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Preprint Nr. 36/2009 — 23. Dezember 2009

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Impressum:

Herausgeber:

Institut für Mathematik
Universität Augsburg
86135 Augsburg
`http://www.math.uni-augsburg.de/pages/de/forschung/preprints.shtml`

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Characterizing Geometric Designs, II

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Abstract

We provide a characterization of the classical point-line designs $PG_1(n, q)$, where $n \geq 3$, among all non-symmetric $(v, k, 1)$-designs as those with the maximal number of hyperplanes. As an application of this result, we characterize the classical quasi-symmetric designs $PG_{n-2}(n, q)$, where $n \geq 4$, among *all* (not necessarily quasi-symmetric) designs with the same parameters as those having line size $q + 1$ and all intersection numbers at least $q^{n-4} + \ldots + q + 1$. Finally, we also give an explicit lower bound for the number of non-isomorphic designs having the same parameters as $PG_1(n, q)$; in particular, we obtain a new proof for the known fact that this number grows exponentially for any fixed value of q.

1. Introduction

In the predecessor to this paper [15], we considered the problem of finding a good characterization of the classical geometric designs $PG_d(n, q)$ formed by the points and d-dimensional subspaces of the n-dimensional projective space $PG(n, q)$ over the field $GF(q)$ with q elements, where $2 \leq d \leq n - 2$, among all designs with the same parameters. In particular, we proposed the following

Conjecture 1.1 *A design with the parameters of $PG_d(n,q)$, where $2 \leq d \leq n-1$ and where $q \geq 2$ is not necessarily a prime power, is classical (so that q is actually a prime power) if and only if all lines*[1] *have size $q+1$.*

At present, this conjecture is known to hold for the cases $d = n-1$ (the Dembowski-Wagner theorem [7]), the case $d = 2$ (by a result of [17]), and also for the cases $d = 3$ and $d = 4$ (by the results obtained in the author's previous paper [15]).

The present paper started with the attempt to settle Conjecture 1.1 for the case $d = n-2$, that is, for the classical quasi-symmetric designs. Note that, in view of the results just mentioned, this problem is only open for $n \geq 7$. Unfortunately, we managed to reach the desired conclusion only by adding an additional hypothesis concerning the intersection numbers of the designs in question; removing this hypothesis seems to be difficult. Still, the resulting characterization is of interest and certainly appears stronger than previous characterizations.

For the convenience of the reader, we first recall some basic facts about these designs. Let Π denote $PG(n,q)$, the n-dimensional projective space over the field $GF(q)$ with q elements, and assume $n \geq 4$. Then the points and $(n-2)$-spaces of Π form a 2-(v,k,λ) design $\mathcal{D} = PG_{n-2}(n,q)$ with parameters

$$v = q^n + \ldots + q + 1, \quad k = q^{n-2} + \ldots + q + 1, \quad \lambda = \frac{(q^{n-1}-1)(q^{n-2}-1)}{(q^2-1)(q-1)},$$

$$r = \frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)} \quad \text{and} \quad b = \frac{(q^{n+1}-1)(q^n-1)}{(q^2-1)(q-1)}.$$

It is easy to see that these particular classical designs are indeed *quasi-symmetric*, that is, they have just two intersection numbers, namely

$$x = q^{n-4} + \ldots + q + 1 \quad \text{and} \quad y = q^{n-3} + \ldots + q + 1. \tag{1}$$

Furthermore, the lines of the design $\mathcal{D}$ are just the lines of Π; in particular, all lines of $\mathcal{D}$ have cardinality $q+1$. All these facts are well-known.

Note that we wish to characterize these classical quasi-symmetric designs among *all* (not necessarily quasi-symmetric) designs with the same parameters. Usually, there is a multitude of non-isomorphic designs with the

[1] Recall that the *line* determined by two points of a design is defined as the intersection of all blocks containing these two points. See [12, 13] for background on finite projective spaces, [2] for background on designs in general, and [23] for a monograph on quasi-symmetric designs.

same parameters. Indeed, Theorem 3.1 of [17] and its subsequent discussion show that the number of non-isomorphic designs with the parameters of $PG_{n-2}(n,q)$ grows exponentially with linear growth of n (for any fixed q). Moreover, there may even be non-isomorphic *quasi-symmetric* designs with the same parameters: by a recent result of Tonchev and the present author [16], this holds at least for the special case $n = 4$.

We know of just three previous characterization results for the classical quasi-symmetric designs. A general theorem due to Lefèvre-Percsy [18] characterizes all designs $PG_d(n,q)$ with $d \geq 2$ and $q \geq 4$; in particular, her result shows that a smooth[2] design with the parameters of $PG_{n-2}(n,q)$, where $n, q \geq 4$, but q not necessarily a prime power, is classical if and only if all lines have size at least $q+1$.

