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Extrinsic Upper Bounds for λ_1

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0. Introduction

Let M be a compact connected n-dimensional Riemannian manifold isometrically immersed into $\mathbb{R}^n$ and $\lambda_1(M)$ the first eigenvalue of the Laplace operator of M. It has been shown by Reilly [R] that

$$\lambda_1(M) \leqq \frac{n}{\operatorname{vol} M} \int_M H^2,$$

where $H = \frac{1}{n}|\eta|$ is the mean curvature, $\eta = \operatorname{tr}\alpha$ the mean curvature vector field and α the second fundamental form of M in $\mathbb{R}^n$. Reilly's estimate improves the earlier estimate

$$\lambda_1 \leqq \frac{1}{\operatorname{vol}} \int_M |\alpha|^2$$

of Bleecker and Weiner [B–W]. Note that

$$nH^2 = \frac{1}{n}\left|\sum_{i=1}^n \alpha(e_i, e_i)\right|^2 \leqq \frac{1}{n}\left(\sum_{i=1}^n |\alpha(e_i, e_i)|\right)^2$$
$$\leqq \sum_{i=1}^n |\alpha(e_i, e_i)|^2 \leqq \sum_{i,j=1}^n |\alpha(e_i, e_j)|^2 = |\alpha|^2,$$

where $e_1, \ldots, e_n$ is an orthonormal basis of the tangent space.

The purpose of this paper is to extend Reilly's inequality to other ambient spaces $\bar{M}$. This is done in several ways.

Theorem 1.1. *If $\bar{M}$ is compact then there exists $c \in \mathbb{R}$ such that*

$$\lambda_1(M) \leqq c + \frac{n}{\operatorname{vol} M} \int_M H^2$$

for all compact connected n-dimensional isometrically immersed submanifolds M of $\bar{M}$.

Theorem 1.1 actually follows by a simple observation from Reilly's result and the Nash imbedding theorem. It gives rise to the following definition: Let

$$c_n(\bar{M}) := \sup_M \left(\lambda_1(M) - \frac{n}{\operatorname{vol} M} \int_M H^2 \right)$$

where the supremum is taken over all compact connected n-dimensional isometrically immersed submanifolds M of $\bar{M}$ and $1 \leqq n \leqq \dim \bar{M}$.

Up to now, not too much is known about these Riemannian invariants. Of course $c_n(\bar{M})$ is the optimal constant in the above upper bound for λ_1 and computing $c_n(\bar{M})$ is equivalent to finding the sharp inequality. About $c_1(\bar{M})$ with $\bar{M}$ compact we have more information. $c_1(\bar{M}) \geqq \max\left(\max K_{\bar{M}}, \frac{4\pi^2}{l^2} \right)$ and hence $c_1(\bar{M}) \geqq \frac{\pi^2}{i(\bar{M})^2}$ where $K_{\bar{M}}$ is the sectional curvature of $\bar{M}$, l the length of the shortest closed geodesic and $i(\bar{M})$ the injectivity radius of $\bar{M}$. Furthermore $c_1(\bar{M}) = \frac{\pi^2}{i(\bar{M})^2}$ if $K_{\bar{M}} \leqq 0$ (Theorem 1.2).

The proof of Theorem 1.1 actually gives a method for computing $c_n(\bar{M})$ if $\bar{M}$ can be imbedded nicely into some euclidean space. This probably works for R-symmetric spaces which can be imbedded as extrinsic symmetric spaces in the sense of Ferus [F]. We carry this out for the sphere and the real and complex projective spaces and obtain $c_n(S^{\bar{n}}_\delta) = n\delta$ for a sphere of curvature δ, $c_n(\mathbb{R}P^{\bar{n}}) = 2(n+1)$ (Theorem 1.3), $c_n(\mathbb{C}P^N) = 2(n+2)$ if n is even and $c_n(\mathbb{C}P^N) = 2\left(n+2-\frac{1}{n}\right)$ if n is odd; in particular $\lambda_1(M) \leqq 2(n+2)$ for any complex submanifold of $\mathbb{C}P^N$ (Theorem 1.4).

Our second approach – and this is the main part of the paper – is to extend the idea of Reilly's proof to more general ambient spaces $\bar{M}$. By this method it turns out that for Hadamard manifolds, i.e. simply connected complete manifolds of non positive curvature

$$\lambda_1(M) \leqq \frac{n}{\operatorname{vol} M} \int H^2$$

is still valid. More generally we have

Theorem 2.1. *If $K_{\bar{M}} \leqq \delta$ for some $\delta \geqq 0$ and if furthermore M lies in a convex ball, of radius $r \leqq \pi/4\sqrt{\delta}$ in case $\delta > 0$, then*

$$\lambda_1(M) \leqq n\delta + \frac{n}{\operatorname{vol} M} \int_M H^2 .$$

If equality holds then M is minimally immersed into some geodesic sphere.

As examples show already in the hyperbolic plane, Theorem 2.1 is not true if $\delta < 0$. But the following weaker result holds in that case.

Theorem 2.3. *If $K_{\bar{M}} \leqq \delta$ for some $\delta \leqq 0$ and M lies in a convex ball then*

$$\lambda_1(M) \leqq n\delta + n \max H^2 .$$

An immidiate consequence of Theorem 2.3 is a result of Barbosa et al. [B–C–E] which states that stable immersed hypersurfaces of constant mean curvature in hyperbolic space are geodesic spheres. Here stable means that the second variation is non positive for all variations which fix the "enclosed volume".

Using the variational argument in the other direction we finally prove:

Proposition 3.2. *Let M be a compact hypersurface of constant mean curvature of $\bar{M}$ which bounds. If there exists a 1-parameters group of isometries of $\bar{M}$ leaving M not invariant then*

$$\lambda_1(M) \leqq \max_M \operatorname{Ricc}_{\bar{M}} + \max_M |\alpha|^2 .$$

If equality holds, M is stable.

