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Model for Laminar and Turbulent Flows of Viscous and Nonlinear Viscous non-Newtonian Fluids

Preprint Nr. 20/2010 — 13. Dezember 2010

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Impressum:

Herausgeber:
Institut für Mathematik
Universität Augsburg
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`http://www.math.uni-augsburg.de/pages/de/forschung/preprints.shtml`

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Key words. Laminar flow, turbulent flow, local Reynolds number, existence theorem, approximate solution.

1. **Introduction**

Theories of flow of viscous Newtonian's and nonlinear viscous fluids are based on the Navier–Stokes equations and on modifications of them in which the viscosity depends on the second invariant of the rate of strain tensor, see [2, 27]. These models describe satisfactorily slow laminar flows, but they are not fit to compute and explore flows with large gradients and turbulent flows.

Let us consider flows of viscous and nonlinear viscous fluids in the circular tube and some characteristics of turbulent flows.

The constitutive equation of the power model of the nonlinear viscous fluid is the following:

$$\sigma_{ij}(p,u) = -p\,\delta_{ij} + 2\,k(2I(u))^{\frac{m-1}{2}}\,\varepsilon_{ij}(u). \tag{1.1}$$

Here $\sigma_{ij}(p,u)$ are the components of the stress tensor which depend on the pressure p and the velocity vector $u = (u_1,\dots,u_n)$,

$$\delta_{ij} = \begin{cases} 1, \text{ if } & i=j \\ 0, \text{ if } & i\neq j \end{cases}, \quad i,j=1,\dots,n, \quad n=2 \text{ or } 3,$$

k and m are positive constants, $\varepsilon_{ij}(u)$ components of the rate of strain tensor,

$$\varepsilon_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \tag{1.2}$$

and $I(u)$ is the second invariant of the rate of strain tensor

$$I(u) = \sum_{i,j=1}^{n} (\varepsilon_{ij}(u))^2. \tag{1.3}$$

In this case, the viscosity function φ is defined by

$$\varphi(I(u)) = k(2I(u))^{\frac{m-1}{2}}. \tag{1.4}$$

At $m=1$ the fluid is the Newtonian one. If $m<1$, the fluid is pseudoplastic, the viscosity decreases as the shear rate increases. At $m>1$ the fluid is dilatant, the viscosity increases with a rise of the shear rate.

Under increase of the shear rate, the structure of fluid, as a rule, is destroyed and the viscosity decreases. Because of this, the most part of real fluids are pseudoplastic. Melted and dissolved polymers, oils, paints, pastes, blood are examples of pseudoplastic fluids. For the power model (1.1), the problem on rectilinear flow of the fluid in the circular tube is exactly solved [2], and its solution is the following:

$$v(r) = \frac{m}{m+1}\Big|\frac{1}{2k}\,\frac{dp}{dz}\Big|^{\frac{1}{m}} R^{\frac{m+1}{m}}\Big[1-\Big(\frac{r}{R}\Big)^{\frac{m+1}{m}}\Big]. \tag{1.5}$$

Here $v(r)$ is the velocity of the fluid at a distance r from the axis of the tube, $\frac{dp}{dz}=$ constant <0 the drop of pressure per unit of the length of the tube, R the radius of the tube.

2000 Mathematics Subject Classification: Primary 76F02, 76A05, 76D03: Secondary: 76M10, 35D05.

Relation (1.5) changes to the well–known Poiseuille formula for the Newtonian fluid at $m = 1$.

The velocity profiles computed by formula (1.5) are shown in Figure 1. Line 1 is the profile of the Newtonian fluid $m = 1$, line 2 a pseudoplastic fluid $m = \frac{1}{3}$, line 3 a dilatant fluid $m = 2$, lines 4 and 5 are the limits as $m \to 0$ and $m \to \infty$, the corresponding profiles are rectangular and triangular.

FIGURE 1

In accordance with an experiment, see [30], pp. 625–628, [8], p. 23, the relations between the mean velocity v_m in the circular tube,

$$v_m = \frac{2}{R^2} \int_0^R v(r) r \, dr, \tag{1.6}$$

and the pressure p in the inflow of the tube have the forms shown in Figure 2.

FIGURE 2

Here line 1 is for the Newtonian fluid, 2 for a pseudoplastic one, a_1 and a_2 are the points of transition to the turbulent flow. When the flow becomes turbulent, whirlwind flows appear and the relation $\frac{p}{v_m}$ increases as v_m is increased, while for the pseudoplastic fluid in the range $[0, a_2]$ the relation $\frac{p}{v_m}$ decreases as v_m is increased, and for the Newtonian fluid $\frac{p}{v_m} =$ constant for $v_m \in [0, a_1]$.

The velocity profiles obtained by experiment for the flow of the Newtonian fluid in the circular tube are shown in Figure 3, see [30], page 588, lines 1,2,3 correspond to $v_m = b_1, b_2, b_3$. In this case $b_1 < b_2 < b_3$ and b_1 corresponds to the laminar flow (line 1 is the parabola), lines 2 and 3 define the profiles of averaged velocities for the turbulent flows.

FIGURE 3

For Newtonian's fluids, the transition from the laminar flow to turbulent occurs when the Reynolds number, defined as

$$Re = \frac{v_m D \rho}{\mu}, \tag{1.7}$$

is equal to Re_c. Here D is the diameter of the tube, ρ and μ are the density and the viscosity of the fluid. Re_c is said to be the critical value of the Reynolds number.

We mention that for a flow which is different from the flow in circular tube, the values v_m and D in (1.7) are changed for a characteristic velocity and a characteristic length. However, these values are not strictly specified, and in many cases there is a large arbitrariness in deciding on these values.

For turbulent flows of the Newtonian fluid in the circular tube, the profiles of averaged velocities are analogous to the velocity profiles of pseudoplastic fluids for the laminar flow (see Figures 1 and 3) and $\frac{v_m}{v(0)} \to 1$ as $v_m \to \infty$, i.e. the velocity profile tends to the rectangular one as $Re \to \infty$.

In 1877 Boussinesq set up the hypothesis that the constitutive equation of the Newtonian fluid for turbulent flows is identical to that for laminar flows, only the normal viscosity is changed for the turbulent viscosity, i.e.

$$\sigma_{ij}(p,u) = -p\delta_{ij} + 2\,\varphi_t\varepsilon_{ij}(u), \tag{1.8}$$

where φ_t is the turbulent viscosity.

Experiment show (see [30], p. 625 and Figure 2) that the turbulent viscosity is far greater than the laminar one, it may be more than the laminar viscosity by a factor 10^5, and the turbulent viscosity increases as the mean velocity v_m and accordingly Re rises. Experiments also show [30], p. 627, that the turbulent viscosity increases with the increase of the distance to the hard wall, but the shearing rate decreases with the increase of this distance (see Figure 3), in addition, in a small vicinity of the hard wall, the flow is laminar.

A rich variety of models for turbulent flows of the Newtonian fluid were suggested. Reviews of these models are contained in [3, 30, 32]. The models of Boussinesq and Prandtl, and the "$k-\varepsilon$" model appear to be the most used. The Boussinesq model is used in hydraulics, meteorology, oceanology. In this case, one assumes that that the turbulent viscosity increases with the increase of the distance to the hard boundary and empirical relations are used.

In 1925 Prandtl constructed the so-called "mixing length theory“ [38] and, on the basis of it, obtained the following formula for the shear stress τ at turbulent flows:

$$\tau = \rho l^2 \left|\frac{du}{dy}\right| \frac{du}{dy}, \tag{1.9}$$

where l is the mixing length and $\frac{du}{dy}$ is the velocity gradient.

In line with (1.9), the turbulent Prandtl viscosity has the form

$$\varphi_t(I(u)) = \rho\, l^2 (2I(u))^{\frac{1}{2}}. \tag{1.10}$$

Comparing (1.10) with (1.4), we can see that φ_t in (1.10) is the viscosity of the power model of fluid at $m = 2$. Such fluid is dilatant, its velocity profile is the line 3 in Figure 1, and the form of the profile is independent of the mean velocity v_m.

However, experiments show that the velocity profile of the Newtonian fluid in the circular tube at the turbulent flow is identical to that of a pseudoplastic fluid, and it tends to rectangular one as v_m tends to infinity, i.e. it has a form of the line 2 in Figure 1 and tends to the form of the line 4 there.

Yet, the Prandtl viscosity (1.10) describes the super-lineal increase in the resistance to flow with the increase of the Reynolds number at turbulent flows, see Figure 2. Because of this, formulas (1.9) and (1.10) are used for calculations of great variety of turbulent motions.

In the "$k-\varepsilon$" model, a system of the Reynolds equations, of the equation of incompressibility, of the transport equation for turbulent fluctuations, and of the equation of dissipation of the fluctuations is solved, see [3, 33, 41].

Six empirical constants are contained in this model, and it was successfully employed for two-dimensional flows of the Newtonian fluid in a vicinity of the hard plane boundary when the flow was close to rectilinear.

Modifications of the "$k-\varepsilon$" model, which take into account the curvature of the hard boundary, were suggested. However, they did not furnish the desired result even at small curvatures, see [32].

Presently, the Large Eddy Simulation or LES is widely used for approximation of solutions of the Navier-Stokes equations at large Reynolds numbers.

In LES the functions of velocity u and pressure p in the Navier-Stokes equations are represented in the following form:

$$u = \overline{u} + u', \quad p = \overline{p} + p',$$

where $\overline{u}$ and $\overline{p}$ are space averaged functions u and p, u' and p' fluctuations.

By averaging the Navier-Stokes equations, the following relations are obtained:

$$\rho\Big(\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial \overline{u}_i}{\partial x_j}\overline{u}_j\Big) - \mu\Delta\overline{u}_i + \frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \alpha_{ij}(u)}{\partial x_j} = K_i, \quad i = 1, ..., n, \tag{1.11}$$

$$\sum_{i=1}^{n} \frac{\partial \overline{u}_i}{\partial x_i} = 0, \tag{1.12}$$

where $\alpha_{ij}(u)$ are components of the Reynolds stress tensor $\alpha(u)$, $\alpha_{ij}(u) = \rho(\overline{u_i u_j} - \overline{u}_i\,\overline{u}_j)$, K_i components of the volume force vector K, see [3]

In (1.11) and below the Einstein convention on summation over repeated index is applied.

The tensor $\alpha(u)$ is approximated in LES as follows:

$$\alpha_{ij}(u) \approx -2\varphi_t(\overline{u},\gamma)\varepsilon_{ij}(\overline{u}) + \frac{1}{n}\alpha_{kk}(u)\delta_{ij}. \tag{1.13}$$

Here γ is the radius of the averaging kernel (mollifier).

One of the most popular LES models is the Smagorinsky model [42], in which

$$\varphi_t(\overline{u},\gamma) = \rho\,(C_S\gamma)^2\,(2I(\overline{u}))^{\frac{1}{2}}\,, \tag{1.14}$$

where C_S is the Smagorinsky constant.

By (1.13), (1.14) the motion equations (1.11) take the form

$$\rho\Big(\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial \overline{u}_i}{\partial x_j}\overline{u}_j\Big) - \mu\Delta\overline{u}_i - 2^{\frac{3}{2}}\rho\,(C_S\gamma)^2\frac{\partial}{\partial x_j}\Big((I(\overline{u}))^{\frac{1}{2}}\varepsilon_{ij}(\overline{u})\Big)$$
$$+\frac{\partial \check{p}}{\partial x_i} = K_i, \quad i = 1, ..., n, \tag{1.15}$$

where $\check{p} = \overline{p} + \frac{1}{n}\alpha_{kk}(u)$.

Comparing (1.10) with (1.14), we see that the Smagorinsky turbulent viscosity is the turbulent viscosity of Prandtl in which $l = C_S\gamma$.

The velocity profile of the fluid described by (1.15) in the circular tube is intermediate between the profiles 1 and 3 in the Figure 1, and it tends to the profile 1 of the Newtonian fluid as γ tends to zero.

