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Amplitude Equations for SPDEs with Cubic Nonlinearities

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Abstract

For a quite general class of SPDEs with cubic nonlinearities we derive rigorously amplitude equations describing the essential dynamics using the natural separation of time-scales near a change of stability. Typical examples are the Swift-Hohenberg equation, the Ginzburg-Landau (or Allen-Cahn) equation and some model from surface growth.

We discuss the impact of degenerate noise on the dominant behavior, and see that additive noise has the potential to stabilize the dynamics of the dominant modes. Furthermore, we discuss higher order corrections to the amplitude equation.

1 Introduction

Stochastic partial differential equations (SPDEs) with cubic nonlinearity appear in several applications, for instance the Swift-Hohenberg equation, which was first used as a toy model for the convective instability of fluids in the Rayleigh-Bénard problem (see [3] or [7]). The simplest example is the well know real valued Ginzburg-Landau equation, which depending on the underlying application is also called Allen-Cahn, Chaffee-Infante or nonlinear Heat equation. Moreover, we briefly discuss a model from surface growth proposed by Lai & Das Sarma (cf. [8] and see also [9]).

All equations considered in this article are parabolic nonlinear SPDEs perturbed by additive forcing. Near a change of stability, we can use the natural separation of time-scales, in order to derive simpler equations for the evolution of the dominant pattern. As these equations describe the amplitudes of dominant pattern, they are referred to as amplitude equations. When the order of the noise strength is comparable to the order of the distance from the change of stability, the impact of noise can be seen. See for example **[1]** and the references therein.

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Recently the impact of degenerate noise not acting directly on the dominant pattern was studies for equations of Burgers type formally [10] and later rigorously [2]. Here noise is transported via nonlinear interaction to the dominant modes.

Our current research was initiated by an observation of Axel Hutt and collaborators [4, 5, 6]. Using numerical simulations and a formal argument based on center manifold theory, they showed that noise constant in space leads to a deterministic amplitude equation, which is stabilized by the impact of additive noise. This leads to a significant shift of the first pattern forming instability. The aim of this paper is to make these results rigorous.

Moreover, we want to study higher order corrections to the amplitude equation, in order to see the fluctuations induced by the impact of the noise on the dominant pattern. Related results in this direction are discussed by Roberts & Wei [11], nevertheless their setting is slightly different, and they use averaging techniques that do not lead to explicit error estimates.

The general prototype of equations under consideration is of the type

$$du(t) = \big[\mathcal{A}u(t) + \varepsilon^2\mathcal{L}u(t) + \mathcal{F}(u(t))\big]\,dt + \varepsilon dW(t), \tag{1}$$

where $\mathcal{A}$ is non-positive self-adjoint operator with finite dimensional kernel, $\varepsilon^2\mathcal{L}u$ is a small deterministic perturbation, $\mathcal{F}$ is a cubic nonlinearity, and W is some finite dimensional Gaussian noise with small noise strength $\varepsilon > 0$. Note that the small deterministic part, that reflects the distance from bifurcation, scales with ε. Different scalings are possible, but the one chosen here, is exactly the one where noise and linear instability will interact in an interesting way. For simplicity of presentation, we will work in some Hilbert space $\mathcal{H}$ equipped with scalar product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$. Other norms like the supremum-norm or the L^p-norm would lead to similar results.

Our aim of this paper is to establish rigorously an amplitude equation and their higher order corrections for this quite general class of SPDEs with cubic nonlinearities given by (1). In the examples we will show that additive degenerate noise leads to stabilization of the solutions.

The paper is organized as follows. In the next section, we discuss the formal derivation of our results, while giving the precise assumptions and statements of the main results in Section 3. Section 4 gives bounds on the non-dominant modes, while Section 5 provides averaging results, in order to remove the impact of the higher modes on the dominant ones. In Section 6, we study the approximation via amplitude equations, which is in the final Section 7 extended to higher order corrections.

2 Formal Derivation

Before we proceed to give detailed assumptions, we present a short formal derivation and motivation of the main results. We will denote the kernel of $\mathcal{A}$ by $\mathcal{N} := \ker \mathcal{A}$. These are the dominant modes or the pattern that change stability. By $S = \mathcal{N}^\perp$ we denote the orthogonal complement in $\mathcal{H}$. Furthermore, denote by P_c the orthogonal projection $P_c : \mathcal{H} \to \mathcal{N}$ onto $\mathcal{N}$ and define $P_s := \mathcal{I} - P_c$, where $\mathcal{I}$ is the identity operator on $\mathcal{H}$.

Here we study the behavior of solutions u of (1) on the natural slow time-scale of order ε^{-2}, given by the distance from bifurcation.

So, we consider u on the slow time and split it into the dominant part $a \in \mathcal{N}$ and the orthogonal part $\psi \in S$.

$$u(t) = \varepsilon a(\varepsilon^2 t) + \varepsilon \psi(\varepsilon^2 t) \tag{2}$$

Rescaling a and ψ to the slow time-scale $T = \varepsilon^2 t$, leads to the following system of equations:

$$da = \left[\varepsilon^{-2}\mathcal{A}_c a + \mathcal{L}_c a + \mathcal{L}_c \psi + \mathcal{F}_c(a+\psi)\right] dT + \varepsilon^{-1} d\tilde{W}_c \ , \tag{3}$$

and

$$d\psi = \left[\varepsilon^{-2}\mathcal{A}_s \psi + \mathcal{L}_s a + \mathcal{L}_s \psi + \mathcal{F}_s(a+\psi)\right] dT + \varepsilon^{-1} d\tilde{W}_s \ , \tag{4}$$

where $\tilde{W}(T) := \varepsilon W(\varepsilon^{-2}T)$ is a rescaled version of the driving Wiener process W. For short-hand notation, we use the subscripts c and s for projection onto $\mathcal{N}$ and $\mathcal{S}$, i.e., $\mathcal{A}_c = P_c\mathcal{A}$ and $\mathcal{A}_s = P_s\mathcal{A}$, for short.

Let us suppose that the projections P_c and P_s commute not only with $\mathcal{A}$, but also with $\mathcal{L}$. Moreover suppose that the noise is degenerate and acts only on S. Then the system (3)-(4) takes the form

$$da = \left[\mathcal{L}_c a + \mathcal{F}_c(a+\psi)\right] dT, \tag{5}$$

and

$$d\psi = \left[\varepsilon^{-2}\mathcal{A}_s \psi + \mathcal{L}_s \psi + \mathcal{F}_s(a+\psi)\right] dT + \varepsilon^{-1} d\tilde{W}_s \ . \tag{6}$$

Formally, in first approximation we immediately see that ψ is a fast Ornstein-Uhlenbeck process (OU, for short) given by the linear equation

$$d\psi = \varepsilon^{-2}\mathcal{A}_s \psi dT + \varepsilon^{-1} d\tilde{W}_s \ .$$

The rigorous statement can be found in Lemma 13.

Thus we can eliminate ψ in Equation (5) by explicitly averaging over the fast modes. In order to derive error estimates this procedure will be based on the Itô-Formula (see Lemma 17). Usually, in most applications of averaging, we can only hope for weak convergence in law without any error bound.

2.1 The Impact of Noise

Let us discuss the averaging and the impact of the noise in some more detail here. Consider for simplicity of the argument here instead of ψ some real valued fast OU-process Z given by

$$Z(T) := \alpha\varepsilon^{-1} \int_0^T e^{-\varepsilon^{-2}\lambda(T-\tau)} d\tilde{\beta}(\tau), \tag{7}$$

where $\tilde{\beta}(T) := \varepsilon\beta(\varepsilon^{-2}T)$ denotes a rescaled version of a Brownian motion β on the fast time-scale.

We apply Itô formula to Z and Z^2, in order to obtain

$$Z dT = \frac{\alpha\varepsilon}{\lambda} d\tilde{\beta} - \frac{\varepsilon^2}{\lambda} dZ.$$

and

$$Z^2 dT = \frac{\alpha^2}{2\lambda} dT + \frac{\varepsilon\alpha}{\lambda} Z d\tilde{\beta} - \frac{\varepsilon^2}{2\lambda} dZ^2$$

Thus, on the slow time-scale T we can suppose that in integrals the process Z is small due to averaging, and a square of Z can be replaced by a constant. See Lemma 17 for a rigorous statement. Note that the next order corrections (order ε) are always stochastic integrals and thus martingales.

We see later in Lemma 14 that for the fast OU-processes $Z = \mathcal{O}(\varepsilon^{-\kappa_0})$ for arbitrarily small $\kappa_0 > 0$. Thus we obtain formally that Z is a white noise on the slow time scale:

$$Z(T) = \varepsilon \frac{\alpha}{\lambda} \partial_T \tilde{\beta} + \text{error},$$

where this error is small only in the sense of distributions, for example in $\mathcal{H}^{-1}$.

2.2 Amplitude Equation

One main result of the paper is the following approximation by amplitude equations. Suppose for simplicity that the initial condition is sufficiently small, then we obtain for u

$$u(t) \simeq \varepsilon b(\varepsilon^2 t) + \varepsilon \mathcal{Z}(\varepsilon^2 t) + \mathcal{O}(\varepsilon^{2-}), \tag{8}$$

where $\mathcal{Z}$ is a fast OU-process and b is the solution of the amplitude equation on the slow time-scale

$$\partial_T b = \mathcal{L}_c b + \mathcal{F}_c(b) + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \mathcal{F}_c(b, e_k, e_k) \ . \tag{9}$$

The exact form of the additional linear terms is discussed later. The OU process $\mathcal{Z}$ is noise of order ε, as discussed in Section 2.1 before.

To illustrate this approximation result stated later in Theorem 9, we discuss here (similar to [6] the Swift-Hohenberg equation subject to periodic boundary conditions on $[0, 2\pi]$ forced by spatially constant noise:

$$\partial_t u = -(1 + \partial_x^2)^2 u + \nu \varepsilon^2 u - u^3 + \varepsilon \alpha \partial_t \beta. \tag{10}$$

Rescaling the solution u of (10) to the slow time-scale by $u(t) = \varepsilon v(\varepsilon^2 t)$, our main theorem in this case (cf. Theorem 9) states that v is of the type

$$v \simeq \gamma_1 \sin + \gamma_{-1} \cos + \varepsilon \frac{\alpha}{\sqrt{2\pi}} \partial_T \tilde{\beta} + \mathcal{O}(\varepsilon^{1-}),$$

where γ_1 and γ_{-1} are the solutions of the amplitude equations

$$\partial_T \gamma_i = (\nu - \tfrac{3\alpha^2}{4\pi}) \gamma_i - \tfrac{3}{4} \gamma_i (\gamma_1^2 + \gamma_{-1}^2) \quad \text{for } i = \pm 1.$$

We note that if α is large compared to ν, then $(\nu - \frac{3\alpha^2}{4\pi})$ is negative. In this case the degenerate additive noise stabilizes the dynamics of the dominant modes.

2.3 Higher order Corrections

The second main results studies the higher order correction for the solution of equation (1). As indicated for the fast OU-process in Section 2.1, we obtain additional Martingale terms that lead to additive noise in an equation for the higher order correction of the amplitude, but the strength of the noise depends on the first order approximation. Unfortunately, as we rely on a Martingale representation argument of [2], we are limited in the final argument to one-dimensional dominant spaces, i.e. $\dim \mathcal{N} = 1$. Nevertheless, it is possible to carry over the results to higher dimensional $\mathcal{N}$, if we only ask for weak convergence of the approximation.

If we consider higher order corrections to (8), we obtain additional martingale terms of order ε in (9) from the Itô-formula argument. These terms depend on b and the fast OU-process. Further averaging arguments are necessary.

We now improve the approximation of (1) from (8) by including a higher order term:

$$u(t) \simeq \varepsilon b_1(\varepsilon^2 t) + \varepsilon^2 b_2(\varepsilon^2 t) + \varepsilon \mathcal{Z}(\varepsilon^2 t) + \mathcal{O}(\varepsilon^{3-}), \tag{11}$$

where b_1 is again the solution of the amplitude equation (9). Later we will see that b_2 is the solution of

$$db_2 = [\mathcal{L}_c b_2 + 3\mathcal{F}_c(b_2, b_1, b_1) + \sum_{k=2}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(b_2, e_k, e_k)]dT + d\tilde{M}_{b_1}, \tag{12}$$

where $\tilde{M}_{b_1}(T)$ is a martingale defined by

$$\tilde{M}_{b_1}(T) = \int_0^T \Big(\sum_{k=2}^{N} g_k(b_1) \Big)^{\frac{1}{2}} dB(s) \ , \tag{13}$$

where the integration is against a one-dimensional Brownian motion B arising from a martingale representation argument (cf. Lemma 34). The g_k's are polynomials of degree 4 in b_1 given later in (74).

3 Assumptions and main results

This section summarizes all assumptions necessary for our results. For the linear operator $\mathcal{A}$ in (1) on the Hilbert-space $\mathcal{H}$ we assume the following:

Assumption 1 *(Linear operator $\mathcal{A}$) Suppose $\mathcal{A}$ is a non-positive operator on $\mathcal{H}$ with eigenvalues $0 \le \lambda_1 \le \ldots \le \lambda_k \le \ldots$ and $\lambda_k \ge Ck^m$ for all sufficiently large k, and a complete orthonormal system of eigenvectors $\{e_k\}_{k=1}^{\infty}$ such that $\mathcal{A}e_k = -\lambda_k e_k$. Suppose that $\mathcal{N} := \ker \mathcal{A}$ has finite dimension n with basis $(e_1, \ldots, e_n)$.*

As before, we denote by P_c the orthogonal projection onto $\mathcal{N}$ and by P_s the orthogonal projection onto the orthogonal complement $\mathcal{S} = \mathcal{N}^\perp$.