The last section contains a list of some open problems.

1. The Invariants $c_n(\bar{M})$

The following notation will be used throughout the paper. M denotes a compact connected n-dimensional Riemannian manifold isometrically immersed into $\bar{M}$, a complete Riemannian manifold of dimension $\bar{n}$. α denotes the second fundamental form of M in $\bar{M}$ and $H := \frac{1}{n}|\eta|$ the mean curvature where $\eta = \operatorname{tr}\alpha$ is the mean curvature vector field. Finally, K_M and $K_{\bar{M}}$ denote the sectional curvature of M and $\bar{M}$, respectively.

Theorem 1.1. *If $\bar{M}$ is compact then there exists $c \in \mathbb{R}$ such that*

$$\lambda_1(M) \leqq c + \frac{n}{\operatorname{vol} M} \int_M H^2$$

for all n-dimensional compact connected isometrically immersed submanifolds M of $\bar{M}$.

Proof. The proof follows from a combination of Reilly's inequality with the Nash imbedding theorem. By Nash we may assume that $\bar{M}$ lies in euclidean space $\mathbb{R}^N$. Hence, M is isometrically immersed into $\mathbb{R}^N$, too and thus

$$\lambda_1(M) \leqq \frac{n}{\operatorname{vol} M} \int_M \tilde{H}^2 ,$$

where $\tilde{H}$ is the mean curvature of M in $\mathbb{R}^N$. If $\tilde{\alpha}$, $\alpha_{\bar{M}}$ denote the second fundamental forms of M and $\bar{M}$ in $\mathbb{R}^N$, respectively, then $\tilde{\alpha} = \alpha + \alpha_{\bar{M}}$ and hence

$$\tilde{\eta} = \sum_{i=1}^{n} \tilde{\alpha}(e_i, e_i) = \eta + \sum_{i=1}^{n} \alpha_{\bar{M}}(e_i, e_i),$$

where $\tilde{\eta}$ is the mean curvature vector field of M in $\mathbb{R}^N$ and $e_1, \dots, e_n$ is an orthonormal basis of the tangent space. Thus,

$$\tilde{H}^2 = H^2 + \left| \frac{1}{n} \sum_{i=1}^{n} \alpha_{\bar{M}}(e_i, e_i) \right|^2$$

and

$$\lambda_1(M) \leqq \frac{1}{n \operatorname{vol} M} \int_M \left| \sum_{i=1}^{n} \alpha_{\bar{M}}(e_i, e_i) \right|^2 + \frac{n}{\operatorname{vol} M} \int_M H^2 \leqq c + \frac{n}{\operatorname{vol} M} \int_M H^2$$

where $c := \max_{v \in S\bar{M}} |\alpha_{\bar{M}}(v, v)|$ and $S\bar{M}$ denotes the unit tangent bundle. This completes the proof of Theorem 1.1.

Definition. Let $c_n(\bar{M}) := \sup_M \left(\lambda_1(M) - \frac{n}{\operatorname{vol} M} \int_M H^2 \right)$ where $1 \leqq n \leqq \bar{n}$ and the supremum is taken over all n-dimensional compact, connected isometrically immersed submanifolds M of $\bar{M}$.

$c_n(\bar{M})$ is the smallest constant such that

$$\lambda_1(M) \leqq c_n(\bar{M}) + \frac{n}{\operatorname{vol} M} \int_M H^2.$$

By Theorem 1.1, $c_n(\bar{M}) < \infty$ for compact $\bar{M}$. In general $c_n(\bar{M})$ need not be finite, e.g. if $\bar{M}$ contains arbitrarily short closed geodesics.

But $c_n(\mathbb{R}^{\bar{n}}) = 0$ as follows from Reilly's inequality and the observation that

$$\lim_{r \to \infty} \frac{n}{\operatorname{vol} M} \int_M H^2 = 0$$

for (round) spheres of radius r. Furthermore, $c_n(\bar{M}) \leqq 0$ for any Hadamard manifold, i.e. simply connected complete manifold of non positive curvature by Theorem 2.1 below.

In case $n = 1$ we have the following additional information.

Theorem 1.2. *Let $\bar{M}$ be compact, l the length of the shortest closed geodesic and $i(\bar{M})$ the injectivity radius of $\bar{M}$. Then*

(i) $c_1(\bar{M}) \geqq \max\left(\max K_{\bar{M}}, \frac{4\pi^2}{l^2}\right)$,

(ii) $c_1(\bar{M}) \geqq \frac{\pi^2}{i^2(\bar{M})}$,

(iii) $c_1(\bar{M}) = \frac{\pi^2}{i^2(\bar{M})}$ *if* $K_{\bar{M}} \leqq 0$.

Remarks. 1. $c_1(\bar{M}) \geqq \max K_{\bar{M}}$ comes from a purely local argument and therefore holds also for noncompact $\bar{M}$.

2. For "short" curves the bound $\max K_{\bar{M}}$ is optimal. This follows from 1. and Theorem 2.1 below.

3. For curves not homotopic to zero the bound $\frac{4\pi^2}{l'^2}$, l' the length of the shortest non contractible closed geodesic, is optimal. This is due to the fact that λ_1 does not decrease if the curve is replaced by a shortest closed geodesic in the same homotopy class.

4. In the definition of $c_n(\bar{M})$ the supremum can not be restricted to minimal submanifolds, even if $\bar{M}$ is compact. This follows from part (i) of the Theorem if one chooses $\bar{M}$ to be a compact manifold with $\max K_{\bar{M}} > \frac{4\pi^2}{l^2}$, e.g. an ellipsoid of

appropriate axes. On the other hand it might be true if $\bar{M}$ is compact and has non positive curvature [cf. (iii)].

