



Universität Augsburg

Institut für
Mathematik

Stella David, Lothar Heinrich

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Empirical Second-Order Product Density and Goodness-of-fit Tests
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Preprint Nr. 17/2010 — 06. Dezember 2010

Institut für Mathematik, Universitätsstraße, D-86135 Augsburg

<http://www.math.uni-augsburg.de/>

Impressum:

Herausgeber:

Institut für Mathematik

Universität Augsburg

86135 Augsburg

<http://www.math.uni-augsburg.de/pages/de/forschung/preprints.shtml>

ViSdP:

Stella David

Institut für Mathematik

Universität Augsburg

86135 Augsburg

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Central limit theorem for the integrated squared error of the empirical second-order product density and goodness-of-fit tests for stationary point processes

Stella David and Lothar Heinrich

Abstract

Spatial point processes are mathematical models for irregular or random point patterns in the d -dimensional space, where usually $d = 2$ or $d = 3$ in applications. The second-order product density and its isotropic analogue, the pair correlation function, are important tools for analyzing stationary point processes. In the present work we derive central limit theorems for the integrated squared error (ISE) of the empirical second-order product density and for the ISE of the empirical pair correlation function for expanding observation windows. The proof techniques are based on higher-order cumulant measures and the Brillinger-mixing property of the underlying point processes. The obtained Gaussian limits are used for constructing asymptotic goodness-of-fit tests for checking point process hypotheses even in the non-Poissonian case.

AMS 2000 Mathematics Subject Classification: Primary: 60G55, 62M30, 60F05; Secondary: 62G10, 62G20

Keywords: Second-order analysis of point processes, pair correlation function, Brillinger-mixing, cumulant measures, asymptotic variance, model specification test

1 Introduction

An important aim of point process statistics is to find a mathematical model that gives a satisfactory description of an observed point pattern. With such mathematical models one can, for instance, draw conclusions about properties of certain materials or tissues. For stationary point processes mainly second-order statistics such as Ripley's K -function and the pair correlation function are used for verifying or rejecting hypothetical point process models by visual inspection or simulation tests, see e.g. Baddeley et al. [1], Cressie [3], Diggle [6], Illian et al. [13], and Stoyan et al. [19]. Often these investigations focus on complete spatial randomness, see e.g. Grabarnik and Chiu [7], Ho and Chiu [12], and Zimmerman [22]. Most tests used in applications are based on heuristic considerations rather than on mathematical models. This is mainly due to the latter models' complexity caused by dimension and by stochastic dependencies of neighboring areas. In the present work we will use the second-order product density and its isotropic analogue, the pair correlation function, to construct goodness-of-fit tests for a wide class of stationary point processes. Based on one realization of a point process in a convex observation window expanding in every direction we study the integrated squared error (ISE) of the estimated second-order product density and the ISE of the estimated pair correlation function. The asymptotic behavior of the ISE of probability density estimators has been studied e.g. by Hall [9] who derived central limit

theorems (CLTs) for the ISE for independent random variables and by Takahata and Yoshihara [21] who extended Hall's result to absolutely regular random sequences. We will derive CLTs for the ISE of the empirical second-order product density and for the ISE of the empirical pair correlation function in the setting of Brillinger-mixing point processes. The limiting normal distribution will solely depend on the underlying hypothetical second-order quantity, the intensity of the point process, and the kernel function. This allows the construction of distribution-free testing procedures.

Firstly, we will define some basic notions. Let $[N, \mathcal{N}]$ denote the measurable space of all locally finite counting measures on the d -dimensional Euclidean space \mathbb{R}^d equipped with its σ -algebra $\mathfrak{B}(\mathbb{R}^d)$ of Borel sets. A point process (shortly PP) on \mathbb{R}^d is defined as a measurable mapping Ψ from a probability space $[\Omega, \mathcal{A}, \mathbb{P}]$ into $[N, \mathcal{N}]$. Throughout in this paper we assume that Ψ is simple, i.e. $\mathbb{P}(\Psi(\{x\}) \leq 1)$ for all $x \in \mathbb{R}^d$, and stationary, i.e. $\Psi(B+x)$ and $\Psi(B)$ have the same distribution for any $x \in \mathbb{R}^d$ and $B \in \mathfrak{B}(\mathbb{R}^d)$ (for a test on stationarity see Guan [8]). Let \mathbb{E} and Var denote expectation and variance, respectively, with respect to \mathbb{P} . Let $P = \mathbb{P} \circ \Psi^{-1}$ denote the probability measure on $[N, \mathcal{N}]$ induced by Ψ and we will briefly write $\Psi \sim P$. If $\mathbb{E}\Psi^k(B) < \infty$ for all bounded Borel sets B , then there exist the k th-order factorial moment measure $\alpha^{(k)}$ and the k th-order factorial cumulant measure $\gamma^{(k)}$ on $[(\mathbb{R}^d)^k, \mathfrak{B}((\mathbb{R}^d)^k)]$ defined by

$$\alpha^{(k)}\left(\times_{j=1}^k B_j\right) := \int_N \sum_{x_1, \dots, x_k \in \psi}^* \prod_{j=1}^k \mathbb{1}_{B_j}(x_j) P(d\psi)$$

and

$$\gamma^{(k)}\left(\times_{j=1}^k B_j\right) := \sum_{\ell=1}^k (-1)^{\ell-1} (\ell-1)! \sum_{\substack{K_1 \cup \dots \cup K_\ell \\ = \{1, \dots, k\}}} \prod_{j=1}^{\ell} \alpha^{(\#K_j)}\left(\times_{k_j \in K_j} B_{k_j}\right)$$

with $B_1, \dots, B_k \in \mathfrak{B}(\mathbb{R}^d)$, respectively. Here the abbreviation “ $x \in \psi$ ” means “ $x \in \mathbb{R}^d : \psi(\{x\}) > 0$ ”. Further, \sum^* denotes summation over summands with index tuples having pairwise distinct components. The sum $\sum_{\substack{K_1 \cup \dots \cup K_\ell \\ = \{1, \dots, k\}}}$ is taken over all partitions of the set $\{1, 2, \dots, k\}$ into ℓ disjoint non-empty subsets K_j and $\#K_j$ denotes the cardinality of K_j . If $\Psi \sim P$ is stationary with intensity $\lambda > 0$ the k th-order reduced factorial moment measure $\alpha_{\text{red}}^{(k)}$ is implicitly defined by the disintegration

$$\alpha^{(k)}\left(\times_{j=1}^k B_j\right) = \lambda \int_{B_k} \alpha_{\text{red}}^{(k)}\left(\times_{j=1}^{k-1} (B_j - x)\right) dx,$$

see Daley and Vere-Jones [4, p. 238]. Analogously, the disintegration

$$\gamma^{(k)}\left(\times_{j=1}^k B_j\right) = \lambda \int_{B_k} \gamma_{\text{red}}^{(k)}\left(\times_{j=1}^{k-1} (B_j - x)\right) dx$$

defines the k th-order reduced factorial cumulant measure $\gamma_{\text{red}}^{(k)}$.

The total variation measure $|\gamma_{\text{red}}^{(k)}|$ is defined by $|\gamma_{\text{red}}^{(k)}|(\cdot) = (\gamma_{\text{red}}^{(k)})^+(\cdot) + (\gamma_{\text{red}}^{(k)})^-(\cdot)$, where the measures $(\gamma_{\text{red}}^{(k)})^+$ and $(\gamma_{\text{red}}^{(k)})^-$ are given by the Jordan decomposition $\gamma_{\text{red}}^{(k)}(\cdot) = (\gamma_{\text{red}}^{(k)})^+(\cdot) - (\gamma_{\text{red}}^{(k)})^-(\cdot)$. The total variation of $\gamma_{\text{red}}^{(k)}$ is defined by $\|\gamma_{\text{red}}^{(k)}\| := |\gamma_{\text{red}}^{(k)}|((\mathbb{R}^d)^{k-1})$.

A stationary PP $\Psi \sim P$ in \mathbb{R}^d satisfying $\mathbb{E}\Psi^k([0, 1]^d) < \infty$ for some $k \geq 2$ is said to be B_k -mixing if

$$\|\gamma_{\text{red}}^{(j)}\| = \int_{(\mathbb{R}^d)^{j-1}} |\gamma_{\text{red}}^{(j)}(d(x_1, \dots, x_{j-1}))| < \infty \quad \text{for } j = 2, \dots, k.$$

If Ψ is B_k -mixing for all $k \geq 2$, Ψ is called *Brillinger-mixing* or B_∞ -mixing, see Brillinger [2] (for $d = 1$) or Ivanoff [14]. Heinrich [10] and Heinrich and Schmidt [11] state conditions on several classes of PPs for being B_∞ -mixing.

If the k th-order reduced factorial moment measure $\alpha_{\text{red}}^{(k)}$ is absolutely continuous with respect to the Lebesgue measure on $[(\mathbb{R}^d)^{k-1}, \mathfrak{B}((\mathbb{R}^d)^{k-1})]$, then its Lebesgue density $\varrho^{(k)}$ is given by

$$\alpha_{\text{red}}^{(k)}\left(\times_{j=1}^{k-1} B_j\right) = \int_{B_1} \cdots \int_{B_{k-1}} \varrho^{(k)}(x_1, \dots, x_{k-1}) dx_1 \cdots dx_{k-1},$$

where $B_1, \dots, B_{k-1} \in \mathfrak{B}(\mathbb{R}^d)$, and is called the *k th-order reduced product density*, henceforth abbreviated as *k th-order product density*.

If the k th-order reduced factorial cumulant measure $\gamma_{\text{red}}^{(k)}$ is absolutely continuous with respect to the Lebesgue measure on $[(\mathbb{R}^d)^{k-1}, \mathfrak{B}((\mathbb{R}^d)^{k-1})]$, then its Lebesgue density $c^{(k)}$ is given by

$$\gamma_{\text{red}}^{(k)}\left(\times_{j=1}^{k-1} B_j\right) = \int_{B_1} \cdots \int_{B_{k-1}} c^{(k)}(x_1, \dots, x_{k-1}) dx_1 \cdots dx_{k-1},$$

where $B_1, \dots, B_{k-1} \in \mathfrak{B}(\mathbb{R}^d)$, and is called the *k th-order reduced cumulant density*, henceforth abbreviated as *k th-order cumulant density*.

In this paper we will focus on the second-order product density $\varrho^{(2)}$, henceforth abbreviated as product density ϱ , and its isotropic analogue, the *pair correlation function* (PCF), defined by

$$g(r) := \frac{\varrho(x)}{\lambda},$$

where $r = \|x\|$, $x \in \mathbb{R}^d$, and λ is the intensity of the stationary PP.

The rest of the paper is organized as follows. Section 2 introduces the estimators for the product density and the PCF and their ISEs. In Section 3 we derive CLTs for these ISEs. In Section 4, these results are used for constructing asymptotic goodness-of-fit tests. The proofs of the results in Section 4 are carried out in Section 5.

2 Integrated squared error of the empirical product density and of the empirical pair correlation function

In this section we will present the estimators for the product density and the PCF and their ISEs and formulate some conditions needed for our asymptotic results in the next sections.

Let $\rho(W) := \sup\{r \geq 0 : b(x, r) \subset W, x \in \mathbb{R}^d\}$ denote the *inradius of the set* $W \subset \mathbb{R}^d$, where $b(x, r) := \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$ is the ball with radius $r \geq 0$ centered at $x \in \mathbb{R}^d$. Let $|\cdot|$ denote the Lebesgue measure on $[\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d)]$ and let $\omega_d = |b(o, 1)|$. The following condition is needed for $s = 1$ or $s = d$.

Condition $(\mathcal{C}(s))$. (i) The sequence of observation windows $(W_n)_{n \in \mathbb{N}}$ is an increasing sequence of convex and compact sets in \mathbb{R}^d with $\rho(W_n) \xrightarrow[n \rightarrow \infty]{} \infty$,

(ii) the sequence of bandwidths $(b_n)_{n \in \mathbb{N}}$ is a decreasing sequence of positive real numbers satisfying $b_n \xrightarrow[n \rightarrow \infty]{} 0$ and $b_n^s |W_n| \xrightarrow[n \rightarrow \infty]{} \infty$, and

(iii) the kernel function $k : \mathbb{R}^s \rightarrow \mathbb{R}$ is bounded with bounded support, symmetric (i.e., $k(x) = k(-x)$ for every $x \in \mathbb{R}^s$), and satisfies $\int_{\mathbb{R}^s} k(x) dx = 1$.

The following definition of a kernel-type estimator for the product density goes back to Krickeberg [17]. The speed of convergence of this estimator has been studied in Jolivet [16].