Definition 2 *(spaces $\mathcal{H}^\alpha$) For $\alpha \in \mathbb{R}$, we define the space $\mathcal{H}^\alpha$ as*

$$\mathcal{H}^\alpha = \Big\{ \sum_{k=1}^{\infty} \gamma_k e_k \ : \ \sum_{k=1}^{\infty} \gamma_k^2 k^{2\alpha} < \infty \Big\} \text{ with norm } \Big\| \sum_{k=1}^{\infty} \gamma_k e_k \Big\|_\alpha^2 = \Big(\sum_{k=1}^{\infty} \gamma_k^2 k^{2\alpha} \Big)^{1/2}.$$

The operator $\mathcal{A}$ given by Assumption 1 generates an analytic semigroup $\{e^{t\mathcal{A}}\}_{t \ge 0}$ defined by

$$e^{\mathcal{A}t} \Big(\sum_{k=1}^{\infty} \gamma_k e_k \Big) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \gamma_k e_k \quad \forall \ t \ge 0,$$

and has the following property for all $t > 0$, $\beta \geq \alpha$, $\lambda_n < c \leq \lambda_{n+1}$ and all $u \in \mathcal{H}^\beta$

$$\left\|e^{t\mathcal{A}} P_s u\right\|_\alpha \leq M t^{-\frac{\alpha-\beta}{m}} e^{-ct} \left\|P_s u\right\|_\beta \ , \tag{14}$$

where M depends only on α, β and c.

Assumption 3 *(Operator $\mathcal{L}$) Let $\mathcal{L} : \mathcal{H}^\alpha \to \mathcal{H}^{\alpha-\beta}$ for some $\beta \in [0, m)$ be a linear continuous mapping that commutes with P_c and P_s.*

For the nonlinearity $\mathcal{F}$ we assume that:

Assumption 4 *(nonlinearity $\mathcal{F}$) Assume that $\mathcal{F} : (\mathcal{H}^\alpha)^3 \to \mathcal{H}^{\alpha-\beta}$ with β as in Assumption 3 is trilinear, symmetric and satisfies the following conditions, for some $C > 0$,*

$$\|\mathcal{F}(u, v, w)\|_{\alpha-\beta} \leq C\|u\|_\alpha \|v\|_\alpha \|w\|_\alpha \quad \forall\ u, v, w \in \mathcal{H}^\alpha, \tag{15}$$

$$\langle \mathcal{F}_c(u), u\rangle \leq 0 \quad \forall\ u \in \mathcal{N}, \tag{16}$$

and

$$\langle \mathcal{F}_c(u, u, w), w\rangle \leq 0 \quad \forall \ \ u,\ w \in \mathcal{N}. \tag{17}$$

We use $\mathcal{F}(u) = \mathcal{F}(u, u, u)$ and $\mathcal{F}_c = P_c\mathcal{F}$ for shorthand notation.

For the noise we suppose:

Assumption 5 *(Wiener process W) Let W be a Wiener process in $\mathcal{H}$ over some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose for $t \geq 0$,*

$$W(t) = \sum_{k=n+1}^{N} \alpha_k \beta_k(t) e_k \quad \textit{for some } N \geq n + 1,$$

where $(\beta_k)_k$ are independent, standard Brownian motions in $\mathbb{R}$ and $(\alpha_k)_k$ are real numbers.

Remark 6 *We take $N < \infty$ in the above assumption for simplicity of presentation. Nevertheless most results are still true for $N = \infty$, if we control the convergence of various infinite series, i.e. for α_k decaying sufficiently fast for $k \to \infty$.*

We define the fast OU processes $\mathcal{Z}$ and $\mathcal{Z}_k(T)$ by

$$\mathcal{Z}_k(T) := \alpha_k \varepsilon^{-1} \int_0^T e^{-\varepsilon^{-2}\lambda_k(T-\tau)} d\tilde{\beta}_k(\tau), \tag{18}$$

for $k \in \{n + 1, \dots, N\}$ and

$$\mathcal{Z}(T) := \sum_{k=n+1}^{N} \mathcal{Z}_k(T) e_k \ , \tag{19}$$

where $\tilde{\beta}_k(T) := \varepsilon\beta_k(\varepsilon^{-2}T)$ is a rescaled version of the Brownian motion.

For our result we rely on a cut off argument. We consider only solutions $u = (a, \psi)$ that are not too large, as given by the next definition.

Definition 7 *(stopping time) For the $\mathcal{N} \times S$-valued stochastic process (a, ψ) defined in (2) we define, for some $T_0 > 0$ and $\kappa \in (0, \frac{1}{12})$, the stopping time τ^* as*

$$\tau^* := T_0 \wedge \inf \left\{ T > 0 : \|a(T)\|_\alpha > \varepsilon^{-\kappa} \text{ or } \|\psi(T)\|_\alpha > \varepsilon^{-\kappa} \right\}. \tag{20}$$

Definition 8 *($\mathcal{O}$-notation) For a real-valued family of processes $\{X_\varepsilon(t)\}_{t\geq 0}$ we say $X_\varepsilon = \mathcal{O}(f_\varepsilon)$, if for every $p \geq 1$ there exists a constant C_p such that*

$$\mathbb{E} \sup_{t\in[0,\tau^*]} |X_\varepsilon(t)|^p \leq C_p f_\varepsilon^p. \tag{21}$$

We use also the analogous notation for time-independent random variables.

The main theorem for the first approximation result is:

Theorem 9 *(Approximation) Under Assumptions 1, 3, 4 and 5 let u be a solution of (1) defined in (2) with the initial conditions $u(0) = \varepsilon a(0) + \varepsilon\psi(0)$ where $a(0) \in \mathcal{N}$ and $\psi(0) \in S$, and b is a solution of (9) with $b(0) = a(0)$.*

Then for all $p > 1$ and $T_0 > 0$ and all $\kappa \in (0, \frac{1}{12})$, there exists $C > 0$ such that for $\|u(0)\|_\alpha \leq \delta_\varepsilon\varepsilon$ for $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$ we have

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \Big\| u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon \mathcal{Q}(\varepsilon^2 t) \Big\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa} \Big) \leq C\varepsilon^p, \tag{22}$$

where

$$\mathcal{Q}(T) = e^{\varepsilon^{-2}T\mathcal{A}_s}\psi(0) + \mathcal{Z}(T), \tag{23}$$

with $\mathcal{Z}(T)$ defined in (19).

The proof will be given in Section 6 later. We see that the first part of $\mathcal{Q}$ in (23) decays exponentially fast on the fast time-scale $\mathcal{O}(\varepsilon^2)$. The second part is an OU-process $\mathcal{Z}$, which is a small noise, as discussed in the formal derivation.

An immediate consequence is the following corollary.

Corollary 10 *Under Assumptions of Theorem 9 and for arbitrary initial condition $u(0)$ we obtain*

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \Big\| u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon \mathcal{Q}(\varepsilon^2 t) \Big\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa} \Big) \leq C\varepsilon^p + \mathbb{P}(\|u(0)\|_\alpha > \delta_0\varepsilon). \tag{24}$$

The proof is straightforward. It is given at the end of Section 6.1 for completeness. For the higher order correction the main result is:

Theorem 11 *(higher order correction) Under Assumptions 1, 3, 4 and 5 with all $\alpha_k = \sigma$ and $n = 1$. Let u be a solution of (1) defined in (2) with the initial condition $u(0) = \varepsilon a(0) + \varepsilon\psi(0)$ where $a(0) \in \mathcal{N}$ and $\psi(0) \in \mathcal{S}$. Let b_1 and b_2 are solutions of (9) and (12), respectively, with $b_1(0) = a(0)$ and $b_2(0) = 0$.*

Then for all $p > 1$, $T_0 > 0$, and $\kappa \in (0, \frac{1}{7})$, there exists $C > 0$ such that for $\|u(0)\|_\alpha \leq \delta_\varepsilon\varepsilon$ for $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$ we have

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \Big\| u(t) - \varepsilon b_1(\varepsilon^2 t) - \varepsilon^2 b_2(\varepsilon^2 t) - \varepsilon \mathcal{Q}(\varepsilon^2 t) \Big\|_\alpha > \varepsilon^{\frac{7}{3}-7\kappa} \Big) \leq C\varepsilon^p, \tag{25}$$

for all $\varepsilon > 0$ sufficiently small.

The proof of this theorem will be given in Section 7 later. Again, with the same proof as the previous corollary, we obtain:

Corollary 12 *Under Assumptions of Theorem 11 and for arbitrary initial condition $u(0)$ we obtain*

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\big\|u(t)-\varepsilon b_1(\varepsilon^2 t)-\varepsilon^2 b_2(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\big\|_\alpha > \varepsilon^{\frac{7}{3}-7\kappa}\Big)$$
$$\leq \mathbb{P}(\|u(0)\|_\alpha > \delta_0\varepsilon) + C\varepsilon^p. \tag{26}$$

4 Bounds for the high modes

In this section, we show that the non-dominant modes ψ are well approximated by a fast OU-process. As $\psi(0)$ is not small, we also have to include an exponentially fast decaying term depending on the initial conditions.

Lemma 13 *Under Assumption 1, 3 and 4, for $\kappa > 0$ from the definition of τ^* and $p \geq 1$, there is a constant $C > 0$ such that,*

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\psi(T)-\mathcal{Q}(T)\Big\|_\alpha^p \leq C\varepsilon^{2p-3p\kappa}, \tag{27}$$

where $\mathcal{Q}(T)$ is defined in (23). I.e., $\psi = \mathcal{Q} + \mathcal{O}(\varepsilon^{2-3\kappa})$.

Proof. The mild solution of (6) is for $T \leq \tau^*$

$$\psi(T) = e^{\varepsilon^{-2}T\mathcal{A}_s}\psi(0) + \int_0^T e^{\varepsilon^{-2}(T-\tau)\mathcal{A}_s}\left[\mathcal{L}_s\psi + \mathcal{F}_s(a+\psi)\right](\tau)\,d\tau + \mathcal{Z}(T).$$

Using triangle inequality

$$\begin{aligned}
\Big\|\psi(T)-\mathcal{Q}(T)\Big\|_\alpha &\leq \Big\|\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}\mathcal{L}_s\psi\,(\tau)\,d\tau\Big\|_\alpha \\
&\quad + \Big\|\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}\mathcal{F}_s(a\,(\tau)+\psi\,(\tau))d\tau\Big\|_\alpha \\
&:= I_1 + I_2\ .
\end{aligned}$$

We now bound these two terms separately. For the first term, we obtain by using (14) for the semigroup

$$\begin{aligned}
I_1 &\leq C\varepsilon^{\frac{2\beta}{m}}\int_0^T e^{-\varepsilon^{-2}c(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\left\|\mathcal{L}_s\psi(\tau)\right\|_{\alpha-\beta}d\tau \\
&\leq C\varepsilon^{\frac{2\beta}{m}}\int_0^T e^{-\varepsilon^{-2}c(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\left\|\psi(\tau)\right\|_\alpha d\tau \\
&\leq C\varepsilon^2\sup_{\tau\in[0,\tau^*]}\left\|\psi(\tau)\right\|_\alpha\int_0^{\varepsilon^{-2}cT} e^{-\eta}\eta^{-\frac{\beta}{m}}d\eta \\
&\leq C\varepsilon^{2-\kappa}
\end{aligned}$$

where we used the definition of τ^*. For the second term, we obtain by using Assumption 4 for $\mathcal{F}$

$$\begin{aligned}
I_2 &\leq C\varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}c(T-\tau)}(T-\tau)^{-\frac{\beta}{m}} \left\|\mathcal{F}_s(a(\tau)+\psi(\tau))\right\|_{\alpha-\beta} d\tau \\
&\leq C\varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}c(T-\tau)}(T-\tau)^{-\frac{\beta}{m}} \left\|a(\tau)+\psi(\tau)\right\|_\alpha^3 d\tau \\
&\leq C\varepsilon^2 \sup_{\tau\in[0,\tau^*]} \left\|a(\tau)+\psi(\tau)\right\|_\alpha^3 \int_0^{\varepsilon^{-2}cT} e^{-\eta}\eta^{-\frac{\beta}{m}} d\eta \\
&\leq C\varepsilon^2 \Big(\sup_{[0,\tau^*]} \|a\|_\alpha^3 + \sup_{[0,\tau^*]} \|\psi\|_\alpha^3 \Big) \\
&\leq C\varepsilon^{2-3\kappa},
\end{aligned}$$

where we used again the definition of τ^*. Combining all results, yields (27). $\square$

Let us now provide bounds on $\mathcal{Z}$ and thus later on ψ. These are also used to show that ψ is not too large, even at time τ^*. The following lemma shows that $\mathcal{Z} = \mathcal{O}(\varepsilon^{-\kappa_0})$ for any $\kappa_0 > 0$.

Lemma 14 *Under Assumption 1 and 5, there is a constant $C > 0$, depending on $p > 1$, α_k, λ_k, $\kappa_0 > 0$ and T_0, such that*

$$\mathbb{E} \sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^p \leq C\varepsilon^{-\kappa_0},$$

and

$$\mathbb{E} \sup_{T\in[0,T_0]} \|\mathcal{Z}(T)\|_\alpha^p \leq C\varepsilon^{-\kappa_0},$$

where $\mathcal{Z}_k(T)$ and $\mathcal{Z}(T)$ are defined in (18) and (19), respectively.