A more recent characterization of the geometric designs $PG_2(4,q)$ in terms of *good blocks*[3] – a notion introduced in [21] – is due to Mavron, McDonough and Shrikhande [20]. Their result characterizes the geometric design $PG_2(4,q)$ among all quasi-symmetric designs with the same parameters and with intersection numbers 1 and $q+1$ by the property that all blocks of the design are good. Subsequently, this result was extended to $PG_{n-2}(n,q)$ in general by Baartmans and Sane [1] who also gave a characterization under somewhat weaker assumptions for the special case $d = 2$; in this case, it suffices to assume that all the blocks passing through a fixed point p are good.

Finally, by the established cases of Conjecture 1.1 discussed before, the classical quasi-symmetric designs $PG_2(4,q)$, $PG_3(5,q)$ and $PG_4(6,q)$ are characterized among all designs with the same parameters as those having line size $q+1$. Note that this result considerably improves upon the characterization given in [18] and [1]: smoothness and the good block property, respectively, are much more severe requirements than line size; moreover, in [1] quasi-symmetry is assumed in addition.

As mentioned before, our attempts to settle the case $d = n-2$ of Conjecture 1.1 in general failed up to now; we need an additional hypothesis, namely that any two blocks of the designs in question intersect in at least x points,

[2]Recall that the *plane* determined by three non-collinear points of a design is defined as the intersection of all blocks containing the three given points. In general, planes may be properly contained in other planes. This undesirable phenomenon is excluded if one requires the design to be *smooth*, that is, if one assumes that any three non-collinear points are contained in a constant number of blocks, which is then usually denoted by ϱ. See [2] for details.

[3] In any quasi-symmetric design with intersection numbers x and y, where $0 \leq x < y$, a block B is said to be *good* if, for any block C with $|B \cap C| = y$ and any point $p \notin C$, there is a (unique) block containing both p and $B \cap C$.

where x is the smaller intersection number of the classical design $PG_{n-2}(n, q)$ given in (1).

The proof of this result will make use of a new characterization of the classical point-line designs $PG_1(n, q)$, where $n \geq 3$, among all non-symmetric $(v, k, 1)$-designs. Note that Conjecture 1.1 would make no sense for the case $d = 1$, as it would then not ask for anything beyond the design property. Indeed, already in the smallest case, namely $PG_1(3, 2)$, there are 80 non-isomorphic designs with the same parameters; see, for instance, [2, Table A.1.1].

Actually, except for the classical Veblen-Young axioms for projective spaces (see, for instance, [2, §XII.1]), no general characterization of the designs $PG_1(n, q)$ among all $(v, k, 1)$-designs or all linear spaces[4] seems to be known. There are some related results, though: Doyen and Hubaut [9] gave a common characterization of the designs $AG_1(n, q)$ and $PG_1(n, q)$ with $n \geq 4$ among all $(v, k, 1)$-designs, and Teirlinck [24] used a notion of "hyperplanes" to characterize the lattice of all subspaces of some projective space $PG(n, q)$ among the lattices of subspaces of 2-coverings. The results just mentioned are not truly combinatorial in nature, as they use structural requirements: for instance, in [9] it is assumed that the given design looks locally like a projective plane or space, and in [24] one of the conditions used is similar to the good block property discussed before.

In this paper, we shall provide a combinatorial characterization of $PG_1(n, q)$ among all non-symmetric $(v, k, 1)$-designs as those designs with the largest number of hyperplanes, that is, subspaces of the maximum conceivable size. This extends work of Dehon [6] who obtained the result in question for the special case $q = 2$ by characterizing $PG_1(n, 2)$ among all Steiner triple systems; see Remark 2.7 for a more detailed discussion of that case.

For our characterization, we shall require some background on subspaces. A *subspace* of a linear space Σ is a subset S of the point set with the property that each line intersecting S in at least two points is entirely contained in S; thus the lines of Σ induce a linear space on S. The subspace *spanned* by a subset U of the point set of a linear space Σ is, of course, just the smallest subspace S of Σ containing U.

Our proofs will repeatedly appeal to two simple, but extremely useful results concerning subspaces. The first of these gives a bound on the cardinality of a proper subspace, see [2, I.8.4]. As we shall only require the case where Σ has constant line size k (so that Σ is actually a 2-design), we merely state

[4] Recall that a *linear space* is just a pairwise balanced design with joining number $\lambda = 1$; therefore one speaks of *lines* instead of *blocks* in this context.

this special case:

Lemma 1.2 (Subspace lemma) *Let S be a proper subspace of a $2-(v,k,1)$-design Σ. Then the cardinality of S satisfies the bound $|S| \leq (v-1)/(k-1)$.*

The second result concerns linear spaces with two distinct subspaces; it is due to Doyen [8], see also [2, I.8.16]. Again, we only give the special case of 2-designs.