Definition 2.1. Let $(W_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ and k satisfy Condition $\mathcal{C}(d)$. Let the PP $\Psi \sim P$ in \mathbb{R}^d be stationary and assume its product density ϱ to exist. Then we define

$$\hat{\varrho}_n(t) := \frac{1}{b_n^d |W_n|} \sum_{x_1, x_2 \in \Psi}^* \mathbb{1}_{W_n}(x_1) k\left(\frac{x_2 - x_1 - t}{b_n}\right)$$

as an estimator for $\lambda \varrho(t)$ for $t \in \mathbb{R}^d$.

Definition 2.2. Let $(W_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ and k satisfy Condition $\mathcal{C}(1)$. Let the PP $\Psi \sim P$ in \mathbb{R}^d be stationary and assume its PCF g to exist. Then we define

$$\hat{g}_n(r) = \frac{1}{b_n |W_n| d \omega_d} \sum_{x_1, x_2 \in \Psi}^* \frac{\mathbb{1}_{W_n}(x_1)}{\|x_2 - x_1\|^{d-1}} k\left(\frac{\|x_2 - x_1\| - r}{b_n}\right)$$

as an estimator for $\lambda^2 g(r)$ for $r \in [0, \infty)$.

For a discussion of estimators for the PCF with regard to bias and variance see Stoyan and Stoyan [20].

The integrated squared error (ISE) of the product density estimator is defined by

$$I_n(K) := \int_K (\hat{\varrho}_n(t) - \lambda \varrho(t))^2 dt,$$

where $K \in \mathfrak{B}(\mathbb{R}^d)$, $|K| > 0$, is a bounded set. Likewise, the ISE of the PCF estimator is defined by

$$J_n(K) := \int_K (\hat{g}_n(r) - \lambda^2 g(r))^2 dr,$$

where $K \in \mathfrak{B}((0, \infty))$, $|K| > 0$, is a bounded set.

Condition $(\mathcal{C}_\varrho(K))$. Let $\Psi \sim P$ be a stationary PP in \mathbb{R}^d with product density ϱ such that, for some $K \in \mathfrak{B}(\mathbb{R}^d)$ and some $\varepsilon > 0$,

(i) the first-order partial derivatives of the product density ϱ are uniformly Lipschitz-continuous in $K \oplus b(o, \varepsilon)$ and

(ii) the third- and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ exist and satisfy

$$\sup_{u,v \in K \oplus b(o,\varepsilon)} |c^{(3)}(u,v)| < \infty \quad \text{and} \quad \sup_{u,v \in K \oplus b(o,\varepsilon)} \int_{\mathbb{R}^d} |c^{(4)}(u,w,v+w)| dw < \infty.$$

Condition ($\mathcal{C}_g(K)$). Let $\Psi \sim P$ be a stationary PP in \mathbb{R}^d with PCF g such that, for some $K \in \mathfrak{B}((0, \infty))$ and some $\varepsilon > 0$,

(i) the first derivative of the PCF g is uniformly Lipschitz-continuous in $K \oplus (-\varepsilon, \varepsilon)$ and

(ii) the third- and fourth-order cumulant densities $c^{(3)}$ and $c^{(4)}$ exist and satisfy

$$\sup_{\substack{u,v \in \mathbb{R}^d: \\ \|u\|, \|v\| \in K \oplus (-\varepsilon, \varepsilon)}} |c^{(3)}(u,v)| < \infty \quad \text{and} \quad \sup_{\substack{u,v \in \mathbb{R}^d: \\ \|u\|, \|v\| \in K \oplus (-\varepsilon, \varepsilon)}} \int_{\mathbb{R}^d} |c^{(4)}(u,w,v+w)| dw < \infty.$$

3 Central limit theorems

In this section we will present asymptotic representations of the mean and the variance of the ISE $I_n(K)$ of the product density estimator. Then we will state a CLT for $I_n(K)$ for B_∞ -mixing PPs. The proofs can be found in Section 5. The results for the ISE $J_n(K)$ of the PCF estimator can be shown analogously and will be given without proof.

The following lemma gives an asymptotic representation of the mean of the ISE $I_n(K)$ of the product density estimator.

Lemma 3.1. Let $\Psi \sim P$ be a B_4 -mixing PP in \mathbb{R}^d with intensity λ and product density ϱ satisfying Condition $\mathcal{C}_\varrho(K)$. Let $(W_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ and k satisfy Condition $\mathcal{C}(d)$. Then we have

$$b_n^d |W_n| \mathbb{E} \int_K (\hat{\varrho}_n(t) - \lambda \varrho(t))^2 dt = \lambda \int_K \varrho(t) dt \int_{\mathbb{R}^d} k^2(x) dx + \mathcal{O}(b_n^{d \wedge 2}) + \mathcal{O}(b_n^{d+4} |W_n|)$$

as $n \rightarrow \infty$ for all bounded $K \in \mathfrak{B}(\mathbb{R}^d)$, $|K| > 0$.

To express the asymptotic variance of the ISE $I_n(K)$ in the next lemma we need the convolution $k * k$ of the kernel function k with itself.

Lemma 3.2. Let $\Psi \sim P$ be a B_8 -mixing PP in \mathbb{R}^d with intensity λ and product density ϱ satisfying Condition $\mathcal{C}_\varrho(K)$. Let $(W_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ and k satisfy Condition $\mathcal{C}(d)$. Furthermore let $b_n^{d+4} |W_n| \xrightarrow{n \rightarrow \infty} 0$. Then we have

$$\text{Var} \left(b_n^{d/2} |W_n| \int_K (\hat{\varrho}_n(t) - \lambda \varrho(t))^2 dt \right) \xrightarrow{n \rightarrow \infty} \sigma^2$$

with

$$\sigma^2 := 2\lambda^2 \left(\int_K \varrho^2(t) dt + \int_{K \cap (-K)} \varrho^2(t) dt \right) \int_{\mathbb{R}^d} (k * k)^2(t) dt$$

for all bounded $K \in \mathfrak{B}(\mathbb{R}^d)$, $|K| > 0$.

Now we state a CLT for the ISE of the product density estimator in the setting of B_∞ -mixing PPs. The result will be proved in Section 5 by showing the cumulants of order $k \geq 3$ of the suitably scaled ISE to converge to zero.

The notation $\xrightarrow[n \rightarrow \infty]{d}$ stands for weak convergence and $\mathcal{N}(\mu, \sigma^2)$ for the univariate normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$.

Theorem 3.3. *Let $\Psi \sim P$ be a B_∞ -mixing PP in \mathbb{R}^d with intensity λ and product density ϱ satisfying Condition $\mathcal{C}_\varrho(K)$. Let all cumulant densities $c^{(k)}$, $k \geq 2$, exist. Let $(W_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ and k satisfy Condition $\mathcal{C}(d)$, and, in addition, $b_n^{d+4}|W_n| \xrightarrow[n \rightarrow \infty]{} 0$. Then we have*

$$b_n^{d/2}|W_n|(I_n(K) - \mathbb{E}I_n(K)) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, \sigma^2)$$

for all bounded $K \in \mathfrak{B}(\mathbb{R}^d)$, $|K| > 0$, with σ^2 given in Lemma 3.2.

Now we present an asymptotic representation for the mean of the ISE $J_n(K)$ of the PCF estimator and a CLT for the centered and suitably scaled ISE $J_n(K)$.

Lemma 3.4. *Let $\Psi \sim P$ be a B_4 -mixing PP in \mathbb{R}^d with intensity λ and PCF g satisfying Condition $\mathcal{C}_g(K)$. Let $(W_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ and k satisfy Condition $\mathcal{C}(1)$. Then we have*

$$b_n|W_n|\mathbb{E} \int_K (\hat{g}_n(r) - \lambda^2 g(r))^2 dr = 2\lambda^2 \int_K \frac{g(r)}{d\omega dr^{d-1}} dr \int_{\mathbb{R}} k^2(x) dx + \mathcal{O}(b_n) + \mathcal{O}(b_n^5|W_n|)$$

as $n \rightarrow \infty$ for all bounded $K \in \mathfrak{B}((0, \infty))$, $|K| > 0$.

Theorem 3.5. *Let $\Psi \sim P$ be a B_∞ -mixing PP in \mathbb{R}^d with intensity λ and PCF g satisfying Condition $\mathcal{C}_g(K)$. Let all cumulant densities $c^{(k)}$, $k \geq 2$, exist. Let $(W_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ and k satisfy Condition $\mathcal{C}(1)$, and, in addition, $b_n^5|W_n| \xrightarrow[n \rightarrow \infty]{} 0$. Then we have*

$$b_n^{1/2}|W_n|(J_n(K) - \mathbb{E}J_n(K)) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, \tau^2)$$

with

$$\tau^2 := 8\lambda^4 \int_K \left(\frac{g(r)}{d\omega dr^{d-1}} \right)^2 dr \int_{\mathbb{R}} (k * k)^2(x) dx$$

for all bounded $K \in \mathfrak{B}((0, \infty))$, $|K| > 0$.

4 Asymptotic goodness-of-fit tests

Given a realization of a PP $\Psi \sim P$ in a sufficiently large observation window, one is interested in whether a hypothetical distribution P_0 of a PP is a good fit for the unknown true distribution P (see e.g. Diggle [5]). In this section we will use the CLTs in Section 3 for constructing asymptotic goodness-of-fit tests for PPs in order to get a decision rule for the non-parametric test problem $H_0 : P = P_0$ versus $H_1 : P \neq P_0$. The test statistic is based on the ISE which uses only the intensity and the product density (or the PCF) in the set K as information from the PP Ψ .

Although second-order quantities do not characterize the distribution of the PP they still give a rather informative description of the point pattern and are therefore an appropriate basis for goodness-of-fit tests.

Compared with existing goodness-of-fit tests, our tests have the advantage that they are theoretically motivated (not based on simulations) and, at the same time, can be applied to a wide class of PPs (not only Poisson processes).

Now we will sketch how Theorem 3.3 and Theorem 3.5 are used for constructing asymptotic goodness-of-fit tests. For $d \in \{1, 2, 3\}$ we can use Lemma 3.1 to get a simple representation for the mean of the ISE $I_n(K)$. The choice of a symmetric set K will simplify the variance. For the simplification of the mean of the ISE $J_n(K)$ of the PCF estimator, see Lemma 3.4, there is no restriction to certain dimensions necessary. In the following, z_q denotes the q -quantile of the standard normal distribution.

Testing $H_0 : P = P_0$ versus $H_1 : P \neq P_0$ using $I_n(K)$

Let $\Psi \sim P$ be a PP in \mathbb{R}^d , $d \in \{1, 2, 3\}$, and let the hypothetical PP $\Psi_0 \sim P_0$ with intensity λ_0 and product density ϱ_0 satisfy the assumptions of Theorem 3.3. Furthermore let $b_n^{d/2+4}|W_n| \xrightarrow[n \rightarrow \infty]{} 0$ and let $K \in \mathfrak{B}(\mathbb{R}^d)$ be bounded and symmetric with $|K| > 0$. Then

$$T_n = \frac{b_n^{d/2}|W_n|}{\sigma_0^2} \left(\int_K (\hat{\varrho}_n(t) - \lambda_0 \varrho_0(t))^2 dt - \frac{\lambda_0}{b_n^d |W_n|} \int_K \varrho_0(t) dt \int_{\mathbb{R}} k^2(x) dx \right)$$

with

$$\sigma_0^2 = 4\lambda_0^2 \int_K \varrho_0^2(t) dt \int_{\mathbb{R}^d} (k * k)^2(t) dt$$

lies in $[-z_{1-\alpha/2}, z_{1-\alpha/2}]$ with probability $1 - \alpha$ approximately. Given a significance level $\alpha \in (0, 1)$ we reject the null hypothesis $H_0 : P = P_0$ if T_n lies outside $[-z_{1-\alpha/2}, z_{1-\alpha/2}]$.

Testing $H_0 : P = P_0$ versus $H_1 : P \neq P_0$ using $J_n(K)$

Let $\Psi \sim P$ be a PP in \mathbb{R}^d and let the hypothetical PP $\Psi_0 \sim P_0$ with intensity λ_0 and PCF g_0 satisfy the assumptions of Theorem 3.5. Furthermore let $b_n^{9/2}|W_n| \xrightarrow[n \rightarrow \infty]{} 0$ and let $K \in \mathfrak{B}((0, \infty))$ be bounded with $|K| > 0$. Then

$$T_n = \frac{b_n^{1/2}|W_n|}{\tau_0^2} \left(\int_K (\hat{g}_n(r) - \lambda_0^2 g_0(r))^2 dr - \frac{2\lambda_0^2}{b_n |W_n|} \int_K \frac{g_0(r)}{d\omega_d r^{d-1}} dr \int_{\mathbb{R}} k^2(x) dx \right)$$

with

$$\tau_0^2 = 8\lambda_0^4 \int_K \left(\frac{g_0(r)}{d\omega_d r^{d-1}} \right)^2 dr \int_{\mathbb{R}} (k * k)^2(x) dx$$

lies in $[-z_{1-\alpha/2}, z_{1-\alpha/2}]$ with probability $1 - \alpha$ approximately. Given a significance level $\alpha \in (0, 1)$ we reject the null hypothesis $H_0 : P = P_0$ if T_n lies outside $[-z_{1-\alpha/2}, z_{1-\alpha/2}]$.