Proof. In order to prove the first part, we define

$$\delta(T) = e^{-\lambda_\varepsilon T} \quad \text{and} \quad \gamma(T) = \int_0^T e^{2\lambda_\varepsilon \tau} d\tau = \frac{1}{2\lambda_\varepsilon}(\delta(T)^{-2} - 1),$$

where $\lambda_\varepsilon = \varepsilon^{-2}\lambda_k$, and

$$Y(T) := \alpha_k \varepsilon^{-1} \delta(T) \cdot \beta(\gamma(T)).$$

Note that $\mathcal{Z}_k(T)$ and $Y(T)$ are Gaussian stochastic process with

$$\mathbb{E}\mathcal{Z}_k(T) = \mathbb{E}Y(T) = 0,$$

and

$$\mathbb{E}\mathcal{Z}_k(T)\mathcal{Z}_k(S) = \mathbb{E}Y(T)Y(S) = \alpha_k^2 \varepsilon^{-2} \delta(T+S)\gamma(S).$$

Thus $\mathcal{Z}_k(T)$ is a version of $Y(T)$, and

$$\begin{aligned}
\mathbb{E} \sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^p &= \mathbb{E} \sup_{T\in[0,T_0]} |Y(T)|^p = \left(\alpha_k\varepsilon^{-1}\right)^p \mathbb{E} \sup_{T\in[0,T_0]} |\delta(T)\cdot\beta(\gamma(T))|^p \\
&\leq \left(\alpha_k\varepsilon^{-1}\right)^p \sum_{i=0}^{n-1} \mathbb{E} \sup_{T\in[T_i,T_{i+1}]} |\delta(T)|^p \, |\beta(\gamma(T))|^p ,
\end{aligned}$$

where $(T_i)_{i=0}^n$ is an equidistant decomposition of $[0,T_0]$. Using Doob's theorem, we obtain

$$\begin{aligned}
\mathbb{E}\sup_{T\in[0,T_0]}|\mathcal{Z}_k(T)|^p &\leq C_{p,\alpha_k}\varepsilon^{-p}\sum_{i=0}^{n-1}\delta(T_i)^p\gamma(T_{i+1})^{\frac{p}{2}} \\
&\leq C_{p,\alpha_k}\varepsilon^{-p}\lambda_\varepsilon^{-p/2}\sum_{i=0}^{n-1}\left[\frac{\delta(T_i)}{\delta(T_{i+1})}\right]^p \\
&= C_{p,\alpha_k}\lambda_k^{-p/2}\sum_{i=0}^{n-1}e^{p\lambda_\varepsilon h} = C_{p,\alpha_k}\lambda_k^{-p/2}\frac{T_0}{h}e^{p\lambda_\varepsilon h},
\end{aligned}$$

where $h = T_{i+1}-T_i$. Taking $h=\frac{1}{\lambda_\varepsilon}$, we obtain

$$\mathbb{E}\sup_{T\in[0,T_0]}|\mathcal{Z}_k(T)|^p \leq C\varepsilon^{-2}. \tag{28}$$

By Hölder inequality we derive for all $p\geq 1$ and sufficiently large $q>\frac{2}{\kappa_0}$

$$\mathbb{E}\sup_{T\in[0,T_0]}|\mathcal{Z}_k(T)|^p \leq \Big(\mathbb{E}\sup_{T\in[0,T_0]}|\mathcal{Z}_k(T)|^{pq}\Big)^q \leq C\varepsilon^{-\kappa_0}.$$

In order to prove the second part,

$$\begin{aligned}
\mathbb{E}\sup_{T\in[0,T_0]}\|\mathcal{Z}(T)\|_\alpha^p &\leq C_p\Big(\mathbb{E}\sup_{T\in[0,T_0]}\sum_{k=n+1}^{N}k^{2\alpha}\mathcal{Z}_k^2(T)\Big)^{p/2} \\
&\leq C_p\Big(\sum_{k=n+1}^{N}k^{2\alpha}\mathbb{E}\sup_{T\in[0,T_0]}\mathcal{Z}_k^2(T)\Big)^{p/2}.
\end{aligned}$$

Using Hölder inequality for all q and (28) to obtain

$$\mathbb{E}\sup_{T\in[0,T_0]}\mathcal{Z}_k^2(T) \leq \Big(\mathbb{E}\sup_{T\in[0,T_0]}\mathcal{Z}_k^{2q}(T)\Big)^{1/q} \leq C\varepsilon^{-2/q}.$$

Hence

$$\mathbb{E}\sup_{T\in[0,T_0]}\|\mathcal{Z}(T)\|_\alpha^p \leq C\varepsilon^{-p/q} \leq C\varepsilon^{-\kappa_0},$$

for q large enough. □

The following corollary states that $\psi(T)$ is with high probability much smaller than $\varepsilon^{-\kappa}$ as asserted by the Definition 7 for $T\leq\tau^*$. To be more precise, $\psi=\mathcal{O}(\delta_\varepsilon+\varepsilon^{-\kappa_0})$ for any $\kappa_0>0$ and $\delta_\varepsilon\in(0,\varepsilon^{-\frac{1}{3}\kappa})$. We will use this later to show that $\tau^*\geq T_0$ with high probability (cf. Remark 24 and proof of Theorem 9).

Corollary 15 *Under the assumptions of Lemmas 13 and 14 with $\kappa<\frac{2}{3}$. For $p>0$ and for $\kappa_0>0$ there exist a constant $C>0$ such that for $\|\psi(0)\|_\alpha\leq\delta_\varepsilon$ one has*

$$\mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\|\psi(T)\|_\alpha^p\Big) \leq C(\delta_\varepsilon^p+\varepsilon^{-\kappa_0}) . \tag{29}$$

Proof. From (27), by triangle inequality and Lemma 14, we obtain

$$\mathbb{E}\Big(\sup_{T\in[0,\tau^*]} \|\psi(T)\|_\alpha^p \Big) \le C\delta_\varepsilon^p + C\varepsilon^{-\kappa_0} + C\varepsilon^{2p-3p\kappa},$$

for $\kappa < \frac{2}{3}$ we obtain (29). $\square$

Lemma 16 *If Assumption 1 holds, then for $q \ge 1$ there exists a constant $C > 0$ such that for $\|\psi(0)\|_\alpha \le \delta_\varepsilon$ one has*

$$\int_0^T \left\| e^{\tau\varepsilon^{-2}\mathcal{A}_s}\psi(0) \right\|_\alpha^q d\tau \le C\delta_\varepsilon^q \varepsilon^2.$$

Proof. Using (14) we obtain

$$\int_0^T \left\| e^{\varepsilon^{-2}\mathcal{A}_s\tau}\psi(0) \right\|_\alpha^q d\tau \le c\int_0^T e^{-q\varepsilon^{-2}c\tau}\, \|\psi(0)\|_\alpha^q \, d\tau \le \frac{\varepsilon^2}{qc}\, \|\psi(0)\|_\alpha^q \, .$$

$\square$

5 Averaging over the fast OU-process

Let us not turn to the averaging result. First in 17, we provide the first order approximation, while in Lemma 18 we state all corrections of order ε.

Lemma 17 *Let X be a real valued stochastic process such that for some $r \ge 0$ we have $X(0) = \mathcal{O}(\varepsilon^{-r})$. Fix any $\kappa_0 > 0$. If $dX = GdT$ with $G = \mathcal{O}(\varepsilon^{-r})$, then, for any non-negative integers n_1 ,n_2 ,n_3 not all zero and for all triples of different indices $k_1,\ k_2,\ k_3 \in \{n+1,\dots,N\}$, we obtain*

$$\begin{aligned}\int_0^T X\mathcal{Z}_{k_1}^{n_1}\mathcal{Z}_{k_2}^{n_2}\mathcal{Z}_{k_3}^{n_3} d\tau \quad &= \quad \sum_{i=1}^3 \frac{n_i(n_i-1)\alpha_{k_i}^2}{2(n_1\lambda_{k_1}+n_2\lambda_{k_2}+n_3\lambda_{k_3})} \int_0^T X\mathcal{Z}_{k_1}^{n_1}\mathcal{Z}_{k_2}^{n_2}\mathcal{Z}_{k_3}^{n_3}\mathcal{Z}_{k_i}^{-2} d\tau \\ &\quad + \mathcal{O}(\varepsilon^{1-r-(n_1+n_2+n_3)\kappa_0}), \end{aligned} \tag{30}$$

where the fast OU-process $\mathcal{Z}_k$ is defined in (18).

Proof. We note first that

$$\mathbb{E}\sup_{[0,T_0]}|X|^p \le C\mathbb{E}\sup_{[0,T_0]}|G|^p \le C\varepsilon^{-pr}.$$

Applying Itô formula to $X\mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3}$ and integrating from 0 to T in order to obtain (note that the β's are independent, thus $d\beta_k d\beta_l = 0$ if $k \neq l$)

$$\begin{aligned}
&(n_1\lambda_k + n_2\lambda_l + n_3\lambda_j)\int_0^T X\mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3}d\tau \\
= \quad & -\varepsilon^2 X(T)\mathcal{Z}_k^{n_1}(T)\mathcal{Z}_l^{n_2}(T)\mathcal{Z}_j^{n_3}(T) + \varepsilon^2\int_0^T \mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3}G d\tau \\
&+n_1\alpha_k\varepsilon\int_0^T X\mathcal{Z}_k^{n_1-1}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3}d\tilde{\beta}_k + n_2\alpha_l\varepsilon\int_0^T X\mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2-1}\mathcal{Z}_j^{n_3}d\tilde{\beta}_l \\
&+n_3\alpha_j\varepsilon\int_0^T X\mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3-1}d\tilde{\beta}_j + \frac{n_1(n_1-1)\alpha_k^2}{2}\int_0^T X\mathcal{Z}_k^{n_1-2}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3}d\tau \\
&+\frac{n_2(n_2-1)\alpha_l^2}{2}\int_0^T X\mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2-2}\mathcal{Z}_j^{n_3}d\tau + \frac{n_3(n_3-1)\alpha_j^2}{2}\int_0^T X\mathcal{Z}_k^{n_1}\mathcal{Z}_l^{n_2}\mathcal{Z}_j^{n_3-2}d\tau.
\end{aligned}$$

Taking the absolute value and using Burkholder-Davis-Gundy theorem yields (30). $\square$

We can also give the higher order correction terms.

Lemma 18 *Under assumption of Lemma 17, we have*

$$\begin{aligned}
\int_0^T X\mathcal{Z}_{k_1}^{n_1}\mathcal{Z}_{k_2}^{n_2}\mathcal{Z}_{k_3}^{n_3}d\tau \quad = \quad & \sum_{i=1}^{3}\frac{n_i(n_i-1)\alpha_{k_i}^2}{2(n_1\lambda_{k_1}+n_2\lambda_{k_2}+n_3\lambda_{k_3})}\int_0^T X\mathcal{Z}_{k_1}^{n_1}\mathcal{Z}_{k_2}^{n_2}\mathcal{Z}_{k_3}^{n_3}\mathcal{Z}_{k_i}^{-2}d\tau \\
&+\varepsilon\sum_{i=1}^{3}\frac{n_i\alpha_{k_i}}{n_1\lambda_{k_1}+n_2\lambda_{k_2}+n_3\lambda_{k_3}}\int_0^T X\mathcal{Z}_{k_1}^{n_1}\mathcal{Z}_{k_2}^{n_2}\mathcal{Z}_{k_3}^{n_3}\mathcal{Z}_{k_i}^{-1}d\tilde{\beta}_{k_i} \\
&+\mathcal{O}(\varepsilon^{2-r-(n_1+n_2+n_3)\kappa_0}).
\end{aligned}$$

Proof. We follow the same proof, as in the previous Lemma. $\square$

Remark 19 *Both Lemmas above are still true, in case X is a stochastic process in $\mathcal{N}$ or $\mathbb{C}$.*

6 First order estimates

This section is devoted to the proof of the first main result of Theorem 9. In the second part of this section we give some applications for this approximation result.

6.1 Proof of the main result

Let us first check, that we can apply the averaging lemma to (5).