The motion equations (1.15) are the regularized Navier-Stokes equations, in which the Prandtl turbulent viscosity serves as a regularizer. That is, LES leads to the regularization of the Navier-Stokes equations, wherein the averaging radius γ serves as a parameter of regularization.

Mathematical problems for dilatant fluids with the constitutive equation

$$\sigma_{ij}(p,u) = -p\delta_{ij} + 2\mu\varepsilon_{ij}(u) + 2k(I(u))^{\gamma}\,\varepsilon_{ij}(u),$$

where μ, k and γ are positive constants, $\gamma \geq \frac{1}{4}$, were investigated in [23, 24]; in so doing it was assumed that the velocity is equal to zero on the whole of the boundary.

We point out that in any real flow, there are areas of the boundary of the domain of flow, in which fluid flows into and out. Therefore, the velocity is not equal to zero on the whole of the boundary in any real flow.

Problems on flow of nonlinear viscous fluids, in which the viscosity is a relatively general function of the second invariant of the rate of strain tensor, are analyzed in [27] under inhomogeneous Dirichlet and mixed boundary conditions, where velocities and surface forces are given on different parts of the boundary. However, the nonlinear terms in the inertial forces are not taken into account there.

Considerable recent attention has been focused on the LANS-α turbulent model, or the Lagrangian-averaged Navier-Stokes α model, see [12], [13], [15]. The LANS-α model equations are the following:

$$\rho\Big(\frac{\partial v_i}{\partial t}+\frac{\partial v_i}{\partial x_j}u_j+\frac{\partial u_j}{\partial x_i}v_j\Big)-\mu\Delta v_i+\frac{\partial \tilde{p}}{\partial x_i}=K_i,\quad i=1,2,3, \tag{1.16}$$

$$v=u-\alpha^2\Delta u,\quad \operatorname{div} u=\operatorname{div} v=0 \tag{1.17}$$

Here v is the Lagrangian-averaged velocity, u the Eulerian-averaged velocity, which is smoother than v, u is the transport velocity and v the transported velocity, α the scale parameter, $\mu>0$ the constant viscosity, $\tilde{p}$ the modified pressure,

$$\tilde{p}=p-\frac{1}{2}\rho(|u|^2+\alpha^2|\nabla u|^2), \tag{1.18}$$

while p is the pressure.

(1.16)–(1.18) imply that u satisfies the following conditions:

$$\begin{aligned}\rho\frac{\partial}{\partial t}(u_i-\alpha^2\Delta u_i)+\frac{\partial(u_i-\alpha^2\Delta u_i)}{\partial x_j}u_j+\frac{\partial u_j}{\partial x_i}(u_j-\alpha^2\Delta u_j)&\\ -\mu\Delta(u_i-\alpha^2\Delta u_i)+\frac{\partial\tilde{p}}{\partial x_i}=K_i,\quad i=1,2,3,&\end{aligned} \tag{1.19}$$

$$\operatorname{div} u=0. \tag{1.20}$$

At $\alpha=0$ the equations (1.19) and (1.18) transfer to the Navier-Stokes equations. (1.19) is a system of equations of the fourth order. Therefore, one has to prescribe two boundary conditions. However, the LANS-α equations were obtained provided that there is no a boundary of a domain of flow.

The LANS-α equations are usefully employed in the ocean-climate modeling [11, 37], where the domain under consideration is very large, and the boundary has a slight impact on the dominant flow.

Global existence result for LANS-α equations for flow with periodic boundary conditions was obtained in [7]. The global regularity of the solution to LANS-α equations in a three-dimensional bounded domain at smooth initial velocity and zero volume forces, and zero boundary conditions is proved in [31]

It is shown in [7], [35] that solutions to LANS-α equations converge to the solution of the Navier-Stokes equations as α goes to zero. Because of this, for any Reynolds number, the velocity profile of the LANS-α fluid in the circular tube is close to parabolic for small α. However, this is not compatible with experimental evidence, see Figure 1.

Turbulent flows are also under investigation from the position of the statistical hydromechanics, see [9] and references there.

In complicated fluid flows there exist regions of laminar flow together with regions of turbulent flow. Because of this, the models which cover both laminar and turbulent flows are of particular interest. In [28, 29] nonlocal models were suggested which describe laminar and turbulent flows of viscous and nonlinear viscous non-Newtonian fluids. In these models, the domain of flow is divided into subdomains, and the solution of the problem on fluid flow determines which flows laminar or turbulent are in these subdomains and describes them. However, this solution depends on the partition of the domain of flow.

Below we introduce and investigate a new model that covers laminar and turbulent flows of viscous and nonlinear viscous fluids. We introduce a characteristic which we call a local Reynolds number and which is calculated at each point of the domain of flow. The local Reynolds number defines the value of the turbulent viscosity . The viscosity of the fluid is the sum of the laminar and turbulent viscosities. In the case that the local Reynolds number does not exceed some value, the turbulent viscosity is equal to zero.

In Section 2 we introduce the model of the fluid and set out the basic equations. Formulations of the problems under consideration and main results are contained in Section 3. We consider stationary and nonstationary flow problems with inhomogeneous boundary conditions where velocities are given on the whole of the boundary and where velocities and surface forces are given on different parts of the boundary. Section 4 contains auxiliary results. In Sections 5 and 7, we prove existence results for the stationary and nonstationary flow problems. Numerical methods for solving these problems are investigated in Sections 6 and 8.

2. **Model of the fluid and basic equations.**

2.1. **Local Reynolds number.** We consider two reference frames in $\mathbb{R}^n$. Denote points of the first frame by $x = (x_1, \dots, x_n)$ and points of the second frame by $x' = (x'_1, \dots, x'_n)$.

Let Ω be a domain in which a fluid flows. We suppose that the reference frame x is immovable relative to the domain of flow Ω, and the frame x moves with respect to the frame x', which is considered as immovable.

For example, if a fluid flows in a canal, the reference frame x is rigidly bound with the body of the canal. The canal can be located in a moving object, e.g. in a car or in a plane. The reference frame x' is rigidly bound with some immovable object which is situated on the Earth.

In this case x is the actual frame that moves relative to the frame x' that is an inertial frame.

Let S be a boundary of Ω. We denote a hard part of the boundary by S_1. In the case that a fluid flows in a canal, the hard boundary of the canal is S_1, and the boundary S of Ω is the join of S_1 and $\overline{S}_{21}$, and $\overline{S}_{22}$, where S_{21}, and S_{22} are open subsets of S corresponding to the regions of inflow and outflow of the fluid, see Figure 4.

Once a fluid flows around a hard body, the boundary of the hard body is S_1.

S_1 S_{21} S_{22}

FIGURE 4

We suppose that the domain of flow satisfies the following conditions:

(C1): Ω is a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with a Lipschitz continuous boundary S. S_1 is an open subset of S of the class C^2, and the absolute values of the principal curvatures of S_1 at points of S_1 are bounded.

It is significant that these conditions can be weakened. We can assume that S_1 is Lipschitz continuous, $\overline{S}_1 = \bigcup_{i=1}^{k} \overline{S}_{1i}$, where S_{1i} are open subsets of S_1 of the class C^2 and such that $S_{1i} \bigcap S_{1j} = \emptyset$ at $i \neq j$, and the absolute values of the principal curvatures of S_{1i} at points of S_{1i} are bounded, $i = 1, \ldots, k$. However, for the sake of simplicity, we consider the condition (C1).

Let $u = (u_1, \ldots, u_n)$ be a vector of fluid velocity in the reference frame x. By analogy with the Reynolds number, that is defined by (1.7), we introduce the following local Reynolds number:

$$(R_l(u))(x,t) = \frac{|u(x,t)|d(k(u,x,t), r(x,S_1))\rho}{\mu(x,t)}, \quad x \in \Omega, \quad t \geq 0. \tag{2.1}$$

Here

$$|u(x,t)| = \Big(\sum_{i=1}^{n} (u_i(x,t))^2\Big)^{\frac{1}{2}}, \tag{2.2}$$

t is the time variable, ρ the density, $\mu(x,t)$ the laminar viscosity of the fluid. For the Newtonian fluid $\mu = \text{constant} > 0$, and

$$\mu(x,t) = (\varphi(I(u)))(x,t), \quad x \in \Omega, \quad t \geq 0 \tag{2.3}$$

for the nonlinear viscous fluid. In (2.1) d is a function of conventional distance, that serves instead of the diameter of the tube D in (1.7). The arguments of d are $k(u,x,t)$ and $r(x,S_1)$, k is a function of a curvature of S_1, and $r(x,S_1)$ is the distance between x and S_1,

$$r(x,S_1) = \inf_{\check{x} \in S_1} \Big[\sum_{i=1}^{n} (x_i - \check{x}_i)^2\Big]^{\frac{1}{2}}. \tag{2.4}$$

In the case that S_1 is planar, we take

$$d(k(u,x,t),\, r(x,S_1)) = r(x,S_1). \tag{2.5}$$

It has been found experimentally, see [36, 39], that in a vicinity of convex boundary, the turbulent stresses are smaller than those in a vicinity of plane boundary. For a concave boundary, the situation is inverse, the turbulent stresses in a vicinity of concave boundary are greater than those in a vicinity of plane boundary.

We suppose that curvatures of S_1, at points s of S_1 are not large. Then these curvatures have an influence on the fluid flow only at points x from Ω which are spaced in a not large vicinity of S_1.

Let B be such a vicinity,

$$B = \{x | x \in \overline{\Omega},\, r(x,S_1) \leq b_0\}. \tag{2.6}$$

Since b_0 is not large, for an arbitrary x from B there exists a unique point $s = (s_1, \ldots, s_n)$ of S_1 such that

$$r(x,S_1) = \Big(\sum_{i=1}^{n} (x_i - s_i)^2\Big)^{\frac{1}{2}}. \tag{2.7}$$

We denote this point s by $s(x)$. Consider the case where $\Omega \subset \mathbb{R}^3$.

Let $x \in B$ and $P_{s(x)}u(x,t)$ be the projection of the velocity vector $u(x,t)$ on the plane that is tangential to S_1 at the point $s(x)$. We define $k(u,x,t)$ as the curvature at the point $s(x)$ of the curve of intersection of the surface S_1 and the plane that goes through the normal to S_1 at the point $s(x)$ and the tangential vector $P_{s(x)}u(x,t)$.

In the case where $\Omega \subset \mathbb{R}^2$, $k(u,x,t)$ is defined as the curvature of S_1 as the point $s(x)$. Thus

$$k(u,x,t) = c(s(x)),$$

where $c(s(x))$ is the corresponding curvature of S_1 at the point $s(x)$.

The following should be stressed: We assumed that the curvature of S_1 influenced the fluid flow at points of Ω which were at a distance up to b_0 from S_1, where b_0 was a constant. However, in the general case, the distance under consideration b_0 should be a function of the curvature of S_1 at points $s \in S_1$ in the above sense.

Relation (2.1) is appropriate in the case that the fluid velocity is equal to zero on the whole of the hard boundary, i.e. the hard boundary is immovable in the reference frame x. However, in specific cases, the boundary of the domain of flow contains a movable hard part along with an immovable one. For instance, this is the case when a fluid moves between two coaxial cylinders such that one cylinder is immovable, whereas the second rotates around its axis.

Swirl flows are widely used in modern practice, see [32]. For example, such a flow is created when a fluid is situated between an immovable cylinder and a screw that rotates inside of the cylinder, the axis of rotation of the screw coinciding with the axis of the cylinder.

If the reference frame x is immovable with respect to the cylinder, the domain of flow is a periodical function of the angle of rotation of the screw with the period 2π. And the domain of flow is time-independent provided that the reference frame x is fixed in relation to the rotating screw.

Let us define the local Reynolds number for such an event. Denote the immovable and movable hard parts of the boundary by S_{11} and S_{12}, respectively. Then $S_1 = S_{11} \bigcup S_{22}$. The velocities of the points of S_{12} are assumed to be given.