Lemma 1.3 (Double subspace lemma) *Let S and T be two proper subspaces of a $2-(v,k,1)$-design Σ. Then the cardinality of $S \cap T$ satisfies the bound*

$$(k-1)|S \cap T| \geq |S| + (k-1)|T| - v.$$

It is well-known that the number of 2-designs with the parameters of a classical point-hyperplane design $PG_{n-1}(n,q)$ grows exponentially. This result was originally established by the author in [14], whose bounds were subsequently somewhat improved. Recently, together with Tonchev [17], the author proved an exponential bound on the number of non-isomorphic designs having the same parameters as the classical geometric design $PG_d(n,q)$ for any $2 \leq d \leq n-1$. In the final section, we will provide an analogous result for the case $d=1$, that is, we will show that the number of non-isomorphic designs having the same parameters as $PG_1(n,q)$ grows exponentially with linear growth of n. Of course, this result is a special case of the following theorem due to Wilson [25] concerning designs in general:

Result 1.4 *Let $k \geq 3$ and λ be positive integers. Then there exist constants $c(k,\lambda) > 1$ and $v_0(k,\lambda)$ such that the number of isomorphism classes of (v,k,λ)-designs is at least c^{v^2} for all admissible $v \geq v_0$.*

As our construction is quite simple and actually gives an explicit bound which provides rather strong results even for small parameter sets (whereas Wilson's proof leads to rather large value of v_0), it seems worth including anyway. We also note that Wilson's result only applies to designs with the parameters of $PG_d(n,q)$ if we fix both q and d, as it requires constant block size; for instance, it says nothing about the case $d = n-2$ discussed before.

2. A characterization of $PG_1(n, q)$

As mentioned before, except for the classical Veblen-Young axioms for projective spaces (see, for instance, [2, §XII.1]), no general characterization of the classical point-line designs $PG_1(n, q)$ among all $(v, k, 1)$-designs seems to be known. In this section, we shall provide such a characterization in terms of "hyperplanes".

Let Σ be a $(v, k, 1)$-design. By the subspace lemma 1.2, the maximum possible size of a proper subspace of Σ is just $r = (v-1)/(k-1)$. In the classical point-line design $PG_1(n, q)$, the largest subspaces are simply the hyperplanes of the associated projective space $PG(n, q)$. Therefore it is natural to call a subspace H of Σ a *hyperplane* if and only if it has cardinality r. The proof of the following simple result may be left to the reader:

Lemma 2.1 *Let Σ be a $(v, k, 1)$-design, and let H be a proper subspace of Σ. Then H is a hyperplane if and only if, for every point $p \notin H$, each line through p intersects H.* □

Lemma 2.1 was first observed by Teirlinck [24], who actually used the equivalent property stated there as his definition of a *projective hyperplane*, even in the very general setting of 2-covers instead of $(v, k, 1)$-designs. We will require the following auxiliary result:

Lemma 2.2 *Let Σ be a $(v, k, 1)$-design. Then any two hyperplanes of Σ intersect in a subspace of cardinality $s := (r-1)/(k-1)$.*

Proof. Let H and H' be two hyperplanes of Σ, and write $U = H \cap H'$. Since U is a proper subspace of the $(r, k, 1)$-design induced on H, the subspace lemma 1.2 gives $|U| \leq s$. On the other hand, the double subspace lemma 1.3 gives

$$(k-1)|U| \geq r + (k-1)r - v,$$

which reduces to $|U| \geq s$, as $r(k-1) = v-1$. □

Theorem 2.3 *Any $(v, k, 1)$-design Σ contains at most v hyperplanes. Equality holds if and only if Σ is symmetric (that is, a projective plane) or a classical point-line design $PG_1(n, q)$.*

Proof. Let us call any subspace of cardinality $s = (r-1)/(k-1)$ a *large subspace*. Our proof uses induction on b, the number of lines of Σ. We may assume that Σ is non-trivial, that is, $v > k$. Then $b \geq v$, by Fisher's inequality. In the case of equality, Σ is symmetric and hence a projective plane of order $n = k - 1 = r - 1$, and the hyperplanes of Σ are simply the lines. Hence both assertions hold in this case.

Now let $b > v$. We claim that any large subspace U lies in at most k hyperplanes of Σ. Indeed, there are precisely $v - s$ points not in U, and each hyperplane through U has to contain exactly $r - s$ of these points. Now

$$\frac{v-s}{r-s} = \frac{v(k-1)-(r-1)}{r(k-1)-(r-1)} = \frac{k(v-r)+(kr-v-r+1)}{v-r} = k.$$

Using the induction hypothesis, any given hyperplane H of Σ contains at most r large subspaces U; as we have just seen, each such subspace can be on at most $k - 1$ further hyperplanes. By Lemma 2.2, any hyperplane $H' \neq H$ meets H in such a subspace U; thus the total number of hyperplanes is indeed bounded by $r(k - 1) + 1 = v$.