Lemma 20 *Assume that Assumption 3 and 4 hold. Let X be a stochastic process in $\mathcal{N}$ and $dX = GdT$. If $X = \mathcal{F}_c(a, e_k, e_l)$ or $X = \mathcal{F}_c(a, a, e_k)$, then $G = \mathcal{O}(\varepsilon^{-3\kappa})$ or $G = \mathcal{O}(\varepsilon^{-4\kappa})$, respectively.*

Proof. If $X = \mathcal{F}_c(a, e_k, e_k)$, then

$$dX = \mathcal{F}_c(da, e_k, e_l) = \mathcal{F}_c(\mathcal{L}_c a + \mathcal{F}_c(a+\psi), e_k, e_l)dT.$$

Let

$$G = \mathcal{F}_c(\mathcal{L}_c a + \mathcal{F}_c(a+\psi), e_k, e_l).$$

Taking the $\mathcal{H}^\alpha$ norm, using Assumption 4 and the fact all $\mathcal{H}^\alpha$-norms are equivalent on $\mathcal{N}$, to obtain

$$\begin{aligned} \|G\|_\alpha &\leq C\|\mathcal{L}_c a + \mathcal{F}_c(a+\psi)\|_\alpha \leq C\|a\|_\alpha + C\|\mathcal{F}_c(a+\psi)\|_{\alpha-\beta} \\ &\leq C\|a\|_\alpha + C\|a+\psi\|_\alpha^3 \leq C\|a\|_\alpha + C\|a\|_\alpha^3 + C\|\psi\|_\alpha^3 . \end{aligned}$$

Using the definition of τ^*, we obtain for $p > 0$

$$\mathbb{E}\sup_{[0,\tau^*]}\|G\|_\alpha^p \leq C\varepsilon^{-3p\kappa}.$$

Analogously, if $X = \mathcal{F}_c(a, a, e_k)$, then

$$dX = 2\mathcal{F}_c(da, a, e_k) = 2\mathcal{F}_c(\mathcal{L}_c a + \mathcal{F}_c(a+\psi), a, e_k)dT.$$

Define

$$G := 2\mathcal{F}_c(\mathcal{L}_c a + \mathcal{F}_c(a+\psi), a, e_k),$$

in order to obtain

$$\mathbb{E}\sup_{[0,\tau^*]}\|G\|_\alpha^p \leq C\varepsilon^{-4p\kappa}.$$

$\square$

Lemma 21 *If Assumptions 1, 3, 4 and 5 hold and $\|\psi(0)\|_\alpha \leq \delta_\varepsilon$ for $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$, for $\kappa \in (0, \frac{1}{2})$ from the definition of τ^*, then*

$$a(T) = a(0) + \int_0^T \mathcal{L}_c a(\tau)d\tau + \int_0^T \mathcal{F}_c(a)d\tau + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k}\int_0^T \mathcal{F}_c(a, e_k, e_k)d\tau + R(T), \tag{31}$$

where

$$R = \mathcal{O}(\varepsilon^{1-5\kappa}). \tag{32}$$

Proof. Recall Lemma 13, which states

$$\psi = y_\varepsilon + \mathcal{Z} + \mathcal{O}(\varepsilon^{2-3\kappa}), \tag{33}$$

where

$$y_\varepsilon(T) = e^{\varepsilon^{-2}T\mathcal{A}_s}\psi(0).$$

Substituting from (33) into (5) we obtain for $\kappa < 2/3$ using the bounds for $a = \mathcal{O}(\varepsilon^{-\kappa})$, $\mathcal{Z} = \mathcal{O}(\varepsilon^{-\kappa_0})$, and $y_\varepsilon = \mathcal{O}(\delta_\varepsilon \varepsilon^2)$

$$\begin{aligned}
da &= [\mathcal{L}_c a + \mathcal{F}_c(a + y_\varepsilon + \mathcal{Z})]\,dT + \mathcal{O}(\varepsilon^{2-5\kappa})dT \\
&= [\mathcal{L}_c a + \mathcal{F}_c(a) + 3\mathcal{F}_c(a,a,\mathcal{Z}) + 3\mathcal{F}_c(a,\mathcal{Z},\mathcal{Z}) + \mathcal{F}_c(\mathcal{Z}) \\
&\quad +3\mathcal{F}_c(a,a,y_\varepsilon) + 6\mathcal{F}_c(a,\mathcal{Z},y_\varepsilon) + 3\mathcal{F}_c(\mathcal{Z},\mathcal{Z},y_\varepsilon) \\
&\quad +3\mathcal{F}_c(a,y_\varepsilon,y_\varepsilon) + 3\mathcal{F}_c(\mathcal{Z},y_\varepsilon,y_\varepsilon) + \mathcal{F}_c(y_\varepsilon)]dT + \mathcal{O}(\varepsilon^{2-5\kappa})dT.
\end{aligned}$$

Integrating from 0 to T yields for $T \le \tau^*$

$$\begin{aligned}
a(T) &= a(0) + \int_0^T \mathcal{L}_c a(\tau)d\tau + \int_0^T \mathcal{F}_c(a)d\tau + 3\sum_{k=n+1}^{N}\int_0^T \mathcal{Z}_k \mathcal{F}_c(a,a,e_k)d\tau \\
&\quad +3\sum_{k=n+1}^{N}\int_0^T \mathcal{Z}_k^2 \mathcal{F}_c(a,e_k,e_k)d\tau + 3\sum_{k=n+1}^{N}\sum_{l\neq k}^{N}\int_0^T \mathcal{Z}_k\mathcal{Z}_l \mathcal{F}_c(a,e_k,e_l)d\tau \\
&\quad + \sum_{k,l,j=n+1}^{N}\int_0^T \mathcal{F}_c(\mathcal{Z}_k e_k, \mathcal{Z}_l e_l, \mathcal{Z}_j e_j)d\tau + R_1 + \mathcal{O}(\varepsilon^{2-5\kappa}),
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
R_1 &= 3\int_0^T \mathcal{F}_c(a,a,y_\varepsilon)d\tau + 6\int_0^T \mathcal{F}_c(a,\mathcal{Z},y_\varepsilon)d\tau + 3\int_0^T \mathcal{F}_c(a,y_\varepsilon,y_\varepsilon)d\tau \\
&\quad +3\int_0^T \mathcal{F}_c(\mathcal{Z},y_\varepsilon,y_\varepsilon)d\tau + 3\int_0^T \mathcal{F}_c(\mathcal{Z},\mathcal{Z},y_\varepsilon)d\tau + 3\int_0^T \mathcal{F}_c(y_\varepsilon)d\tau \\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{35}$$

Now we use Assumption 3, the definition of τ^*, and the equivalence of $\mathcal{H}^\alpha$-norms on $\mathcal{N}$ to bound R_1. We bound all terms in (35) separately. For the first term in (35)

$$\|I_1\|_\alpha \le C\int_0^T \|a\|_\alpha^2 \|y_\varepsilon\|_\alpha \,d\tau \le C \sup_{[0,T_0]} \|a\|_\alpha^2 \int_0^T \|y_\varepsilon\|_\alpha \,d\tau.$$

Using Lemma 16 for $q = 1$, we obtain

$$I_1 = \mathcal{O}(\delta_\varepsilon \varepsilon^{2-2\kappa}).$$

Analogous results hold for all other terms. To be more precise:

$$\begin{gathered}
I_2 = \mathcal{O}(\delta_\varepsilon \varepsilon^{2-\kappa-\kappa_0}),\ I_3 = \mathcal{O}(\delta_\varepsilon^2 \varepsilon^{2-\kappa}),\ I_4 = \mathcal{O}(\delta_\varepsilon^2 \varepsilon^{2-\kappa_0}), \\
I_5 = \mathcal{O}(\delta_\varepsilon \varepsilon^{2-2\kappa_0}),\ I_6 = \mathcal{O}(\delta_\varepsilon^3 \varepsilon^2)\ .
\end{gathered}$$

Collecting all results we obtain for $\kappa_0 \le \kappa$, where $\kappa_0 > 0$ is arbitrary from Lemma 14,

$$R_1 = \mathcal{O}((1+\delta_\varepsilon^2)\varepsilon^{2-2\kappa}). \tag{36}$$

Finally, applying Lemmas 17 and 20 to (34), we obtain (31). $\square$

Lemma 22 *Let Assumptions 1, 3 and 4 hold. Define b in $\mathcal{N}$ as the solution of (9). If the initial condition satisfies $\mathbb{E}\,|b(0)|^p \leq \delta_\varepsilon^p$ for $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$, then for all $T_0 > 0$ there exists a constant $C > 0$ such that*

$$\sup_{T\in[0,T_0]} \|b(T)\| \leq C|b(0)|$$

$$\mathbb{E} \sup_{T\in[0,T_0]} |b(T)|^p \leq C^p \delta_\varepsilon^p. \tag{37}$$

Proof. Taking the scalar product $\langle \cdot, b\rangle$ on both sides of (9) yields

$$\frac{1}{2}\partial_T |b|^2 = \langle \mathcal{L}_c b, b\rangle + \langle \mathcal{F}_c(b), b\rangle + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \langle \mathcal{F}_c(b, e_k, e_k), b\rangle\,.$$

Using Cauchy-Schwarz inequality and Assumption 4, we obtain

$$\frac{1}{2}\partial_T |b|^2 \leq C\,|b|^2\,.$$

We apply now a comparison argument to deduce for all $T \in [0, T_0]$

$$|b(T)| \leq |b(0)|\, e^{CT_0}. \tag{38}$$

Taking expectation after supremum on both sides yields (37). □

In the following we are no longer able to calculate moments of error terms. Thus we restrict ourselves to a sufficiently large subset of Ω, where our estimates go through.

Definition 23 *Given $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$ with $\kappa > 0$ from the definition of τ^*. Define the set $\Omega^* \subset \Omega$ of all $\omega \in \Omega$ such that all these estimates*

$$\sup_{[0,\tau^*]} \|\psi - \mathcal{Q}\|_\alpha < C\varepsilon^{2-4\kappa}\ , \tag{39}$$

$$\sup_{[0,\tau^*]} \|\psi\|_\alpha < \delta_0 + \varepsilon^{-\frac{1}{2}\kappa}\ , \tag{40}$$

$$\sup_{[0,\tau^*]} |R| < \varepsilon^{1-6\kappa}\ , \tag{41}$$

and

$$\sup_{[0,\tau^*]} |b| < \delta_0 \varepsilon^{-\frac{1}{2}\kappa}\ , \tag{42}$$

hold.

Remark 24 *The set Ω^* has approximately probability 1. For this consider*

$$\begin{aligned} \mathbb{P}(\Omega^*) \quad &\geq \quad 1 - \mathbb{P}(\sup_{[0,\tau^*]} \|\psi - \mathcal{Q}\|_\alpha \geq \varepsilon^{2-4\kappa}) - \mathbb{P}(\sup_{[0,\tau^*]} \|\psi\|_\alpha \geq \delta_\varepsilon + \varepsilon^{-\frac{1}{2}\kappa}) \\ &\qquad -\mathbb{P}(\sup_{[0,\tau^*]} |b| \geq \delta_\varepsilon \varepsilon^{-\frac{1}{2}\kappa}) - \mathbb{P}(\sup_{[0,\tau^*]} |R| \geq \varepsilon^{1-6\kappa}). \end{aligned}$$

Using Chebychev inequality and Lemmas 13, 21, 22 and Corollary 15 with $\delta_\varepsilon < \varepsilon^{-\frac{1}{3}\kappa}$, and some $\kappa_0 \leq \frac{1}{3}\kappa$, we obtain for sufficient large q

$$\mathbb{P}(\Omega^*) \geq 1 - C[\varepsilon^{q\kappa} + \varepsilon^{\frac{1}{2}q\kappa - q\kappa_0} + \varepsilon^{\frac{1}{2}q\kappa} + \varepsilon^{q\kappa}] \geq 1 - C\varepsilon^{\frac{1}{6}q\kappa} \geq 1 - C\varepsilon^p. \tag{43}$$

Theorem 25 *Assume that Assumptions 1, 3, 4 and 5 hold and suppose $|a(0)| \leq \delta_\varepsilon$ and $\|\psi(0)\|_\alpha \leq \delta_\varepsilon$. Let b be a solution of the amplitude equation (9) and a as defined in (2). If the initial conditions satisfy $a(0) = b(0)$, then*

$$\sup_{T\in[0,\tau^*]} |a(T) - b(T)| \leq C(1+\delta_\varepsilon^2)\varepsilon^{1-12\kappa}, \tag{44}$$

and for $\kappa < \frac{1}{12}$

$$\sup_{T\in[0,\tau^*]} |a(T)| \leq C(1+\delta_\varepsilon^2), \tag{45}$$

on Ω^*.

Proof. Define $\varphi := a - R$, where R is defined in (32). From (31) we obtain

$$\varphi(T) = a(0) + \int_0^T \mathcal{L}_c[\varphi+R]d\tau + \int_0^T \mathcal{F}_c(\varphi+R)d\tau + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \int_0^T \mathcal{F}_c(\varphi+R, e_k, e_k)d\tau. \tag{46}$$

Subtracting (46) from the amplitude equation (9) and defining $h := b - \varphi$, we obtain

$$\begin{aligned} h(T) &= \int_0^T \mathcal{L}_c h d\tau - \int_0^T \mathcal{L}_c R d\tau + \int_0^T [\mathcal{F}_c(b) - \mathcal{F}_c(b-h+R)]\, d\tau \\ &\quad + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \int_0^T \mathcal{F}_c(h-R, e_k, e_k)d\tau. \end{aligned}$$

Thus

$$\partial_T h = \mathcal{L}_c h - \mathcal{L}_c R + \mathcal{F}_c(b) - \mathcal{F}_c(b-h+R) + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \mathcal{F}_c(h-R, e_k, e_k). \tag{47}$$

Taking the scalar product $\langle \cdot, h\rangle$ on both sides of (47), we have

$$\begin{aligned} \frac{1}{2}\partial_T |h|^2 &= \langle \partial_T h, h\rangle = \langle \mathcal{L}_c h, h\rangle - \langle \mathcal{L}_c R, h\rangle + \langle \mathcal{F}_c(b) - \mathcal{F}_c(b-h+R), h\rangle \\ &\quad + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \langle \mathcal{F}_c(h, e_k, e_k), h\rangle - \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \langle \mathcal{F}_c(R, e_k, e_k), h\rangle. \end{aligned}$$

Using Cauchy-Schwarz inequality and Assumption 4, we obtain the following differential inequality

$$\partial_T |h|^2 \leq C[|h|^2 + |h|^4] + C\left[|R|^4 + |b|^2|R|^2 + |b|^4|R|^2 + |b|^2|R|^4\right].$$

Using (41) and (42) in the definition of Ω^*, we obtain for $T \leq \tau^*$

$$\partial_T |h|^2 \leq C[|h|^2 + |h|^4] + C(1+\delta_\varepsilon^4)\varepsilon^{2-24\kappa} \quad \text{on} \quad \Omega^*.$$

As long as $|h| \leq 1$, we obtain

$$\partial_T |h|^2 \leq 2C[|h|^2 + C(1+\delta_\varepsilon^4)\varepsilon^{2-24\kappa}.$$

Using Gronwall's Lemma, we obtain for $T \leq \tau^* \leq T_0$

$$|h(T)|^2 \leq C(1+\delta_\varepsilon^4)\varepsilon^{2-24\kappa} \leq 1,$$

for $\delta_\varepsilon < \varepsilon^{-\frac{1}{3}\kappa}$ with $\kappa < \frac{1}{12}$ and $\varepsilon > 0$ sufficiently small. Thus

$$\sup_{[0,\tau^*]} |h| \leq C(1+\delta_0^2)\varepsilon^{1-12\kappa} \quad \text{on} \quad \Omega^*. \tag{48}$$

We finish the first part by using (41), (48) and

$$\sup_{[0,\tau^*]} |a-b| = \sup_{[0,\tau^*]} |h-R| \leq \sup_{[0,\tau^*]} |h| + \sup_{[0,\tau^*]} |R|\,.$$

For the second part of the theorem consider

$$\sup_{[0,\tau^*]} |a| \leq \sup_{[0,\tau^*]} |a-b| + \sup_{[0,\tau^*]} |b|.$$

Using the first part and (42), we obtain (45) as $\kappa < \frac{1}{12}$. $\square$

Now, we can use the results previously obtained to prove the main result of Theorem 9 for the approximation of the solution (8) of the SPDE (1).