An important question concerning the applicability of our asymptotic goodness-of-fit tests is how large the observation window has to be for a satisfactory approximation in the CLT. An answer

may be found through simulation studies. The approximation will depend on several factors such as the distribution of the underlying PP—in particular the intensity and the product density or PCF—the choice of the bandwidth and the kernel function, and the choice of the set K . Given a hypothetical distribution P_0 and the associated test problem $H_0 : P = P_0$ versus $H_1 : P \neq P_0$ it is obvious how to investigate the type-I error (that is, the probability of rejecting the null hypothesis when it is actually true) by simulation studies. The type-II error (that is, the probability of not rejecting the null hypothesis when the alternative hypothesis is actually true) is difficult to handle since the true distribution P can differ from P_0 in many different ways. Hence the type-II error can only be studied for some special cases. For example, if $P = \Pi_\lambda$ and $P_0 = \Pi_{\lambda_0}$ with $\lambda \neq \lambda_0$, an investigation of the type-II error for different combinations of λ and λ_0 is a sensitivity analysis of the test procedure with respect to the intensity of the underlying Poisson process. Another example of such a sensitivity analysis is given in Grabarnik and Chiu [7] who consider the null hypothesis of a Poisson process and the alternative hypothesis of a mixture of a conditional Strauss PP and Matérn’s cluster process.

Note that the intensity λ_0 must be known—if $I_n(K)$ is replaced by $\int_K (\hat{\varrho}_n(t) - \widehat{(\lambda_0)_n} \varrho(t))^2 dt$,

the limiting distribution may differ (as it is the case for the Kolmogorov-Smirnov statistic if the parameters are estimated). However, the intensity λ_0 occurring in the mean and the variance of T_n can be replaced by a consistent estimator $\widehat{(\lambda_0)_n}$ due to Slutsky’s theorem. Another problem might arise if the product density ϱ_0 of the hypothetical PP Ψ_0 is not known explicitly. Nevertheless, our tests can be applied if ϱ_0 is replaced by an estimator $\widehat{(\varrho_0)_n}$ achieved by simulation of the null hypothesis model. It should be ensured that $\widehat{(\varrho_0)_n}$ meets Condition $\mathcal{C}_\varrho(K)(i)$, e.g., by using a kernel function that satisfies $\mathcal{C}_\varrho(K)(i)$. Analogous considerations apply to the test based on the PCF.

5 Proofs

The normal convergence of the centered and suitably scaled ISE $I_n(K)$ is proved by showing all cumulants of order three and higher to converge to zero. Lemmas 5.1 and 5.2 will lead to a representation for the cumulants of the ISE of the estimated product density. This representation can also be used for deriving the asymptotic variance of the ISE. We start with the proof for the asymptotic representation of the mean of the ISE $I_n(K)$.

Proof of Lemma 3.1. By Fubini’s theorem we have

$$\mathbb{E} \int_K (\hat{\varrho}_n(t) - \lambda \varrho(t))^2 dt = \int_K \text{Var}(\hat{\varrho}_n(t)) dt + \int_K (\mathbb{E} \hat{\varrho}_n(t) - \lambda \varrho(t))^2 dt.$$

For the second summand we get

$$b_n^d |W_n| \int_K (\mathbb{E} \hat{\varrho}_n(t) - \lambda \varrho(t))^2 dt = b_n^d |W_n| \lambda^2 \int_K \left(\int_{\mathbb{R}^d} (\varrho(t + b_n z) - \varrho(t)) k(z) dz \right)^2 dt.$$

Using Taylor’s expansion of the product density ϱ in $t = (t_1, \dots, t_d)'$ we get

$$\varrho(t + b_n z) = \varrho(t) + b_n \sum_{i=1}^d z_i \frac{\partial}{\partial t_i} \varrho(t + \theta_i b_n z)$$

$$= \varrho(t) + b_n \sum_{i=1}^d z_i \frac{\partial}{\partial t_i} \varrho(t) + b_n \sum_{i=1}^d z_i \left(\frac{\partial}{\partial t_i} \varrho(t + \theta_i b_n z) - \frac{\partial}{\partial t_i} \varrho(t) \right)$$

with $\theta_i = \theta_i(t) \in [0, 1]$, $i = 1, \dots, d$. The symmetry of the kernel function k and the uniform Lipschitz-continuity of the first-order partial derivatives of the product density (with L being the maximum of the Lipschitz constants) entail

$$\begin{aligned} \left| \int_{\mathbb{R}^d} (\varrho(t + b_n z) - \varrho(t)) k(z) dz \right| &= b_n \left| \sum_{i=1}^d \int_{\mathbb{R}^d} z_i \left(\frac{\partial}{\partial t_i} \varrho(t + \theta_i b_n z) - \frac{\partial}{\partial t_i} \varrho(t) \right) k(z) dz \right| \\ &\leq b_n^2 L \sum_{i=1}^d \int_{\mathbb{R}^d} |z_i|^2 k(z) dz. \end{aligned}$$

This implies

$$b_n^d |W_n| \int_K (\mathbb{E} \hat{\varrho}_n(t) - \lambda \varrho(t))^2 dt = \mathcal{O}(b_n^{d+4} |W_n|)$$

as $n \rightarrow \infty$. Now we will prove the asymptotic representation

$$b_n^d |W_n| \int_K \text{Var}(\hat{\varrho}_n(t)) dt = \lambda \int_K \varrho(t) dt \int_{\mathbb{R}^d} k^2(x) dx + \mathcal{O}(b_n^{d \wedge 2}).$$

Using the representation (4.17) in Heinrich [10] we obtain

$$\begin{aligned} b_n^{2d} |W_n|^2 \text{Var}(\hat{\varrho}_n(t)) &= \int_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x) k^2 \left(\frac{y-x-t}{b_n} \right) \alpha^{(2)}(d(x, y)) \\ &\quad + \int_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(y) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{x-y-t}{b_n} \right) \alpha^{(2)}(d(x, y)) \\ &\quad + \int_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{z-x-t}{b_n} \right) \alpha^{(3)}(d(x, y, z)) \\ &\quad + \int_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(y) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{z-y-t}{b_n} \right) \alpha^{(3)}(d(x, y, z)) \\ &\quad + \int_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(z) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{x-z-t}{b_n} \right) \alpha^{(3)}(d(x, y, z)) \\ &\quad + \int_{(\mathbb{R}^d)^3} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(z) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{y-z-t}{b_n} \right) \alpha^{(3)}(d(x, y, z)) \\ &\quad + \int_{(\mathbb{R}^d)^4} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(z) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{v-z-t}{b_n} \right) \\ &\quad \quad \quad [\alpha^{(4)}(d(x, y, z, v)) - \alpha^{(2)}(d(x, y)) \alpha^{(2)}(d(z, v))], \end{aligned}$$

where

$$\begin{aligned}
& \alpha^{(4)}(d(x, y, z, v)) - \alpha^{(2)}(d(x, y))\alpha^{(2)}(d(z, v)) \\
&= \gamma^{(4)}(d(x, y, z, v)) + \gamma^{(1)}(dx)\gamma^{(3)}(d(y, z, v)) + \gamma^{(1)}(dy)\gamma^{(3)}(d(x, z, v)) \\
&\quad + \gamma^{(1)}(dz)\gamma^{(3)}(d(x, y, v)) + \gamma^{(1)}(dv)\gamma^{(3)}(d(x, y, z)) + \gamma^{(2)}(d(x, z))\gamma^{(2)}(d(y, v)) \\
&\quad + \gamma^{(2)}(d(x, v))\gamma^{(2)}(d(y, z)) + \gamma^{(2)}(d(x, z))\gamma^{(1)}(dy)\gamma^{(1)}(dv) + \gamma^{(2)}(d(x, v))\gamma^{(1)}(dy)\gamma^{(1)}(dz) \\
&\quad + \gamma^{(2)}(d(y, z))\gamma^{(1)}(dx)\gamma^{(1)}(dv) + \gamma^{(2)}(d(y, v))\gamma^{(1)}(dx)\gamma^{(1)}(dz).
\end{aligned}$$

First we consider the two integrals with respect to the second-order factorial moment measure $\alpha^{(2)}$. For the first one we obtain

$$\begin{aligned}
\frac{1}{b_n^d |W_n|} \int_K \int_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x) k^2 \left(\frac{y-x-t}{b_n} \right) \alpha^{(2)}(d(x, y)) dt &= \lambda \int_K \int_{\mathbb{R}^d} k^2(y) \varrho(b_n y + t) dy dt \\
&= \lambda \int_K \varrho(t) dt \int_{\mathbb{R}^d} k^2(y) dy + \mathcal{O}(b_n^2)
\end{aligned}$$

as $n \rightarrow \infty$ which can be seen by Taylor's expansion of $\varrho(b_n y + t)$ in t , using the uniform Lipschitz-continuity of the first-order partial derivatives of the product density, and the boundedness and the symmetry of the kernel function k which entails boundedness and symmetry of k^2 . For the second integral with respect to the second-order factorial moment measure $\alpha^{(2)}$ we have

$$\begin{aligned}
& \frac{1}{b_n^d |W_n|} \int_K \int_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(y) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{x-y-t}{b_n} \right) \alpha^{(2)}(d(x, y)) dt \\
&= b_n^d \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\mathbb{1}_{W_n} \cap (W_n - b_n y - b_n t)|}{|W_n|} k(y) k(y-2t) \varrho(b_n y + b_n t) \mathbb{1}_K(b_n t) dy dt \\
&= \mathcal{O}(b_n^d)
\end{aligned}$$

as $n \rightarrow \infty$ due to the continuity of the product density ϱ in $K \oplus b(o, \varepsilon)$ for some $\varepsilon > 0$ if $o \in K \oplus b(o, \varepsilon)$ (otherwise the integral vanishes eventually).

Now we consider the integrals with respect to the third-order factorial moment measure $\alpha^{(3)}$. For the first of these integrals we have

$$\begin{aligned}
& \frac{1}{b_n^d |W_n|} \int_{(\mathbb{R}^d)^4} \mathbb{1}_K(t) \mathbb{1}_{W_n}(x) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{z-x-t}{b_n} \right) dt \alpha^{(3)}(d(x, y, z)) \\
&= \frac{1}{b_n^d |W_n|} \int_{(\mathbb{R}^d)^4} \mathbb{1}_K(t) \mathbb{1}_{W_n}(x) k \left(\frac{y-x-t}{b_n} \right) k \left(\frac{z-x-t}{b_n} \right) dt [\gamma^{(3)}(d(x, y, z)) + \\
&\quad \lambda dx \gamma^{(2)}(d(y, z)) + \lambda dy \gamma^{(2)}(d(x, z)) + \lambda dz \gamma^{(2)}(d(x, y)) + \lambda^3 dx dy dz] \\
&= b_n^d \lambda \int_{(\mathbb{R}^d)^3} \mathbb{1}_K(t) k(y) k(z) c^{(3)}(b_n y + t, b_n z + t) dy dz dt \\
&\quad + b_n^d \lambda^2 \int_{(\mathbb{R}^d)^3} \mathbb{1}_K(t) k(y) k(y+z) c^{(2)}(b_n z) dy dz dt \\
&\quad + 2b_n^d \lambda^2 \int_{(\mathbb{R}^d)^3} \mathbb{1}_K(t) k(y) k(z) c^{(2)}(b_n z + t) dy dz dt
\end{aligned}$$

$$\begin{aligned}
& + b_n^d \lambda^3 \int_{(\mathbb{R}^d)^3} \mathbb{1}_K(t) k(y) k(z) dy dz dt \\
& = \mathcal{O}(b_n^d)
\end{aligned}$$

as $n \rightarrow \infty$. For deriving this asymptotic order we have used Lebesgue's dominated convergence theorem which is applicable due to $|K| < \infty$, the boundedness assumptions on the kernel function, the continuity of $c^{(2)}$, and $\sup_{u,v \in K \oplus b(o,\varepsilon)} |c^{(3)}(u,v)| < \infty$ for some $\varepsilon > 0$. By analogous arguments we can show the asymptotic order of the other integrals with respect to the third-order factorial moment measure $\alpha^{(3)}$ to be $\mathcal{O}(b_n^d)$, too.