Proof of Theorem 1.2. (i) $c_1(\bar{M}) \geqq \frac{4\pi^2}{l^2}$ since $\lambda_1 = \frac{4\pi^2}{l^2}$ for a closed curve of length L.

To prove $c_1(\bar{M}) \geqq \max K_{\bar{M}}$ let $p \in \bar{M}$ and $\sigma \subset T_p\bar{M}$ be a plane of maximal curvature at this point. Let $K := K_{\bar{M}}(\sigma)$, u, v an orthonormal basis of σ and

$$\gamma_r := \{\exp_p r(\cos tu + \sin tv)/t \in \mathbb{R}\}$$

the geodesic circle of radius r in the plane $\Sigma := \exp_p(\sigma)$. Then, $L(\gamma_r) = 2\pi r\left(1 - \frac{K}{6}r^2 + O(r^3)\right)$ and by a short calculation using polar coordinates $\kappa(r) = \frac{1}{r} - \frac{K}{3}r + O(r^2)$ where $\kappa(r)$ is the geodesic curvature of γ_r in Σ. Let α_γ be the second fundamental form of Σ in $\bar{M}$. Then, $\alpha_\Sigma = 0$ at p and as follows from the Codazzi equations and the maximality of $K_{\bar{M}}(\sigma)$, $D\alpha_\Sigma = 0$ at p, too. Thus, $H(r)$, the geodesic curvature of γ_r in $\bar{M}$ differs from $\kappa(r)$ only by terms of second order in r. Hence,

$$\begin{aligned} c_1(\bar{M}) &\geqq \lambda_1(\gamma_r) - \frac{1}{L(\gamma_r)} \int_{\gamma_r} H^2(r) = \frac{1}{r^2}\left(1 + \frac{K}{3}r^2 + O(r^3)\right) \\ &\quad - \frac{1}{r^2}\left(1 - \frac{2K}{3}r^2 + O(r^3)\right) = K + O(r). \end{aligned}$$

This proves (i).

(ii) follows from (i) by Klingenberg's injectivity radius estimate $i(\bar{M}) \geqq \min(\frac{1}{2}, \pi/\sqrt{\max K_{\bar{M}}})$ (where $\pi/\sqrt{\max K_{\bar{M}}}$ has to be deleted if $K_{\bar{M}} \leqq 0$).

(iii) If $K_{\bar{M}} \leqq 0$ then $\lambda_1(\gamma) \leqq \frac{1}{l(\gamma)} \int_\gamma H^2$ for closed curves γ homotopic to zero as follows from Theorem 2.1 below. Thus $c_1(\bar{M}) = \frac{4\pi^2}{l^2} = \frac{\pi^2}{i^2(\bar{M})}$ by Remark 3.

This completes the proof of Theorem 1.2.

From the proof of Theorem 1.1 we have the estimate

$$\lambda_1(M) \leqq \frac{1}{n \cdot \operatorname{vol} M} \int_M \left|\sum_{i=1} \alpha_{\bar{M}}(e_i, e_i)\right|^2 + \frac{n}{\operatorname{vol} M} \int_M H^2$$

and hence by the Gauss equations.

$$\lambda_1(M) \leqq \frac{1}{n \operatorname{vol} M} \int_M \left(\sum_{i \neq j} K_{\bar{M}}(e_i, e_j) + \sum_{i,j} |\alpha_{\bar{M}}(e_i, e_j)|^2\right) + \frac{n}{\operatorname{vol} M} \int_M H^2, \tag{1.1}$$

where $K_{\bar{M}}$ denotes sectional curvature of $\bar{M}$. (1.1) may be used for explicit computations if $\bar{M}$ is embedded nicely, e.g. as an extrinsic symmetric space in the sense of Ferus [F], i.e. as a submanifold which is invariant under the reflections at it's normal spaces. We carry this out for $\bar{M}$ a sphere, real and complex projective space.

If $\bar{M} = S^n_\delta$ is the sphere of constant curvature δ then $|\alpha_{\bar{M}}(v, w)|^2 = \delta \cdot \langle v, w\rangle^2$ for the standard imbedding and hence (cf. [S–W])

$$\lambda_1(M) \leqq n\delta + \frac{n}{\operatorname{vol} M} \int_M H^2.$$

Since equality holds for totally geodesic spheres the inequality is sharp. Thus, $c_n(S_\delta^n)=n\delta$.

Next, we consider the case $\bar{M}=\mathbb{R}P^n$ (of curvature 1).

Theorem 1.3. *For any compact n-dimensional isometrically immersed submanifold M of real projective space* $\mathbb{R}P^{\bar{n}}$

$$\lambda_1(M)\leqq 2(n+1)+\frac{n}{\operatorname{vol} M}\int_M H^2.$$

Equality is attained e.g. for totally geodesic $\mathbb{R}P^n$*'s. Hence,* $c_n(\mathbb{R}P^{\bar{n}})=2(n+1)$.

Proof. The mapping $f:S^{\bar{n}}\to\mathbb{R}^{(\bar{n}+1)^2}$, $f(X_0,\dots,X_{\bar{n}})=\frac{2}{\sqrt{2}}(X_i\cdot X_j)$, induces an isometric embedding (essentially the Veronese embedding) of $\mathbb{R}P^{\bar{n}}$ into $\mathbb{R}^N$, $N=(\bar{n}+1)^2$. Computing the second derivative of $f\circ c$ where c is a geodesic of $S^{\bar{n}}$ yields $\alpha_{\bar{M}}(v,v)$ and by polarization $\alpha_{\bar{M}}(v,w)$. This gives $|\alpha_{\bar{M}}(e_i,e_i)|=2$ and for $i\neq j\, |\alpha_{\bar{M}}(e_i,e_j)|=1$. Hence the Theorem follows from (1.1).