Proof of Theorem 9. For the stopping time, we note that provided $\delta_\varepsilon < \varepsilon^{-\frac{1}{3}\kappa}$

$$\Omega \supset \{\tau^* = T_0\} \supseteq \{\sup_{T\in[0,T_0]} |a(T)| < \varepsilon^{-\kappa}, \sup_{T\in[0,T_0]} \|\psi(T)\|_\alpha < \varepsilon^{-\kappa}\} \supseteq \Omega^*,$$

where the last inclusion holds due to (40) and Theorem 25. Now let us turn to the approximation result. Using (2) and the triangle inequality yields on Ω^*

$$\begin{aligned}\sup_{T\in[0,\tau^*]} \left\|u(\varepsilon^{-2}T) - \varepsilon b(T) - \varepsilon\mathcal{Q}(T)\right\|_\alpha &\leq \varepsilon \sup_{[0,\tau^*]} \|a-b\|_\alpha + \varepsilon \sup_{[0,\tau^*]} \left\|\psi - \mathcal{Q}\right\|_\alpha \\ &\leq C(1+\delta_\varepsilon^2)\varepsilon^{2-12\kappa} + C\varepsilon^{3-4\kappa}\,.\end{aligned}$$

From (39) and (44), we obtain

$$\begin{aligned}\sup_{t\in[0,\varepsilon^{-2}T_0]} \left\|u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\right\|_\alpha &= \sup_{t\in[0,\varepsilon^{-2}\tau^*]} \left\|u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\right\|_\alpha \\ &\leq C\varepsilon^{2-\frac{38}{3}\kappa} \qquad \text{on} \quad \Omega^*.\end{aligned}$$

Thus

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \left\|u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\right\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big) \leq 1 - \mathbb{P}(\Omega^*).$$

Using (43), yields (22). $\square$

Proof of Corollary 10. Define $\Omega_0 \subset \Omega$ as

$$\Omega_0 = \{\omega \in \Omega \;:\; \|u(0)\|_\alpha \leq \delta_0\varepsilon\},$$

and define

$$\hat{u}(0) = \begin{cases} 0 & \text{on} \;\; \Omega_0^c \\ u(0) & \text{on} \;\; \Omega_0. \end{cases}$$

Hence, the solutions u and $\hat{u}$ corresponding to the initial conditions $u(0)$ and $\hat{u}(0)$ coincide on Ω_0. Thus

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\|u(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big)\\
&= \mathbb{P}\Big(\big\{\sup_{t\in[0,\varepsilon^{-2}T_0]}\|\hat{u}(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\big\}\cap\Omega_0\Big)\\
&\qquad +\mathbb{P}\Big(\big\{\sup_{t\in[0,\varepsilon^{-2}T_0]}\|u(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\big\}\cap\Omega_0^c\Big)\\
&\le \mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\|\hat{u}(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big)+\mathbb{P}\Big(\Omega_0^c\Big)\\
&\le C\varepsilon^p+\mathbb{P}\left(\|u(0)\|_\alpha>\delta_\varepsilon\varepsilon\right),
\end{aligned}$$

where we used (22) for the solution $\hat{u}$. □

6.2 Applications

In the literature there are numerous examples of equations with cubic nonlinearities where our theory does apply. Examples are Swift-Hohenberg equation, Ginzburg-Landau / Allen-Cahn equation and some Surface growth model. In all these examples we obtain that adding noise stabilizes the dynamics of the dominant modes and the amplitude equation is always the following type

$$\partial_T A = \nu A - C_\alpha A - C_{\mathcal{F}} A|A|^2,$$

where A is the amplitude of the dominant modes in $\mathcal{N}$. The constant C_α depends explicitly on the noise strength, while $C_{\mathcal{F}}$ depends only on the nonlinearity and the linear operators in the equation.

6.2.1 Swift-Hohenberg equation

The Swift-Hohenberg equation was already defined in the introduction (cf. (10)). It has been used as a toy model for the convective instability in Rayleigh-Bénard problem (see [3] or [7]). Now it is one of the celebrated models in the theory of pattern formation [3]. For this model note that

$$\mathcal{A}=-(1+\partial_x^2)^2,\quad \mathcal{L}=\nu\mathcal{I},\ \mathcal{F}(u)=-u^3.$$

If we take the orthonormal basis

$$e_k(x)=\begin{cases}\frac{1}{\sqrt{\pi}}\sin(kx) & \text{if}\quad k>0,\\ \frac{1}{\sqrt{2\pi}} & \text{if}\quad k=0,\\ \frac{1}{\sqrt{\pi}}\cos(kx) & \text{if}\quad k<0,\end{cases}$$

and the spaces

$$\mathcal{H}=L^2([0,2\pi])\quad\text{and}\quad \mathcal{N}=\text{span}\{\sin,\ \cos\},$$

then the eigenvalues of $-\mathcal{A}=(1+\partial_x^2)^2$ are $\lambda_k=(1-k^2)^2$ for $k\in\mathbb{Z}$. So, it is easy to check that, after rearranging the indices, Assumption 1 is true with $m=4$.

If we consider $u = u_1 \sin + u_{-1} \cos$ and $w = w_1 \sin + w_{-1} \cos \in \mathcal{N}$, then we can easily verify Assumption 4 as follows:

$$\langle \mathcal{F}_c(u), u\rangle = -\frac{3\pi}{4}(u_1^2 + u_{-1}^2)^2 \leq 0,$$

where we used

$$\mathcal{F}_c(u) = -\tfrac{3}{4}(u_1^3 + u_1 u_{-1}^2) \sin - \tfrac{3}{4}(u_{-1}^3 + u_1^2 u_{-1}) \cos,$$

Moreover,

$$\langle \mathcal{F}_c(u, u, w), w\rangle = -\frac{3\pi}{4}\left(u_1^2 w_1^2 + w_1^2 u_{-1}^2 + w_{-1}^2 u_{-1}^2 + w_{-1}^2 u_1^2\right) \leq 0.$$

and with $\alpha = 1$ and $\beta = 0$ it holds that

$$\|\mathcal{F}(u, v, w)\|_{\mathcal{H}^1} = \|-uvw\|_{\mathcal{H}^1} \leq C\|u\|_{\mathcal{H}^1} \|v\|_{\mathcal{H}^1} \|w\|_{\mathcal{H}^1}.$$

For Assumption 5 we consider several cases:
First case. The noise is a constant in the space (i.e. $W(t) = \frac{\alpha_0}{\sqrt{2\pi}}\beta_0(t)$).
Our main theorem states that the rescaled solution of (10)

$$u(t, x) = \varepsilon v(\varepsilon^2 t, x),$$

is of the type

$$v(T, x) \simeq \gamma_1(T)\sin(x) + \gamma_{-1}(T)\cos(x) + \varepsilon\frac{\alpha_0}{\sqrt{2\pi}}\partial_T\tilde{\beta}_0(T) + \mathcal{O}(\varepsilon^{1-}),$$

where γ_1 and γ_{-1} are the solutions of the following two-dimensional amplitude equations:

$$\partial_T\gamma_i = (\nu - \tfrac{3\alpha_0^2}{4\pi})\gamma_i - \tfrac{3}{4}\gamma_i(\gamma_1^2 + \gamma_{-1}^2) \quad \text{for } i = \pm 1\ .$$

Second case. If the noise acts only on $\sin(kx)$ (or $\cos(kx)$) for one single $k \in \{2, 3, \dots, N\}$, then the amplitude equations for (10) are

$$\partial_T\gamma_i = (\nu - \tfrac{3\alpha_k^2}{2\pi(k^2-1)^2})\gamma_i - \tfrac{3}{4}\gamma_i(\gamma_1^2 + \gamma_{-1}^2) \quad \text{for } i = \pm 1.$$

Third case. If the noise takes the form $W(t) = \sum_{k=2}^{N}\frac{\alpha_k}{\sqrt{\pi}}\beta_k(t)\sin(kx)$, then the amplitude equations for (10) are

$$\partial_T\gamma_i = (\nu - \sum_{k=2}^{N}\tfrac{3\alpha_k^2}{2\pi(k^2-1)^2})\gamma_i - \tfrac{3}{4}\gamma_i(\gamma_1^2 + \gamma_{-1}^2) \quad \text{for } i = \pm 1,$$

and our main theorem states that the rescaled solution of (10)

$$u(t, x) = \varepsilon v(\varepsilon^2 t, x),$$

is of the type

$$v(T, x) \simeq \gamma_1(T)\sin(x) + \gamma_{-1}(T)\cos(x) + \varepsilon\sum_{k=2}^{N}\frac{\alpha_k}{\sqrt{\pi}}\partial_T\tilde{\beta}_k(T)\sin(kx) + \mathcal{O}(\varepsilon^{1-}).$$

In the following examples we consider the noise takes the form $W(t) = \sum_{k=2}^{N}\sigma_k\beta_k(t)e_k$ where $\sigma_k = \delta\alpha_k$ for $k \in \{2, 3, \dots, N\}$ and δ will defined later.

6.2.2 Ginzburg-Landau / Allen-Cahn equation

The second example is the Ginzburg-Landau or Allen-Cahn equation

$$\partial_t u = (\partial_x^2 + 1)u + \nu\varepsilon^2 u - u^3 + \varepsilon\partial_t W(t), \tag{49}$$

subject to Dirichlet boundary conditions on the interval $[0, \pi]$. We note that

$$\mathcal{A} = \partial_x^2 + 1, \quad \mathcal{L} = \nu\mathcal{I}, \ \mathcal{F}(u) = -u^3.$$

If we take

$$\mathcal{H} = L^2([0,\pi]), \quad e_k(x) = \sqrt{\tfrac{2}{\pi}}\sin(kx) = \delta\sin(kx) \quad \text{and} \ \ \mathcal{N} = \text{span}\{\sin\},$$

then the Assumption 1 is true, where the eigenvalues of $-\mathcal{A} = -\partial_x^2 - 1$ are $\lambda_k = k^2 - 1$ with $m = 2$ and $\lim_{k\to\infty}\lambda_k = \infty$. The condition (15) is satisfied for $\alpha = 1$ and $\beta = 0$. Furthermore, for $u = \gamma_1 \sin$ and $w = \gamma_2 \sin \in \mathcal{N}$ the condition (16) is satisfied as follows:

$$\langle \mathcal{F}_c(u), u\rangle = -\frac{3\pi}{8}\gamma_1^4 \le 0,$$

where

$$\mathcal{F}_c(u) = -\frac{3}{4}\gamma_1^3 \sin,$$

and

$$\langle \mathcal{F}_c(u,u,w), w\rangle = -\frac{3\pi}{8}\gamma_1^2\gamma_2^2 \le 0,$$

For Assumption 5, we consider two cases:
First case. The noise acting only on $\sin(2x)$.

In this case the amplitude equation (Landau equation) of (49) takes the form

$$\partial_T\gamma = \Big(\nu - \frac{\sigma^2}{4}\Big)\gamma - \frac{3}{4}\gamma^3. \tag{50}$$

Second case. The noise acting on $\sin(2x),\ \sin(3x),\ \ldots,\ \sin(Nx)$.

In this case the amplitude equation of (49) takes the form

$$\partial_T\gamma = \Big(\nu - \frac{3}{4}\sum_{k=2}^{N}\frac{\sigma_k^2}{k^2-1}\Big)\gamma - \frac{3}{4}\gamma^3, \tag{51}$$

If we assume that $\sigma_2 = \sigma_3 = \ldots = \sigma_N = \sigma$, then this takes the form

$$\partial_T\gamma = \Big(\nu - \frac{9\sigma^2}{16} + \frac{3\sigma^2(2N+1)}{8N(N+1)}\Big)\gamma - \frac{3}{4}\gamma^3,$$

where $\mathcal{F}_c(u, e_k, e_k) = -\frac{1}{\pi}u$.

The main theorem states that the rescaled solution of (49)

$$u(t) = \varepsilon v(\varepsilon^2 t),$$

takes the form

$$v(T) \simeq \gamma(T)\sin + \varepsilon\sum_{k=2}^{N}\frac{\sigma_k}{k^2-1}\partial_T\tilde{\beta}_k(T)\sin(kx) + \mathcal{O}(\varepsilon^{1-}),$$

where γ is the solution of the amplitude equation (51).