Let us now consider the integrals with respect to the factorial cumulant measures. Due to the finiteness of the total variations of order two and three the asymptotic order of the integrals with respect to $\gamma^{(2)}$ and $\gamma^{(3)}$ is $\mathcal{O}(b_n^d)$. The integral with respect to $\gamma^{(4)}$ is

$$\begin{aligned}
& \frac{1}{b_n^d |W_n|} \int_{(\mathbb{R}^d)^5} \mathbb{1}_K(t) \mathbb{1}_{W_n}(x) \mathbb{1}_{W_n}(z) k\left(\frac{y-x-t}{b_n}\right) k\left(\frac{v-z-t}{b_n}\right) \gamma^{(4)}(d(x,y,z,v)) dt \\
& = b_n^d \lambda \int_{(\mathbb{R}^d)^4} \frac{|W_n \cap (W_n - z)|}{|W_n|} \mathbb{1}_K(t) k(y) k(v) c^{(4)}(b_n y + t, z, b_n v + z + t) dy dz dv dt.
\end{aligned}$$

Due to $\sup_{u,v \in K \oplus b(o,\varepsilon)} \int_{\mathbb{R}^d} |c^{(4)}(u,w,v+w)| dw < \infty$ for some $\varepsilon > 0$ we find that this integral is of asymptotic order $\mathcal{O}(b_n^d)$. This completes the proof. \square

Lemma 5.1 will show that the k th cumulant of certain random variables (including the ISE $I_n(K)$) is a sum of integrals that are indecomposable, in the sense that they cannot be represented as a product of two integrals. The rigorous definition of decomposability is as follows.

Let $f_i : (\mathbb{R}^d)^{p_i} \rightarrow \mathbb{R}$ be fixed measurable functions, let $k \in \mathbb{N}$ and $p_i \in \mathbb{N}$ with $i \in I = \{1, \dots, k\}$ be fixed and set

$$\Psi^{(p_i)}(f_i) := \sum_{x_1, \dots, x_{p_i} \in \Psi} f_i(x_1, \dots, x_{p_i}).$$

Let $\mathbb{E} [|\Psi^{(p_i)}(f_i)|^k] < \infty$ for all $i \in I$. We will now find a representation of the mixed moment $M(\Psi^{(p_1)}(f_1), \dots, \Psi^{(p_k)}(f_k)) := \mathbb{E} \left[\prod_{i=1}^k \Psi^{(p_i)}(f_i) \right]$ as a sum of integrals defined as follows.

For arbitrary $T \subseteq I$, $q \in \{1, \dots, p_T\}$ with $p_T := \sum_{i \in T} p_i$, $r \in \{1, \dots, q\}$, and decompositions $\mathcal{P}_T = \{P_1, \dots, P_q\}$ of $\{1, \dots, p_T\}$ and $\mathcal{Q} = \{Q_1, \dots, Q_r\}$ of $\{1, \dots, q\}$ we define the integral

$$\begin{aligned}
& I_{\mathcal{P}_T, \mathcal{Q}}(f_i : i \in T) \\
& := \int_{(\mathbb{R}^d)^q} \prod_{b=1}^q \prod_{a \in P_b} \mathbb{1}_{\{x_a = z_b\}} f_{i_1}(x_1, \dots, x_{p_{i_1}}) \\
& \quad \times f_{i_2}(x_{p_{i_1}+1}, \dots, x_{p_{i_1}+p_{i_2}}) \cdots f_{i_{\#T}}(x_{\sum_{j=1}^{\#T-1} p_{i_j}+1}, \dots, x_{p_T}) \prod_{c=1}^r \gamma^{(\#Q_c)}(d\mathbf{z}_{Q_c}),
\end{aligned}$$

where $\{i_1, \dots, i_{\#T}\} = T$ with $1 \leq i_1 < i_2 < \dots < i_{\#T} \leq k$ and $\mathbf{z}_{Q_c} = (z_q)_{q \in Q_c}$. The elements of a set P_b are the indices of the arguments of the functions $f_{i_1}, \dots, f_{i_{\#T}}$ that are identical and

distinct from all the arguments in every other set $P_c \neq P_b$. In the above-mentioned integral this is indicated by the term $\prod_{b=1}^q \prod_{a \in P_b} \mathbb{1}_{\{x_a = z_b\}}$. For the special case $T = I$ we have

$$\begin{aligned} I_{\mathcal{P}_I, \mathcal{Q}}(f_1, \dots, f_k) &= \int_{(\mathbb{R}^d)^q} \prod_{b=1}^q \prod_{a \in P_b} \mathbb{1}_{\{x_a = z_b\}} f_1(x_1, \dots, x_{p_1}) \cdots f_k(x_{\sum_{i=1}^{k-1} p_i + 1}, \dots, x_{p_I}) \prod_{c=1}^r \gamma^{(\#Q_c)}(d\mathbf{z}_{Q_c}). \end{aligned}$$

Now the mixed moment $M(\Psi^{(p_1)}(f_1), \dots, \Psi^{(p_k)}(f_k))$ can be represented as

$$\begin{aligned} M(\Psi^{(p_1)}(f_1), \dots, \Psi^{(p_k)}(f_k)) &= \sum_{q=1}^{p_I} \sum_{\substack{P_1 \cup \dots \cup P_q \\ = \{1, \dots, p_I\}}} \int_{(\mathbb{R}^d)^q} \prod_{b=1}^q \prod_{a \in P_b} \mathbb{1}_{\{x_a = z_b\}} \\ &\quad \times f_1(x_1, \dots, x_{p_1}) \cdots f_k(x_{\sum_{i=1}^{k-1} p_i}, \dots, x_{p_I}) \alpha^{(q)}(d(z_1, \dots, z_q)) \\ &= \sum_{q=1}^{p_I} \sum_{\substack{P_1 \cup \dots \cup P_q \\ = \{1, \dots, p\}}} \sum_{r=1}^q \sum_{\substack{Q_1 \cup \dots \cup Q_r \\ = \{1, \dots, q\}}} \int_{(\mathbb{R}^d)^q} \prod_{b=1}^q \prod_{a \in P_b} \mathbb{1}_{\{x_a = z_b\}} \\ &\quad \times f_1(x_1, \dots, x_{p_1}) \cdots f_k(x_{\sum_{i=1}^{k-1} p_i + 1}, \dots, x_{p_I}) \prod_{c=1}^r \gamma^{(\#Q_c)}(d\mathbf{z}_{Q_c}), \end{aligned}$$

see Krickeberg [17]. With the above notation we have

$$M(\Psi^{(p_1)}(f_1), \dots, \Psi^{(p_k)}(f_k)) = \sum_{q=1}^{p_I} \sum_{\substack{P_1 \cup \dots \cup P_q \\ = \{1, \dots, p_I\}}} \sum_{r=1}^q \sum_{\substack{Q_1 \cup \dots \cup Q_r \\ = \{1, \dots, q\}}} I_{\mathcal{P}_I, \mathcal{Q}}(f_1, \dots, f_k).$$

Let $T = \{T_1, T_2\}$ be a decomposition of $I = \{1, \dots, k\}$. An integral $I_{\mathcal{P}_I, \mathcal{Q}}(f_1, \dots, f_k)$ is *decomposable with respect to the decomposition* $T = \{T_1, T_2\}$ if there exist a decomposition $\mathcal{P}^{(1)}$ of $\{1, \dots, p_{T_1}\}$, a decomposition $\mathcal{P}^{(2)}$ of $\{1, \dots, p_{T_2}\}$, $q_1 \in \{1, \dots, p_{T_1}\}$ and $q_2 \in \{1, \dots, p_{T_2}\}$ with $q_1 + q_2 = q$, and decompositions $\mathcal{Q}^{(1)}$ of $\{1, \dots, q_1\}$ and $\mathcal{Q}^{(2)}$ of $\{1, \dots, q_2\}$ such that

$$I_{\mathcal{P}_I, \mathcal{Q}}(f_1, \dots, f_k) = I_{\mathcal{P}_{T_1}, \mathcal{Q}^{(1)}}(f_i : i \in T_1) \cdot I_{\mathcal{P}_{T_2}, \mathcal{Q}^{(2)}}(f_i : i \in T_2).$$

An integral is called *decomposable* if there exists a nontrivial decomposition of I such that this integral is decomposable with respect to this decomposition. An integral which is not decomposable with respect to any nontrivial decomposition is called *indecomposable*.

The following lemma is the key tool for the proof of the CLT for the ISE $I_n(K)$. It gives a representation of the k th cumulant of certain random variables as a sum of indecomposable integrals. Let $\Gamma_k(X)$ denote the k th cumulant of a real-valued random variable X and $\text{Cum}_k(X_1, \dots, X_k)$ denote the mixed cumulant of a random vector $X = (X_1, \dots, X_k)' \in \mathbb{R}^k$, $k \geq 1$.

Lemma 5.1. *Let $\Psi \sim P$ be a PP in \mathbb{R}^d . Let $j, k \in \mathbb{N}$ be fixed, let $C_i \in \mathbb{R}$ be constants for $i = 1, \dots, j$, and set*

$$\Psi^{(p_i)}(f_i) = \sum_{x_1, \dots, x_{p_i} \in \Psi} f_i(x_1, \dots, x_{p_i}),$$

where $f_i : (\mathbb{R}^d)^{p_i} \rightarrow \mathbb{R}$ is a fixed measurable function with $p_i \in \mathbb{N}$, for $i = 1, \dots, j$. Let $\mathbb{E} [|\Psi^{(p_i)}(f_i)|^k] < \infty$ for all $i = 1, \dots, j$. Then we have

$$\Gamma_k \left(\sum_{i=1}^j C_i \Psi^{(p_i)}(f_i) \right) = \sum_{\substack{k_1 + \dots + k_j = k \\ k_1, \dots, k_j \geq 0}} \frac{k!}{k_1! \cdot \dots \cdot k_j!} C_1^{k_1} \cdot \dots \cdot C_j^{k_j} \mu_{k_1, \dots, k_j}^*,$$

where

$$\mu_{k_1, \dots, k_j}^* := \left(\sum_{q=1}^{p_{k_1, \dots, k_j}} \sum_{\substack{P_1 \cup \dots \cup P_q \\ = \{1, \dots, p_{k_1, \dots, k_j}\}}} \sum_{r=1}^q \sum_{\substack{Q_1 \cup \dots \cup Q_r \\ = \{1, \dots, q\}}} \right)^* I_{\mathcal{P}_I, \mathcal{Q}}(\underbrace{f_1, \dots, f_1}_{k_1}, \dots, \underbrace{f_j, \dots, f_j}_{k_j}) \quad (1)$$

and $p_{k_1, \dots, k_j} = \sum_{i=1}^j p_i k_i$. The summation $(\cdot)^*$ is taken only over the indecomposable integrals.

Proof. Due to multilinearity, symmetry, and homogeneity of the mixed cumulants we have

$$\begin{aligned} \Gamma_k \left(\sum_{i=1}^j C_i \Psi^{(p_i)}(f_i) \right) &= \text{Cum}_k \left(\sum_{i=1}^j C_i \Psi_{f_i}, \dots, \sum_{i=1}^j C_i \Psi^{(p_i)}(f_i) \right) \\ &= \sum_{\substack{k_1 + \dots + k_j = k \\ k_1, \dots, k_j \geq 0}} \frac{k!}{k_1! \cdot \dots \cdot k_j!} C_1^{k_1} \cdot \dots \cdot C_j^{k_j} \\ &\quad \times \text{Cum}_k \left(\underbrace{\Psi^{(p_1)}(f_1), \dots, \Psi^{(p_1)}(f_1)}_{k_1}, \dots, \underbrace{\Psi^{(p_j)}(f_j), \dots, \Psi^{(p_j)}(f_j)}_{k_j} \right). \end{aligned}$$

In order to prove the identity

$$\mu_{k_1, \dots, k_j}^* = \text{Cum}_k \left(\underbrace{\Psi^{(p_1)}(f_1), \dots, \Psi^{(p_1)}(f_1)}_{k_1}, \dots, \underbrace{\Psi^{(p_j)}(f_j), \dots, \Psi^{(p_j)}(f_j)}_{k_j} \right)$$

for all $k_1, \dots, k_j \in \{0, \dots, k\}$ with $\sum_{i=1}^j k_i = k$ we will proceed as in Jolivet [15] and Leonov and Shiryayev [18]. Let $k_1, \dots, k_j \in \{0, \dots, k\}$ with $\sum_{i=1}^j k_i = k$ and set $\Psi_i = \Psi^{(p_i)}(g_i)$, where

$$g_i = \begin{cases} f_1 & \text{for } i \in \{1, \dots, k_1\}, \\ f_2 & \text{for } i \in \{k_1 + 1, \dots, k_1 + k_2\}, \\ \vdots & \\ f_j & \text{for } i \in \{k_1 + \dots + k_{j-1} + 1, \dots, k\}. \end{cases}$$

Then we have

$$\begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_{k_1} \\ \Psi_{k_1+1} \\ \vdots \\ \Psi_k \end{pmatrix} = \begin{pmatrix} \Psi^{(p_1)}(f_1) \\ \vdots \\ \Psi^{(p_1)}(f_1) \\ \Psi^{(p_2)}(f_2) \\ \vdots \\ \Psi^{(p_j)}(f_j) \end{pmatrix}.$$

With $M(\Psi_1, \dots, \Psi_k) = \mathbb{E} \left[\prod_{i=1}^k \Psi_i \right]$ and $I = \{1, \dots, k\}$ we have

$$\begin{aligned} \text{Cum}_k(\Psi_1, \dots, \Psi_k) &= M(\Psi_1, \dots, \Psi_k) - \sum_{j=2}^k \sum_{I_1 \cup \dots \cup I_j = I} \prod_{i=1}^j \text{Cum}_{\#I_i}(\Psi_a : a \in I_i) \\ &= \Sigma_{\text{indec}} + \Sigma_{\text{dec}} - \mathcal{C} \end{aligned}$$

(see Leonov and Shiryaev [18]), where Σ_{dec} is the sum over the decomposable integrals from $M(\Psi_1, \dots, \Psi_k)$,

$$\mu_{k_1, \dots, k_j}^* \equiv \Sigma_{\text{indec}} = M(\Psi_1, \dots, \Psi_k) - \Sigma_{\text{dec}}$$

is the sum over all indecomposable integrals from $M(\Psi_1, \dots, \Psi_k)$, and

$$\mathcal{C} = \sum_{j=2}^k \sum_{I_1 \cup \dots \cup I_j = I} \prod_{i=1}^j \text{Cum}_{\#I_i}(\Psi_a : a \in I_i)$$

denotes the remaining term.