Next, we treat submanifolds of complex projective space.

Theorem 1.4. *Let* $\bar{M}=\mathbb{C}P^N$ *with curvature between 1 and 4. Then*

$$\lambda_1(M)\leqq 2\left(n+1+\frac{1}{n\operatorname{vol} M}\int_M |J^T|^2\right)+\frac{n}{\operatorname{vol} M}\int_M H^2\leqq 2(n+2)+\frac{n}{\operatorname{vol} M}\int_M H^2,$$

where $J^T: T_pM\to T_pM$ *is the tangential component of the almost complex structure J of* $\mathbb{C}P^N$ *restricted to* T_pM*. In particular,*

(i) $\lambda_1(M)\leqq 2(n+2)$ *for complex submanifolds (of real dimension n),*

(ii) $\lambda_1(M)\leqq 2(n+1)+\frac{n}{\operatorname{vol} M}\int_M H^2$ *for totally real submanifolds,*

(iii) $\lambda_1(M)\leqq 2\left(n+2-\frac{1}{n}\right)+\frac{n}{\operatorname{vol} M}\int_M H^2$ *if n is odd.*

The inequalities (i)–(iii) *are sharp. Hence,* $c_n(\mathbb{C}P^N)=2(n+2)$ *if n is even and* $c_n(\mathbb{C}P^N)=2\left(n+2-\frac{1}{n}\right)$ *if n is odd.*

Proof. Let S^{2N+1} be the unit sphere in $\mathbb{C}^{N+1}$. Then the mapping $f:S^{2N+1}\to\mathbb{C}^{(N+1)^2}$, $f(z_0,\dots,z_n):=\frac{1}{\sqrt{2}}(z_i\bar{z}_j)_{i,j}$, induces an isometric embedding of $\mathbb{C}P^N$ into $\mathbb{R}^{2(N+1)^2}$. A similar calculation as in the real case yields for orthonormal tangent vectors v,w: $|\alpha_{\bar{M}}(v,v)|^2=4$, $|\alpha_{\bar{M}}(v,w)|^2=1-\langle v,Jw\rangle^2$. Since $K_{\bar{M}}(v,w)=1+3\langle v,Jw\rangle^2$ we get

$$\frac{1}{n}\sum_{i\neq j}K_{\bar{M}}(e_i,e_j)+\frac{1}{n}\sum_{i,j}|\alpha_{\bar{M}}(e_i,e_j)|^2=\frac{1}{n}\sum_{i\neq j}(2+2\langle e_i,Je_j\rangle^2)+4$$

$$=2(n+1)+\frac{2}{n}|J^T|^2\leqq 2(n+2),$$

where as usual $e_1, \ldots, e_n$ is an orthonormal basis of T_pM. This together with (1.1) proves the first two inequalities. Now, (i)–(iii) are due to the minimality of complex submanifolds, to $J^T=0$ for totally real submanifolds and to $\operatorname{codim} J^T(T_pM) \geqq 1$ for odd-dimensional submanifolds, respectively. Equality holds in (i) and (ii) for totally geodesic $\mathbb{C}P^{n/2}$ and $\mathbb{R}P^n$, respectively. Equality in (iii) is more delicate. We claim that a geodesic sphere of radius $\arctan\sqrt{n+2}$ in a totally geodesic $\mathbb{C}P^{\frac{n+1}{2}}$ yields equality. Since the mean curvature does not change if we replace $\bar{M}$ by a totally geodesic submanifold in which M sits we may assume that $n=2N-1$. For a geodesic sphere of radius r in $\mathbb{C}P^N$ we have

$$H=\frac{1}{n}\left((n-1)\frac{\cos r}{\sin r}+\frac{2\cos 2r}{\sin 2r}\right)=\frac{\cos r}{\sin r}-\frac{1}{n}\frac{\sin r}{\cos r}$$

and by a recent result of Barbosa et al. [B–C–E]

$$\lambda_1=\min\left(\frac{1}{\cos^2 r}+\frac{n}{\sin^2 r}, \frac{2(n+1)}{\sin^2 r}\right).$$

This proves the claim and completes the proof of the theorem.

2. An Extension of Reilly's Inequality

Let M, $\bar{M}$, n, $\bar{n}$, α, η, H, K_M, and $K_{\bar{M}}$ as above. $\bar{M}$ need not be compact. The main result of this section is the following generalization of Reilly's inequality.

Theorem 2.1. *If $K_{\bar{M}} \leqq \delta$ for some $\delta \geqq 0$ and if furthermore M lies in a convex ball, of radius $r \leqq \pi/4\sqrt{\delta}$ if $\delta>0$, then*

$$\lambda_1(M) \leqq n\delta+\frac{n}{\operatorname{vol} M}\int_M H^2.$$

Equality implies that M is minimally immersed into some geodesic sphere.

Before we start with the proof we discuss the case $\delta<0$. In that case the theorem is not true as it stands. For example, if γ is a closed curve in hyperbolic plane of length l and geodesic curvature κ it would mean

$$l+\frac{4\pi^2}{l} \leqq \int_\gamma \kappa^2$$

which is not true. For a counter example see Fig. 8 of [L–S]. On the other hand Langer and Singer have proved this inequality (for closed curves in hyperbolic plane) if $l \leqq 2\pi$. One therefore might ask whether Theorem 2.1 is true also in case $\delta<0$ if M is "small" in some sense. We do not know the answer.

Another special case in which Theorem 2.1 is true for $\delta<0$ is the following one.