Now assume that we are in the case of equality, so that Σ has as many points as hyperplanes. We shall investigate the incidence structure $\mathcal{D}$ formed by the points and hyperplanes of Σ. We first show that $\mathcal{D}$ is a 1-design with both block size and replication number r. By definition, hyperplanes indeed have r points, and by our assumption, there are as many points as hyperplanes. Counting flags, we see that the *average* replication number of $\mathcal{D}$ is r. Hence it suffices to show that each point is on at most r hyperplanes. We will establish this using induction again, the case $b = v$ being trivial.

Now let $b > v$, and consider an arbitrary point p. We may assume that p is contained in some hyperplane H of Σ. By the induction hypothesis, p lies in at most $s = (r - 1)/(k - 1)$ large subspaces U_p contained in H, each of which extends to at most $k - 1$ hyperplanes $H' \neq H$. By Lemma 2.2, any hyperplane $H' \neq H$ has to meet H in some large subspace U; hence p lies indeed in at most $s(k - 1) + 1 = r$ hyperplanes.

Again referring to Lemma 2.2, any two hyperplanes of Σ intersect in exactly s points. Hence $\mathcal{D}$ is a symmetric (v, r, s)-design. We now claim that the lines of $\mathcal{D}$ are just the lines of Σ. To see this, note that the intersection of all hyperplanes through two given points is a subspace of Σ and therefore at least a line of Σ. We need to show that no line of $\mathcal{D}$ can be a larger subspace of Σ. But the maximal possible line size in a symmetric (v, r, s)-design is $(v - s)/(r - s)$ (see, for instance, [2, Lemma XII.2.16]), and we have already seen that this fraction equals k. Therefore all lines of $\mathcal{D}$ have size exactly $k = (v - s)/(r - s)$, and the well-known Dembowski-Wagner theorem [7] gives $\mathcal{D} \cong PG_{n-1}(n, q)$ for some $n \geq 3$; see also [2, Theorem XII.2.10]. As the lines of Σ and $\mathcal{D}$ coincide, we conclude $\Sigma \cong PG_1(n, q)$. □

We shall now show that the bound in Theorem 2.3 is quite good. To do so, we consider designs with the parameters of $PG_1(n,q)$ so that $v = q^n + \ldots + q + 1$. In this case, at least the leading term in the bound given by Theorem 2.3 is correct:

Proposition 2.4 *Let q be a prime power, and $n \geq 3$ an integer. Then there exists a 2-design with the same parameters as the classical design $\Pi = PG_1(n,q)$ which contains exactly $q^n - q^{n-1} + q^{n-2} + \ldots + q + 1$ hyperplanes.*

Proof. As a general principle, we may replace the lines in a fixed plane P of Π by the line set of any other projective plane of order q on the same point set P to obtain another design Π' with the same parameters. One way of doing this is to simply apply a permutation α of the point set P to the lines of P; thus we replace every line ℓ of P with the line

$$\ell^\alpha := \{p^\alpha \colon p \in \ell\}, \tag{2}$$

while keeping the point set P itself unchanged. Depending on the choice of α, this construction will destroy certain hyperplanes of Π, while keeping others unchanged.

In order to prove the desired result, we choose α as a transposition, interchanging the two points x and y, say. This choice of α changes just the $2q$ lines of P in the bundles determined by x and y, respectively, with the exception of their common line xy, and keeps all other lines of P. It is now rather obvious that all hyperplanes of Π containing exactly one of these $2q$ lines are no longer subspaces of Π': Any former hyperplane H through such a line ℓ generates a subspace S containing all of H, as we interfered with only one of the points of H, say by replacing the point $x \in \ell$ with $y \in \ell^\alpha$. Since all lines of H except for ℓ remain unchanged, x is still forced to be in S, which follows by considering any line $\ell' \neq \ell$ of H through x. On the other hand, S also has to contain the point $y \in \ell^\alpha$, as $\ell^\alpha \setminus \{y\} \subseteq H$, and hence S is the entire point set of Π', by Lemma 1.2. Finally, all other hyperplanes of Π (including those intersecting P in xy) remain unchanged, as none of their lines is changed by the transposition α.

Note that the number of hyperplanes of Π not containing the plane P and intersecting P in a specified line is simply q^{n-2}. Hence our construction destroys exactly $2q^{n-1}$ hyperplanes of Π and leaves the remaining $q^n - q^{n-1} + q^{n-2} + \ldots + q + 1$ hyperplanes unchanged. $\square$

Of course, Proposition 2.4 immediately poses the question whether or not the non-classical examples constructed there have the maximum possible number of hyperplanes. We will leave this question as a (probably not all that easy) open problem. For the special case $q = 2$, we have a positive answer in view of the results of [10] discussed in Remark 2.7 below.