For $j \in \{2, \dots, k\}$ and a fixed decomposition $\{I_1, \dots, I_j\}$ of $I = \{1, \dots, k\}$, a summand $\prod_{i=1}^j \text{Cum}_{\#I_i}(\Psi_a : a \in I_i)$ of \mathcal{C} factorizes with respect to a decomposition $T = \{T_1, T_2\}$ if for each $i \in \{1, \dots, j\}$ we have either $I_i \subseteq T_1$ or $I_i \subseteq T_2$, that is, if the summand can be written as

$$\prod_{i=1}^j \text{Cum}_{\#I_i}(\Psi_a : a \in I_i) = \prod_{\substack{i=1 \\ I_i \subseteq T_1}}^j \text{Cum}_{\#I_i}(\Psi_a : a \in I_i) \cdot \prod_{\substack{i=1 \\ I_i \subseteq T_2}}^j \text{Cum}_{\#I_i}(\Psi_a : a \in I_i).$$

Note that due to $j \geq 2$ each summand $\prod_{i=1}^j \text{Cum}_{\#I_i}(\Psi_a : a \in I_i)$ factorizes with respect to at least one nontrivial decomposition.

Let P_I be the distribution of the vector $(\Psi_1, \dots, \Psi_k)'$ which is determined by the distribution P of the PP Ψ . For all $S \subseteq I$, let P_S be the distribution of the vector $(\Psi_a)_{a \in S}$. Every term in \mathcal{C} that factorizes with respect to a fixed decomposition $T = \{T_1, T_2\}$ of I is completely determined by the marginals P_{T_1} and P_{T_2} . The same is true for every term in Σ_{dec} that is decomposable with respect to T .

Let $T^{(1)} = \{T_1^{(1)}, T_2^{(1)}\}$ be an arbitrary fixed decomposition of I . The sum over the terms of Σ_{dec} that are decomposable with respect to $T^{(1)}$ is denoted by $\Sigma_{\text{dec}}^{(1)}$, and the sum over the terms of \mathcal{C} that factorize with respect to $T^{(1)}$ is denoted by $\mathcal{C}_{\text{dec}}^{(1)}$. Let

$$\Sigma^{(1)} = \Sigma_{\text{dec}} - \Sigma_{\text{dec}}^{(1)}$$

and

$$\mathcal{C}^{(1)} = \mathcal{C} - \mathcal{C}_{\text{dec}}^{(1)}.$$

Then we have

$$\text{Cum}_k(\Psi_1, \dots, \Psi_k) = \Sigma_{\text{indec}} + \Sigma^{(1)} + \Sigma_{\text{dec}}^{(1)} - \mathcal{C}^{(1)} - \mathcal{C}_{\text{dec}}^{(1)}.$$

Now we will show

$$\Sigma_{\text{dec}}^{(1)} = \mathcal{C}_{\text{dec}}^{(1)}. \quad (2)$$

To this end we set $\tilde{P}_I := P_{T_1^{(1)}} \otimes P_{T_2^{(1)}}$, where $P_{T_1^{(1)}}$ and $P_{T_2^{(1)}}$ are the distributions of the random vectors $(\Psi_a : a \in T_1^{(1)})$ and $(\Psi_a : a \in T_2^{(1)})$, respectively. Let $(\tilde{\Psi}_1, \dots, \tilde{\Psi}_k)'$ be a random vector with distribution \tilde{P}_I , and let $\tilde{\Sigma}_{\text{indec}}$, $\tilde{\Sigma}^{(1)}$, $\tilde{\Sigma}_{\text{dec}}^{(1)}$, $\tilde{\mathcal{C}}^{(1)}$, and $\tilde{\mathcal{C}}_{\text{dec}}^{(1)}$ be defined as Σ_{indec} , $\Sigma^{(1)}$, $\Sigma_{\text{dec}}^{(1)}$, $\mathcal{C}^{(1)}$, and $\mathcal{C}_{\text{dec}}^{(1)}$ above, with $(\Psi_1, \dots, \Psi_k)'$ replaced by $(\tilde{\Psi}_1, \dots, \tilde{\Psi}_k)'$. By construction we have $(\tilde{\Psi}_a : a \in T_i^{(1)}) \sim P_{T_i^{(1)}}$, that is, $(\tilde{\Psi}_a : a \in T_i^{(1)}) \stackrel{d}{=} (\Psi_a : a \in T_i^{(1)})$, $i = 1, 2$. Hence the fact that $\Sigma_{\text{dec}}^{(1)}$ and $\mathcal{C}_{\text{dec}}^{(1)}$ are completely determined by the marginals $P_{T_1^{(1)}}$ and $P_{T_2^{(1)}}$ implies $\tilde{\Sigma}_{\text{dec}}^{(1)} = \Sigma_{\text{dec}}^{(1)}$ and $\tilde{\mathcal{C}}_{\text{dec}}^{(1)} = \mathcal{C}_{\text{dec}}^{(1)}$. In particular we have

$$\text{Cum}_k(\tilde{\Psi}_1, \dots, \tilde{\Psi}_k) = \tilde{\Sigma}_{\text{indec}} + \tilde{\Sigma}^{(1)} + \Sigma_{\text{dec}}^{(1)} - \tilde{\mathcal{C}}^{(1)} - \tilde{\mathcal{C}}_{\text{dec}}^{(1)}. \quad (3)$$

Clearly, $(\tilde{\Psi}_a : a \in T_1^{(1)})$ and $(\tilde{\Psi}_a : a \in T_2^{(1)})$ are independent by construction. This implies the left-hand side in (3) to be equal to zero. Since the mixed moment

$$M(\tilde{\Psi}_1, \dots, \tilde{\Psi}_k) = \mathbb{E} \left[\prod_{i=1}^k \tilde{\Psi}_i \right] = \mathbb{E} \left[\prod_{i \in T_1^{(1)}} \tilde{\Psi}_i \right] \cdot \mathbb{E} \left[\prod_{i \in T_2^{(1)}} \tilde{\Psi}_i \right] = \tilde{\Sigma}_{\text{dec}}^{(1)}$$

is decomposable with respect to the decomposition $T^{(1)}$ we also have $\tilde{\Sigma}_{\text{indec}} = 0$ and $\tilde{\Sigma}^{(1)} = 0$.

Finally the independence of $(\tilde{\Psi}_a : a \in T_1^{(1)})$ and $(\tilde{\Psi}_a : a \in T_2^{(1)})$ yields $\text{Cum}_k(\tilde{\Psi}_a : a \in K) = 0$ for all $K \subseteq \{1, \dots, k\}$ with $K \cap T_1^{(1)} \neq \emptyset$ and $K \cap T_2^{(1)} \neq \emptyset$. Since every summand in $\tilde{\mathcal{C}}^{(1)}$ contains a factor of this type we obtain $\tilde{\mathcal{C}}^{(1)} = 0$.

Altogether this proves (2) by equation (3). As a result we have

$$\text{Cum}_k(\Psi_1, \dots, \Psi_k) = \Sigma_{\text{indec}} + \Sigma^{(1)} - \mathcal{C}^{(1)}.$$

Now we go through all possible decompositions of I in this manner. Since every term of Σ_{dec} is decomposable with respect to some decomposition and every term of \mathcal{C} factorizes with respect to some decomposition, this yields

$$\Sigma_{\text{dec}} = \mathcal{C}$$

and hence

$$\text{Cum}_k(\Psi_1, \dots, \Psi_k) = \Sigma_{\text{indec}}.$$

In summary we have

$$\text{Cum}_k \left(\underbrace{\Psi^{(p_1)}(f_1), \dots, \Psi^{(p_1)}(f_1)}_{k_1}, \dots, \underbrace{\Psi^{(p_j)}(f_j), \dots, \Psi^{(p_j)}(f_j)}_{k_j} \right) = \mu_{k_1, \dots, k_j}^*$$

for all $k_1, \dots, k_j \in \{0, \dots, k\}$ with $\sum_{i=1}^j k_i = k$. This completes the proof. \square

The cumulants of the ISE $I_n(K)$ can be represented by a sum of indecomposable and *irreducible* integrals which will be shown in Lemma 5.2. First we give a definition of an irreducible integral. This definition is closely related to the special form of the functions

$$f_1 : (\mathbb{R}^d)^4 \rightarrow \mathbb{R},$$

$$(x_1, x_2, x_3, x_4) \mapsto \mathbb{1}_{W_n}(x_1)\mathbb{1}_{W_n}(x_3)\mathbb{1}_{\{x_1 \neq x_2, x_3 \neq x_4\}} \int_K k\left(\frac{x_2 - x_1 - t}{b_n}\right) k\left(\frac{x_4 - x_3 - t}{b_n}\right) dt,$$

and

$$f_2 : (\mathbb{R}^d)^2 \rightarrow \mathbb{R}, \quad (x_1, x_2) \mapsto \mathbb{1}_{W_n}(x_1)\mathbb{1}_{\{x_1 \neq x_2\}} \int_K k\left(\frac{x_2 - x_1 - t}{b_n}\right) \lambda \varrho(t) dt.$$

An integral $I_{\mathcal{P}, \mathcal{Q}}(\underbrace{f_1, \dots, f_1}_{k-j}, \underbrace{f_2, \dots, f_2}_j)$, $j = 0, \dots, k-1$, with $\mathcal{P} = \{P_1, \dots, P_q\}$ and $\mathcal{Q} = \{Q_1, \dots, Q_r\}$ (see page 11) is *reducible* if there are indices $a, b \in \{1, \dots, q\}$, $c, d \in \{1, \dots, r\}$, and an odd number $i \in \{1, \dots, 4k-4j\}$ such that $P_a = \{i\}$, $P_b = \{i+1\}$, and $Q_c = \{a, b\}$ or $Q_c = \{a\}$ and $Q_d = \{b\}$. In other words, a reducible integral contains one of the terms

$$\begin{aligned} & \int_{(\mathbb{R}^d)^2} f_1(x_i, x_{i+1}, x, y) \gamma^{(2)}(d(x_i, x_{i+1})), \quad \int_{(\mathbb{R}^d)^2} f_1(x, y, x_i, x_{i+1}) \gamma^{(2)}(d(x_i, x_{i+1})), \\ & \int_{(\mathbb{R}^d)^2} f_1(x_i, x_{i+1}, x, y) \gamma^{(1)}(dx_i) \gamma^{(1)}(dx_{i+1}), \quad \text{or} \quad \int_{(\mathbb{R}^d)^2} f_1(x, y, x_i, x_{i+1}) \gamma^{(1)}(dx_i) \gamma^{(1)}(dx_{i+1}), \end{aligned}$$

with $x, y \notin \{x_i, x_{i+1}\}$, and the remaining functions contain neither x_i nor x_{i+1} . We will call this term the *reducible part* of the integral. An integral can have more than one reducible part. An integral that is not reducible is called *irreducible*. For instance, the integral

$$\begin{aligned} & I_{\{\{1\}, \{2\}, \{3,5\}, \{4,6\}\}, \{\{1,2\}, \{3,4\}\}}(f_1, f_2) \\ & = \int_{(\mathbb{R}^d)^2} \int_{(\mathbb{R}^d)^2} f_1(z_1, z_2, z_3, z_4) f_2(z_3, z_4) \gamma^{(2)}(d(z_1, z_2)) \gamma^{(2)}(d(z_3, z_4)) \end{aligned}$$

is reducible with reducible part $\int_{(\mathbb{R}^d)^2} f_1(z_1, z_2, z_3, z_4) \gamma^{(2)}(d(z_1, z_2))$, whereas the integral

$$\begin{aligned} & I_{\{\{1,5\}, \{2\}, \{3,6\}, \{4\}\}, \{\{1,2\}, \{3,4\}\}}(f_1, f_2) \\ & = \int_{(\mathbb{R}^d)^2} \int_{(\mathbb{R}^d)^2} f_1(z_1, z_2, z_3, z_4) f_2(z_1, z_3) \gamma^{(2)}(d(z_1, z_2)) \gamma^{(2)}(d(z_3, z_4)) \end{aligned}$$

is irreducible.