Let $x \in \Omega$ and $r(x, S_{11})$, and $r(x, S_{12})$ be the distances of x to S_{11} and S_{12}, respectively. In the case where $r(x, S_{11}) \leq r(x, S_{12})$, we compute $R_l(u)$ by the formula (2.1). Once $r(x, S_{11}) > r(x, S_{12})$, we replace the factor $|u(x,t)|$ in (2.1) by $|u(x,t) - \check{u}(s(x),t)|$, where $\check{u}(s(x),t)$ is the velocity of the movable hard part S_{12} at an instant t at the point $s(x)$ that is defined as follows:

$$r(x, S_{12}) = |x - s(x)| = \Big(\sum_{i=1}^{n} (x_i - (s(x))_i)^2 \Big)^{\frac{1}{2}}.$$

Consider the case where $\Omega \subset \mathbb{R}^3$. Let $P_{s(x)}(u(x,t) - \check{u}(s(x),t))$ be the projection of the vector $u(x,t) - \check{u}(s(x),t)$ on the plane that is tangential to S_{12} at the point $s(x)$. We denote by $k_1(u,x,t)$ the curvature at the point $s(x)$ of the curve of intersection of the surface S_{12} and the plane that goes through the normal to S_{12} at the point $s(x)$ and the tangential vector $P_{s(x)}(u(x,t) - \check{u}(s(x),t))$.

For $\Omega \subset \mathbb{R}^2$, we denote the curvature of S_{12} at the point $s(x)$ by $k_1(u,x,t)$.

Thus, in the case that the boundary of the domain of flow contains a movable hard part along with an immovable one, the local Reynolds number is defined as follows:

$$(R_l(u))(x,t) = \begin{cases} |u(x,t)| d(k(u,x,t), r(x, S_{11}))\rho |\, \mu(x,t)|^{-1} \\ \text{at} \quad r(x, S_{11}) \leq r(x, S_{12}), \quad t > 0, \\ |u(x,t) - \check{u}(s(x),t)| d(k_1(u,x,t), r(x, S_{12}))\rho |\mu(x,t)|^{-1} \\ \text{at } r(x, S_{11}) > r(x, S_{12}), \quad t > 0. \end{cases} \tag{2.8}$$

Since the hard boundary is not deformed when it moves, formula (2.8) also is true when the domain of flow depends on the displacement of the hard boundary S_{12}.

We assume

(C2): d is a continuous in $[-b_1, b_2] \times [0, b_3]$ function with values in $\mathbb{R}_+$.

Here, b_1, b_2, b_3 are positive constants, $b_3 \geq b_0$, $\mathbb{R}_+ = \{y | y \in \mathbb{R},\ y \geq 0\}$.

In line with experimental results and the above assumptions, the function d satisfies the following conditions:

$$
\begin{aligned}
&d(0, y_2) = y_2 \quad \text{at} \quad y_2 \in [0, b_3],\\
&d(y_1, y_2) = y_2 \quad \text{at} \quad y_2 \in [b_0, b_3],\ y_1 \in [-b_1, b_2],\\
&d(y_1, 0) = 0 \quad \text{at} \quad y_1 \in [-b_1, b_2],\\
&d(y_1, y_2) \leq y_2 \quad \text{at} \quad y_1 > 0,\ d(y_1, y_2) \geq y_2 \quad \text{at} \quad y_1 < 0.
\end{aligned} \tag{2.9}
$$

2.2. **Constitutive equation.** Define the following constitutive equation for both laminar and turbulent flows:

$$\sigma_{ij}(p, u) = -p\delta_{ij} + 2(\varphi(I(u)) + \varphi_t(R_l(u)))\varepsilon_{ij}(u). \tag{2.10}$$

Here φ is the laminar viscosity that depends on $I(u)$, φ_t is the turbulent viscosity depending on the local Reynolds number.

For the Newtonian fluid

$$\varphi(I(u)) = \mu = \text{constant} > 0. \tag{2.11}$$

We assume that φ_t satisfies the following conditions:

(C3): $\varphi_t : y \to \varphi_t(y)$ is a continuous and nondecreasing mapping of $\mathbb{R}_+$ into $\mathbb{R}_+$ and

$$\varphi_t(y) = 0 \quad \text{at} \quad y \leq b, \qquad \varphi_t(y) > 0 \quad \text{at} \quad y > b. \tag{2.12}$$

Here b is a positive constant, it is the point of transition from the laminar flow to turbulent.

Relations (2.12) are in agreement with experimental results, which were treated above.

It is supposed that the laminar viscosity φ satisfies the following conditions:

(C4): φ is a continuous function in $\mathbb{R}_+$, and the following inequalities hold:

$$a_1 \leq \varphi(y) \leq a_2, \quad y \in \mathbb{R}_+, \tag{2.13}$$

$$(\varphi(y_1^2)y_1 - \varphi(y_2^2)y_2)(y_1 - y_2) \geq a_3(y_1 - y_2)^2, \quad (y_1, y_2) \in \mathbb{R}_+^2, \tag{2.14}$$

where a_1, a_2, a_3 are positive constants.

The inequality (2.13) indicates that the laminar viscosity is bounded from below and above by positive constants, (2.14) means that in the case of simple shear flow, the shear stress increases with increasing shearing rate.

These inequalities are natural from the physical point of view.

The local Reynolds number, like the Reynolds number, depends on the velocity of the fluid in the frame that is immovable relative to the domain of flow. Because of this, the turbulent viscosity φ_t is independent of the motion of the domain of flow with respect to the immovable frame x'. Moreover, the turbulent viscosity is independent of the chosen frame, providing that the frame is immovable relative to Ω.

Indeed, an arbitrary frame which is immovable relative to Ω is given by

$$y = x + r^1 + r^2, \tag{2.15}$$

where $r^1 = (r_1^1, \ldots, r_n^1)$ is a vector of translation, $r^2 = (r_1^2, \ldots, r_n^2)$ is a vector of turn. At $n = 3$ the vector r^2 has the following form:

$$\begin{aligned} r_1^2 &= \omega_2 x_3 - \omega_3 x_2, \\ r_2^2 &= \omega_3 x_1 - \omega_1 x_3, \\ r_3^2 &= \omega_1 x_2 - \omega_2 x_1. \end{aligned} \tag{2.16}$$

Here $\omega = (\omega_1, \omega_2, \omega_3)$ is the angle of the vector of turn.

Let e_1, e_2, e_3 and e_1', e_2', e_3' be unit base vectors in the frames $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Let also $u(x,t) = u_i(x,t)e_i$ be a velocity vector in the frame x, and $u'(y,t) = u_i'(y,t)e_i'$ be the same vector in the frame y, i.e. $u(x,t) = u'(y,t)$. In this case,

$$u_i'(y,t) = \gamma_{ij} u_j(x,t), \quad \gamma_{ij} = (e_i', e_j), \quad i,j = 1,2,3,$$

where (e_i', e_j) is the scalar product of the vectors e_i' and e_j.

It follows from the definition of the local Reynolds number that

$$(R_l(u))(x,t) = (R_l(u'))(y,t). \tag{2.17}$$

Therefore,

$$(\varphi_t(R_l(u)))(x,t) = (\varphi_t(R_l(u')))(y,t), \tag{2.18}$$

and the turbulent viscosity is invariant relative to the transformation (2.15).

In what follows, we consider the following modification (regularization) of the constitutive equation (2.10):

$$\sigma_{ij}(p,u) = -p\delta_{ij} + 2(\alpha(I(u))^{\frac{1+\beta}{2}} + \varphi(I(u)) + \varphi_t(R_l(u)))\varepsilon_{ij}(u). \tag{2.19}$$

Here α and β are small positive constants, $\alpha(I(u))^{\frac{1+\beta}{2}}$ is the regulating term.

We assume that for the nonlinear viscous fluid, the viscosity $\mu(x,t)$ in $R_l(u)$ is defined by a regularized velocity field, i.e. instead of (2.3), we use such relation

$$\mu(x,t) = (\varphi(I(u_\gamma)))(x,t), \tag{2.20}$$

where

$$\begin{gathered} u_\gamma(x,t) = \int_{\mathbb{R}^3} \omega_\gamma(|x-z|)\tilde{u}(z,t)\,dz, \\ \omega_\gamma \in C^\infty(\mathbb{R}_+), \quad \operatorname{supp}\ \omega_\gamma \subset [0,\gamma], \quad \omega_\gamma(y) \geq 0, \quad y \in \mathbb{R}_+, \\ \int_{\mathbb{R}^3} \omega_\gamma(|x|)\,dx = 1. \end{gathered} \tag{2.21}$$

Here γ is a small positive constant, and $\tilde{u}(.,t)$ is an extension of the function $u(.,t)$ to $\mathbb{R}^3$ which preserves the class of smoothness.

Formulas (2.21) and (2.20) define averaged values of the velocities and the laminar viscosity. As a result of regularization, we have $u_\gamma|_{S_1} \neq 0$, while $u|_{S_1} = 0$. However, this has no effect on the validity of our model, since $R_l(u)$ and $\varphi_t(R_l(u))$ are equal to zero in a vicinity of S_1.

Because the averaged velocities and stresses are calculated for turbulent flows, the relation (2.20) is natural from the physical point of view.

We mention that the functions φ, φ_t, and d can be defined by identification. In the case where the fluid is Newtonian and the boundary S_1 is plane, φ_t is identified only.

2.3. **Basic equations.** We denote by $\overset{*}{u}(x,t) = (\overset{*}{u}_1(x,t), \overset{*}{u}_2(x,t), \overset{*}{u}_3(x,t))$ the vector of transfer velocity, $\overset{*}{u}(x,t)$ is the velocity of a point x of the actual frame at an instant t in the immovable (inertial) frame x'.

In the general case, the function $\overset{*}{u}$ is of the form

$$\overset{*}{u}(x,t) = l(t) + w(x,t), \tag{2.22}$$

where $l(t) = (l_1(t), l_2(t), l_3(t))$ is the vector of translation velocity, and $w(x,t) = (w_1(x,t), w_2(x,t), w_3(x,t))$ the vector of rotational velocity,

$$\begin{aligned} w_1(x,t) &= \omega_2(t)x_3 - \omega_3(t)x_2, \\ w_2(x,t) &= \omega_3(t)x_1 - \omega_1(t)x_3, \\ w_3(x,t) &= \omega_1(t)x_2 - \omega_2(t)x_1, \end{aligned} \tag{2.23}$$

$\omega(t) = (\omega_1(t), \omega_2(t), \omega_3(t))$ being the vector of angular velocity.

The absolute velocity of the fluid is u^a,

$$u^a(x,t) = \overset{*}{u}(x,t) + u(x,t). \tag{2.24}$$

In the case under consideration that the actual frame x moves at a velocity $\overset{*}{u}$ given by (2.22) and (2.23), the motion equation is defined as follows:

$$\rho\Big(\frac{Du^a}{Dt} + (\operatorname{grad} u)u\Big) - \operatorname{div}\sigma(p,u) = K \text{ in } Q. \tag{2.25}$$

Here $Q = \Omega \times (0,T)$, $T = \text{constant} > 0$, $K = (K_1, K_2, K_3)$ is the volume force vector, $\frac{Du^a}{Dt}$ the total derivative with respect to time of the function of absolute velocity u^a .

According to the Coriolis theorem on composition of accelerations, we obtain

$$\frac{Du^a}{Dt} = \frac{\partial u}{\partial t} + \frac{dl}{dt} + \frac{d\omega}{dt} \times x + \omega \times (\omega \times x) + 2\omega \times u, \tag{2.26}$$

where $\times$ is the sign of vector product.