It is also interesting to investigate the possible configurations of hyperplanes in non-classical designs with the parameters of $PG_1(n, q)$ in more detail, using the general construction method presented above (and perhaps, more generally, changing the lines in several planes). Regarding this problem, we will just mention the following general result; its proof proceeds exactly as that of Proposition 2.4 and may be left to the reader.

Proposition 2.5 *Let q be a prime power, and $n \geq 3$ an integer. Let P be a plane of the classical design $\Pi = PG_1(n, q)$, and let α be a permutation of the point set of P. Distort Π by replacing every line ℓ of P by the line ℓ^α defined in (2), and assume that exactly c lines of P satisfy $\ell^\alpha \neq \ell$. Then the resulting design $\alpha(\Pi)$ contains exactly $q^n + q^{n-1} + \ldots + q + 1 - cq^{n-2}$ hyperplanes.* □

Let us apply Proposition 2.5 to provide one further class of examples, generalizing those obtained in Proposition 2.4:

Example 2.6 We choose an arbitrary line ℓ_∞ of P and let α fix all points of P, with the exception of d points on ℓ_∞ which we permute in a fixed-point-free manner, say in a cycle, where $2 \leq d \leq q + 1$. Note that the lines remaining unchanged by α are precisely the lines in the $q + 1 - d$ bundles through one of the fixed points on ℓ_∞. Thus α changes exactly qd lines of P, and hence destroys dq^{n-1} hyperplanes of Π. Therefore the number of hyperplanes of $\alpha(P)$ is exactly $q^n - (d - 1)q^{n-1} + q^{n-2} + \ldots + q + 1$.

Note that the special case $d = q + 1$ in the preceding example leaves just $q^{n-2} + q^{n-2} + \ldots + q + 1$ hyperplanes. Of course, using permutations with fewer fixed points, we can get examples with much fewer hyperplanes, too. We leave it to the reader to amuse himself with considering various other types of permutations.

Remark 2.7 As already mentioned, hyperplanes in various types of incidence structures have been considered before. For our purposes, the two most relevant previous references concerning this topic are the papers by Dehon [6] and by Doyen, Hubaut and Vandensavel [10] which appeared in 1977 and 1978, respectively. These authors studied hyperplanes in Steiner triple systems, that is, in $(v, 3, 1)$-designs. (Dehon actually considered also more general Steiner systems, namely $S(t, t+1, v)$'s; see [2] for a definition.) As noted before, the corresponding special case of Theorem 2.3 is already contained in Dehon's paper.

In addition, Doyen, Hubaut and Vandensavel proved that the collection of all hyperplanes of an arbitrary Steiner triple system Σ always carries the structure of a projective space Π over $GF(2)$, and that the 2-rank of Σ (that is, the rank of its incidence matrix over $GF(2)$) is precisely $v - (\dim \Pi + 1)$. Consequently, the 2-rank of a design Σ with the parameters of $PG_1(n, 2)$ always is at least $2^n - n - 1$, with equality if and only if Σ is actually the classical design; note that this establishes Hamada's conjecture [11] in a very special case. (Infinite families of counter-examples to the general conjecture were recently obtained by the author in collaboration with Tonchev and Clark, see [16, 4].)

It is natural to wonder if one might extend not only Dehon's result (as we have done in Theorem 2.3), but also the results of [10] to arbitrary $(v, k, 1)$-designs and therefore establish Hamada's conjecture in a case which is still open. Unfortunately, the examples constructed in Proposition 2.4 show that this approach cannot possibly work: For $q \geq 3$, the number of hyperplanes obtained there does not agree with the number of points of a projective space over $GF(q)$. Note that for $q = 2$, Proposition 2.4 and Example 2.6 yield designs with the parameters of $PG_1(n, 2)$ where the hyperplanes form a projective space of dimension $n - 1$ and $n - 2$, respectively, over $GF(2)$.

3. A characterization of $PG_{n-2}(n, q)$

In this section, we provide the following characterization of the classical quasi-symmetric designs $PG_{n-2}(n, q)$:

Theorem 3.1 *Let $\mathcal{D}'$ be a 2-design with the same parameters as the classical design $\mathcal{D} = PG_{n-2}(n, q)$, where $n \geq 4$ and where $q \geq 2$ is not necessarily a prime power. Then $\mathcal{D}'$ is isomorphic to the classical design (and therefore, in particular, quasi-symmetric) if and only if any two blocks of $\mathcal{D}'$ intersect in at least $q^{n-4} + \ldots + q + 1$ points and all lines of $\mathcal{D}'$ have size $q + 1$.*

Proof. The conditions in the statement of the theorem are obviously necessary. Thus assume that these conditions are satisfied, and denote the linear space induced by the lines of $\mathcal{D}'$ on the point set V of $\mathcal{D}'$ by Σ. Thus Σ is a design with the same parameters as $PG_1(n, q)$; not surprisingly, we want to show that Σ actually is this classical point-line design. This will be achieved using the characterization in Section 2, but this reduction will require some work.