Recall the sum of indecomposable integrals $\mu_{k-j,j}^*$, $j = 0, \dots, k$, see (1). We denote the sum of irreducible integrals in $\mu_{k-j,j}^*$ by $\mu_{k-j,j}^{**}$, $j = 0, \dots, k$. We will write $\mu_{k-(j+r),j+r}^{**a}$, $a = 1, \dots, r$, for the term obtained from $\mu_{k-(j+r),j+r}^{**}$ by replacing a instances of f_2 with \tilde{f}_2 , where the function \tilde{f}_2 is given by

$$\tilde{f}_2(x, y) := \mathbb{1}_{W_n}(x) \int_K \left[k\left(\frac{y-x-t}{b_n}\right) \lambda \left(\int_{\mathbb{R}^d} R_n(z, t) k(z) dz \right) \right] dt$$

with $R_n(z, t) = \sum_{i=1}^d z_i \left(\frac{\partial}{\partial t_i} \varrho(t + \theta_i b_n z) - \frac{\partial}{\partial t_i} \varrho(t) \right)$ and $\theta_i = \theta_i(t) \in [0, 1]$, $i = 1, \dots, d$. As a result, $\mu_{k-(j+r),j+r}^{**a}$ contains only $j+r-a$ instances of f_2 .

Now we can state the lemma giving a representation of the k th cumulant of the ISE in terms of indecomposable and irreducible integrals for $k \geq 2$.

Lemma 5.2. *Let $k \geq 2$, and let $\Psi \sim P$ be a B_{4k} -mixing PP in \mathbb{R}^d with intensity λ and product density ϱ . Let the first-order partial derivatives of the product density ϱ be uniformly Lipschitz-continuous in $K \oplus b(o, \varepsilon)$ for some $\varepsilon > 0$. Let k satisfy Condition $\mathcal{C}(d)$ (iii).*

Then the k th cumulant of the ISE $I_n(K)$ satisfies

$$\Gamma_k(I_n(K)) = \sum_{j=0}^k \binom{k}{j} 2^j b_n^j (b_n^d |W_n|)^{j-2k} \mu_{k-j, j}^{**j}.$$

Proof. In the first part of the proof we apply Lemma 5.1 in order to express the k th cumulant by a sum of indecomposable integrals. Due to the smoothness conditions on the product density this representation can be further simplified. This is shown in the second part of the proof.

Due to the semi-invariance of the cumulants of order two and higher the k th cumulant of $\int_K (\hat{\varrho}_n^2(t) - 2\lambda\varrho(t)\hat{\varrho}_n(t))dt$ is identical to the k th cumulant of $(I_n(K) - \mathbb{E}I_n(K))$ for $k \geq 2$. Therefore we investigate the k th cumulant of $\int_K (\hat{\varrho}_n^2(t) - 2\lambda\varrho(t)\hat{\varrho}_n(t))dt$.

I Representation of the k th cumulant by indecomposable integrals

First we rewrite $\int_K (\hat{\varrho}_n^2(t) - 2\lambda\varrho(t)\hat{\varrho}_n(t))dt$. We have

$$\begin{aligned} & \int_K (\hat{\varrho}_n^2(t) - 2\lambda\varrho(t)\hat{\varrho}_n(t))dt \\ &= \sum_{\substack{x_1, x_2, x_3, x_4 \in \Psi \\ x_1 \neq x_2, x_3 \neq x_4}} (b_n^d |W_n|)^{-2} \mathbb{1}_{W_n}(x_1) \mathbb{1}_{W_n}(x_3) \int_K k\left(\frac{x_2 - x_1 - t}{b_n}\right) k\left(\frac{x_4 - x_3 - t}{b_n}\right) dt \\ & \quad - \sum_{x_1, x_2 \in \Psi}^* 2(b_n^d |W_n|)^{-1} \mathbb{1}_{W_n}(x_1) \int_K k\left(\frac{x_2 - x_1 - t}{b_n}\right) \lambda\varrho(t) dt \\ &= \sum_{x_1, x_2, x_3, x_4 \in \Psi} C_1 f_1(x_1, x_2, x_3, x_4) + \sum_{x_1, x_2 \in \Psi} C_2 f_2(x_1, x_2), \end{aligned}$$

with functions

$$\begin{aligned} f_1 : (\mathbb{R}^d)^4 &\rightarrow \mathbb{R}, \\ (x_1, x_2, x_3, x_4) &\mapsto \mathbb{1}_{W_n}(x_1) \mathbb{1}_{W_n}(x_3) \mathbb{1}_{\{x_1 \neq x_2, x_3 \neq x_4\}} \int_K k\left(\frac{x_2 - x_1 - t}{b_n}\right) k\left(\frac{x_4 - x_3 - t}{b_n}\right) dt, \end{aligned}$$

and

$$f_2 : (\mathbb{R}^d)^2 \rightarrow \mathbb{R}, \quad (x_1, x_2) \mapsto \mathbb{1}_{W_n}(x_1) \mathbb{1}_{\{x_1 \neq x_2\}} \int_K k\left(\frac{x_2 - x_1 - t}{b_n}\right) \lambda\varrho(t) dt,$$

and constants $C_1 := (b_n^d |W_n|)^{-2}$ and $C_2 := -2(b_n^d |W_n|)^{-1}$. Since we have $|K| < \infty$ and since k is bounded with bounded support the moments $\mathbb{E} \left[\left| \sum_{x_1, x_2, x_3, x_4 \in \Psi} f_1(x_1, x_2, x_3, x_4) \right|^k \right]$ and $\mathbb{E} \left[\left| \sum_{x_1, x_2 \in \Psi} f_2(x_1, x_2) \right|^k \right]$ are finite. Hence we can apply Lemma 5.1. Therefore the k th cumulant $\Gamma_k(I_n(K))$ of the ISE $I_n(K)$ satisfies

$$\begin{aligned} \Gamma_k(I_n(K)) &= \sum_{\substack{k_1+k_2=k \\ k_1, k_2 \geq 0}} \frac{k!}{k_1! k_2!} (-1)^{k_2} 2^{k_2} (b_n^d |W_n|)^{-2k_1 - k_2} \mu_{k_1, k_2}^* \\ &= \sum_{j=0}^k \binom{k}{j} (-1)^j 2^j (b_n^d |W_n|)^{j-2k} \mu_{k-j, j}^*. \end{aligned} \quad (4)$$

II Representation of the cumulants by indecomposable and irreducible integrals

The special form of the functions f_1 and f_2 allows a further simplification of the representation for the k th cumulant given in Lemma 5.2. This simplification is based on the approximate identity

$$\begin{aligned} \int_{(\mathbb{R}^d)^2} f_1(x_1, x_2, x_3, x_4) \alpha^{(2)}(d(x_1, x_2)) &= \int_{(\mathbb{R}^d)^2} f_1(x_3, x_4, x_1, x_2) \alpha^{(2)}(d(x_1, x_2)) \\ &\approx b_n^d |W_n| f_2(x_3, x_4) \end{aligned}$$

for $x_3, x_4 \notin \{x_1, x_2\}$, which implies the reducible integrals of $\mu_{k,0}^*$ (except for the error terms) and integrals in $\mu_{k-\ell, \ell}^*$, $\ell = 1, \dots, k$, to cancel.

More precisely we start by combining two reducible integrals in $\mu_{k-j, j}^*$, $j = 0, \dots, k-1$. These integrals differ only by their reducible parts, in two possible ways. Either the two integrals' reducible parts are

$$\int_{(\mathbb{R}^d)^2} f_1(x_i, x_{i+1}, x, y) \gamma^{(2)}(d(x_i, x_{i+1})) \quad \text{and} \quad \int_{(\mathbb{R}^d)^2} f_1(x_i, x_{i+1}, x, y) \gamma^{(1)}(dx_i) \gamma^{(1)}(dx_{i+1})$$

or they are

$$\int_{(\mathbb{R}^d)^2} f_1(x, y, x_i, x_{i+1}) \gamma^{(2)}(d(x_i, x_{i+1})) \quad \text{and} \quad \int_{(\mathbb{R}^d)^2} f_1(x, y, x_i, x_{i+1}) \gamma^{(1)}(dx_i) \gamma^{(1)}(dx_{i+1}).$$

The sum of these two reducible integrals in $\mu_{k-j, j}^*$ is hence an integral which emerges from either of the two aforementioned integrals by replacing the respective reducible parts by

$$\int_{(\mathbb{R}^d)^2} f_1(x_i, x_{i+1}, x, y) \alpha^{(2)}(d(x_i, x_{i+1})) \quad (5)$$

or

$$\int_{(\mathbb{R}^d)^2} f_1(x, y, x_i, x_{i+1}) \alpha^{(2)}(d(x_i, x_{i+1})), \quad (6)$$

depending on the above distinction. If the integral has more than one reducible part, then we iterate the above procedure, eventually obtaining an irreducible integral. In the following, we will

only consider irreducible integrals and integrals which arise from the above-mentioned combination and summation of reducible integrals. The latter integrals are also called reducible parts. Now we simplify one of the reducible parts (5) and (6) of a reducible integral by disintegration and Taylor's expansion, that is,

$$\begin{aligned}
& \int_{(\mathbb{R}^d)^2} \mathbb{1}_{W_n}(x_i) \mathbb{1}_{W_n}(x) \int_K k\left(\frac{x_{i+1} - x_i - t}{b_n}\right) k\left(\frac{y - x - t}{b_n}\right) dt \alpha^{(2)}(d(x_i, x_{i+1})) \\
&= b_n^d |W_n| \mathbb{1}_{W_n}(x) \int_K k\left(\frac{y - x - t}{b_n}\right) \lambda\left(\int_{\mathbb{R}^d} k(x_{i+1}) \varrho(b_n x_{i+1} + t) dx_{i+1}\right) dt \\
&= b_n^d |W_n| f_2(x, y) + b_n b_n^d |W_n| \tilde{f}_2(x, y),
\end{aligned}$$

where

$$\tilde{f}_2(x, y) = \mathbb{1}_{W_n}(x) \int \left[k\left(\frac{y - x - t}{b_n}\right) \lambda\left(\int_{\mathbb{R}^d} R_n(z, t) k(z) dz\right) \right] dt$$

with $R_n(z, t) = \sum_{i=1}^d z_i \left(\frac{\partial}{\partial t_i} \varrho(t + \theta_i b_n z) - \frac{\partial}{\partial t_i} \varrho(t) \right)$ and $\theta_i = \theta_i(t) \in [0, 1]$, $i = 1, \dots, d$. Here we have used the symmetry of the kernel function k so that only $\varrho(t)$ and the error term R_n remain from Taylor's expansion. In the following we will refer to the above simplification by disintegration and Taylor's expansion as *reduction* of the integral. Note that the uniform Lipschitz-continuity of the first-order partial derivatives of the product density yields the upper bound

$$|R_n(z, t)| \leq b_n L \sum_{i=1}^d |z_i|, \tag{7}$$

where L is the maximum of the Lipschitz-constants.