6.2.3 Surface growth model

Another example arising in the theory of surface growth is

$$\partial_t u = -\triangle^2 u - \mu\triangle u + \nabla\cdot(|\nabla u|^2\nabla u) + \varepsilon\partial_t W(t). \tag{52}$$

subject to periodic boundary conditions for simplicity only on the interval $[0, 2\pi]$. In order to get close to the change of stability, we consider $\mu = 1 + \varepsilon^2\nu$. Hence,

$$\mathcal{A} = -\triangle^2 - \triangle,\quad \mathcal{L} = -\nu\triangle \quad\text{and}\quad \mathcal{F}(u) = \nabla\cdot(|\nabla u|^2\nabla u).$$

Consider

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\sin(kx) & \text{if} \quad k > 0,\\ \frac{1}{\sqrt{2\pi}} & \text{if} \quad k = 0,\\ \frac{1}{\sqrt{\pi}}\cos(kx) & \text{if} \quad k < 0, \end{cases}$$

and

$$\mathcal{H} = L^2([0, 2\pi]) \quad\text{and}\quad \mathcal{N} = \operatorname{span}\{1,\ \sin,\ \cos\}.$$

The eigenvalues of $-\mathcal{A}$ are $\lambda_k = k^4 - k^2$ with $m = 4$. So, Assumption 1 is true. Moreover, if $u = \gamma_0 + \gamma_1 \sin + \gamma_{-1}\cos \in \mathcal{N}$, then all conditions of Assumption 4 are satisfied as follows

$$\langle \mathcal{F}_c(u), u\rangle = -\tfrac{3\pi}{4}\left(\gamma_1^2 + \gamma_{-1}^2\right)^2 \leq 0\ ,$$

where

$$\mathcal{F}_c(u) = -\tfrac{3}{4}\left(\gamma_1^3 + \gamma_{-1}^2\gamma_1\right)\sin - \tfrac{3}{4}\left(\gamma_{-1}^3 + \gamma_1^2\gamma_{-1}\right)\cos,$$

and for $\alpha = \beta = 2$ we obtain

$$\|\mathcal{F}(u)\|_{L^2} = \|\partial_x(\partial_x u)^3\|_{L^2} \leq C\|(\partial_x u)^3\|_{\mathcal{H}^1} \leq C\|\partial_x u\|^3_{\mathcal{H}^1} \leq C\|u\|^3_{\mathcal{H}^2}.$$

For Assumption 5, we consider two cases:

First case. Noise acting only on $\sin(2x)$.

In this case the amplitude equation for (52) is a system of ordinary differential equations:

$$\partial_T\gamma_0 = 0,$$
$$\partial_T\gamma_i = \left(\nu - \tfrac{\sigma^2}{4}\right)\gamma_i - \tfrac{3}{4}\gamma_i(\gamma_1^2 + \gamma_{-1}^2) \quad \text{for } i = \pm 1.$$

Second case. Noise acting on $\sin(2x), \sin(3x), \ldots, \sin(Nx)$.

In this case the amplitude equation for (52) is a system of ordinary differential equations:

$$\partial_T\gamma_0 = 0,$$
$$\partial_T\gamma_i = \Big(\nu - \tfrac{3}{4}\sum_{k=2}^{N}\frac{\sigma_k^2}{k^2-1}\Big)\gamma_i - \tfrac{3}{4}\gamma_i\left(\gamma_1^2 + \gamma_{-1}^2\right) \quad \text{for } i = \pm 1.$$

If we assume that $\sigma_2 = \sigma_3 = \ldots = \sigma_N = \sigma$, then the amplitude equation for (52) in this case takes the form

$$\partial_T\gamma_0 = 0,$$
$$\partial_T\gamma_i = \Big(\nu - \tfrac{9\sigma^2}{16} + \tfrac{3\sigma^2(2N+1)}{8N(N+1)}\Big)\gamma_i - \tfrac{3}{4}\gamma_i\left(\gamma_1^2 + \gamma_{-1}^2\right) \quad \text{for } i = \pm 1,$$

where $\mathcal{F}_c(\gamma_0 + \gamma_1\sin + \gamma_{-1}\cos, e_k, e_k) = -\frac{k^2}{2}\frac{1}{\pi}(\gamma_1\sin + \gamma_{-1}\cos)$.

7 Higher order correction

This section is devoted to the improvement of the approximation of (1) from (8) to (11) by adding a higher order order term. In order to get an equation for the higher order terms, we need to approximate martingale term in the equation for a in order to have explicit error bounds. We rely on Lemma 6.1 from [2], which is based on the martingale representation theorem. Thus we are limited in the final argument to $\dim \mathcal{N} = 1$. In the end of this section we give applications to the stochastic Swift-Hohenberg equation and Ginzburg-Landau equation.

7.1 Proof of the main result

For simplicity in this section we assume that $\alpha_k = \sigma$ for all $k \in \mathbb{N}$ in Assumption 5. This means the noise takes the form

$$W(t) = \sum_{k=n+1}^{N} \sigma \beta_k(t) e_k \quad \text{for } N \geq n+1. \tag{53}$$

This assumption is only for simplicity of presentation. The proofs can easily be modified to the general case.

In order to take higher order corrections into account in next definition we modify the stopping time as follows.

Definition 26 *For the $\mathcal{N} \times S$-valued stochastic process (a, ψ) defined in (2) we split a into $a = a_1 + \varepsilon a_2$ with a_1 a solution of the amplitude equation (9) subject to initial condition $a_1(0) = a(0)$.*

For some $T_0 > 0$ and $\kappa \in (0, \frac{1}{7})$ we define the stopping time $\tau^\sharp$ as

$$\tau^\sharp = T_0 \wedge \inf\left\{T > 0 : \|a_1(T)\|_\alpha > 2\varepsilon^{-\kappa} \text{ or } \|a_2(T)\|_\alpha > \varepsilon^{-\kappa} \text{ or } \|\psi(T)\|_\alpha > \varepsilon^{-\kappa}\right\}. \tag{54}$$

First let us state bounds on stochastic integrals over fast OU-processes. Unfortunately, we can not prove explicit averaging results using Itô's formula like in Lemma 17.

Lemma 27 *Let X as in Lemma 17, then*

$$\int_0^T X \mathcal{Z}_k d\tilde{\beta}_l = \mathcal{O}(\varepsilon^{-r}), \tag{55}$$

and

$$\int_0^T \mathcal{Z}_k \mathcal{Z}_l d\tilde{\beta}_j = \mathcal{O}(1). \tag{56}$$

Proof. In order to prove (55) we rely on Burkholder-Davis-Gundy inequality

$$\mathbb{E} \sup_{T\in[0,T_0]} \Big| \int_0^T X \mathcal{Z}_k d\tilde{\beta}_l \Big|^p \leq C_p \mathbb{E}\Big(\int_0^{T_0} |X|^2 \mathcal{Z}_k^2 d\tau \Big)^{\frac{p}{2}}.$$

Using Lemma 17 for some $\kappa_0 < \frac{1}{2}$ and Hölder inequality, yields

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,T_0]}\Big|\int_0^T X\mathcal{Z}_k d\tilde{\beta}_l\Big|^p &\leq C_p\mathbb{E}\Big(\frac{\alpha_k^2}{2}\int_0^{T_0}|X|^2 d\tau + \mathcal{O}(\varepsilon^{1-2r-2\kappa_0})\Big)^{\frac{p}{2}} \\ &\leq C\varepsilon^{-pr}\Big(1+\varepsilon^{\frac{p}{2}(1-2\kappa_0)}\Big) \\ &\leq C\varepsilon^{-pr}.\end{aligned}$$

In order to prove (56) we again use Burkholder-Davis-Gundy inequality to obtain

$$\mathbb{E}\sup_{T\in[0,T_0]}\left|\int_0^T \mathcal{Z}_k\mathcal{Z}_l d\tilde{\beta}_j\right|^p \leq C_p\mathbb{E}\Big(\int_0^{T_0}\mathcal{Z}_k^2\mathcal{Z}_l^2 d\tau\Big)^{\frac{p}{2}}. \tag{57}$$

Using Lemma 17, yields (56). □

First we prove a technical lemma on ordinary differential equations.

Lemma 28 *Let X and R_δ be continuous functions from $[0,\tau]$ to $\mathcal{N}$ with $X(0) = R_\delta(0)$. If X is a solution of*

$$X(T) = \int_0^T Q_a(X)ds + \int_0^T Q_b(X)ds + R_\delta,$$

where Q_a and Q_b are linear and bounded operators on $\mathcal{N}$ such that

$$|Q_a(X)| \leq C_a|X|,\ |Q_b(X)| \leq C_b|X|, \tag{58}$$

and

$$\langle Q_b(X), X\rangle \leq 0, \tag{59}$$

then

$$\sup_{[0,\tau]}|X|^2 \leq [2 + C_0(C_a^2 + C_b^2)]\sup_{[0,\tau]}|R_\delta|^2, \tag{60}$$

where $C_0 = \frac{1}{C_a+1}e^{2[C_a+1]T_0}$.

We note that later in the application of this lemma the constant C_b might grow with ε while C_a is independent of ε. Therefore condition (59) is important in order to have no C_b in the exponent.

Proof. Define $Y = X - R_\delta$, hence

$$\partial_T Y = Q_a(Y) + Q_a(R_\delta) + Q_b(Y) + Q_b(R_\delta).$$

Taking the scalar product $\langle\cdot, Y\rangle$ on both sides, we obtain

$$\frac{1}{2}\partial_T|Y|^2 = \langle Q_a(Y), Y\rangle + \langle Q_b(Y), Y\rangle + \langle Q_a(R_\delta), Y\rangle + \langle Q_b(R_\delta), Y\rangle.$$

Using Cauchy-Schwarz and Young inequalities and (59), yields

$$\partial_T|Y|^2 \leq 2[C_a + 1]|Y|^2 + [C_a^2 + C_b^2]|R_\delta|^2.$$

Applying Gronwall's lemma, yields for all $T \le \tau$

$$\begin{aligned}|Y(T)|^2 &\le [C_a^2 + C_b^2] \int_0^T |R_\delta|^2 \, e^{2[C_a+1](T-s)} ds \\ &\le C_0 [C_a^2 + C_b^2] \sup_{[0,\tau]} |R_\delta|^2 \,. \end{aligned} \tag{61}$$

To prove (60) we use

$$|X|^2 = |Y + R_\delta|^2 \le 2 |Y|^2 + 2 |R_\delta|^2 \,,$$

and (61). □

Let us recall Lemma 21 and look closer at the terms of order ε.

Lemma 29 *Under Assumptions 1, 3, 4 and 5 with all $\alpha_k = \sigma$ for $k \in \{n+1, \dots, N\}$, we obtain*

$$\begin{aligned} a(T) &= a(0) + \int_0^T \mathcal{L}_c a(\tau) d\tau + \int_0^T \mathcal{F}_c(a) d\tau + \sum_{k=n+1}^N \frac{3\sigma^2}{2\lambda_k} \int_0^T \mathcal{F}_c(a, e_k, e_k) d\tau \\ &\quad + \varepsilon M_a(T) + \tilde{R}(T), \end{aligned} \tag{62}$$

where $M_a(T)$ is a martingale and it is defined by

$$M_a(T) = \int_0^T \sum_{k=n+1}^N \ominus_k(a) d\tilde{\beta}_k(s), \tag{63}$$

where all sums are from $n+1$ to N, if it is not explicitly stated otherwise

$$\begin{aligned} \ominus_k(a) &= \frac{3\sigma}{\lambda_k} \mathcal{F}_c(a, a, e_k) + \sum_{l=n+1}^N \frac{6\sigma \mathcal{F}_c(a, e_k, e_l)}{\lambda_k + \lambda_l} \mathcal{Z}_l + \sum_{l=n+1}^N \frac{3\sigma^3 \mathcal{F}_c(e_k, e_l, e_l)}{\lambda_k(\lambda_k + 2\lambda_l)} \\ &\quad + \sum_{l \neq k} \frac{6\sigma \mathcal{F}_c(e_k, e_k, e_l)}{\lambda_l + 2\lambda_k} \mathcal{Z}_k \mathcal{Z}_l + \sum_{l=n+1}^N \sum_{j=n+1}^N \frac{3\sigma \mathcal{F}_c(e_k, e_l, e_j)}{\lambda_k + \lambda_l + \lambda_j} \mathcal{Z}_l \mathcal{Z}_j \end{aligned} \tag{64}$$

and

$$\tilde{R} = R_1 + \mathcal{O}(\varepsilon^{2-5\kappa}),$$

where $R_1 = \mathcal{O}(\delta_\varepsilon^2 \varepsilon^{2-2\kappa})$ is defined in (36).

Proof. In order to obtain (62) we use (34) and use Lemmas 18 and 27. □

Lemma 30 *Under Assumptions 1, 3, 4 and 5 with all $\alpha_k = \sigma$, consider some stochastic process $\xi = \mathcal{O}(\varepsilon^{-r})$ for $r \ge 0$. Then for all $p > 0$ there exists $C > 0$ such that*

$$\mathbb{E}\Big(\sup_{T \in [0,\tau^\sharp]} |M_\xi(T)|^p \Big) \le C \varepsilon^{-2pr}, \tag{65}$$

where M_ξ is defined in (63). If ξ is bounded up to time T_0, then (65) holds with T_0 instead of $\tau^\sharp$.