In (2.25) $\operatorname{grad} u$ is a tensor of the second order of the form

$$\operatorname{grad} u = \Big\{\frac{\partial u_i}{\partial x_j}\Big\}_{i,j=1}^{3},$$

and $\operatorname{div}\sigma(p,u)$ is a vector with components

$$\frac{\partial \sigma_{ij}(p,u)}{\partial x_j}, \quad i = 1,2,3.$$

We assume that l and ω are known functions and denote

$$G = \rho\Big(\frac{dl}{dt} + \frac{d\omega}{dt} \times x + \omega \times (\omega \times x)\Big). \tag{2.27}$$

By using the above formulas, we obtain the following representation of the motion equations:

$$\rho\Big(\frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x_j}u_j + 2(\omega_{i+1}u_{i+2} - \omega_{i+2}u_{i+1})\Big) - \frac{\partial \sigma_{ij}(p,u)}{\partial x_j} \tag{2.28}$$

$$= K_i - G_i \text{ in } Q, \quad i = 1,2,3. \tag{2.29}$$

Here we take $i+k$ equal to $i+k-3$ at $i+k > 3$, $k = 1,2$; the terms

$$2\rho(\omega_{i+1}u_{i+2} - \omega_{i+2}u_{i+1}), \quad i = 1,2,3,$$

are the components of the Coriolis force.

In the special case that the frame x is immovable (inertial), the function G and the Coriolis force are equal to zero, and we have

$$\rho\Big(\frac{\partial u_i}{\partial t}+\frac{\partial u_i}{\partial x_j}u_j\Big)-\frac{\partial \sigma_{ij}(p,u)}{\partial x_j}=K_i \text{ in } Q,\quad i=1,2,3. \tag{2.30}$$

Since the local Reynolds number is independent of the motion of the frame x with respect to any inertial frame, the motion equations (2.28), (2.30) in which $\sigma_{ij}(p,u)$ are defined by (2.10) or by (2.19), are invariant with respect to the Galilei transformation $x\to z$, where $z=(z_1,z_2,z_3)$ is an arbitrary inertial frame, $x=z+Vt$, where V is a constant velocity. In this case, one assumes that R_l is computed in the frame x, and the move of time in the frames x and z is identical, i.e. $t=t'$, t' is the time in the frame z.

It is presupposed that the fluid is incompressible

$$\operatorname{div} u=\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}=0 \text{ in } Q. \tag{2.31}$$

We consider two types of boundary conditions, the mixed conditions and the Dirichlet ones. In the case of mixed conditions, we prescribe the nonslip condition on S_1 and surface forces on S_2, i.e.

$$u|_{S_1\times(0,T)}=0, \tag{2.32}$$

$$\sigma_{ij}(p,u)\nu_j|_{S_2\times(0,T)}=F_i,\qquad i=1,2,3. \tag{2.33}$$

Here F_i and ν_j are components of the vector functions of surface force $F=(F_1,F_2,F_3)$ and unit outward normal $\nu=(\nu_1,\nu_2,\nu_3)$ to S_2, respectively.

In the case of the Dirichlet conditions, we set

$$u|_{S\times(0,T)}=\hat u. \tag{2.34}$$

The initial condition is the following:

$$u(x,0)=u_0(x),\quad x\in\Omega. \tag{2.35}$$

Be virtue of the nonslip condition, we can consider that

$$\hat u|_{S_1\times(0,T)}=0,\quad \text{and}\quad u_0|_{S_1}=0.$$

In investigation of the stationary problem, we suppose that the frame x moves with a constant velocity V relative to the frame x', in particular, it can be $V=0$. In this case, the motion equations have the form

$$\rho\,\frac{\partial u_i}{\partial x_j}u_j-\frac{\partial \sigma_{ij}(p,u)}{\partial x_j}=K_i \text{ in } \Omega,\quad i=1,\dots,n, \tag{2.36}$$

and the mixed boundary conditions are the following:

$$u|_{S_1}=0,\quad \sigma_{ij}(p,u)\nu_j|_{S_2}=F_i,\quad i=1,\dots,n. \tag{2.37}$$

We also consider the Dirichlet condition

$$u|_S=\hat u. \tag{2.38}$$

We mention that in the majority of publications on mathematical problems for the Navier-Stokes equations, the authors assumed that the velocity is equal to zero on the whole of the boundary of the domain of flow. However, in any real flow there are regions of the boundary in which fluid flows into and out, the velocity is nonzero in these regions.

So far as we know, there are no results on the solvability of the Navier-Stokes equations with nonhomogeneous Dirichlet boundary conditions (2.34) and (2.38) in the general case, where $\hat{u} \in W_2^{\frac{1}{2}}(S)^n$, $\int_S \hat{u}_i \nu_i \, ds = 0$. There are results for the cases of stationary flow, where there exists a vector function w such that $\operatorname{curl} w|_s = \hat{u}$, and where the values of $\hat{u}$ are small, see [25], Section 2, Chapter 5, and [27], Sections 3, 4, Chapter 3.

A problem on nonstationary flow of the viscous fluid with mixed boundary conditions (2.32), (2.33) is investigated in [20]. However, the nonlinear terms in the inertial forces are not taken into account in this work.

To the best of our knowledge, the problem with mixed boundary conditions (2.32), (2.33) for the full Navier-Stokes equations was not investigated. Although such boundary conditions are quite important for practical application.

We will now point out some features of the fluid model under consideration.

1. The adhesion (nonslip) condition is usually accepted for viscous fluids. This means that the velocity of a fluid at points of the hard boundary coincides with the velocity of points of the hard boundary. Because of this, the values of the local Reynolds number are small in a vicinity of the hard boundary and the turbulent viscosity equals zero. Hence, the constitutive equations (2.10) and (2.19) predict the existence of a laminar boundary layer in a vicinity of the hard boundary at turbulent flow of the fluid.

2. The equations (2.10) and (2.19) identify the areas of the laminar and the turbulent flow in the domain of flow, they enable us to describe special features of turbulent flows such as a drastic increase in the resistance to flow and the variation of the velocity profile with the increase of the Reynolds number, see Figures 2 and 3.

3. The equations (2.10) and (2.19) incorporate the curvature of the hard boundary and they describe the convex and concave surface curvature effects in wall-bounded turbulent flows, see (2.9). Therefore, these equations can be used for the simulation of curved and swirling turbulent flows.

4. The implementation of the non-inertial reference frame x enables us to solve flow problems in moving domains; in this case, the inertia forces induced by the motion of the domain of flow are taken into account.

5. The constitutive equation (2.19), which is the regularization of the relation (2.10), leads to well posed mathematical problems. We prove the existence of global solutions to stationary and nonstationary flow problems with the nonhomogeneous Dirichlet and mixed boundary conditions.

6. The Smagorinsky model, which is one of the most popular LES models, represents a particular case of our model. Indeed, at $\beta = 0$, $\varphi(I(u)) = \mu$, and $\varphi_t = 0$, we obtain (1.15) from (2.19), (2.30), and (2.31).

The above properties of our model from 1., 2., and 3. cannot be described by other models, in particular, by LANS-α equations.

Essentially, there are no results on solvability of the Navier-Stokes equations with nonhomogeneous boundary conditions for the velocity function. To the best of our knowledge, there are no whatsoever results on solvability of the Navier-Stokes equations for the most important engineering problem with mixed boundary conditions, where surface forces are prescribed on the inflow and the outflow of the fluid, and zero velocities are given on the remainder of the boundary of the domain of flow.

There are also no results on solvability of the LES and LANS-α equations with nonhomogeneous Dirichlet and mixed boundary conditions. However, for our model these results are contained in Section 3.

Our model cannot describe so well the turbulent structure as it does the LANS-α model. But on the other hand, it can describe complicated flows, not only viscous but also nonlinear viscous non-Newtonian fluids as well, in which there are areas of laminar and turbulent flows. It gives boundary effects, the distribution of averaged velocities, forces, the energy consumed in flow, and so on.

The model under consideration lets us obtain good approximations to the solutions of the corresponding problems, see Sections 6 and 8.

Thus, our new approach can compliment earlier approaches, in particular, LANS-α approach, for modeling complicated flows in real-world, specifically , engineering applications.

3. **Formulations of the problems and the main results.**

3.1. **Stationary problems.** We use the following spaces:

$$V = \{u | u = (u_1, \dots, u_n) \in W^1_{3+\beta}(\Omega)^n,\ \beta \in (0,1),\ u|_{S_1} = 0\}, \tag{3.1}$$

$$V_1 = \{u | u \in V,\ \operatorname{div} u = 0\}, \tag{3.2}$$

$$V_2 = \{u | u \in W^1_{3+\beta}(\Omega)^n,\ u|_S = 0\}, \tag{3.3}$$

$$V_3 = \{u | u \in V_2,\ \operatorname{div} u = 0\}. \tag{3.4}$$

In the sequel, we will use the following notations:

If Y is a normed space, we denote by Y^* the dual of Y, and by (f,h) the duality between Y^* and Y, where $f \in Y^*$, $h \in Y$. In particular, if $f \in L_2(\Omega)$ or $f \in L_2(\Omega)^n$, then (f,h) is the scalar product in $L_2(\Omega)$ or in $L_2(\Omega)^n$, respectively. $\mathcal{L}(X,Y)$ is the space of linear continuous mappings of a normed space X into Y.

Once $B(\Omega)$ is a normed space of functions which are given in Ω, we denote by $B_{loc}(\Omega)$ the set of all functions f given in Ω, which satisfy the condition: For an arbitrary subdomain $\Lambda \subset \Omega$ such that $\overline{\Lambda} \subset \Omega$, the restriction of f to Λ belongs to the space $B(\Lambda)$.

The sign $\rightharpoonup$ denotes weak convergence in Banach spaces.

Define an operator $N : V \to V^*$ as follows:

$$(N(u),h) = 2\int_\Omega \Big(\alpha(I(u))^{\frac{1+\beta}{2}} + \varphi(I(u)) + \varphi_t(R_l(u))\Big)\varepsilon_{ij}(u)\varepsilon_{ij}(h)\,dx$$
$$+\rho\int_\Omega u_j \frac{\partial u_i}{\partial x_j} h_i\,dx, \quad u,h \in V. \tag{3.5}$$

The functions of volume and surface forces are assumed to satisfy the following conditions:

$$K = (K_1, \dots, K_n) \in L_r(\Omega)^n, \tag{3.6}$$

$$F = (F_1, \dots, F_n) \in L_r(S_2)^n, \quad r > 1. \tag{3.7}$$

We consider the problem: Find a pair of functions u, p satisfying

$$(u,p) \in V_1 \times L_{\frac{3+\beta}{2+\beta}}(\Omega), \tag{3.8}$$

$$(N(u),h) - \int_\Omega p \operatorname{div} h\,dx = (G^1,h), \quad h \in V, \tag{3.9}$$

where

$$(G^1,h) = \int_\Omega K_i h_i\,dx + \int_{S_2} F_i h_i\,ds. \tag{3.10}$$

By using the Green formula, one can verify that the pair u,p resolving the problem (3.8), (3.9) is a weak solution of the problem (2.36), (2.31) wherein Q is changed for Ω, and (2.37) with $\sigma_{ij}(p,u)$ defined by (2.19).

In the case that the boundary condition is given by (2.38), we assume that there exists a function $\tilde{u}$ that satisfies the conditions

$$\tilde{u} \in W^1_{3+\beta}(\Omega)^n, \quad \operatorname{div} \tilde{u} = 0, \quad \tilde{u}|_S = \hat{u}. \tag{3.11}$$

With the proviso that the boundary S is out of the class C^2, for an arbitrary $\hat{u}$ such that

$$\hat{u} \in W^{1-\frac{1}{3+\beta}}_{3+\beta}(S)^n, \qquad \int_S \hat{u}_i \nu_i \, ds = 0, \tag{3.12}$$

there exists a function $\tilde{u}$ that meets the conditions (3.11), see Lemma 4.2 below.

We introduce an operator $L : V_2 \to V_2^*$ of the form

$$\begin{aligned}(L(v), h) =& 2\int_\Omega (\alpha(I(\tilde{u}+v))^{\frac{1+\beta}{2}} + \varphi(I(\tilde{u}+v)) + \varphi_t(R_l(\tilde{u}+v)))\varepsilon_{ij}(\tilde{u}+v)\varepsilon_{ij}(h)\, dx \\ &+\rho \int_\Omega (\tilde{u}_j + v_j)\frac{\partial(\tilde{u}_i + v_i)}{\partial x_j} h_i \, dx, \qquad v, h \in V_2. \end{aligned} \tag{3.13}$$

Consider the problem: Find a pair v, p such that

$$v \in V_3, \qquad p \in L_{\frac{3+\beta}{2+\beta}}(\Omega), \tag{3.14}$$

$$(L(v), h) - \int_\Omega p \operatorname{div} h \, dx = \int_\Omega K_i h_i \, dx, \quad h \in V_2. \tag{3.15}$$

If v, p is a solution of problem (3.14), (3.15), then the pair $u = \tilde{u}+v$, p is a weak solution of the problem (2.36), (2.31) wherein Q is changed for Ω, and (2.38).