Consider an arbitrary block B of $\mathcal{D}'$. As any two points of B define a unique line of $\mathcal{D}'$, the lines contained in B induce a linear space Σ_B with constant line size $q+1$ on B. In particular, the blocks of $\mathcal{D}'$ are subspaces of Σ. Of course, we hope that all these subspaces extend to hyperplanes of Σ. Note, however, that it is a priori not even clear if there are *any* hyperplanes in Σ. These questions will have to be settled before we can hope to apply Theorem 2.3. We shall now split our argument into a series of smaller steps.

Step 1. Any two blocks of Σ intersect in a subspace U whose cardinality satisfies

$$x := q^{n-4} + \ldots + q + 1 \;\leq\; |U| \;\leq\; q^{n-3} + \ldots + q + 1 =: y.$$

Here the upper bound follows by an application of the subspace lemma to the linear space Σ_B associated to one of the given blocks, say B, and the lower bound is just one of our two conditions. Thus the two intersection numbers x and y of the classical quasi-symmetric design $\mathcal{D}$ indeed bound the possible intersection sizes for $\mathcal{D}'$.

Step 2. $\mathcal{D}'$ is quasi-symmetric with intersection numbers x and y as in (1).

To see this, let us fix a block B_0 and write $x_B = |B_0 \cap B|$ for each of the $b-1$ blocks $B \neq B_0$. The assertion will be established using the standard method of square counting. To this end, we will evaluate the sum

$$\sum_B (y - x_B)(x_B - x), \text{ where } B \text{ runs over all blocks } B \neq B_0.$$

Note that this sum is non-negative, since each of the terms involved is non-negative by Step 1. Therefore our assertion amounts to proving that the sum equals 0. Standard counting arguments give the equations

$$\sum_B x_B \;=\; k(r-1) \quad \text{and} \quad \sum_B x_B(x_B - 1) \;=\; k(k-1)(\lambda - 1).$$

Of course, the sum in question could now be computed by brute force, but we prefer to use a little trick which allows us to avoid the rather unpleasant computations. From what we have seen, we clearly have

$$\sum_B (y - x_B)(x_B - x) \;=\; f(v, k, \lambda, x, y)$$

for some function f of the parameters of $\mathcal{D}'$ and the integers x and y given in (1). Thus the sum has to take the same value for both $\mathcal{D}'$ and $\mathcal{D}$. But we know that the classical example $\mathcal{D}$ is quasi-symmetric with intersection numbers (1), so that the sum has to be equal to 0 in this special case. Therefore it also equals 0 for our given design $\mathcal{D}'$, and we are done.

Step 3. Let B_0 be a fixed block of $\mathcal{D}'$. Then there are exactly

$$\eta := (q^2+q)(q^{n-2}+\ldots+q+1)$$

blocks intersecting B_0 in a subspace U of cardinality y.

To see this, denote the number of blocks intersecting B_0 in a subspace of cardinality x by ξ. In view of Step 2, $\xi = b-1-\eta$. Then the first count in the proof of Step 2 simplifies to

$$(b-1-\eta)x+\eta y = k(r-1),$$

from which we could compute the value of η. Again, this computation only involves the parameters of the design, so that we may instead consider the classical example $\mathcal{D}$. There we have exactly $q^{n-2}+\ldots+q+1$ choices for the subspace U (namely the number of $(n-3)$-dimensional subspaces of $\Sigma \cong PG_1(n,q)$ contained in the $(n-2)$-dimensional subspace B_0), each of which extends to an $(n-2)$-subspace in exactly q^2+q+1 ways (one of which is the fixed block B_0).

Step 4. Let B_0 be a fixed block of $\mathcal{D}'$. Then B_0 contains exactly $k = q^{n-2}+\ldots+q+1$ subspaces U of cardinality $y = q^{n-3}+\ldots+q+1$. Moreover, for a fixed such subspace U_0, there are precisely q^2+q blocks intersecting B_0 in U_0, and these blocks partition the points in $V \setminus B_0$.

First note that B_0 contains at most k subspaces U of cardinality $y = q^{n-3}+\ldots+q+1$, by Theorem 2.3, as these subspaces are just the hyperplanes of Σ_{B_0}. Given any particular U_0, it extends to at most q^2+q+1 blocks: there are precisely $q^n+q^{n-1}+q^{n-2}$ points not in U_0, and each block through U_0 has to contain exactly q^{n-2} of these points. Hence we obtain *at most* $(q^2+q)(q^{n-2}+\ldots+q+1)$ blocks B intersecting B_0 in a subspace U of cardinality y; but by Step 3, there are *precisely* that many blocks with this property. Hence we have to have equality in both of our preceding estimates, which proves the assertion.

Step 5. Let B be any block of $\mathcal{D}'$. Then $\Sigma_B \cong PG_1(n-2,q)$.

This is an immediate consequence of Step 4 together with Theorem 2.3.