Proposition 2.2. *Let M be homeomorphic to S^2 and $K_{\bar{M}} \leqq \delta$ for some $\delta \in \mathbb{R}$. Then*

$$\lambda_1(M) \leqq 2\delta+\frac{2}{\operatorname{vol} M}\int_M H^2.$$

Proof. By Hersch [H] and the Gauss-Bonnet theorem

$$\lambda_1(M) \leqq \frac{8\pi}{\operatorname{vol} M} = \frac{2}{\operatorname{vol} M} \int_M K_M,$$

where K_M denotes the curvature of M. But $K_M \leqq K_{\bar{M}} + H^2 \leqq \delta + H^2$ as follows from the Gauss equations.

The case $M = S^2$ is really an extraordinary one. Although there is a generalization of Hersch's inequality by Yang and Yau [Y–Y] to other compact surfaces. Proposition 2.2 is not true in that case. To see this one simply may take $\bar{M} = M$ with a metric of constant curvature.

Our most general result in case $\delta < 0$ is the following one.

Theorem 2.3. *If $K_{\bar{M}} \leqq \delta$ for some $\delta < 0$ and M lies in a convex ball then*

$$\lambda_1(M) \leqq n\delta + n \max H^2.$$

Theorem 2.3 has a nice application to stable hypersurfaces of constant mean curvature in hyperbolic space, see Sect. 3.

The proof of Theorems 2.1 and 2.3 are inspired by Reilly's proof (the case $\bar{M} = \mathbb{R}^n$) which essentially consists in using the coordinate functions of $\mathbb{R}^n$ as test functions in the Rayleigh quotient and in applying Minkowski's formula. Recall that by Rayleigh's principle

$$\lambda_1(M) \cdot \int_M f^2 \leqq \int_M |\operatorname{grad} f|^2$$

for any sufficiently smooth function f with $\int_M f = 0$. Thus if 0 is the center of mass of $M \subset \mathbb{R}^n$ and x_i are the coordinate functions then

$$\lambda_1 \int_M |X|^2 = \lambda_1 \int_M \sum_{i=1}^n x_i^2 \leqq \int_M \sum_{i=1}^n |\operatorname{grad}_M x_i|^2 = n \cdot \operatorname{vol} M,$$

where X is the position vector field. Now, Reilly's inequality $\lambda_1 \leqq \frac{n}{\operatorname{vol} M} \int_M H^2$ follows directly from $\operatorname{vol}^2 M \leqq \int_M |X|^2 \cdot \int_M H^2$ which in turn follows by Cauchy-Schwarz from Minkowski's formula $\int_M \langle X, \eta \rangle = -n \operatorname{vol} M$. Recall also that Minkowski's formula is obtained from $\int_M \operatorname{div} X^T = 0$ where X^T is the component of X tangent to M. We will extend these results to arbitrary manifolds $\bar{M}$ with upper curvature bound $\delta \in \mathbb{R}$. This will be done in a series of lemmas.

Let s_δ be the solution of $y'' + \delta y = 0$ with $y(0) = 0$, $y'(0) = 1$ and put $c_\delta := s'_\delta$. Then $c'_\delta = -\delta s_\delta$ and $c_\delta^2 + \delta s_\delta^2 = 1$. We will use $\frac{s_\delta(r)}{r} \cdot x_i$ and in case $\delta > 0$ also $\frac{c_\delta(r) - c}{\sqrt{\delta}}$, c a constant, as test functions in the Rayleigh quotient where the x_i are normal coordinates of $\bar{M}$ centered at some point $p_0 \in \bar{M}$ and where $r = d(p_0, \cdot)$ is the distance to p_0. We assume, that M lies in a convex ball around p_0 of radius less or equal $\pi/2\sqrt{\delta}$ if $\delta > 0$. In particular $c_\delta \geqq 0$. Let $X := s_\delta(r) \operatorname{grad} r$, where the gradient is taken in $\bar{M}$. Then

$$\lambda_1(M)\int_M s_\delta^2(r)=\lambda_1(M)\cdot\int_M |X|^2=\lambda_1(M)\cdot\int_M\sum_{i=1}^n\left(\frac{s_\delta(r)}{r}\cdot x_i\right)^2$$
$$\leqq\int_M\sum_{i=1}^n\left|\operatorname{grad}_M\frac{s_\delta(r)}{r}x_i\right|^2 \tag{2.1}$$

and in case $\delta>0$

$$\lambda_1(M)\int\frac{(c_\delta(r)-c)^2}{\delta}\leqq\int\frac{1}{\delta}|\operatorname{grad}_M c_\delta(r)|^2 \tag{2.2}$$

if $\int_M\frac{s_\delta(r)}{r}\cdot x_i=\int_M(c_\delta(r)-c)=0$. Here grad_M denotes the gradient in M, i.e. the tangential component of the gradient in $\bar{M}$. Note, that $\operatorname{grad} c_\delta=-\delta X$ and hence $\operatorname{grad}_M c_\delta=-\delta X^T$ where T denotes the tangential part.

For a vector field Y on $\bar{M}$, $\operatorname{div}_M Y$ is defined by $\operatorname{div}_M Y(p):=\sum_{i=1}^n\langle\bar{\nabla}_{e_i}Y,e_i\rangle$ if $p\in M$ and $e_1,\dots,e_n\in T_pM$ is an orthonormal basis. $\bar{\nabla}$ denotes covariant differentiation in $\bar{M}$.

The proofs of the next lemma and Lemma 2.7 are influenced by a paper of Chavel [Ch].

Lemma 2.4. (i) $\operatorname{div}_M X\geqq n\cdot c_\delta$,
(ii) $\operatorname{div}_M X^T\geqq nc_\delta+\langle X,\eta\rangle$.
If $K_M\equiv\delta$ equality holds.