An integral in $\mu_{k-j,j}^*$ is called *r-reducible* if it can be reduced exactly r times (that is, if reduction as defined above can be applied exactly r times), with $r \in \{0, \dots, k-j\}$. Reducing an r -reducible integral r times yields a sum of two parts. The first part is an integral in $\mu_{k-(j+r),j+r}^{**}$ multiplied by $(b_n^d |W_n|)^r$ while the second part is a sum of integrals containing the error terms from all Taylor expansions performed in the reductions. Note that within this iterative scheme reductions can also be applied to error terms obtained from earlier reductions. We illustrate this procedure by an example involving three reductions of a 3-reducible integral in $\mu_{3,0}^*$:

$$\begin{aligned}
& \int_{(\mathbb{R}^d)^8} f_1(x_1, x_2, x_3, x_4) f_1(x_1, x_2, x_5, x_6) f_1(x_1, x_2, x_7, x_8) \\
& \quad \gamma^{(2)}(d(x_1, x_2)) \alpha^{(2)}(d(x_3, x_4)) \alpha^{(2)}(d(x_5, x_6)) \alpha^{(2)}(d(x_7, x_8)) \\
&= b_n^{3d} |W_n|^3 \left(\int_{(\mathbb{R}^d)^2} (f_2(x_1, x_2))^3 \gamma^{(2)}(d(x_1, x_2)) \right. \\
& \quad + 3b_n \int_{(\mathbb{R}^d)^4} (f_2(x_1, x_2))^2 \tilde{f}_2(x_1, x_2) \gamma^{(2)}(d(x_1, x_2)) \\
& \quad \left. + 3b_n^2 \int_{(\mathbb{R}^d)^4} f_2(x_1, x_2) (\tilde{f}_2(x_1, x_2))^2 \gamma^{(2)}(d(x_1, x_2)) \right)
\end{aligned}$$

$$+ b_n^3 \int_{(\mathbb{R}^d)^4} (\tilde{f}_2(x_1, x_2))^3 \gamma^{(2)}(d(x_1, x_2)).$$

In the remaining terms a instances of the function f_2 are replaced by \tilde{f}_2 , $a = 1, \dots, r$. For each integral in $\mu_{k-(j+r), j+r}^{**a}$ the number of r -reducible integrals in $\mu_{k-j, j}^*$ leading to this integral is $2^r \binom{k-j}{r}$. Hence we obtain the representation

$$\mu_{k-j, j}^* = \sum_{r=0}^{k-j} 2^r \binom{k-j}{r} (b_n^d |W_n|)^r \sum_{a=0}^r b_n^a \binom{r}{a} \mu_{k-(j+r), j+r}^{**a} \quad (8)$$

for $j = 0, \dots, k$. The main terms are $\mu_{k-(j+r), j+r}^{**} \equiv \mu_{k-(j+r), j+r}^{**0}$, $r = 0, \dots, k-j$, and the remaining terms are $\mu_{k-(j+r), j+r}^{**a}$, $a = 1, \dots, r$. Equations (4) and (8) imply

$$\begin{aligned} \Gamma_k(I_n(K)) &= \sum_{i=0}^k (-1)^i \binom{k}{i} 2^i (b_n^d |W_n|)^{i-2k} \mu_{k-i, i}^* \\ &= \sum_{i=0}^k \sum_{r=0}^{k-i} \sum_{a=0}^r (-1)^i \frac{k!}{i! a! (r-a)! (k-(i+r))!} 2^{i+r} (b_n^d |W_n|)^{i+r-2k} b_n^a \mu_{k-(i+r), i+r}^{**a} \\ &= \sum_{j=0}^k \sum_{i=0}^j \sum_{a=0}^{j-i} (-1)^i \frac{k!}{i! a! (j-i-a)! (k-j)!} 2^j (b_n^d |W_n|)^{j-2k} b_n^a \mu_{k-j, j}^{**a} \\ &= \sum_{j=0}^k \binom{k}{j} 2^j b_n^j (b_n^d |W_n|)^{j-2k} \mu_{k-j, j}^{**j}. \end{aligned} \quad (9)$$

For the last line we consider the summands indexed by $j \in \{0, \dots, k\}$. This yields

$$\begin{aligned} &\sum_{i=0}^j \sum_{a=0}^{j-i} (-1)^i \frac{k!}{i! a! (j-i-a)! (k-j)!} 2^j (b_n^d |W_n|)^{j-2k} b_n^a \mu_{k-j, j}^{**a} \\ &= \frac{k!}{(k-j)!} 2^j (b_n^d |W_n|)^{j-2k} \sum_{a=0}^j \frac{1}{a!} b_n^a \mu_{k-j, j}^{**a} \sum_{i=0}^{j-a} (-1)^i \frac{1}{i! (j-a-i)!}. \end{aligned}$$

Due to $\sum_{i=0}^{j-a} (-1)^i \frac{1}{i! (j-a-i)!} = 0$ for $a = 0, \dots, j-1$ and $\sum_{i=0}^0 (-1)^i \frac{1}{i! (j-i)!} = 1$ the identity (9) follows and the proof is complete. \square

Now we are ready to prove the asymptotic representation of the variance of the ISE of the product density estimator.

Proof of Lemma 3.2. We use the representation of the second cumulant of the scaled ISE

$$\Gamma_2(b_n^{d/2} |W_n| I_n(K)) = b_n^d |W_n|^2 \Gamma_2(I_n(K)) = \sum_{j=0}^2 \binom{2}{j} 2^j b_n^j (b_n^d)^{j-3} (|W_n|)^{j-2} \mu_{2-j, j}^{**j} \quad (10)$$

derived in Lemma 5.2. Now we will determine the asymptotic order of $b_n^j(b_n^d)^{j-3}(|W_n|)^{j-2}\mu_{2-j,j}^{**j}$, $j = 0, 1, 2$. The highest-order terms in $\mu_{2,0}^{**0}$ are

$$\begin{aligned} & \int_{(\mathbb{R}^d)^4} f_1(x_1, x_2, x_3, x_4) [f_1(x_1, x_2, x_3, x_4) + f_1(x_3, x_4, x_1, x_2) \\ & \quad + f_1(x_2, x_1, x_4, x_3) + f_1(x_4, x_3, x_2, x_1)] \\ & \quad [\gamma^{(2)}(d(x_1, x_2))\gamma^{(2)}(d(x_3, x_4)) + \gamma^{(2)}(d(x_1, x_2))\gamma^{(2)}(d(x_3, x_4)) \\ & \quad + \gamma^{(1)}(dx_1)\gamma^{(1)}(dx_2)\gamma^{(2)}(d(x_3, x_4)) + \gamma^{(1)}(dx_3)\gamma^{(1)}(dx_4)\gamma^{(2)}(d(x_3, x_4)) \\ & \quad + \gamma^{(1)}(dx_1)\gamma^{(1)}(dx_2)\gamma^{(1)}(dx_3)\gamma^{(1)}(dx_4)]. \end{aligned}$$

Combining the factorial cumulant measures to factorial moment measures and multiplying with the scaling factor $(b_n^{3d}|W_n|^2)^{-1}$ we obtain

$$\begin{aligned} & \frac{1}{b_n^{3d}|W_n|^2} \int_{(\mathbb{R}^d)^4} f_1(x_1, x_2, x_3, x_4) [f_1(x_1, x_2, x_3, x_4) + f_1(x_3, x_4, x_1, x_2) \\ & \quad + f_1(x_2, x_1, x_4, x_3) + f_1(x_4, x_3, x_2, x_1)] \alpha^{(2)}(d(x_1, x_2)) \alpha^{(2)}(d(x_3, x_4)) \\ & = 2\lambda^2 \int_{(\mathbb{R}^d)^2} \mathbb{1}_K(b_n t_1 + t_2) \mathbb{1}_K(t_2) \left(\int_{\mathbb{R}^d} k(x) k(x+t_1) \varrho(b_n x + b_n t_1 + t_2) dx \right)^2 dt_1 dt_2 \\ & \quad + 2\lambda^2 \int_{(\mathbb{R}^d)^2} \mathbb{1}_K(b_n t_1 - t_2) \mathbb{1}_K(t_2) \\ & \quad \times \left(\int_{\mathbb{R}^d} \frac{|W_n \cap (W_n - b_n x + b_n t_1 - t_2)|}{|W_n|} k(x) k(x+t_1) \varrho(b_n x + b_n t_1 - t_2) dx \right)^2 dt_1 dt_2 \\ & \xrightarrow{n \rightarrow \infty} 2\lambda^2 \int_{\mathbb{R}^d} (k * k)^2(t) dt \left(\int_K \varrho^2(t) dt + \int_{K \cap (-K)} \varrho^2(t) dt \right). \end{aligned}$$

The remaining part of $\mu_{2,0}^{**0}$, scaled with $(b_n^{3d}|W_n|^2)^{-1}$, is of order $O(b_n^d + (b_n^d|W_n|)^{-1})$ as $n \rightarrow \infty$. For integrals in $\mu_{2,0}^{**0}$ containing an integration with respect to $\gamma^{(5)}$, $\gamma^{(6)}$, $\gamma^{(7)}$ and $\gamma^{(8)}$, this is due to the finiteness of these measures' total variation. For the other integrals one uses the assumptions on the cumulant densities up to order four or the finiteness of the total variations $\|\gamma_{\text{red}}^{(k)}\|$, $k = 2, 3, 4$. For example, if we do not assume the existence of the fourth-order cumulant density, the integral

$$\begin{aligned} & \frac{1}{b_n^{3d}|W_n|^2} \int_{(\mathbb{R}^d)^8} f_1(x_1, x_2, x_3, x_4) f_1(x_5, x_6, x_7, x_8) \gamma^{(4)}(d(x_1, x_2, x_5, x_6)) \gamma^{(4)}(d(x_3, x_4, x_7, x_8)) \\ & = \frac{1}{b_n^{3d}} \int_{(\mathbb{R}^d)^8} \frac{|W_n \cap (W_n - x_5)| |W_n \cap (W_n - x_7)|}{|W_n|^2} \mathbb{1}_K(t_1) \mathbb{1}_K(t_2) \\ & \quad \times k\left(\frac{x_2 - t_1}{b_n}\right) k\left(\frac{x_4 - t_1}{b_n}\right) k\left(\frac{x_6 - x_5 - t_2}{b_n}\right) k\left(\frac{x_8 - x_7 - t_2}{b_n}\right) \\ & \quad dt_1 dt_2 \gamma_{\text{red}}^{(4)}(d(x_2, x_5, x_6)) \gamma_{\text{red}}^{(4)}(d(x_4, x_7, x_8)) \end{aligned}$$

occurring in $(b_n^{3d}|W_n|^2)^{-1}\mu_{2,0}^{**0}$ can only be shown to be of asymptotic order $\mathcal{O}(b_n^{-d})$. Assuming

the fourth-order cumulant density $c^{(4)}$ to exist, the above integral turns into

$$b_n^d \int_{(\mathbb{R}^d)^8} \frac{|W_n \cap (W_n - x_5)| |W_n \cap (W_n - x_7)|}{|W_n|^2} \mathbb{1}_K(t_1) \mathbb{1}_K(t_2) k(x_2) k(x_4) k(x_6) k(x_8) \\ \times c^{(4)}(b_n x_2 + t_1, x_5, b_n x_6 + x_5 + t_2) c^{(4)}(b_n x_4 + t_1, x_7, b_n x_8 + x_7 + t_2) \\ dt_1 dt_2 dx_2 dx_4 dx_5 dx_6 dx_7 dx_8$$

by substitution. Due to $\sup_{u,v \in K \oplus b(o,\varepsilon)} \int_{\mathbb{R}^d} |c^{(4)}(u, w, v + w)| dw < \infty$ for some $\varepsilon > 0$ and since the kernel function is bounded with bounded support this term is of asymptotic order $\mathcal{O}(b_n^d)$. Likewise, the assumption $\sup_{u,v \in K \oplus b(o,\varepsilon)} |c^{(3)}(u, v)| < \infty$ is needed for showing that the integral

$$\frac{1}{b_n^{3d} |W_n|^2} \int_{(\mathbb{R}^d)^6} f_1(x_1, x_2, x_3, x_4) f_1(x_1, x_5, x_3, x_6) \gamma^{(3)}(d(x_1, x_2, x_5)) \gamma^{(3)}(d(x_3, x_4, x_6))$$

is of asymptotic order $\mathcal{O}(b_n^d)$ as $n \rightarrow \infty$.

The term $\mu_{1,1}^{**1}$ is of asymptotic order $\mathcal{O}(b_n^{2d+1} |W_n|)$ and $\mu_{0,2}^{**2}$ is of asymptotic order $\mathcal{O}(b_n^{2d+2} |W_n|)$ which can be shown by using the finiteness of the total variations $\|\gamma_{\text{red}}^{(k)}\|$, $k = 2, \dots, 6$, the upper bound (7) for the error term $|R_n|$ occurring in the function \tilde{f}_2 , and the boundedness conditions on the kernel function. Together with the representation (10) this leads to the asymptotic representation

$$b_n^d |W_n|^2 \Gamma_2(I_n(K)) = 2\lambda^2 \int_{\mathbb{R}^d} (k * k)^2(t) dt \left(\int_K \varrho^2(t) dt + \int_{K \cap (-K)} \varrho^2(t) dt \right) \\ + \mathcal{O}(b_n^{d \wedge 2}) + \mathcal{O}((b_n^d |W_n|)^{-1}) + \mathcal{O}(b_n^{d+4} |W_n|).$$

Now the assumption $b_n^{d+4} |W_n| \rightarrow 0$ implies the assertion. \square

Proof of Theorem 3.3. The asymptotic variance of $b_n^{d/2} |W_n| (I_n(K) - \mathbb{E}I_n(K))$ has already been determined in Lemma 3.2. We will prove asymptotic normality by showing that the k th cumulant of $b_n^{d/2} |W_n| (I_n(K) - \mathbb{E}I_n(K))$ converges to zero for all $k \geq 3$.