Proof. To prove (65) we take $|\cdot|^p$ and expectation after supremum on both sides of (64) and use Assumptions 4, Lemma 27 and Burkholder-Davis-Gundy inequality. □

Lemma 31 *Under Assumptions 1, 3, 4 and 5 with all $\alpha_k = \sigma$. If we define a as $a = a_1 + \varepsilon a_2$ such that a_1 is a solution of the amplitude equation*

$$da_1 = [\mathcal{L}_c a_1 + \mathcal{F}_c(a_1) + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(a_1, e_k, e_k)]dT, \tag{66}$$

then a_2 is a solution of

$$da_2 = [\mathcal{L}_c a_2 + 3\mathcal{F}_c(a_1, a_1, a_2) + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(a_2, e_k, e_k)]dT + dM_{a_1} + dR_2, \tag{67}$$

where

$$\begin{aligned} R_2 \quad = \quad & \varepsilon^{-1}\tilde{R} + 3\varepsilon \int_0^T \mathcal{F}_c(a_1, a_2, a_2)d\tau + \varepsilon^2 \int_0^T \mathcal{F}_c(a_2)d\tau \\ & + \varepsilon \sum_{k=n+1}^{N} \frac{6\sigma}{\lambda_k} \int_0^T \mathcal{F}_c(a_1, a_2, e_k)d\tilde{\beta}_k + \varepsilon^2 \sum_{k=n+1}^{N} \frac{3\sigma}{\lambda_k} \int_0^T \mathcal{F}_c(a_2, a_2, e_k)d\tilde{\beta}_k \\ & + \varepsilon \sum_{k=n+1}^{N} \sum_{l=1}^{N} \frac{6\sigma}{\lambda_k + \lambda_l} \int_0^T \mathcal{F}_c(a_2, e_k, e_l)\mathcal{Z}_l d\tilde{\beta}_k, \end{aligned} \tag{68}$$

with

$$R_2 = \mathcal{O}(\varepsilon^{1-5\kappa}). \tag{69}$$

Proof. The equation for a_2 is a straightforward calculation using (62) and (66). To bound R_2, we take $\|\cdot\|_\alpha^p$ on both sides of (68) and use Assumption 4, Lemma 27, Burkholder-Davis-Gundy inequality and the definition of $\tau^\sharp$ (cf. (54)). □

Lemma 32 *Under assumptions of Lemma 31. Let a_1 be a solution of (66) with initial condition $a_1(0)$ such that $|a_1(0)| \le \delta_\varepsilon$. Define ζ in $\mathcal{N}$ as the solution of*

$$d\zeta = [\mathcal{L}_c \zeta + 3\mathcal{F}_c(a_1, a_1, \zeta) + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(\zeta, e_k, e_k)]dT + dM_{a_1}(T) \tag{70}$$

with $\zeta(0) = 0$.

If $|a_1(0)| \le \delta_\varepsilon$ for some $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$, then for all $T_0 > 0$ and $p > 0$ there exist a constant $C > 0$ such that

$$\sup_{T\in[0,T_0]} |a_1(T)|^p \le C\delta_\varepsilon^p, \tag{71}$$

and

$$\sup_{T\in[0,T_0]} |\zeta(T)| \le C(1 + \delta_\varepsilon) \sup_{T\in[0,T_0]} |M_{a_1}(T)|. \tag{72}$$

Proof. The bound on a_1 follows directly from Lemma 22.

To bound ζ we define

$$Q_a(\zeta) = \mathcal{L}_c\zeta + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(\zeta, e_k, e_k) \quad \text{and} \quad Q_b(\zeta) = 3\mathcal{F}_c(a_1, a_1, \zeta),$$

and we obtain from Lemma 28

$$\sup_{T\in[0,T_0]} |\zeta(T)|^2 \le [2 + C(1+\delta_\varepsilon^2)] \sup_{T\in[0,T_0]} |M_{a_1}(T)|^2.$$

Taking the square root on both sides, yields (72). □

Remark 33 *Note that, from now on, we consider dim($\mathcal{N}$) = 1 and identify $\mathcal{N}$ with $\mathbb{R}$ using the natural isomorphism $\gamma \cdot e_1 \mapsto \gamma$. Thus for example $\mathcal{F}_c$ is defined as $\langle \mathcal{F}, e_1\rangle$ and $\mathcal{F}_c^2$ is $\langle \mathcal{F}, e_1\rangle^2$. Moreover it is easy to check that the quadratic variation of M_{a_1} as a real valued process $\langle M_{a_1}, e_1\rangle$ is given by $\sum_{k=2}^N \int_0^T \ominus_k^2(a_1)d\tau$.*

Before we prove the main result let us deduce the approximation g_k of the quadratic variation function $\ominus_k^2$.

Taking the square for both sides of (64) and using Lemma 17, we obtain for some small $\kappa_0 > 0$

$$\int_0^T \ominus_k^2(a_1)d\tau = \int_0^T g_k(a_1)d\tau + \mathcal{O}((1+\delta_\varepsilon^2)\varepsilon^{1-4\kappa_0}), \tag{73}$$

where

$$\begin{aligned} g_k(b_1) \quad &= \quad \frac{9\sigma^2}{\lambda_k^2}[\mathcal{F}_c(b_1,b_1,e_k)]^2 + \theta_1^{(k)}[\mathcal{F}_c(b_1,b_1,e_k)] \\ &\quad + \sum_{l=2}^N \frac{18\sigma^4}{\lambda_l(\lambda_k+\lambda_l)^2}[\mathcal{F}_c(b_1,e_k,e_l)]^2 + \theta_2^{(k)}, \end{aligned} \tag{74}$$

with constants

$$\theta_1^{(k)} = \sum_{l=2}^N \frac{9\sigma^4 \mathcal{F}_c(e_k,e_l,e_l)}{\lambda_k^2\lambda_l} \ ,$$

and

$$\begin{aligned} \theta_2^{(k)} \quad = \quad & \frac{11\sigma^6\mathcal{F}_c^2(e_k)}{4\lambda_k^4} + \sum_{l\neq k}^N \frac{9\sigma^6(3\lambda_k^2+4\lambda_l\lambda_k+4\lambda_l^2)\mathcal{F}_c^2(e_k,e_l,e_l)}{4\lambda_k^2\lambda_l(\lambda_k+2\lambda_l)^2} \\ & + \sum_{l\neq k}^N \frac{9\sigma^6\mathcal{F}_c^2(e_k,e_k,e_l)}{\lambda_k\lambda_l(\lambda_l+2\lambda_k)^2} + \sum_{l\neq k}^N \sum_{j\notin\{l,k\}}^N \frac{9\sigma^6\mathcal{F}_c^2(e_k,e_l,e_j)}{2\lambda_l\lambda_j(\lambda_k+\lambda_l+\lambda_j)^2} \\ & + \sum_{l\neq k}^N \frac{\sigma^6(6\lambda_k^2+18\lambda_l+3\lambda_k)\mathcal{F}_c(e_k,e_k,e_l)\mathcal{F}_c(e_k)}{2\lambda_l\lambda_k^3(\lambda_k+2\lambda_l)} \\ & + \sum_{l\neq k}^N \sum_{j\notin\{l,k\}}^N \frac{9\sigma^6(4\lambda_l\lambda_j+\lambda_k^2+\lambda_l\lambda_k)\mathcal{F}_c(e_k,e_l,e_l)\mathcal{F}_c(e_k,e_j,e_j)}{4\lambda_k^2\lambda_l\lambda_j(\lambda_k+2\lambda_l)(\lambda_k+2\lambda_j)} \ . \end{aligned}$$

Let us state without proof Lemma 6.1 from [2] to bound $M_{a_1}(T) - \tilde{M}_{a_1}(T)$ where the martingale $M_{a_1}(T)$ is defined in (63) and the martingale $\tilde{M}_{a_1}(T)$ is defined in (13).

Lemma 34 *Let $M_{a_1}(T)$ be a continuous martingale with respect to some filtration $(\mathcal{F}_T)_{T\geq 0}$. Denote the quadratic variation of M_{a_1} by f and let g be an arbitrary $\mathcal{F}_T$-adapted increasing process with $g(0) = 0$. Then, there exists a filtration $\tilde{\mathcal{F}}_T$ with $\mathcal{F}_T \subset \tilde{\mathcal{F}}_T$ and a continuous $\tilde{\mathcal{F}}_T$-martingale $\tilde{M}_{a_1}(T)$ with quadratic variation g such that, for every $r_0 < \frac{1}{2}$ there exists a constant C with*

$$
\begin{aligned}
\mathbb{E} \sup_{T\in[0,T_0]} \Big| M_{a_1}(T) - \tilde{M}_{a_1}(T) \Big|^p \quad &\leq \quad C(\mathbb{E}g(T_0)^{2p})^{1/4} \Big(\mathbb{E} \sup_{T\in[0,T_0]} |f(T) - g(T)|^p \Big)^{r_0} \\
&\quad + \mathbb{E} \sup_{T\in[0,T_0]} |f(T) - g(T)|^{p/2}.
\end{aligned}
$$

Remark 35 *Using the martingale representation theorem, there exists a Brownian motion B with respect to the filtration $\tilde{\mathcal{F}}_T$ such that $\tilde{M}_{a_1}(T)$ is given as the stochastic integral in (13).*

Lemma 36 *Under conditions of Lemma 34, let $M_{a_1}(T)$ and $\tilde{M}_{a_1}(T)$ are martingales defined in (63) and (13) where the Brownian motion is given in Lemma 34 and Remark 35 with $|a(0)| \leq \delta_\varepsilon$ for $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$, respectively.*

Let $f(T) = \int_0^T \sum_{k=2}^N \ominus_k^2(a_1) ds$ be the quadratic variation of $M_{b_1}(T)$ and $g(T) = \int_0^T \sum_{k=2}^N g_k(a_1) ds$ be the quadratic variation of the martingale $\tilde{M}_{a_1}(T)$, then for $r_0 = \frac{1}{3}$ and $\kappa_0 \leq \kappa$ we obtain

$$
\mathbb{E} \sup_{T\in[0,T_0]} \Big| M_{a_1}(T) - \tilde{M}_{a_1}(T) \Big|^p \leq C(1 + \delta_\varepsilon^{\frac{8}{3}p}) \varepsilon^{\frac{1}{3}p - \frac{4}{3}p\kappa}. \tag{75}
$$

Proof. From (73), we obtain

$$
\begin{aligned}
\mathbb{E} \sup_{T\in[0,T_0]} |f(T) - g(T)|^p \quad &= \quad \mathbb{E} \sup_{T\in[0,T_0]} \Big| \int_0^T \sum_{k=2}^N [\ominus_k^2(a_1) - g_k(a_1)] ds \Big|^p \\
&\leq \quad C(1 + \delta_\varepsilon^2)^p \varepsilon^{p - 4p\kappa_0} .
\end{aligned}
$$

Secondly, as the $\theta_i^{(k)}$ are constants, we derive

$$
\begin{aligned}
g(T_0)^{2p} \quad &\leq \quad \sup_{T\in[0,T_0]} |g(T)|^{2p} = \sup_{T\in[0,T_0]} \Big| \int_0^T \sum_{k=2}^N g_k(s) ds \Big|^{2p} \\
&\leq \quad C \sup_{[0,T_0]} |a_1|^{8p} + C \sup_{[0,T_0]} |a_1|^{4p} .
\end{aligned}
$$

Using (71) we obtain

$$
\mathbb{E}g(T_0)^{2p} \leq C\delta_\varepsilon^{8p} .
$$

Applying Lemma 34 yields (75). □

Let us now turn to the proof of the main result.

Definition 37 *Given $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$ with κ from the stopping time $\tau^\sharp$, we define the set $\Omega^{**} \subset \Omega$ as the set of all $\omega \in \Omega$ such that the following estimates hold:*

$$
\sup_{[0,\tau^\sharp]} \|\psi - \mathcal{Q}\|_\alpha < \varepsilon^{2-4\kappa}, \tag{76}
$$

$$
\sup_{[0,\tau^\sharp]} \|\psi\|_\alpha < \delta_0 + \varepsilon^{-\frac{1}{2}\kappa} , \tag{77}
$$

$$\sup_{[0,\tau^\sharp]} |R_2| < \varepsilon^{1-6\kappa}, \tag{78}$$

$$\sup_{[0,\tau^\sharp]} |M_{a_1}| < \varepsilon^{-\frac{1}{2}\kappa}, \tag{79}$$

and

$$\sup_{[0,\tau^\sharp]} |M_{a_1} - \tilde{M}_{a_1}| < (1+\delta_\varepsilon^{\frac{8}{3}})\varepsilon^{\frac{1}{3}-\frac{7}{3}\kappa} \ . \tag{80}$$

We will see later that the set Ω^{**} has approximately probability 1 (cf. proof of Theorem 11 later) and that $\tau^\sharp = T_0$ on Ω^{**}.

The following theorem states that in (70), (67) we have a good approximation when leaving out the error term R_2. We will take care of the martingale part later. Note that here we could still work with $\dim(\mathcal{N}) \geq 1$.

Theorem 38 *We assume that Assumption 1, 3, 4 and 5 with all $\alpha_k = \sigma$ hold. Let a_1 be a solution of (66) and let ζ and a_2 are solution of (70) and (67), respectively. If the initial condition satisfies $a_2(0) = \zeta(0) = 0$ and if $\kappa < \frac{1}{7}$, then there is a constant $C > 0$ such that*

$$\sup_{T\in[0,\tau^\sharp]} |a_2(T) - \zeta(T)| \leq C\varepsilon^{1-7\kappa}, \tag{81}$$

and

$$\sup_{T\in[0,\tau^\sharp]} |a_2(T)| \leq C(1+\delta_\varepsilon)\varepsilon^{-\frac{1}{2}\kappa} \tag{82}$$

on Ω^{**}.