Proof. (i) Let $p\in M$ and $e_1,\dots,e_n\in T_pM$ be an orthonormal basis such that e_n lies in the direction of $\operatorname{grad} r^T$ (if $\operatorname{grad} r^T\neq 0$). Then $e_n=\lambda\operatorname{grad} r+\mu e_n^*$ a unit vector orthogonal to $\operatorname{grad} r$. Now,

$$\operatorname{div}_M\operatorname{grad} r(p)=\sum_{i=1}^n\langle\bar{\nabla}_{e_i}\operatorname{grad} r,e_i\rangle=\sum_{i=1}^{n-1}\langle\bar{\nabla}_{e_i}\operatorname{grad} r,e_i\rangle$$
$$+\mu^2\langle\bar{\nabla}_{e_n^*}\operatorname{grad} r,e_n^*\rangle\geqq(n-1+\mu^2)\frac{c_\delta}{s_\delta}$$

by the standard Jacobi field estimates. Equality holds if $K_{\bar{M}}\equiv\delta$. Hence

$$\operatorname{div}_M X=s_\delta\operatorname{div}_M\operatorname{grad} r+c_\delta|\operatorname{grad} r^T|^2\geqq(n-1+\mu^2)c_\delta+\lambda^2c_\delta=n\cdot c_\delta$$

as required. (ii) follows from $\operatorname{div}_M X^T=\operatorname{div}_M X-\operatorname{div}_M X^\perp=\operatorname{div}_M X+\langle X,\eta\rangle$.

The next lemma is a generalization of Minkowski's formula (Note, that $c_\delta\equiv 1$ if $\delta=0$). It follows by integrating (ii) of Lemma 2.4 and by the Cauchy-Schwarz inequality.

Lemma 2.5. $\int_M c_\delta\leqq-\frac{1}{n}\int_M\langle X,\eta\rangle\leqq\int_M H\cdot s_\delta$.

If $K_M\equiv\delta$ equality holds in the first inequality.

Lemma 2.6. $\delta\int_M|X^T|^2\geqq n\int_M c_\delta^2-n\int_M H\cdot s_\delta\cdot c_\delta$.

Proof. If $\delta=0$ the inequality reduces to Lemma 2.5. If $\delta\neq 0$, $X^T=\mathrm{grad}_M f$ where $f=-\frac{1}{\delta}c_\delta$ and hence

$$\delta\int_M |X^T|^2=-\delta\int_M f\cdot\Delta f=-\int_M f\,\mathrm{div}_M X^T=\int_M c_\delta\,\mathrm{div}_M X^T$$
$$\geqq n\int_M c_\delta^2+\int_M c_\delta\langle X,\eta\rangle\geqq n\int_M c_\delta^2-n\int_M Hs_\delta c_\delta$$

by Lemma 2.4 and the Cauchy-Schwarz inequality.

Lemma 2.7. $\sum_{i=1}^n \left|\mathrm{grad}_M \frac{s_\delta(r)}{r}x_i\right|^2+\delta|X^T|^2\leqq n.$

If $K_{\bar M}\equiv\delta$ equality holds.

Proof. Let $v:=\exp_{p_0*}\tilde v\in T_p\bar M$ be orthogonal to $\mathrm{grad}\,r$, where $\tilde v\in T_{p_0}\bar M$. Then $|v|^2\geqq\frac{s_\delta^2(r)}{r^2}|\tilde v|^2$ by the standard Jacobi field estimates.

Hence,

$$\frac{s_\delta^2(r)}{r^2}\sum_{i=1}^n\langle\mathrm{grad}\,x_i,v\rangle^2=\frac{s_\delta^2(r)}{r^2}|\tilde v|^2\leqq|v|^2.$$

Note, that $\langle\mathrm{grad}\,x_i,v\rangle=v(x_i)=\tilde v(x_i\circ\exp)$ is the i^{th} component of $\tilde v$ w.r.t. the orthonormal basis defining the normal coordinates. Since $\exp_{p_0}$ is a radial isometry

$$\sum_{i=1}^n\langle\mathrm{grad}\,x_i,v\rangle\langle\mathrm{grad}\,x_i,\mathrm{grad}\,r\rangle=\langle v,\mathrm{grad}\,r\rangle=0.$$

Using $\sum_{i=1}^n x_i\,\mathrm{grad}\,x_i=r\,\mathrm{grad}\,r$, a simple calculation gives

$$\begin{aligned}
&\sum_{i=1}^n\left|\mathrm{grad}_M\frac{s_\delta(r)}{r}x_i\right|^2+\delta|X^T|^2\\
&=\frac{s_\delta^2}{r^2}\sum_{i=1}^n|\mathrm{grad}_M x_i|^2+\left(1-\frac{s_\delta^2}{r^2}\right)|\mathrm{grad}_M r|^2\\
&=\frac{s_\delta^2}{r^2}\sum_{j=1}^n\sum_{i=1}^n\langle\mathrm{grad}\,x_i,e_j\rangle^2+\left(1-\frac{s_\delta^2}{r^2}\right)|\mathrm{grad}_M r|^2\\
&\leqq n-1+\frac{s_\delta^2}{r^2}\sum_{i=1}^n(\langle\mathrm{grad}\,x_i,\lambda\,\mathrm{grad}\,r\rangle+\langle\mathrm{grad}\,x_i,\mu e_n^*\rangle)^2+\left(1-\frac{s_\delta^2}{r^2}\right)\lambda^2\\
&\leqq n-1+\lambda^2\frac{s_\delta^2}{r^2}+\mu^2+\left(1-\frac{s_\delta^2}{r^2}\right)\lambda^2=n,
\end{aligned}$$

where $e_1,\ldots,e_{n-1}$, $e_n=\lambda\,\mathrm{grad}\,r+\mu e_n^*$ is an orthonormal basis as in the proof of Lemma 2.4. If $K_{\bar M}\equiv\delta$ equality holds everywhere. This completes the proof of Lemma 2.7.