Step 6. Consider the subspace S of Σ generated by any three non-collinear points of $\mathcal{D}'$. Then S is a projective plane of order q.

We first check that any three non-collinear points are contained in a common block. To see this, denote the line determined by two of the given points by ℓ, and call the third point p. Also, choose any block B through ℓ. If p is in B, we are done. Otherwise choose any subspace U of cardinality $q^{n-3} + \ldots + q + 1$ of B containing ℓ, which is possible by Step 5. By Step 4, the $q^2 + q$ blocks intersecting B in U partition the points in $V \setminus B$, so that precisely one of these blocks, say B', contains both p and ℓ. By Step 5, $\Sigma_{B'}$ is a projective geometry $PG_1(n, q-2)$, and hence p and ℓ generate a projective plane of order q.

Step 7. $\mathcal{D}'$ is isomorphic to $PG_{n-2}(n, q)$.

Using Step 6, one easily checks that the points and lines of $\mathcal{D}'$ satisfy the Veblen-Young axioms and therefore define a projective space Π; see, for instance, [2, §XII.1]. In view of the parameters of $\mathcal{D}'$, we have $\Sigma \cong PG_1(n, q)$. As the blocks are $(n-2)$-dimensional subspaces of Σ, the assertion follows. □

4. Bounds

In this final section, we use a variation of the construction described in Proposition 2.5 to provide an explicit lower bound for the number of non-isomorphic designs having the same parameters as $PG_1(n, q)$. In particular, we obtain a new proof for the known fact that this number grows exponentially with linear growth of n.

Let us first introduce a notion which will turn out to be useful. Let A be a set of q^n points in a 2-design Σ with the same parameters as the classical design $\Pi = PG_1(n, q)$, where $n \geq 3$. We will call A an *affine subspace* of Σ if the lines of Σ induce an isomorphic copy of the classical affine geometry $AG_1(n, q)$ on A.[5] Let us note the following simple facts:

Proposition 4.1 *Let Σ be a 2-design with the same parameters as the classical design $\Pi = PG_1(n, q)$, where $n \geq 3$ and where $q \geq 2$ is not necessarily a prime power. Then the complement of an affine subspace is necessarily a hyperplane of Σ. Moreover, Σ contains at most $q^n + \ldots + q + 1$ affine subspaces, and equality holds if and only if $\Sigma \cong \Pi$.*

[5] Note that an affine subspace is *not* a subspace as defined before: A line of Σ joining two points of A is not entirely contained in A, but meets A in only q points.

Proof. Let H be the set of $q^{n-1} + \ldots + q + 1$ points not contained in A. By definition, any line of Σ joining two points of A meets A in q points and hence intersects H uniquely. Also, as $PG_1(n, q)$ and $AG_1(n, q)$ have the same replication number, namely $r = q^{n-1} + \ldots + q + 1$, a line joining two points of H cannot meet A. Thus H is a subspace and therefore a hyperplane. Now the remainder of the assertion is clear from Theorem 2.3. □

We shall use the following construction:

Construction 4.2 *Let q be a prime power, and $n \geq 3$ an integer. Let P be a hyperplane of the classical design $\Pi = PG_1(n, q)$, and let α be a permutation of the point set of P. Distort Π by replacing every line ℓ contained in P by the line ℓ^α defined as in (2). Then the resulting incidence structure $\alpha(\Pi)$ is a design with the same parameters as Π.* □

We shall now show that Construction 4.2 leads to a multitude of non-isomorphic designs. Note first that the complementary point set A of P is, of course, an affine subspace of Π. Now let α and β be permutations of the hyperplane P and consider the corresponding designs $\alpha(\Pi)$ and $\beta(\Pi)$. Suppose that there is an isomorphism between these two designs. As both designs are defined on the same point set V, this isomorphism is simply a suitable permutation $\gamma \in S_V$.

By construction, both designs contain the specified affine subspace A, and the lines of each of the designs induce identical copies of $AG_1(n, q)$ on A. (Note that the complementary point set H is a hyperplane in both designs, but the associated copies of $PG_1(n-1, q)$ are, in general, not identical.) We now fix α and ask for a bound on the number of those $\gamma \in S_V$ which yield a design $(\alpha(\Pi))^\gamma$ which is actually of the form $\beta(\Pi)$ for a suitable permutation β of H. Clearly, any such γ must have the property that $(\alpha(\Pi))^\gamma$ contains the specified affine subspace A; in other words, the lines of $(\alpha(\Pi))^\gamma$ induce the given affine geometry on A and a projective geometry on the hyperplane H of $\alpha(\Pi)$ associated with A according to Proposition 4.1. Therefore $A^{\gamma^{-1}}$ has to be some affine subspace A' of $\alpha(\Pi)$, and $H^{\gamma^{-1}}$ has to be the associated hyperplane H'. Again by Proposition 4.1, we have at most $q^n + \ldots + q + 1$ possibilities for A' and H'.