Theorem 3.1. *Suppose that the conditions* $(C1)$–$(C4)$ *and* (3.6), (3.7) *are satisfied. Let the local Reynolds number be defined by* (2.1), *where μ is either a positive constant, or is given by* (2.20). *Then there exists a function u and a unique function p such that, the pair u, p is a solution of the problem* (3.8), (3.9).

Theorem 3.2. *Suppose that the conditions* $(C1)$–$(C4)$ *and* (3.6), (3.11) *are satisfied. Let the local Reynolds number be defined by* (2.1), *where μ is either a positive constant, or is given by* (2.20). *Then there exists a pair v, p which is a solution of the problem* (3.14), (3.15). *In this case p is defined within a constant addend.*

3.2. **Nonstationary problems.** We suppose that the functions of volume and surface forces and the initial data satisfy the conditions

$$K = (K_1, \dots, K_n) \in L_{\frac{3+\beta}{2+\beta}}(0, T; L_r(\Omega)^n), \tag{3.16}$$

$$F = (F_1, \dots, F_n) \in L_{\frac{3+\beta}{2+\beta}}(0, T; L_r(S_2)^n), \quad r > 1, \tag{3.17}$$

$$u_0 \in U, \tag{3.18}$$

where

$$U = \{h | h \in L_2(\Omega)^n, \ \operatorname{div} h = 0\}. \tag{3.19}$$

It is also assumed that the vector of transfer velocity $\overset{*}{u}$, that is defined by (2.22), (2.23), is known, and the function of turbulent viscosity is Lipschitz continuous, i.e.

$$|\varphi_t(y_1) - \varphi_t(y_2)| \le \xi \, |y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R}_+, \quad \xi = \text{constant} \ > 0. \tag{3.20}$$

We consider the problem: Find a pair u, p such that

$$u \in L_{3+\beta}(0,T;V_1), \quad \frac{du}{dt} \in L_{\frac{3+\beta}{2+\beta}}(0,T;V^*), \tag{3.21}$$

$$p \in L_{\frac{3+\beta}{2+\beta}}(Q), \tag{3.22}$$

$$\rho\Big(\frac{du}{dt}, h\Big) + (\tilde{N}u, h) + (N(u), h) - (p, \operatorname{div} h) = (\tilde{G}, h) \text{ in } D'(0,T), \quad h \in V, \tag{3.23}$$

$$u(0) = u_0. \tag{3.24}$$

Here

$$(\tilde{N}u, h) = 2\rho \sum_{i=1}^{3} \int_\Omega (\omega_{i+1}\, u_{i+2} - \omega_{i+2}\, u_{i+1})\, h_i\, dx, \tag{3.25}$$

$$(\tilde{G}, h) = (G^1, h) - (G, h). \tag{3.26}$$

In (3.25), we take $i+k$ equal to $i+k-3$ at $i+k>3$, $k=1,2$. In (3.26) G is defined by (2.27), and we suppose that

$$l \in W^1_{3+\beta}(0,T)^n, \qquad \omega \in W^1_{3+\beta}(0,T)^n. \tag{3.27}$$

The pair of functions u, p, which is a solution of the problem (3.21)–(3.24), is a weak solution of the problem (2.28), (2.31), (2.32), (2.33), (2.35).

Theorem 3.3. *Suppose that the conditions* $(C1)-(C4)$ *and* (3.16)–(3.18), (3.20), (3.27) *are satisfied. Let the local Reynolds number be defined by* (2.1), *where* μ *is either a positive constant, or is given by* (2.20). *Then there exists a solution of the problem* (3.21)–(3.24).

In the case that the boundary condition is given by (2.34), we assume that there exists a function $\tilde{u}$ which complies with the conditions

$$\begin{aligned} &\tilde{u} \in L_{3+\beta}(0,T;W^1_{3+\beta}(\Omega)^n), \\ &\frac{d\tilde{u}}{dt} \in L_{\frac{3+\beta}{2+\beta}}(0,T;V_2^*), \quad \operatorname{div}\tilde{u} = 0, \quad \tilde{u}|_\Gamma = \hat{u}, \end{aligned} \tag{3.28}$$

where $\Gamma = S \times (0,T)$.

We consider the problem: Find a pair v, p such that

$$v \in L_{3+\beta}(0,T;V_3), \quad \frac{dv}{dt} \in L_{\frac{3+\beta}{2+\beta}}(0,T;V_2^*), \tag{3.29}$$

$$p \in L_{\frac{3+\beta}{2+\beta}}(Q), \tag{3.30}$$

$$\rho\Big(\frac{dv}{dt}, h\Big) + (\tilde{N}(\tilde{u}+v), h) + (L(v), h) - (p, \operatorname{div} h) = (G^2, h) \text{ in } D'(0,T), \quad h \in V_2, \tag{3.31}$$

$$v(0) = u_0 - \tilde{u}(0). \tag{3.32}$$

Here

$$(G^2, h) = \int_\Omega K_i h_i\, dx - (G, h) - \Big(\frac{d\tilde{u}}{dt}, h\Big). \tag{3.33}$$

If v, p is a solution of the problem (3.29)–(3.32), then the pair $u = \tilde{u} + v, p$ is a weak solution of the problem (2.28), (2.31), (2.34), and (2.35).

Theorem 3.4. *Suppose that the conditions* $(C1) - (C4)$ *and* (3.16), (3.18), (3.20), (3.27), (3.28) *are satisfied. Let the local Reynolds number be defined by* (2.1), *where* μ *is either a positive constant, or is given by* (2.20). *Then there exists a solution of the problem* (3.29)–(3.32).

For simplicity sake, we do not consider the case where the boundary of the domain of flow contains a movable hard part, but the domain of flow is independent of time. The local Reynolds number is defined by (2.8) in this case. As will be seen from the following presentation, the Theorems 3.1–3.4 can be extended to this case.

4. **Auxiliary results.**

4.1. **Equivalent norms.**

Lemma 4.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$, $n = 2$ *or 3, with a Lipschitz continuous boundary* S. *Let* S_1 *be an open nonempty subset of* S *and* $1 < l < \infty$. *Then the expression*

$$||u||_1 = \Big(\int_\Omega (I(u))^{\frac{l}{2}}\, dx \Big)^{\frac{1}{l}} + \int_{S_1} |u|\, ds \tag{4.1}$$

defines a norm in the space $W^1_l(\Omega)^n$, *that is equivalent to the main norm of* $W^1_l(\Omega)^n$.

Proof. It is known, see [34], that the norm

$$||u||_2 = \Big(\int_\Omega (I(u))^{\frac{l}{2}}\, dx \Big)^{\frac{1}{l}} + ||u||_{L_l(\Omega)^n} \tag{4.2}$$

is equivalent to the main norm of $W^1_l(\Omega)^n$.

Therefore, it is sufficient to show that there exists a constant c such that

$$||u||_{L_l(\Omega)^n} \le c||u||_1, \qquad u \in W^1_l(\Omega)^n. \tag{4.3}$$

Indeed, it follows from (4.3) that there exists a constant c_1 such that

$$||u||_2 \le c_1||u||_1, \qquad u \in W^1_l(\Omega)^n.$$

Therefore,

$$||u||_{W^1_l(\Omega)^n} \le c_2||u||_1, \qquad u \in W^1_l(\Omega)^n.$$

The inverse inequality follows from the triangle inequality.

Suppose that (4.3) is false. Then there exists a sequence $\{u_m\}$ that satisfies the conditions

$$||u_m||_{L_l(\Omega)^n} = 1, \qquad \Big(\int_\Omega (I(u_m))^{\frac{l}{2}}\ dx \Big)^{\frac{1}{l}} \to 0, \qquad \int_{S_1} |u_m|\, ds \to 0. \tag{4.4}$$

It follows from (4.2) and (4.4) that the sequence $\{u_m\}$ is bounded in $W^1_l(\Omega)^n$. Hence, a subsequence $\{u_k\}$ can be extracted such that

$$u_k \to u_0 \quad \text{in} \quad L_l(\Omega)^n, \tag{4.5}$$

and, in addition,

$$\liminf \Big(\int_\Omega (I(u_k))^{\frac{l}{2}}\ dx \Big)^{\frac{1}{l}} \ge \Big(\int_\Omega (I(u_0))^{\frac{l}{2}}\ dx \Big)^{\frac{1}{l}} = 0, \tag{4.6}$$

$$\liminf \int_{S_1} |u_k|\, ds \ge \int_{S_1} |u_0|\, ds = 0. \tag{4.7}$$

By virtue of (4.6), the function u_0 belongs to the well known space of rigid displacements, and it has the following form at $n = 3$:

$$u_0(x) = a + Ax, \tag{4.8}$$

where

$$u_0 = \left\{ \begin{array}{c} u_{01} \\ u_{02} \\ u_{03} \end{array} \right\}, \quad a = \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right\}, \quad A = \left\{ \begin{array}{ccc} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{array} \right\}. \tag{4.9}$$

Here a_i and b_i are constants.

Since S_1 has a positive two-dimensional area, there exists three points $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ of S_1 such that the vectors $x^{(1)} - x^{(2)}$ and $x^{(1)} - x^{(3)}$ are linearly independent. (4.7) and (4.8) imply

$$Ax^{(1)} = Ax^{(2)} = Ax^{(3)} = -a.$$

Therefore

$$A(x^{(1)} - x^{(2)}) = 0, \qquad A(x^{(1)} - x^{(3)}) = 0. \tag{4.10}$$

Thus, the rank of the matrix A does not exceed unit, and all the minors of A of the second order are equal to zero. It follows here from that $b_1 = b_2 = b_3 = 0$, and (4.7), (4.8) yield $a_1 = a_2 = a_3 = 0$, i.e. $u_0 = 0$. However, (4.4) and (4.5) imply that $||u_0||_{L_l(\Omega)^n} = 1$. The two last relations are contradictory. Therefore, (4.3) is true, and our lemma is proved.

We assign the following norm in V

$$||u||_V = \Big(\int_\Omega (I(u))^{\frac{3+\beta}{2}} \, dx \Big)^{\frac{1}{3+\beta}}. \tag{4.11}$$

It follows from Lemma 4.1 that the norm $||.||_V$ is equivalent to the main norm of the space $W^1_{3+\beta}(\Omega)^n$.

4.2. **Renewal of functions.**

Lemma 4.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with a boundary S of the class C^2. Let g be a function from the space $W_{3+\beta}^{1-\frac{1}{3+\beta}}(S)^n$ that meets the condition*

$$\int_S g_i \nu_i \, ds = 0. \tag{4.12}$$

Then there exists a function v such that

$$\begin{gathered} v \in W^1_{3+\beta}(\Omega)^n, \qquad \operatorname{div} v = 0, \qquad v|_S = g, \\ ||v||_{W^1_{3+\beta}(\Omega)^n} \le c||g||_{W_{3+\beta}^{1-\frac{1}{3+\beta}}(S)^n}, \end{gathered} \tag{4.13}$$

where c is independent of g.

Proof. We consider the problem: Find $\psi \in W^2_{3+\beta}(\Omega)$ satisfying

$$\Delta\psi = \operatorname{div} \operatorname{grad} \psi = 0 \quad \text{in} \quad \Omega, \tag{4.14}$$

$$\frac{\partial \psi}{\partial \nu} = g_\nu \quad \text{on} \quad S, \tag{4.15}$$

where $g_\nu = g \cdot \nu = g_i \nu_i$.

