_3 = 12\,\log 3\;,\quad c_5 = \frac{10\,(2+\sqrt{5})}{3}\Big[2\,\log(2+\sqrt{5}) - (8-3\sqrt{5})\,\log 5\Big]$$

$$c_4 = \frac{16}{3}\Big[3\,\log(1+\sqrt{2}) + 1 - \sqrt{2}\Big]\;,\quad c_6 = \frac{4}{3}\Big[11\,\log 3 - 2\,\log(2+\sqrt{3}) - 4(2-\sqrt{3})\Big]$$

$$c_8 = \frac{16}{3}\Big[(5+2\sqrt{2})\,\log\Big(1+\sqrt{4+2\sqrt{2}}\Big) - (8+\sqrt{2})\,\log(1+\sqrt{2}) - (1+\sqrt{2})(\sqrt{4-2\sqrt{2}}-1)\Big]$$
$$+\ \frac{8}{3}\Big[(2+\sqrt{2})\,\log\Big(\sqrt{2}-1+\sqrt{4-2\sqrt{2}}\Big) - (3\sqrt{2}-4)\,\log\Big(1+\sqrt{2}+\sqrt{4+2\sqrt{2}}\Big)\Big]$$

Numerical values of the ratio $c_N = L(\partial P_N)\, I_2(P_N)/A^2(P_N)$

N	c_N	N	c_N	N	c_N	N	c_N	N	c_N
3	13.183347	11	10.812517	19	10.715294	27	10.690730	40	10.677629
4	11.892838	12	10.789036	20	10.710545	28	10.689041	50	10.673683
5	11.412043	13	10.770821	21	10.706460	29	10.687524	60	10.671539
6	11.172030	14	10.756401	22	10.702921	30	10.686156	70	10.670247
7	11.033211	15	10.744788	23	10.699834	31	10.684919	80	10.669408
8	10.945188	16	10.735297	24	10.697126	32	10.683796	90	10.668832
9	10.885692	17	10.727439	25	10.694736	33	10.682773	100	10.668421
10	10.843523	18	10.720859	26	10.692617	34	10.681840	1000	10.666684

We can prove that $c_N > 32/3$ for $N \geq 3$ and $c_N \downarrow 32/3$ as $N \to \infty$.

Taking into account the 'discrete isoperimetric inequality' for $N-$gons K_N, namely $4\,N\,\tan\varphi_N\,A(K_N) \leq L^2(\partial K_N)$, see [2] for a survey, and the above formulas for $I_2(R_{ab})$, $I_2(\Delta_{abc})$, $I_2(P_N)$ (and also as consequence of numerical experiments) we have enough reasons for

Conjecture II Among all convex N-gons K_N with fixed perimeter $L(\partial K_N)$ the ratio $I_2(K_N)/A^2(K_N)$ attains its minimum for the regular N-gon P_N with edge length $L(\partial K_N)/N$. More precisely, the 'discrete' variant of (3)

$$\frac{c_N}{L(\partial K_N)} \leq \frac{I_2(K_N)}{A^2(K_N)} \leq \frac{c_N}{\sqrt{4\,N\,\tan\varphi_N\,A(K_N)}} \tag{8}$$

should be possible to prove with the constants c_N defined in Theorem 4 . Equality on each side holds iff K_N is regular $N-$gons P_N.

Remark 2 The right-hand inequality is the 'discrete' counterpart of Carleman's inequality. To the best of the author's knowledge a rigorous proof of it seems to be unknown.

5 An Extension to Higher Dimensions

Let there be given a stationary and isotropic Poisson hyperplane tessellation $T_\lambda^{(d)} = \bigcup_{i\in\mathbb{Z}^1} H(P_i, V_i)$ in $\mathbb{R}^d$ driven by an independently marked stationary Poisson process $\Pi_\lambda = \{[P_i, V_i] : i \in \mathbb{Z}^1\}$ on the real line with intensity λ and typical mark V_0 uniformly distributed on the upper hemisphere $\mathbb{S}_+^{d-1}$ of the unit ball $B_1^{(d)}$, see [3] for details. An unoriented hyperplane in $\mathbb{R}^d$ is defined by $H(p, v) = \{x \in \mathbb{R}^d : \langle v, x\rangle = p\}$ for any $p \in \mathbb{R}^1$ and $v \in \mathbb{S}_+^{d-1}$. Further, by $\mathcal{H}_d$ resp. $\mathcal{H}_{d-1}$ we denote the $d-$ resp. $(d-1)-$dimensional Hausdorff measure in $\mathbb{R}^d$ and put $\kappa_d = \mathcal{H}_d(B_1^{(d)})$.