Proof. To prove (81) subtract (67) from (70) and define $\eta := \zeta - a_2$ to obtain

$$d\eta = [\mathcal{L}_c\eta + 3\mathcal{F}_c(a_1, a_1, \eta) + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k}\mathcal{F}_c(\eta, e_k, e_k)]dT + dR_2.$$

If we take

$$Q_a(\eta) = \mathcal{L}_c\eta + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k}\mathcal{F}_c(\eta, e_k, e_k) \quad \text{and} \quad Q_b(\eta) = 3\mathcal{F}_c(a_1, a_1, \eta),$$

then we obtain from Lemma 28 using the bound on a_1 given by $\tau^\sharp$

$$\sup_{[0,\tau^\sharp]} |\eta|^2 \leq C\varepsilon^{-2\kappa} \sup_{[0,\tau^\sharp]} |R_2|^2 \quad \text{on} \quad \Omega^{**}. \tag{83}$$

From (78) we obtain

$$\sup_{[0,\tau^\sharp]} |\zeta - a_2| = \sup_{[0,\tau^\sharp]} |\eta| \leq C\varepsilon^{1-7\kappa} \quad \text{on} \quad \Omega^{**}.$$

For the second part of the Theorem (cf. (82)), consider

$$\sup_{[0,\tau^\sharp]} |a_2| \leq \sup_{[0,\tau^\sharp]} |\zeta - a_2| + \sup_{[0,\tau^\sharp]} |\zeta| \quad \text{on} \quad \Omega^{**}.$$

Using (79) together with (72) and (81), yields (82) for $\kappa < \frac{1}{7}$. □

In the following theorem we approximate the martingale part $\tilde{M}_{a_1}$, that still depends on the fast OU-process. Here we need $n = 1$, as otherwise only weak convergence of the approximation is possible.

Theorem 39 *Under assumptions of Theorem 38. Let a_1 be a solution of (66) with $a_1(0) = a(0)$ such that $|a(0)| \leq \delta_\varepsilon$ and let ζ be a solution of (70). Define b_2 in $\mathcal{N}$ as a solution of*

$$db_2 = [\mathcal{L}_c b_2 + 3\mathcal{F}_c(a_1, a_1, b_2) + \sum_{k=2}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(b_2, e_k, e_k)]dT + d\tilde{M}_{a_1}, \qquad (84)$$

where $\tilde{M}_{a_1}$ is defined in (13). If the initial condition satisfies $\zeta(0) = b_2(0) = 0$, then for every $p > 0$ and every $\kappa \in (0, \frac{1}{7})$ from the definition of $\tau^\sharp$ there exists a constant $C > 0$ such

$$\sup_{T\in[0,\tau^\sharp]} |b_2(T) - \zeta(T)| \leq C(1 + \delta_\varepsilon^{\frac{11}{3}})\varepsilon^{\frac{1}{3}-\frac{7}{3}\kappa}. \qquad (85)$$

Proof. Subtracting (70) from (84) and defining $\phi = b_2 - \zeta$ we obtain

$$\begin{aligned}\phi(T) &= \int_0^T \mathcal{L}_c \phi d\tau + 3\int_0^T \mathcal{F}_c(\phi, a_1, a_1)d\tau \\ &\quad + \sum_{k=2}^{N} \frac{3\sigma^2}{2\lambda_k} \int_0^T \mathcal{F}_c(\phi, e_k, e_k)]d\tau + \tilde{M}_{a_1}(T) - M_{a_1}(T).\end{aligned}$$

Let

$$Q_a(\phi) = \mathcal{L}_c \phi + \sum_{k=2}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(\phi, e_k, e_k) \quad \text{and} \quad Q_b(\phi) = 3\mathcal{F}_c(a_1, a_1, \phi),$$

then all conditions of Lemma 28 are satisfied as follows

$$|Q_a(\phi)| \leq C|\phi|, \ |Q_b(\phi)| \leq |a_1|^2|\phi| \leq C\delta_\varepsilon^2|\phi| \ \text{ on } \Omega^{**},$$

and from Assumption 4

$$\langle Q_b(\phi), \phi\rangle \leq 0.$$

Hence, we apply Lemma 28 to obtain

$$\sup_{[0,\tau^\sharp]} |\phi|^2 \leq C(1 + \delta_\varepsilon^2) \sup_{[0,\tau^\sharp]} |\tilde{M}_{a_1}(T) - M_{a_1}(T)|^2.$$

Using (80) to finish the proof. $\square$

Finally, we use the results previously obtained to prove the main result of Theorem 11 for the approximation of the solution of the SPDE (1).

Proof of Theorem 11. We note that provided $\delta_\varepsilon < \varepsilon^{-\frac{1}{3}\kappa}$

$$\begin{aligned}\Omega &\supseteq \{\tau^\sharp = T_0\} \\ &\supseteq \Big\{ \sup_{[0,T_0]} \|a_1\|_\alpha < 2\varepsilon^{-\kappa}, \ \sup_{[0,T_0]} \|a_2\|_\alpha < \varepsilon^{-\kappa}, \sup_{[0,T_0]} \|\psi\|_\alpha < \varepsilon^{-\kappa} \Big\} \\ &\supseteq \Omega^{**},\end{aligned}$$

where the last inclusion holds due to (77) with Lemma 32 and Theorem 38. Moreover $\Omega^* \supset \Omega^{**}$ by definition, as $a = a_1 + \varepsilon a_2$. Hence,

$$\begin{aligned}\mathbb{P}(\Omega^{**}) &\geq 1 - \mathbb{P}\Big(\sup_{[0,\tau^\sharp]} \|\psi - \mathcal{Q}\|_\alpha \geq \varepsilon^{2-4\kappa}\Big) - \mathbb{P}\Big(\sup_{[0,\tau^\sharp]} \|\psi\|_\alpha \geq \varepsilon^{-\frac{1}{2}\kappa}\Big) \\ &\quad - \mathbb{P}\Big(\sup_{[0,\tau^\sharp]} \|R_2\|_\alpha \geq \varepsilon^{1-6\kappa}\Big) - \mathbb{P}\Big(\sup_{[0,\tau^\sharp]} |M_{a_1} - \tilde{M}_{a_1}| \geq \varepsilon^{\frac{1}{3}-\frac{29}{3}\kappa}\Big) \\ &\quad - \mathbb{P}\Big(\sup_{[0,\tau^\sharp]} |M_{a_1}| \geq \varepsilon^{-\frac{\kappa}{2}}\Big).\end{aligned}$$

Using Chebychev inequality and Lemmas 13, 30, 32, 36 and Corollary 15, we obtain for sufficiently small κ_0

$$\mathbb{P}(\Omega^{**}) \geq 1 - C[\varepsilon^{q\kappa} + \varepsilon^{\frac{1}{2}q\kappa - q\kappa_0} + \varepsilon^{\frac{1}{2}q\kappa}] \geq 1 - C\varepsilon^{\frac{1}{4}q\kappa} \geq 1 - C\varepsilon^{p}, \tag{86}$$

if q is sufficiently large. Now let us turn to the approximation result. Using (2) and triangle inequality, yields

$$\begin{aligned}
&\sup_{[0,\tau^\sharp]} \|u(\varepsilon^{-2}\cdot) - \varepsilon a_1 - \varepsilon^2 b_2 - \varepsilon \mathcal{Q}\|_\alpha \\
&\quad = \sup_{[0,\tau^\sharp]} \|\varepsilon^2 a_2 - \varepsilon^2 b_2 + \varepsilon\psi - \varepsilon\mathcal{Q}\|_\alpha \\
&\quad \leq \varepsilon^2 \sup_{[0,\tau^\sharp]} \|a_2 - b_2\|_\alpha + \varepsilon \sup_{[0,\tau^\sharp]} \|\psi - \mathcal{Q}\|_\alpha \\
&\quad \leq \varepsilon^2 \sup_{[0,\tau^\sharp]} \|a_2 - \zeta\|_\alpha + \varepsilon^2 \sup_{[0,\tau^*]} \|\zeta - b_2\|_\alpha + \varepsilon \sup_{[0,\tau^\sharp]} \|\psi - \mathcal{Q}\|_\alpha .
\end{aligned}$$

From (76), (81) and (85), we obtain

$$\begin{aligned}
&\sup_{t\in[0,\varepsilon^{-2}T_0]} \|u(t) - \varepsilon a_1(\varepsilon^2 t) - \varepsilon^2 b_2(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\|_\alpha \\
&\quad = \sup_{t\in[0,\varepsilon^{-2}\tau^\sharp]} \|u(t) - \varepsilon a_1(\varepsilon^2 t) - \varepsilon^2 b_2(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\|_\alpha \\
&\quad \leq C\varepsilon^{\frac{7}{3}-7\kappa} \quad \text{on} \quad \Omega^{**}.
\end{aligned}$$

Thus

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \Big\|u(t) - \varepsilon a_1(\varepsilon^2 t) - \varepsilon^2 b_2(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > C\varepsilon^{\frac{7}{3}-7\kappa}\Big) \leq 1 - \mathbb{P}(\Omega^{**}).$$

Using (86), yields (25). □

7.2 Applications

To apply our main theorem, we will consider two examples. The first one is the Swift-Hohenberg equation (10) but now with respect to Neumann boundary conditions on the interval $[0,\pi]$. The second one is the Ginzburg-Landau or Allen-Cahn equation (49). We will discuss several cases depending on the form of the noise.

7.2.1 Swift-Hohenberg equation

For Neumann boundary conditions we consider the orthonormal basis of eigenfunctions

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}} & \text{if} \quad k = 0, \\ \sqrt{\frac{2}{\pi}}\cos(kx) & \text{if} \quad k > 0. \end{cases}$$

The spaces are given by

$$\mathcal{H} = L^2([0,\pi]) \ \text{ and } \ \mathcal{N} = \mathrm{span}\{\cos\},$$

In this case our main theorem states that the solution of (10) is

$$u(t,x) \simeq \varepsilon\gamma_1(\varepsilon^2 t)\cos(x) + \varepsilon^2\gamma_2(\varepsilon^2 t)\cos(x) + \varepsilon\mathcal{Z}_k(\varepsilon^2 t)\cos(kx) + \mathcal{O}(\varepsilon^{3-}),$$

where γ_1 and γ_2 are the solution of the amplitude equation given below. We will discuss three cases depending on the noise.
First case. If the noise is a constant in the space, i.e.

$$W(t) = \sigma\beta_0(t) \ ,$$

then

$$\partial_T\gamma_1 = \Big(\nu - \frac{3\sigma^2}{2}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = [\Big(\nu - \frac{3\sigma^2}{2}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2]dT + \frac{3\sigma^2}{\sqrt{2}}\gamma_1 dB.$$

Second case: If the noise acting on $\cos(kx)$ for one $k \in \{2,4,5,6,\ldots,N\}$, then

$$\partial_T\gamma_1 = \Big(\nu - \frac{3\sigma^2}{2(k^2-1)^2}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = [\Big(\nu - \frac{3\sigma^2}{2(k^2-1)^2}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2]dT + \frac{3\sigma^2}{2\sqrt{2}(k^2-1)^3}\gamma_1 dB.$$

Third case: If the noise takes the form

$$W(t) = \sigma\beta_3(t)\cos(3x) \ ,$$

then

$$\partial_T\gamma_1 = \Big(\nu - \frac{3\sigma^2}{128}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = [\Big(\nu - \frac{3\sigma^2}{128}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2]dT + \frac{3\sigma}{256}\gamma_1\sqrt{(\gamma_1^2 + \frac{\sigma^2}{32})}dB.$$

7.2.2 Ginzburg-Landau / Allen-Cahn equation

Our main theorem states that the solution of (49) takes the form

$$u(t,x) \simeq \varepsilon\gamma_1(\varepsilon^2 t)\sin(x) + \varepsilon^2\gamma_2(\varepsilon^2 t)\sin(x) + \varepsilon\mathcal{Z}_k(\varepsilon^2 t)\sin(kx) + \mathcal{O}(\varepsilon^{3-}),$$

where γ_1 and γ_2 are the solution of the amplitude equations given below. We will discuss three cases depending on the noise.
First case. Noise acting on $\sin(kx)$ for one $k \in \{2,4,5,6,\ldots,N\}$. In this case

$$\partial_T\gamma_1 = \Big(\nu - \frac{3\sigma^2}{4(k^2-1)}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = [\Big(\nu - \frac{3\sigma^2}{4(k^2-1)}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2]dT + \frac{3\sigma^2}{2\sqrt{2(k^2-1)^3}}\gamma_1 dB.$$

Second case. Noise acting only on $\sin(3x)$. In this case

$$\partial_T\gamma_1 = \Big(\nu - \frac{3\sigma^2}{32}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = [\Big(\nu - \frac{3\sigma^2}{32}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2]dT + \frac{3\sigma}{32}\gamma_1\sqrt{(\gamma_1^2 + \frac{\sigma^2}{16})}dB.$$

Third case. The noise is of the form

$$W(t) = \sum_{k=2}^{3} \sigma\beta_k(t)e_k \ .$$

In this case

$$\partial_T\gamma_1 = \Big(\nu - \frac{11\sigma^2}{32}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = [\Big(\nu - \frac{11\sigma^2}{32}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2]dT + d\tilde{M},$$

where

$$d\tilde{M} = \frac{3\sigma}{32}\left(\gamma_1^4 + \frac{1289\sigma^2}{128}\gamma_1^2 + \frac{89\sigma^4}{147}\right)^{1/2} dB.$$

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