We now require an estimate on the number of permutations γ leading to the same affine subspace A' and the same hyperplane H' of $\alpha(\Pi)$ in the manner just described. To this end, let $\delta \in S_V$ be a further permutation and assume $A^{\gamma^{-1}} = A^{\delta^{-1}}$. Then $A^{\gamma^{-1}\delta} = A$, and hence the permutation $\gamma^{-1}\delta$ has to induce collineations of the affine space $AG_1(n, q)$ defined on A and of the projective space $PG_1(n-1, q)$ defined on H. This shows that at most

$|P\Gamma L(n,q)||A\Gamma L(n,q)|$ permutations in S_V can lead to a fixed affine subspace A of $\alpha(\Pi)$. Thus we have established the following bound:

Theorem 4.3 *The number of non-isomorphic designs with the parameters of $PG_1(n,q)$, namely*

$$v = q^n + \ldots + q + 1, \quad k = q+1 \quad \text{and} \quad \lambda = 1,$$

where $n \geq 3$, obtained via Construction 4.2 is greater than or equal to

$$\frac{(q-1)\,(q^{n-1}+\ldots+q+1)!}{(q^{n+1}-1)|P\Gamma L(n,q)||A\Gamma L(n,q)|} = \frac{(q^{n-1}+\ldots+q+1)!}{(q^{n+1}-1)s^2 q^{n^2}\prod_{i=2}^{n}(q^i-1)^2}. \quad (3)$$

Let us look at one specific example:

Example 4.4 Applied to the parameters $n=3,\ q=4$, the lower bound (3) implies that the number of non-isomorphic 2-$(85,5,1)$ designs is at least

$$\frac{21!}{255 \cdot 2^{20} \cdot 15^2 \cdot 63^2} = \frac{19 \cdot 13 \cdot 11 \cdot 7 \cdot 5 \cdot 3^2}{2^2} > 213963.$$

We note that our construction actually gives more examples than guaranteed by (3), as all properly distorted designs $\alpha(\Sigma)$ have fewer than $q^n + \ldots + q + 1$ hyperplanes, so that our estimate is indeed too pessimistic. This also explains that the quotient can turn out to be non-integral.

The previously published lower bound on the number of non-isomorphic 2-$(85,5,1)$ designs was 10, see [3]; in the second edition of this handbook [5], a much larger bound is stated, referring to unpublished work of Mathon and Rosa who plan to write up a general construction giving a considerably stronger bound than the one in Theorem 4.3 [19].

As mentioned before, the asymptotic exponential rate of growth of the number of non-isomorphic designs with the parameters of $PG_1(n,q)$ (for any fixed prime power q) is known: it is a special case of Wilson's result 1.4. As we shall show now, this exponential growth also follows easily from Theorem 4.3; actually, a very crude estimate suffices to establish the desired result.

Let $q = p^s$, where p is a prime. Then we have

$$|P\Gamma L(n,q)| = sq^{\frac{n(n-1)}{2}} \prod_{i=2}^{n}(q^i - 1) \leq sq^{n^2-1},$$

and

$$|A\Gamma L(n,q)| = sq^{\frac{n(n+1)}{2}} \prod_{i=1}^{n} (q^i - 1) \leq sq^{n^2+n},$$

and thus the denominator in (3) is smaller than $s^2 q^{2n^2+2n}$.

The numerator in (3) is bounded from below by

$$(q^{n-1})! \geq \left(\frac{q^{n-1}}{e}\right)^{q^{n-1}} \geq q^{(n-3)q^{n-1}},$$

and hence the expression (3) is bounded from below by

$$\frac{1}{s^2} q^{(n-3)q^{n-1} - 2n^2 - 2n}. \tag{4}$$

Thus, we have the following

Corollary 4.5 *For any prime power q, the number of non-isomorphic designs having the same parameters as $PG_1(n,q)$ grows exponentially with linear growth of n.* □

Of course, one may obtain stronger estimates than the one given in formula (3) in specific cases. For instance, we may also first replace the lines of the hyperplane P by the lines of some design with the same parameters as but not isomorphic to $PG_1(n-1,q)$ in our construction, before applying permutations. Already in the 3-dimensional case, this allows at least to multiply our bound by the number of isomorphism classes of projective planes of order q. More precisely, using a given plane Π_0, we obtain the term corresponding to (3), but with $|P\Gamma L(n,q)|$ replaced by $|\mathrm{Aut}\,\Pi_0|$.

Similarly, any given bound in the case of dimension n can be used to get stronger results for dimension $n+1$. In the interest of conserving journal space and in view of the forthcoming paper by Mathon and Rosa [19], we shall not discuss such improvements in detail.

Acknowledgment. The author wishes to thank Vladimir Tonchev for pointing out the relevance of reference [10] to the present study of hyperplanes.

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