The random number of vertices $\Psi_V(\varrho K)$ generated by $T_\lambda^{(d)}$ in ϱK is approximately normally distributed for large ϱ, see Thm. 3.1 in [3]:

$$\sqrt{\varrho}\,\Big(\frac{\Psi_V(\varrho K)}{\mathcal{H}_d(\varrho K)} - \kappa_d\,\Big(\lambda\,\frac{\kappa_{d-1}}{d\,\kappa_d}\Big)^d\Big) \underset{\varrho\to\infty}{\Longrightarrow} \mathcal{N}\Big(0\,,\,\frac{\kappa_{d-1}}{d-1}\,\Big(\lambda\,\frac{\kappa_{d-1}}{d\,\kappa_d}\Big)^{2d-1}\,\frac{J_d(K)}{\mathcal{H}_d^2(K)}\Big)$$

with

$$J_d(K) = \frac{2}{(d-1)\kappa_{d-1}} \int\limits_{\mathbb{R}^1}\int\limits_{\mathbb{S}_+^{d-1}} \mathcal{H}_{d-1}^2(K\cap H(p,v))\,\mathcal{H}_{d-1}(\mathrm{d}v)\mathrm{d}p = \int\limits_K\int\limits_K \frac{\mathrm{d}x\mathrm{d}y}{\|x-y\|}.$$

The right-hand equality can be shown by combining the affine Blaschke- Petkantschin formula of Thm. 7.2.7 with Thm. 8.6.4 in [5]. As a by-product it turns out that $J_d(K)$ coincides (up to a constant) with the dth-order CPI of the convex body $K\subset\mathbb{R}^d$. Taking into account that $L(\partial K) = \pi\, b_2(K)$ for any convex disc $K\in\mathbb{R}^2$, where $b_d(K)$ denotes the *mean breadth* of a convex body $K\in\mathbb{R}^d$, we formulate as a natural extension of (3) the following

Conjecture III For any convex body $K\subset\mathbb{R}^d$ with inner points, it holds

$$\frac{(\,2^d\,d!\,)^2\,\kappa_{d-1}}{(d-1)\,(2\,d)!\,\kappa_d}\,\frac{2}{b_d(K)} \le \frac{J_d(K)}{\mathcal{H}_d^2(K)} \le \frac{(\,2^d\,d!\,)^2\,\kappa_{d-1}}{(d-1)\,(2\,d)!\,\kappa_d}\left(\frac{\kappa_d}{\mathcal{H}_d(K)}\right)^{\frac{1}{d}} \tag{9}$$

with equality on each side iff K is a $d-$ball $B_r^{(d)}$ with some radius $r>0$.

Due to the generalized Carleman inequality for CPIs, see Thm. 8.6.6 in [5], the upper bound is known for long time, whereas the lower bound of (9) seems to be still unproved. In view of the well-known inequality $2/b_d(K) \le \big(\,d\,\kappa_d/\mathcal{H}_{d-1}(\partial K)\,\big)^{1/(d-1)}$, see e.g. [5], one could strenghten the lower bound of (9) in an obvious way. However, replacing $2/b_d(K)$ by $\big(\,d\,\kappa_d/\mathcal{H}_{d-1}(\partial K)\,\big)^{1/(d-1)}$ in (9) is impossible even in case of ellipsoids in $\mathbb{R}^3$. This is seen from the below formulae derived for a three-dimensional ellipsoid

$$E_{abc} = \Big\{(x_1,x_2,x_3)\in\mathbb{R}^3 : \frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}\le 1\Big\}$$