Lemma 2.8. *If $\delta \leqq 0$, then*

$$\int_M s_\delta \int_M s_\delta c_\delta \leqq \int_M s_\delta^2 \int_M c_\delta .$$

Proof. $\left(\int_M s_\delta \int_M s_\delta c_\delta\right)^2 \leqq \left(\int_M s_\delta\right)^2 \cdot \int_M s_\delta^2 \int_M c_\delta^2$

$$= \operatorname{vol} M \left(\int_M s_\delta\right)^2 \int_M s_\delta^2 - \delta \left(\int_M s_\delta\right)^2 \left(\int_M s_\delta^2\right)^2$$

$$\leqq \left(\int_M s_\delta^2\right)^2 \left((\operatorname{vol} M)^2 - \delta \left(\int_M s_\delta\right)^2\right) \leqq \left(\int_M s_\delta^2 \cdot \int_M c_\delta\right)^2 .$$

The first and second inequalities follow from the Cauchy-Schwarz inequality while the last is a consequence of $\left|\int_M f\right| \leqq \int_M |f|$ where $f: M \to \mathbb{R}^2$ is given by $f(p) = f(p) = (1, \sqrt{|\delta|}\, s_\delta(r))$.

After these preparations we can start with the proofs of the theorems.

Proof of Theorem 2.1 and 2.3. By a standard argument [Ch] there exists $p_0 \in M$ with $\int_M \frac{s_\delta(r)}{r} x_i = 0$, $i = 1, \ldots, \bar{n}$, where $r = d(p_0, \cdot)$ and the x_i are normal coordinates w.r.t. p_0. Namely, if M lies in the convex ball B the vector field Y defined in a neighborhood of B by

$$Y_q := \int_M \frac{s_\delta(d(q,p))}{d(q,p)} \exp_q^{-1}(p) dp \in T_q \bar{M}$$

points at the boundary into the interior of B. Thus it has necessarely a zero in B, say at p_0. If $\delta > 0$, B has radius $\leqq \pi/4\sqrt{\delta}$ so that M lies in a ball of radius $\leqq \pi/2\sqrt{\delta}$ around p_0 and $c_\delta \geqq 0$ on M. We now treat the cases $\delta = 0$, $\delta < 0$, and $\delta > 0$ separately.

If $\delta \leqq 0$ we use $\frac{s_\delta(r)}{r} \cdot x_i$, $i = 1, \ldots, \bar{n}$, as test functions in the Rayleigh quotient. Thus, if $\delta = 0$ we obtain from (2.1), Lemma 2.7 and Lemma 2.5

$$\lambda_1 \int_M s_\delta^2 \leqq n \operatorname{vol} M \leqq \frac{n}{\operatorname{vol} M} \left(\int_M H \cdot s_\delta\right)^2 \leqq \frac{n}{\operatorname{vol} M} \int_M H^2 \cdot \int_M s_\delta^2$$

which proves the desired inequality.

If $\delta < 0$ we obtain from (2.1), Lemma 2.7, Lemma 2.6, Lemma 2.8 and Lemma 2.5

$$\lambda_1 \int_M s_\delta^2 \leqq n \operatorname{vol} M - \delta \int_M |X^T|^2 \leqq n \cdot \operatorname{vol} M - n \int_M c_\delta^2 + n \int_M H s_\delta c_\delta$$

$$= n\delta \int_M s_\delta^2 + n \int_M H s_\delta c_\delta \leqq n\delta \int_M s_\delta^2 + n \cdot \max |H| \cdot \int_M s_\delta^2$$

$$\times \frac{\int_M c_\delta}{\int_M s_\delta} \leqq n\delta \int_M s_\delta^2 + n \cdot \max H^2 \cdot \int_M s_\delta^2 .$$

This proves Theorem 2.3.

If $\delta > 0$ we use $\frac{s_\delta(r)}{r} x_i$ and $\frac{c_\delta(r) - c}{\sqrt{\delta}}$ with $c := \frac{1}{\operatorname{vol} M} \int_M c_\delta(r)$ as test functions and obtain from (2.1), (2.2) and Lemma 2.7

$$\lambda_1(M) \cdot \int_M \left(s_\delta^2(r) + \frac{(c_\delta(r) - c)^2}{\delta} \right) \leqq n \operatorname{vol} M$$

and hence

$$\lambda_1(1 - c^2) \leqq n\delta.$$

Now,

$$(1 - c^2)\left(1 + \frac{1}{\delta \operatorname{vol} M} \int_M H^2\right) \geqq 1 + \frac{1}{\delta \operatorname{vol} M} \int_M H^2 - \frac{1}{(\operatorname{vol} M)^2} \int_M s_\delta^2 \int_M H^2$$

$$- \frac{1}{\delta (\operatorname{vol} M)^2} \int_M c_\delta^2 \int_M H^2 = 1$$

as follows from Lemma 2.5 and the Cauchy-Schwarz inequality. This proves the inequality also in case $\delta > 0$.

Euality implies in all cases equality in Lemma 2.5. Thus, η is a constant multiple of $X = s_\delta(r) \operatorname{grad} r$ and M is (at least outside p_0) orthogonal to $\operatorname{grad} r$ (if $\eta \equiv 0$ note, that $c_\delta \equiv 0$ by Lemma 2.5, hence $\delta > 0$ and r constant). Hence r is constant on M and M lies in a sphere around p_0. Since η has no component tangent to this sphere, M is minimal in there. This completes the proof.