By virtue of (4.12) and (4.14), there exists a solution of the problem (4.14), (4.15), which is defined within a constant addend and such that, see [40], Theorem 5.3.1,

$$||\nabla\psi||_{W^1_{3+\beta}(\Omega)^n} \le c_1 ||g_\nu||_{W^{1-\frac{1}{3+\beta}}_{3+\beta}(S)}. \tag{4.16}$$

Consider the problem: Find $h \in W^1_{3+\beta}(\Omega)^n$ which satisfies the conditions:

$$\operatorname{div} h = 0, \qquad h|_S = g - \nabla\psi|_S. \tag{4.17}$$

(4.15) and (4.16) yield

$$(g - \nabla\psi|_S)\cdot\nu = 0, \quad (g - \nabla\psi|_S) \in W^{1-\frac{1}{3+\beta}}_{3+\beta}(S)^n. \tag{4.18}$$

By virtue of the known results, see [25], Chapter 1, Section 2, relations (4.18) imply that there exists a function θ which complies with the following conditions:

$$\theta \in W^2_{3+\beta}(\Omega)^n, \quad \operatorname{curl}\theta|_S = g - \nabla\psi|_S, \tag{4.19}$$

$$||\operatorname{curl}\theta||_{W^1_{3+\beta}(\Omega)^n} \le c_2 ||g - \nabla\psi|_S||_{W^{1-\frac{1}{3+\beta}}_{3+\beta}(S)^n}. \tag{4.20}$$

The function $h = \operatorname{curl}\theta$ is a solution of the problem (4.17). It follows from (4.14), (4.16), (4.19), and (4.20) that the function $v = \nabla\psi + \operatorname{curl}\theta$ meets the conditions (4.13). This completes the proof.

4.3. **Operators div and grad.** We introduce the following spaces:

$$Y = \{f | f \in V_2^*,\ (f,u) = 0,\ u \in V_3\}, \tag{4.21}$$

$$\overset{\circ}{L}_e(\Omega) = \{q | q \in L_e(\Omega),\ \int_\Omega q(x)\,dx = 0\}, \tag{4.22}$$

where $e \in (1,\infty)$.

Lemma 4.3. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with a boundary S of the class C^2 . Then the operator* div *is an isomorphism of the factor space V_2/V_3 onto $\overset{\circ}{L}_{3+\beta}(\Omega)$, and the operator* grad*, that is adjoint to the operator* div*, is an isomorphism of $\overset{\circ}{L}_{\frac{3+\beta}{2+\beta}}(\Omega)$ onto Y.*

Proof. Let $v \in V_2$. The Green formula yields

$$\int_\Omega \operatorname{div} v\,dx = \int_S v_i\nu_i\,ds = 0.$$

Thus

$$\operatorname{div} \in \mathcal{L}(V_2, \overset{\circ}{L}_{3+\beta}(\Omega)).$$

Let us show that the operator div maps V_2/V_3 onto the whole of $\overset{\circ}{L}_{3+\beta}(\Omega)$.

Let g be a function from $\overset{\circ}{L}_{3+\beta}(\Omega)$. Since the boundary S is of the class C^2, there exists $\theta \in W^2_{3+\beta}(\Omega)$ such that

$$\Delta\theta = g \ \text{ in } \ \Omega, \qquad \frac{\partial\theta}{\partial\nu}\Big|_S = 0. \tag{4.23}$$

The function $h = \nabla\theta$ belongs to the space $W^1_{3+\beta}(\Omega)^n$ and satisfies the conditions

$$\operatorname{div} h = \Delta\theta = g \ \text{ in } \ \Omega, \quad h|_S\cdot\nu = 0, \quad h|_S \in W^{1-\frac{1}{3+\beta}}_{3+\beta}(S)^n. \tag{4.24}$$

By Lemma 4.2, there exists a function v satisfying

$$v \in W^1_{3+\beta}(\Omega)^n, \quad \operatorname{div} v = 0, \quad v|_S = h|_S. \tag{4.25}$$

The function $u = h - v$ belongs to V_2 and $\operatorname{div} u = g$.

Therefore, the operator div maps V_2 onto $\overset{\circ}{L}_{3+\beta}(\Omega)$, and it is a one-to-one mapping of V_2/V_3 onto $\overset{\circ}{L}_{3+\beta}(\Omega)$.

It follows from the Banach theorem on inverse operator, see e.g. [44], Chapter 2, Section 5, that the inverse operator div^{-1} is a continuous mapping of $\overset{\circ}{L}_{3+\beta}(\Omega)$ onto V_2/V_3.

Thus, the operator div is an isomorphism of V_2/V_3 onto $\overset{\circ}{L}_{3+\beta}(\Omega)^n$. It follows here from, see [18], Chapter 3, Theorem 5.30, that the operator grad^{-1}, that is the inverse of the operator grad, is a linear continuous mapping of Y onto $\overset{\circ}{L}_{\frac{3+\beta}{2+\beta}}(\Omega)$. Therefore, the operator grad is an isomorphism of $\overset{\circ}{L}_{\frac{3+\beta}{2+\beta}}(\Omega)$ onto Y. ■

Lemma 4.4. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with a boundary S of the class C^2 . Then the operator* div *is an isomorphism of the factor space V/V_1 onto $L_{3+\beta}(\Omega)$, and the operator* div^*, *that is adjoint of* div, *is an isomorphism of $L_{\frac{3+\beta}{2+\beta}}(\Omega)$ onto W, where*

$$W = \{f | f \in V^*, \ (f, v) = 0, \ v \in V_1\}. \tag{4.26}$$

Moreover, there exists a constant $\zeta > 0$ such that

$$\inf_{\chi \in L_{\frac{3+\beta}{2+\beta}}(\Omega)} \sup_{v \in V} \frac{\int_\Omega \chi \operatorname{div} v \, dx}{||v||_V ||\chi||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)}} \geq \zeta, \tag{4.27}$$

$$|| \operatorname{div}^{-1} ||_{\mathcal{L}(L_{3+\beta}(\Omega), V/V_1)} \leq \frac{1}{\zeta}, \tag{4.28}$$

$$||(\operatorname{div}^*)^{-1}||_{\mathcal{L}(W, L_{\frac{3+\beta}{2+\beta}}(\Omega))} \leq \frac{1}{\zeta}, \tag{4.29}$$

where div^{-1} *and* $(\operatorname{div}^*)^{-1}$ *are the inverse operators of* div *and* div^*, *respectively.*

Proof. Let us show that there exists a function w satisfying

$$w \in V, \quad \operatorname{div} w = 1. \tag{4.30}$$

Take a function g such that

$$g \in V, \quad \operatorname{supp} g \subset \Omega \cup S_2, \quad \int_{S_2} g_i \nu_i \, ds = a \neq 0. \tag{4.31}$$

By the Green formula and (4.31), we obtain

$$\int_\Omega \Big(\frac{\operatorname{mes} \Omega}{a} \operatorname{div} g - 1 \Big) \, dx = 0. \tag{4.32}$$

Lemma 4.3 and (4.31), (4.32) imply that there exists a function u which meets the conditions

$$u \in V_2, \quad \operatorname{div} u = \frac{\operatorname{mes} \Omega}{a} \operatorname{div} g - 1. \tag{4.33}$$

Therefore, the function

$$w = \frac{\operatorname{mes}\Omega}{a} g - u \tag{4.34}$$

complies with (4.30).

By virtue of Lemma 4.3 and (4.30), the operator div maps V onto $L_{3+\beta}(\Omega)$. Owing to this, the inequality (4.28) follows from the Banach theorem on inverse operator. Thus the operator div is an isomorphism of V/V_1 onto $L_{3+\beta}(\Omega)$.

The space $(V/V_1)^*$ can be identified with W. Taking into account the following equalities:

$$\begin{gathered}(\operatorname{div}^{-1})^* = (\operatorname{div}^*)^{-1},\\ ||\operatorname{div}^{-1}||_{\mathcal{L}(L_{3+\beta}(\Omega),\,V/V_1)} = ||(\operatorname{div}^{-1})^*||_{\mathcal{L}(W,L_{\frac{3+\beta}{2+\beta}}(\Omega))},\end{gathered} \tag{4.35}$$

see [17], Chapter 12, Section 2, or [14], Section 6.5, we obtain (4.29) from (4.28).

Hence

$$||\operatorname{div}^*\chi||_W = \sup_{v\in V/V_1} \frac{(v,\operatorname{div}^*\chi)}{||v||_{V/V_1}} \geq \zeta||\chi||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)}, \quad \chi \in L_{\frac{3+\beta}{2+\beta}}(\Omega), \tag{4.36}$$

and (4.27) follows from (4.36). ■

Lemma 4.5. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with a Lipschitz continuous boundary S. Let $p, R, b,$ and g be four functions such that*

$$p \in L_{\frac{3+\beta}{2+\beta},\mathrm{loc}}(\Omega), \quad R = \{R_{ik}\}_{i,k=1}^n, \quad R_{ik} = R_{ki}, \quad R_{ik} \in L_{\frac{3+\beta}{2+\beta}}(\Omega), \tag{4.37}$$

$$b = \{b_{ik}\}_{i,k=1}^n, \quad b_{ik} \in L_{\frac{3+\beta}{2+\beta}}(\Omega), \quad g = (g_1,\dots,g_n) \in L_e(\Omega)^n, \quad e > 1, \tag{4.38}$$

$$\begin{gathered}\int_\Omega R_{ik}\varepsilon_{ik}(h)\,dx + \int_\Omega b_{ik}\frac{\partial h_i}{\partial x_k}\,dx - \int_\Omega p\operatorname{div} h\,dx = \int_\Omega g_i h_i\,dx,\\ h \in W^1_{3+\beta}(\Omega)^n, \quad \operatorname{supp} h \subset \Omega.\end{gathered} \tag{4.39}$$

Then $p \in L_{\frac{3+\beta}{2+\beta}}(\Omega)$ and

$$\begin{aligned}||p||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)} \leq c\Big(&\sum_{i,k=1}^n ||R_{ik}||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)} + \sum_{i,k=1}^n ||b_{ik}||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)}\\ &+ \sum_{i=1}^n ||g_i||_{L_e(\Omega)} + ||p||_{L_{\frac{3+\beta}{2+\beta}}(\Omega\setminus\overline{\Omega}_\delta)}\Big),\end{aligned} \tag{4.40}$$

where

$$\Omega_\delta = \{x | x \in \Omega,\ r(x,S) < \delta\}, \tag{4.41}$$

δ is a small positive constant, $r(x,S)$ is defined by (2.4) at $S = S_1$, and c depends on δ, but independent of R, b, and g.

Proof. The following lemma is proved in [26]: Let Ω be a bounded domain in $\mathbb{R}^3$, with a Lipschitz continuous boundary S. Let p, $A = \{A_{ik}\}_{i,k=1}^3$, $g = \{g_i\}_{i=1}^3$ be three functions such that

$$p \in W^1_{2,\mathrm{loc}}(\Omega),\ \ A_{ik} \in W^1_2(\Omega),\ \ g_i \in L_{\frac{6}{5}}(\Omega),\ \ i,k = 1,2,3, \tag{4.42}$$

$$\frac{\partial p}{\partial x_i} = \frac{\partial A_{ik}}{\partial x_k} + g_i \ \text{ in } \Omega, \quad i = 1,2,3. \tag{4.43}$$

Then $p \in L_2(\Omega)$ and

$$||p||_{L_2(\Omega)} \le c_\delta \Big(\sum_{i,k=1}^{3} ||A_{ik}||_{L_2(\Omega)} + \sum_{i=1}^{3} ||g_i||_{L_{\frac{6}{5}}(\Omega)} + ||p||_{L_2(\Omega \setminus \overline{\Omega}_\delta)} \Big), \tag{4.44}$$

where c_δ is independent of A_{ik} and g_i.