with semi-axes $a,b,c>0$. For the unit vectors $v_{\alpha,\beta} = (\sin\alpha\,\cos\beta, \sin\alpha\,\sin\beta, \cos\beta)$, $0\le\alpha\le\pi$, $0\le\beta\le 2\pi$, the distances of the support planes $H(p_{\alpha,\beta}, v_{\alpha,\beta})$ of E_{abc} from the origin $\mathbf{o}$ (which just gives the support function $h(v_{\alpha,\beta})$ of E_{abc}) are equal to

$$|p_{\alpha,\beta}| = \sqrt{a^2\,\sin^2\alpha\,\cos^2\beta + b^2\,\sin^2\alpha\,\sin^2\beta + c^2\,\cos^2\alpha}$$

providing the mean breadth

$$b_3(E_{abc}) = \frac{4}{\pi}\int\limits_0^{\pi/2}\int\limits_0^{\pi/2} |p_{\alpha,\beta}|\,\sin\alpha\,\mathrm{d}\alpha\,\mathrm{d}\beta = \frac{1}{2\pi}\int\limits_{\mathbb{S}_+^2}\big(\,h(-v)+h(v)\,\big)\,\mathcal{H}_2(\mathrm{d}v)\,.$$

Further, after a lengthy calculation including the determination of the semi-axes of the ellipse $E_{abc} \cap H(p, v_{\alpha,\beta})$ for $0 \leq p \leq |p_{\alpha,\beta}|$, it turns out that the ovoid functional $J_3(E_{abc})$ can be expressed as follows [1]:

$$J_3(E_{abc}) = \frac{64}{15}\,\pi\, a^2\, b^2\, c^2 \int_0^{\pi/2}\!\!\int_0^{\pi/2} \frac{\sin\alpha}{|p_{\alpha,\beta}|}\,\mathrm{d}\alpha\,\mathrm{d}\beta = \frac{6}{5\pi}\,\mathcal{H}_3^2(E_{abc}) \int_{\mathbb{S}_+^2} \frac{\mathcal{H}_2(\mathrm{d}v)}{h(-v)+h(v)}\,.$$

Hence, by the Cauchy-Schwarz inequality, it is immediately seen that

$$\frac{b_3(E_{abc})\,J_3(E_{abc})}{\mathcal{H}_3^2(E_{abc})} = \frac{3}{5\,\pi^2}\int_{\mathbb{S}_+^2} \frac{\mathcal{H}_2(\mathrm{d}v)}{h(-v)+h(v)} \int_{\mathbb{S}_+^2} \big(h(-v)+h(v)\big)\mathcal{H}_2(\mathrm{d}v) \geq \frac{3}{5\,\pi^2}\,\mathcal{H}_2^2(\mathbb{S}_+^2) = \frac{12}{5}\,,$$

which is nothing else, but the left-hand part of (9) for $d = 3$ and $K = E_{abc}$. In the particular case of a prolate spheroid, e.g. $a \geq b = c$, we get the closed-term expressions

$$\frac{J_3(E_{abb})}{\mathcal{H}_3^2(E_{abb})} = \frac{6}{5\,a\,\varepsilon}\,\log\frac{a}{b}\Big(1+\varepsilon\Big) \quad \text{and} \quad \mathcal{H}_2(\partial E_{abb}) = 2\pi\, b^2 + 2\pi\, a\, b\,\frac{\arcsin\varepsilon}{\varepsilon}$$

with numerical excentricity $\varepsilon = \sqrt{a^2 - b^2}/a$ so that we have

$$\sqrt{\mathcal{H}_2(\partial E_{abb})}\;\frac{J_3(E_{abb})}{\mathcal{H}_3^2(E_{abb})} \sim \frac{6\,\pi\,\sqrt{b}\,\log a}{5\,\sqrt{a}} \underset{a\to\infty}{\longrightarrow} 0 \quad \text{as} \quad a \to \infty\,.$$

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[1]The details of this calculation were figured out by stud. math. Christian Bräu, Univ. of Augsburg