3. Hypersurfaces of Constant Mean Curvature

A bounded domain $\Omega \subset \bar{M}$ is a *solution of the isoperimetric problem* if $\operatorname{vol} \partial\Omega \leqq \operatorname{vol} \partial\Omega'$ for all other domains $\Omega' \subset \bar{M}$ with $\operatorname{vol} \Omega = \operatorname{vol} \Omega'$. If $M := \partial\Omega$ is smooth, M has constant mean curvature (as follows from the first variation formula) and the second variation is non negative. This is given by (cf. [S, p. 535] or [B–C])

$$I(f) = - \int_M \{f \cdot \Delta f + (\operatorname{Ricc}_{\bar{M}}(N, N) + |\alpha|^2) f^2\},$$

where N is a unit normal field, $\operatorname{Ricc}_{\bar{M}}$ is the Ricci curvature of $\bar{M}$ and f is an arbitrary smooth function with $\int_M f = 0$. The last condition corresponds to the fact that the variation has to fix the enclosed volume. If X is the variation vector field then $f = \langle X, N \rangle$.

By definition, a compact isometrically immersed hypersurface M of constant mean curvature is *stable* if $I(f) \geqq 0$ for all smooth f with $\int_M f = 0$.

Let M be a compact stable hypersurface of constant mean curvature isometrically immersed into $\bar{M} = \mathbb{R}^{n+1}, S^{n+1}$ or H^{n+1}. If f is a first eigenfunction of the Laplace operator, then

$$\int_M (\lambda_1 - \operatorname{Ricc}_{\bar{M}}(N, N) - |\alpha|^2) f^2 = I(f) \geqq 0.$$

On the other hand $\lambda_1 - \operatorname{Ricc}_{\bar{M}} \leqq nH^2$ by Reilly's inequality, the remark before Theorem 1.3 and by Theorem 2.3. Since $nH^2 \leqq |\alpha|^2$ (cf. the introduction) this implies $nH^2 \equiv |\alpha|^2$ and M must be totally umbilic (cf. the proof of $nH^2 \leqq |\alpha|^2$). This proves the following result of Barbosa, do Carmo and Eschenburg.

Theorem 3.1 [B–C, B–C–E]. *Let M be a compact isometrically immersed stable hypersurface of constant mean curvature of $\mathbb{R}^{n+1}$, S^{n+1}, or H^{n+1}. Then M is a geodesic sphere.*

By reversing the second variation argument we obtain the following upper bound for λ_1.

Proposition 3.2. *Let M be a compact hypersurface of constant mean curvature of $\bar{M}$ which bounds. If there exists a 1-parameter group of isometries of $\bar{M}$ leaving M not invariant then*

$$\lambda_1(M) \leqq \max_M \mathrm{Ricc}_{\bar{M}} + \max_M |\alpha|^2 .$$

If equality holds M is stable.

It might be interesting to note, that this inequality is sharp (e.g. for geodesic spheres of $\mathbb{R}^{n+1}$, S^{n+1}, or H^{n+1}) and that none of the assumptions can be delected: That M bounds is necessary to rule out for example closed geodesics on a flat 2-torus. The existence of the 1-parameter group rules out closed geodesics on a flat cylinder which has been closed by two spherical caps. Disturbing this geodesic slightly shows, that the assumption $H = \text{const}$ can not be deleted either. Note also, that $\max|\alpha|^2$ in the inequality cannot be replaced by $n \cdot \max H^2$ or even $\frac{n}{\mathrm{vol}\,M} \int_M H^2$ as geodesic spheres in $\mathbb{C}P^N$ show. This follows easily from the computations at the end of Sect. 1.

Proof of Proposition 3.2. The 1-parameter family of isometries gives rise to a variation of M which fixes the enclosed volume. If X denotes the variation vector field and N the outer unit normal, then $f := \langle X, N \rangle$ does not vanish identically since M is not left invariant by the 1-parameter group. Now, $I(f) = 0$ since the volume of the hypersurfaces is constant through the variation and hence

$$0 = I(f) = -\int_M \{f \Delta f + (\mathrm{Ricc}_{\bar{M}}(N,N) + |\alpha|^2) f^2\} \geqq \int_M (\lambda_1 - \mathrm{Ricc}_{\bar{M}}(N,N) - |\alpha|^2) f^2 .$$

This proves the inequality. If

$$\lambda_1 = \max_M \mathrm{Ricc}_{\bar{M}} + \max_M |\alpha|^2$$

then

$$I(g) \geqq \int_M (\lambda_1 - \mathrm{Ricc}_{\bar{M}}(N,N) - |\alpha|^2) g^2 \geqq 0$$

for any $g \in C^\infty(M)$ with $\int_M g = 0$. This completes the proof of the Proposition.

4. Some Open Problems

a) What is $c_n(H^{\bar{n}})$ if $1 < n < \bar{n}$?

b) Compute $c_n(\bar{M})$ for compact symmetric R-spaces.

c) Is $c_1(\bar{M}) \geqq \frac{\pi^2}{i(\bar{M})^2}$ also true for non compact $\bar{M}$? (cf. Theorem 1.2).

d) Does $\lambda_1(M) \leqq n \cdot (\max K_{\bar{M}}) + \frac{n}{\mathrm{vol}\,M} \int_M H^2$ hold for "small" submanifolds M, if $K_{\bar{M}} < 0$? See the discussion after Theorem 2.1.

e) Determine a lower bound for $c_n(\bar{M})$ analogous to $c_1(\bar{M}) \geqq \max K_{\bar{M}}$ by considering "n-dimensional" geodesic spheres of radius r with $r \to 0$. What is the Taylor expansion of the first eigenvalue of these spheres at $r=0$ (at least what is the constant term)?

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