In Lemma 5.2 we derived a representation of the k th cumulant of $I_n(K)$ by indecomposable and irreducible integrals. Now we will show that the k th cumulant of $b_n^{d/2} |W_n| (I_n(K) - \mathbb{E}I_n(K))$ is of order $\mathcal{O}((b_n^d)^{k/2-1} + b_n^{4+\frac{k}{2}d} |W_n|)$ as $n \rightarrow \infty$ for $k \geq 2$. This implies the cumulants of order three and higher to converge to zero.

We will use the representation $\Gamma_k(I_n(K))$ derived in Lemma 5.2 and determine the asymptotic order of the terms $\mu_{k-j,j}^{**j}$ for $j = 0, \dots, k$. It is essential that the integrals in $\mu_{k-j,j}^{**j}$ are neither decomposable nor reducible.

Consider an integral $I_{\mathcal{P},\mathcal{Q}}(\cdot)$ in $\mu_{k-j,j}^{**j}$, $j = 0, \dots, k$, see (1). Let V be the set of integration variables occurring in the integral and define the set of *argument pairs*

$$\mathcal{V} := \{ \{v, w\} \subseteq V : \text{the integrand of } I_{\mathcal{P},\mathcal{Q}}(\cdot) \text{ contains} \\ \text{a term } f_1(v, w, \dots), \text{ a term } f_1(\dots, v, w), \text{ or a term } f_2(v, w) \}.$$

Now we define a *linkage* relation on \mathcal{V} . Two argument pairs $\{v, w\}, \{x, y\} \in \mathcal{V}$ are said to be *linked* (notation: $\{v, w\} \sim \{x, y\}$) if at least one of the following conditions is satisfied:

- (i) The argument pairs $\{v, w\}, \{x, y\}$ have a common element, that is, $\{v, w\} \cap \{x, y\} \neq \emptyset$.
- (ii) The integral $I_{\mathcal{P}, \mathcal{Q}}(\cdot)$ involves an integration $\gamma^{(i)}(d(v_1, \dots, v_i))$ for some $i \geq 2$ and some $v_1, \dots, v_i \in V$ such that $\{v, w\} \cap \{v_1, \dots, v_i\} \neq \emptyset$ and $\{x, y\} \cap \{v_1, \dots, v_i\} \neq \emptyset$.
- (iii) The integral $I_{\mathcal{P}, \mathcal{Q}}(\cdot)$ involves an integration $\gamma^{(1)}(dv_0) \gamma^{(i)}(d(v_1, \dots, v_i))$ for some $i \geq 1$ and $v_0, \dots, v_i \in V$ such that $\{v, w\} \cap \{v_0, \dots, v_i\} \neq \emptyset$ and $\{x, y\} \cap \{v_0, \dots, v_i\} \neq \emptyset$.

Note that the relation \smile is reflexive and symmetric.

The maximal asymptotic order of each integration of linked argument pairs with ℓ arguments is $O((b_n^d)^{\lceil \frac{\ell}{2} \rceil} |W_n|)$. After reduction of the factorial cumulant measures we make use of the existence of the cumulant densities. There are at least $\lceil \frac{\ell}{2} \rceil$ kernel functions k . By substitution of the arguments of the kernel functions k we get a factor b_n^d for each function. Furthermore there is exactly one variable occurring only in the indicator functions $\mathbb{1}_{W_n}$ (this is due to the integral's indecomposability and irreducibility). Integration over this variable yields the factor $|W_n|$. Because of the boundedness of the total variations the integrals over the cumulant densities are also bounded. Therefore we obtain the order $\mathcal{O}((b_n^d)^{\lceil \frac{\ell}{2} \rceil} |W_n|)$ for each integration over ℓ linked argument pairs. Note that without the existence of the cumulant densities one can only derive the order $\mathcal{O}(|W_n|)$. For determining the order of the whole integral we also have to take into account that some of the arguments t of the functions $\mathbb{1}_K(t)$ can be substituted, where each substitution produces a factor b_n^d . Thus the highest-order terms are those in which as many argument pairs as possible are not linked.

We will now use the concept of a *cyclic linkage*. Consider a product

$$\prod_{i=1}^m f_1(p_i, q_i)$$

occurring in the integrand of $I_{\mathcal{P}, \mathcal{Q}}(\cdot)$ and involving the argument pairs $p_1, q_1, \dots, p_m, q_m \in \mathcal{V}$. (Here $f_1(p, q)$ with argument pairs $p = \{u, v\}$, $q = \{x, y\}$ is understood as $f_1(u, v, x, y)$.) This product is said to be *cyclically linked* if there are an enumeration r_1, \dots, r_{2m} of $\{p_1, q_1, \dots, p_m, q_m\}$ and a permutation π of $\{1, \dots, m\}$ such that $\{r_{2i-1}, r_{2i}\} = \{p_{\pi(i)}, q_{\pi(i)}\}$ for all $i = 1, \dots, m$ and such that

$$r_{2i} \smile r_{2i+1 \bmod 2m} \quad \text{for all } i \in \{1, \dots, m\}$$

is an exhaustive list of the links between the argument pairs $p_1, q_1, \dots, p_m, q_m$.

We will now investigate the highest-order terms in $\mu_{k-j, j}^{**j}$ for $j = 0, \dots, k$.

Let $j = 0$. Then the integrands of all highest-order integrals in $\mu_{k, 0}^{**0}$ are cyclically linked. As an example consider the integral

$$\int_{(\mathbb{R}^d)^{2k}} \prod_{\substack{a=1 \\ a \text{ odd}}}^{2k-3} f_1(x_a, x_{a+1}, x_{a+2}, x_{a+3}) f_1(x_{2k-1}, x_{2k}, x_1, x_2) \prod_{\substack{a=1 \\ a \text{ odd}}}^{2k-1} \gamma^{(2)}(d(x_a, x_{a+1})).$$

By disintegration and substitution we get

$$\begin{aligned}
& \lambda^k |W_n|^k \int_{(\mathbb{R}^d)^k} \prod_{\substack{a=1 \\ a \text{ odd}}}^{2k-3} \int_K k\left(\frac{x_{a+1} - t_a}{b_n}\right) k\left(\frac{x_{a+3} - t_a}{b_n}\right) dt_a \\
& \quad \times \int_K k\left(\frac{x_{2k} - t_{2k-1}}{b_n}\right) k\left(\frac{x_2 - t_{2k-1}}{b_n}\right) dt_{2k-1} \prod_{\substack{a=1 \\ a \text{ odd}}}^{2k-1} \gamma_{\text{red}}^{(2)}(dx_{a+1}) \\
& = (b_n^d)^k |W_n|^k \lambda^k \int_{(\mathbb{R}^d)^{2k}} \prod_{\substack{a=1 \\ a \text{ odd}}}^{2k-3} \mathbb{1}_K(t_a) k(x_{a+1}) k\left(x_{a+3} + \frac{t_{a+2} - t_a}{b_n}\right) \mathbb{1}_K(t_{2k-1}) k(x_{2k}) \\
& \quad \times k\left(x_2 + \frac{t_1 - t_{2k-1}}{b_n}\right) \prod_{\substack{a=1 \\ a \text{ odd}}}^{2k-1} c^{(2)}(b_n x_{a+1} + t_a) \\
& \quad dx_2 dx_4 \dots dx_{2k} dt_1 dt_3 \dots dt_{2k-1}.
\end{aligned}$$

By substituting $\tilde{t}_{a+2} = \frac{t_{a+2} - t_a}{b_n}$, $a = 3, 5, \dots, 2k-1$, we see that this is equal to

$$\begin{aligned}
& (b_n^d)^{2k-1} |W_n|^k \lambda^k \int_{(\mathbb{R}^d)^{2k}} \prod_{\substack{a=1 \\ a \text{ odd}}}^{2k-3} \mathbb{1}_K(b_n t_{a+2} + t_1) k(x_{a+1}) k(x_{a+3} + t_{a+2} - t_a) \mathbb{1}_K(t_1) k(x_{2k}) \\
& \quad \times k(x_2 - t_{2k-1}) \prod_{\substack{a=3 \\ a \text{ odd}}}^{2k-1} c^{(2)}(b_n x_{a+1} + b_n t_a + t_1) c^{(2)}(b_n x_2 + t_1) \\
& \quad dx_2 dx_4 \dots dx_{2k} dt_1 dt_3 \dots dt_{2k-1}.
\end{aligned}$$

The second-order cumulant density $c^{(2)}$ is continuous since the product density is continuous in $K \oplus b(o, \varepsilon)$ for some $\varepsilon > 0$. Hence the above-mentioned integral is of order $O((b_n^d)^{2k-1} |W_n|^k)$. Analogous arguments apply to the other terms in $\mu_{k,0}^{**}$.

Now let $j = 1$. Then each integrand of a highest-order term in $\mu_{k-1,1}^{**1}$ is a product of two parts: First, a cyclically linked product of $k-1$ instances of f_1 , and second, one instance of the function \tilde{f}_2 whose argument pair is linked to at least one argument pair from the first part. One of these highest-order integrals is

$$\int_{(\mathbb{R}^d)^{2k-2}} \prod_{\substack{a=1 \\ a \text{ odd}}}^{2k-5} f_1(x_a, x_{a+1}, x_{a+2}, x_{a+3}) f_1(x_{2k-3}, x_{2k-2}, x_1, x_2) \tilde{f}_2(x_1, x_2) \prod_{\substack{a=1 \\ a \text{ odd}}}^{2k-3} \alpha^{(2)}(d(x_a, x_{a+1})).$$

By applying disintegration and substitution as above and taking advantage of the upper bound (7) for $|R_n|$ due to the uniform Lipschitz-continuity of the first-order partial derivatives of ϱ in $K \oplus b(o, \varepsilon)$ for some $\varepsilon > 0$, one finds the above-mentioned integral to be of asymptotic order $O(b_n (b_n^d)^{2k-2} |W_n|^{k-1})$. Analogous arguments apply to the remaining integrals.

Next let $j = 2$. Then each integrand of a highest-order term in $\mu_{k-2,2}^{**2}$ is a product of two parts: First, a cyclically linked product of $k-2$ instances of f_1 , and second, a product of two instances of the function \tilde{f}_2 whose argument pairs are both linked to argument pairs from the first part. For example, the integral

$$\int_{(\mathbb{R}^d)^{2k-4}} \prod_{\substack{a=1 \\ a \text{ odd}}}^{2k-7} f_1(x_a, x_{a+1}, x_{a+2}, x_{a+3}) \tilde{f}_2(x_{2k-5}, x_{2k-4}) \tilde{f}_2(x_1, x_2) \prod_{\substack{a=1 \\ a \text{ odd}}}^{2k-5} \alpha^{(2)}(d(x_a, x_{a+1}))$$

is of asymptotic order $\mathcal{O}(b_n^2(b_n^d)^{2k-2}|W_n|^{k-1})$ and hence one of the highest-order terms for the case $j = 2$.

For $j = 3, \dots, k - 1$ one obtains the asymptotic order $\mathcal{O}(b_n^j(b_n^d)^{2k-j}|W_n|^{k-j+1})$ by analogous considerations.

Finally, in the case $j = k$ all integrands of the integrals in $\mu_{0,k}^{**k}$ are products of k instances of the function \tilde{f}_2 . Since these integrals are indecomposable the argument pairs occurring in the integrand can be enumerated as p_1, \dots, p_k such that $p_i \sim p_{i+1}$ for $i = 1, \dots, k - 1$. Hence the term $\mu_{0,k}^{**k}$ is of order $\mathcal{O}(b_n^k(b_n^d)^k|W_n|)$.

Altogether we have

$$\begin{aligned}\mu_{k,0}^{**0} &= \mathcal{O}((b_n^d)^{2k-1}|W_n|^k), \\ \mu_{k-1,1}^{**1} &= \mathcal{O}(b_n(b_n^d)^{2k-2}|W_n|^{k-1}),\end{aligned}$$

and

$$\mu_{k-j,j}^{**j} = \mathcal{O}(b_n^j(b_n^d)^{2k-j}|W_n|^{k-j+1}) \quad \text{for } j = 2, \dots, k.$$

Together with Lemma 5.2 the k th cumulant hence satisfies

$$\Gamma_k(I_n(K)) = \mathcal{O}(b_n^{-d}|W_n|^{-k}) + 2k\mathcal{O}(b_n^{2-d}|W_n|^{-k}) + \sum_{j=2}^k \binom{k}{j} 2^j \mathcal{O}(b_n^{2j}|W_n|^{1-k}).$$

As a result the k th cumulant of $b_n^{d/2}|W_n|(I_n(K) - \mathbb{E}I_n(K))$ is of order $\mathcal{O}((b_n^d)^{k/2-1} + b_n^{4+\frac{k}{2}d}|W_n|)$ for $k \geq 2$. Due to the assumption $b_n^{d+4}|W_n| \xrightarrow{n \rightarrow \infty} 0$ the k th cumulant of $b_n^{d/2}|W_n|(I_n(K) - \mathbb{E}I_n(K))$ converges to zero for every $k \geq 3$. This proves normal convergence. \square

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