It follows from the proof of this lemma and the Calderon-Zygmund theorem on singular integrals (see [5], [16], Chapter 1, Section 2) and the Sobolev theorem on integrals with weak singularity, that the statement of this lemma remains true in the cases where $n = 2$ and $n = 3$, and the index 2 in (4.42) and (4.44) is changed for an arbitrary q such that $1 < q < \infty$, and the space $L_{\frac{6}{5}}(\Omega)$ for g_i is changed for the space $L_e(\Omega)$ such that $L_e(\Omega) \subset W^{-1}_q(\Omega)$. The relation $L_e(\Omega) \subset W^{-1}_q(\Omega)$ denotes that we identify an element $f \in L_e(\Omega)$ with the functional $\tilde{f} \in W^{-1}_q(\Omega)$ that is given by $(\tilde{f}, h) = \int_\Omega f h \, dx$, $h \in W^1_{\frac{q}{q-1}}(\Omega)$, $\quad h|_S = 0$.

We consider the case that $q = \frac{3+\beta}{2+\beta}$ and $e > 1$.

Let (4.37), (4.38), and (4.39) be satisfied, and let p_γ, $R_{ik\gamma}$, $b_{ik\gamma}$, and $g_{i\gamma}$ be the regularized functions p, R_{ik}, b_{ik}, and g_i which are defined as it is in (2.21).

Let also $\{\Omega_j\}_{j=1}^\infty$ be a sequence of subdomains of Ω satisfying

$$\overline{\Omega}_\delta \subset \Omega_j, \quad \overline{\Omega}_j \subset \Omega, \quad \overline{\Omega}_j \subset \Omega_{j+1}, \quad \cup_{j=1}^\infty \Omega_j = \Omega.$$

For any j there exists $\gamma_j > 0$ such that $r(x,S) > \gamma_j$ for all $x \in \overline{\Omega}_j$.

Since the operators of regularization and differentiation commute with each other, we obtain from (4.39) that

$$\frac{\partial p_\gamma}{\partial x_i} = \frac{\partial R_{ik\gamma}}{\partial x_k} + \frac{\partial b_{ik\gamma}}{\partial x_k} + g_{i\gamma} \ \text{ in } \ \Omega_j \text{ at } \gamma < \gamma_j, \quad i = 1,\dots,n. \tag{4.45}$$

Thus, the functions p_γ, $R_{ik\gamma}$, $b_{ik\gamma}$, $g_{i\gamma}$ satisfy the above conditions in Ω_j, and hence,

$$\begin{aligned} ||p_\gamma||_{L_{\frac{3+\beta}{2+\beta}}(\Omega_j)} \le c\Big(&\sum_{i,k=1}^{n} ||R_{ik\gamma}||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)} + \sum_{i,k=1}^{n} ||b_{ik\gamma}||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)} \\ &+ \sum_{i=1}^{n} ||g_{i\gamma}||_{L_e(\Omega)} + ||p_\gamma||_{L_{\frac{3+\beta}{2+\beta}}(\Omega \setminus \overline{\Omega}_\delta)} \Big). \end{aligned}$$

We pass here to the limit as γ tends to zero. This gives

$$\begin{aligned} ||p||_{L_{\frac{3+\beta}{2+\beta}}(\Omega_j)} \le c\Big(&\sum_{i,k=1}^{n} ||R_{ik}||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)} + \sum_{i,k=1}^{n} ||b_{ik}||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)} \\ &+ \sum_{i=1}^{n} ||g_i||_{L_e(\Omega)} + ||p||_{L_{\frac{3+\beta}{2+\beta}}(\Omega \setminus \overline{\Omega}_\delta)} \Big), \quad j \in \mathbb{N}. \end{aligned}$$

Therefore, (4.40) holds and our lemma is proved.

Theorem 4.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with a Lipschitz continuous boundary S. Then the operator* div *is an isomorphism of V_2/V_3 onto $\overset{\circ}{L}_{3+\beta}(\Omega)$, and the operator* $\mathrm{grad} = \mathrm{div}^*$ *is an isomorphism of $\overset{\circ}{L}_{\frac{3+\beta}{2+\beta}}(\Omega)$ onto $Y = (V_2/V_3)^*$. Moreover, there exists a constant $\zeta_1 > 0$ such that*

$$\inf_{\chi\in\overset{\circ}{L}_{\frac{3+\beta}{2+\beta}}(\Omega)} \sup_{v\in V_2} \frac{\int_\Omega \chi \,\mathrm{div}\, v\, dx}{||v||_V ||\chi||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)}} \geq \zeta_1,$$

$$||\,\mathrm{div}^{-1}\,||_{\mathcal{L}(\overset{\circ}{L}_{3+\beta}(\Omega), V_2/V_3)} \leq \frac{1}{\zeta_1},$$

$$||\,\mathrm{grad}^{-1}\,||_{\mathcal{L}((V_2/V_3)^*,\, \overset{\circ}{L}_{\frac{3+\beta}{2+\beta}}(\Omega))} \leq \frac{1}{\zeta_1}. \tag{4.46}$$

Proof. Let us show that for an arbitrary $f \in (V_2/V_3)^*$ there exists a unique function $p \in \overset{\circ}{L}_{\frac{3+\beta}{2+\beta}}(\Omega)$ satisfying

$$(f, h) = \int_\Omega p \,\mathrm{div}\, h\, dx, \quad h \in V_2. \tag{4.47}$$

There exists a sequence $\{\Omega_i\}$ of subdomains of Ω such that

$$\overline{\Omega}_i \subset \Omega, \quad \Omega_i \subset \Omega_{i+1}, \quad \cup_{i=1}^\infty \Omega_i = \Omega, \tag{4.48}$$

and the boundaries of Ω_i are of the class C^2.

Let $\{V_2^i\}$ and $\{V_3^i\}$ be sequences of subspaces of V_2 and V_3, which are given as follows:

$$V_2^i = \{v | v \in V_2,\ \mathrm{supp}\, v \subset \overline{\Omega}_i\}, \quad V_3^i = \{v | v \in V_3,\ \mathrm{supp}\, v \subset \overline{\Omega}_i\}. \tag{4.49}$$

Let f be an arbitrary element of $(V_2/V_3)^*$. We apply Lemma 4.3 in which Ω, V_2 and V_3 are changed for Ω_i, V_2^i, and V_3^i. This gives the existence of a unique function $p_i \in \overset{\circ}{L}_{\frac{3+\beta}{2+\beta}}(\Omega_i)$ such that

$$(f, h) = (p_i, \mathrm{div}\, h), \quad h \in V_2^i, \quad i \in \mathbb{N}. \tag{4.50}$$

In this case $p_i = p_{i+j}|_{\Omega_i} + c_{i+j}$, where c_{i+j} is a constant, $j \geq 1$.

By (4.48) there exists a function $\tilde{p} \in L_{\frac{3+\beta}{2+\beta},\mathrm{loc}}(\Omega)$, satisfying

$$\tilde{p}|_{\Omega_i} = p_i + \tilde{c}_i, \tag{4.51}$$

$$(f, h) = (\tilde{p}, \mathrm{div}\, h), \quad h \in V_2, \quad \mathrm{supp}\, h \subset \Omega, \tag{4.52}$$

$\tilde{c}_i$ being a constant.

The functional f can be presented in the form (see [1], Theorem 3.8)

$$(f, h) = \int_\Omega b_{0i} h_i\, dx + \int_\Omega b_{ik} \frac{\partial h_i}{\partial x_k}\, dx, \quad h \in V_2, \tag{4.53}$$

where b_{0i} and b_{ik} are elements of $L_{\frac{3+\beta}{2+\beta}}(\Omega)$ which meet the condition $(f, h) = 0$ for any $h \in V_3$. We mention that the representation (4.53) is not unique.

By (4.48), there exists $l \in \mathbb{N}$ whereby $\Omega_l \supset (\Omega \setminus \Omega_\delta)$, where Ω_δ is a subdomain of Ω defined in (4.41). Taking (4.52), (4.53) into account, and applying Lemma 4.5, we obtain that $\tilde{p} \in L_{\frac{3+\beta}{2+\beta}}(\Omega)$. Therefore, there exists a unique $p \in \overset{\circ}{L}_{\frac{3+\beta}{2+\beta}}(\Omega)$, $p = \tilde{p} + \tilde{c}$, $\tilde{c}$ is a constant, such that (4.47) holds. In this case, the operator $A : f \to Af = p$ is a linear continuous mapping of $(V_2/V_3)^*$ into $\overset{\circ}{L}_{\frac{3+\beta}{2+\beta}}(\Omega)$.

Conversely, the equality (4.47) defines a mapping $\overset{\circ}{L}_{\frac{3+\beta}{2+\beta}}(\Omega) \ni p \to f \in (V_2/V_3)^*$. Therefore, the operator A maps $(V_2/V_3)^*$ onto the whole of $\overset{\circ}{L}_{\frac{3+\beta}{2+\beta}}(\Omega)$. Because of this, the Banach theorem on inverse operator implies that the operator A is an isomorphism of $(V_2/V_3)^*$ onto $\overset{\circ}{L}_{\frac{3+\beta}{2+\beta}}(\Omega)$.

It is obvious that $A = \text{grad}^{-1}$. Therefore, the third inequality of (4.46) holds. The other inequalities of (4.46) are proved in the same way, as it is done in the proof of Lemma 4.4. ■

Remark. It can be proved that the statement of Theorem 4.1 remains true in the case where the indices $3+\beta$ and $\frac{3+\beta}{2+\beta}$ in the spaces V_2, V_3 and in the statement are changed for any $l \in (1,\infty)$ and $l_1 = l/(l-1)$ respectively.

Theorem 4.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with a Lipschitz continuous boundary S. Then the operator* div *is an isomorphism of V/V_1 onto $L_{3+\beta}(\Omega)$, and the operator* div^*, *is an isomorphism of $L_{\frac{3+\beta}{2+\beta}}(\Omega)$ onto $(V/V_1)^*$. In addition, the inequalities* (4.27), (4.28), *and* (4.29) *are satisfied.*

Proof. By the Theorem 4.1, we have $\text{div}\{V_2\} = \overset{\circ}{L}_{3+\beta}(\Omega)$, in addition, there exists a function w that satisfies (4.30). Therefore, the operator div maps V onto $L_{3+\beta}(\Omega)$, and div is an isomorphism of V/V_1 onto $L_{3+\beta}(\Omega)$. By analogy with the proof of the Lemma 4.4, we obtain that the operator div^* is an isomorphism of $L_{\frac{3+\beta}{2+\beta}}(\Omega)$ onto $(V/V_1)^*$ and (4.27), (4.28), (4.29) are satisfied. ■

Remark. The statement of Theorem 4.2 remains true in the case that the indices $3+\beta$ and $\frac{3+\beta}{2+\beta}$ in the spaces V, V_1 and in the statement are changed for any $l \in (1,\infty)$ and $l_1 = l/(l-1)$, respectively.

4.4. **Operators N_i.** We consider the following operators $N_i : V \to V^*$, $i = 1,2,3,4$.

$$(N_1(v), h) = 2\alpha \int_\Omega (I(v))^{\frac{1+\beta}{2}} \varepsilon_{ij}(v)\varepsilon_{ij}(h)\, dx, \tag{4.54}$$

$$(N_2(v), h) = 2 \int_\Omega \varphi(I(v))\varepsilon_{ij}(v)\varepsilon_{ij}(h)\, dx, \tag{4.55}$$

$$(N_3(v), h) = 2 \int_\Omega \varphi_t(R_l(v))\varepsilon_{ij}(v)\varepsilon_{ij}(h)\, dx, \tag{4.56}$$

$$(N_4(v), h) = \rho \int_\Omega v_j \frac{\partial v_i}{\partial x_j} h_i\, dx. \tag{4.57}$$

It is obvious that

$$N = \sum_{i=1}^{4} N_i. \tag{4.58}$$

Lemma 4.6. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with a Lipschitz continuous boundary S. Then the operator N_1 is a bounded, monotone, and continuous mapping of V into V^*.*

Proof. By using the inequality

$$|\varepsilon_{ij}(v)\varepsilon_{ij}(h)| \le (I(v))^{\frac{1}{2}}(I(h))^{\frac{1}{2}} \tag{4.59}$$

and the Hölder inequality with the numbers $\frac{3+\beta}{2+\beta}$ and $3+\beta$, we obtain from (4.54) that

$$\begin{aligned}|(N_1(v),h)| &\le 2\alpha \int_\Omega \big(I(v)\big)^{\frac{2+\beta}{2}} \big(I(h)\big)^{\frac{1}{2}}\, dx \le 2\alpha\Big(\int_\Omega \big(I(v)\big)^{\frac{3+\beta}{2}}\, dx\Big)^{\frac{2+\beta}{3+\beta}} \\ &\times\Big(\int_\Omega \big(I(h)\big)^{\frac{3+\beta}{2}}\, dx\Big)^{\frac{1}{3+\beta}} = 2\alpha\, ||v||_V^{2+\beta}\, ||h||_V. \end{aligned}\tag{4.60}$$

Therefore, N_1 is a bounded mapping of V into V^*.

The application of the inequality (4.59) gives

$$\begin{aligned}(N_1(v)-N_1(w),v-w) &= 2\alpha\int_\Omega \Big[\big(I(v)\big)^{\frac{1+\beta}{2}} I(v) + \big(I(w)\big)^{\frac{1+\beta}{2}} I(w) \\ &-\big(I(v)\big)^{\frac{1+\beta}{2}}\varepsilon_{ij}(v)\varepsilon_{ij}(w) - \big(I(w)\big)^{\frac{1+\beta}{2}}\varepsilon_{ij}(w)\varepsilon_{ij}(v)\Big]\, dx \\ &\ge 2\alpha\int_\Omega \Big[\big(I(v)\big)^{\frac{1+\beta}{2}}\big(I(v)\big)^{\frac{1}{2}} - \big(I(w)\big)^{\frac{1+\beta}{2}}\big(I(w)\big)^{\frac{1}{2}}\Big] \\ &\times\Big[\big(I(v)\big)^{\frac{1}{2}} - \big(I(w)\big)^{\frac{1}{2}}\Big]\, dx \ge 0, \quad v,w\in V.\end{aligned}$$

Hence N_1 is a monotone operator.

Taking into account that

$$\Big(\sum_{m=1}^{k} a_m\Big)^q \le \sum_{m=1}^{k}(ka_m)^q, \quad a_m\in\mathbb{R}_+, \quad q\ge 1,$$

we obtain

$$\begin{aligned}\Big|\big((I(v))(x)\big)^{\frac{1+\beta}{2}}(\varepsilon_{lm}(v))(x)\Big| &\le \big((I(v))(x)\big)^{\frac{2+\beta}{2}} \le n^{2+\beta}\sum_{i,j=1}^{n}|(\varepsilon_{ij}(v))(x)|^{2+\beta} \\ &\le c\sum_{i,j=1}^{n}\Big|\frac{\partial v_i}{\partial x_j}(x)\Big|^{2+\beta}, \qquad v\in V, \qquad l,m=1,\dots,n.\end{aligned}\tag{4.61}$$

From (4.61) and the continuity of the Nemytskii operator, see [43], Sections 5.1, 25.1, it follows that the condition $v_m\to v$ in V implies

$$\big(I(v_m)\big)^{\frac{1+\beta}{2}}\varepsilon_{ij}(v_m) \to \big((I(v)\big)^{\frac{1+\beta}{2}}\varepsilon_{ij}(v) \ \text{ in } \ L_{\frac{3+\beta}{2+\beta}}(\Omega), \quad i,j=1,\dots,n.$$

Therefore, N_1 is a continuous mapping of V into V^*.

Lemma 4.7. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n=2$ or 3, with a Lipschitz continuous boundary S. Let also the condition $(C4)$ be satisfied. Then the operator N_2 is a strictly monotone and continuous mapping of V into V^*.*

Proof. Taking (2.14) and (4.59) into account, we obtain from (4.55) that

$$
\begin{aligned}
(N_2(v) - N_2(w), v - w) = 2 \int_\Omega \big[\varphi\big(I(v)\big)I(v) + \varphi\big(I(w)\big)I(w) \\
-\varphi\big(I(v)\big)\varepsilon_{ij}(v)\varepsilon_{ij}(w) - \varphi\big(I(w)\big)\varepsilon_{ij}(w)\varepsilon_{ij}(v)\big]\, dx \\
\geq 2 \int_\Omega \big[\varphi\big(I(v)\big)\big(I(v)\big)^{\frac{1}{2}} - \varphi\big(I(w)\big)\big(I(w)\big)^{\frac{1}{2}}\big]\big[\big(I(v)\big)^{\frac{1}{2}} - \big(I(w)\big)^{\frac{1}{2}}\big]\, dx \\
\geq 2a_3 \int_\Omega \big(\big(I(v)\big)^{\frac{1}{2}} - \big(I(w)\big)^{\frac{1}{2}}\big)^2\, dx \geq 0, \quad v, w \in V. \qquad (4.62)
\end{aligned}
$$

Let now

$$
(N_2(v) - N_2(w), v - w) = 0. \tag{4.63}
$$

By (4.62), we have

$$
I(v) = I(w) \;\text{ a.e. in } \Omega, \quad \varphi(I(v)) = \varphi(I(w)) \;\text{ a.e. in } \Omega. \tag{4.64}
$$

Taking (2.13), (4.55), and (4.64) into account, we derive from (4.63) that

$$
I(v - w) = 0 \;\text{ a.e. in } \Omega.
$$

Therefore $||v - w||_V = 0$, and the operator N_2 is strictly monotone.

The continuity of the operator N_2 follows from the continuity of the Nemytskii operator.■

Lemma 4.8. *Suppose that the conditions* $(C1)$, $(C2)$, $(C3)$, *and* (2.13) *are satisfied. Let* μ *in* (2.1) *be either a positive constant or be given by* (2.20). *Then the terms* $\{v_m\} \subset V$, $v_m \rightharpoonup v$ *in* V *imply* $N_3(v_m) \rightharpoonup N_3(v)$ *in* V^*.

Proof. Let $v_m \rightharpoonup v$ in V. Then we have

$$
\varepsilon_{ij}(v_m) \rightharpoonup \varepsilon_{ij}(v) \text{ in } L_{3+\beta}(\Omega), \quad i, j = 1, ..., n, \tag{4.65}
$$

$$
v_m \to v \;\text{ in } C(\overline{\Omega})^n, \tag{4.66}
$$

$$
d(k(v_m, \cdot),\; r(\cdot, S_1)) \to d(k(v, \cdot),\; r(\cdot, S_1)) \;\text{ in } C(\overline{\Omega}), \tag{4.67}
$$

in addition,

$$
\varphi(I(v_{m\gamma})) \to \varphi(I(v_\gamma)) \;\text{ in } C(\overline{\Omega}), \tag{4.68}
$$

where $v_{m\gamma}$ and v_γ are the regularized functions v_m and v, that are defined by the formula (2.21).

(4.66)–(4.68) and (2.1) yield

$$
R_l(v_m) \to R_l(v) \;\text{ in } C(\overline{\Omega}). \tag{4.69}
$$

$(C3)$ and (4.69) imply $\varphi_t(R_l(v_m)) \to \varphi_t(R_l(v))$ in $C(\overline{\Omega})$, and the application of the Lebesgue theorem gives

$$
\varphi_t(R_l(v_m))\varepsilon_{ij}(h) \to \varphi_t(R_l(v))\varepsilon_{ij}(h)) \;\text{ in } L_{3+\beta}(\Omega), \quad h \in V, \quad i, j = 1, \ldots, n. \tag{4.70}
$$

Now by (4.65) and (4.70), we obtain

$$
\lim_{m\to\infty} \int_\Omega \Big(\varphi_t(R_l(v_m))\varepsilon_{ij}(v_m) - \varphi_t(R_l(v))\varepsilon_{ij}(v)\Big)\varepsilon_{ij}(h)\, dx = 0, \quad h \in V
$$

This completes the proof.

Lemma 4.9. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with a Lipschitz continuous boundary S. Let $\{v_m\} \subset V$, and $v_m \rightharpoonup v$ in V. Then $N_4(v_m) \to N_4(v)$ in V^* and, in addition,*

$$|(N_4(w), h)| \leq c||w||_V^2\, ||h||_V, \quad w, h \in V. \tag{4.71}$$

Proof. Let

$$v_m \rightharpoonup v \ \text{ in } V. \tag{4.72}$$

It is obvious that

$$|(N_4(v_m) - N_4(v), h)| \leq \rho\, (A_{1m} + A_{2m}), \tag{4.73}$$

where

$$A_{1m} = \Big| \int_\Omega (v_{mj} - v_j) \frac{\partial v_{mi}}{\partial x_j}\, h_i\, dx \Big|, \quad A_{2m} = \Big| \int_\Omega (v_j \Big(\frac{\partial v_{mi}}{\partial x_j} - \frac{\partial v_i}{\partial x_j} \Big)\, h_i\, dx \Big|. \tag{4.74}$$

Here v_{mj} and v_{mi} are components of the vector function v_m.

We have

$$A_{1m} \leq c||v_m - v||_{C(\overline{\Omega})^n} ||v_m||_V\, ||h||_V. \tag{4.75}$$

The application of the Green formula gives

$$A_{2m} \leq \Big| \int_\Omega \Big[\frac{\partial v_j}{\partial x_j} (v_{mi} - v_i) h_i + v_j (v_{mi} - v_i) \frac{\partial h_i}{\partial x_j} \Big]\, dx \Big| + \Big| \int_S v_j\, (v_{mi} - v_i)\, h_i \nu_j\, ds \Big|. \tag{4.76}$$

It follows from (4.72) that

$$v_m \to v \ \text{ in } C(\overline{\Omega})^n. \tag{4.77}$$

(4.75), (4.76), and (4.77) imply

$$A_{1m} + A_{2m} \leq \alpha_m ||h||_V, \quad \lim \alpha_m = 0. \tag{4.78}$$

Now (4.73) and (4.78) yield $N_4(v_m) \to N_4(v)$ in V^*. The inequality (4.71) is evident. ■

We will use the following known result (see [6], [21], Theorem 5.1, Chapter 1):

Lemma 4.10. *Let B_0, B, B_1 be three Banach spaces such that $B_0 \subset B \subset B_1$, B_0 and B_1 are reflexive, and the embedding of B_0 in B is compact. Let also*

$$\mathcal{W} = \{v | v \in L_{q_0}(0, T; B_0),\ \frac{dv}{dt} \in L_{q_1}(0, T; B_1)\}, \tag{4.79}$$

where T is finite, $1 < q_i < \infty$, $i = 0, 1$, and the norm in $\mathcal{W}$ is defined by

$$||v||_{\mathcal{W}} = ||v||_{L_{q_0}(0,T;B_0)} + ||\frac{dv}{dt}||_{L_{q_1}(0,T;B_1)}. \tag{4.80}$$

Then the embedding of $\mathcal{W}$ into $L_{q_0}(0, T; B)$ is compact.

Lemma 4.11. *Let the local Reynolds number $R_l(u)$ be defined by (2.1), where μ is given by (2.20). Suppose that the conditions $(C1) - (C4)$ and (3.20)are satisfied. Let also*

$$v \in L_{3+\beta}(0, T; V),\ \{u_m\} \subset L_{3+\beta}(0, T; V),\ u_m \to u \ \textit{ in } L_{3+\beta}(Q)^n \ \textit{ and a.e. in } Q. \tag{4.81}$$

Then

$$\lim_{m \to \infty} ||(\varphi_t(R_l(u_m)) - \varphi_t(R_l(u)))\varepsilon_{ij}(v)||_{L_{\frac{3+\beta}{2+\beta}}(Q)} = 0, \quad i, j = 1, \ldots, n. \tag{4.82}$$

Proof. By applying (3.20) and the Hölder inequality with the numbers $\frac{2+\beta}{1+\beta}$ and $2+\beta$, we obtain

$$\begin{aligned}
&\Big(\int_Q\Big|\Big(\varphi_t(R_l(u_m))-\varphi_t(R_l(u))\Big)\varepsilon_{ij}(v)\Big|^{\frac{3+\beta}{2+\beta}}\,dx\,dt\Big)^{\frac{2+\beta}{3+\beta}}\\
&\le\xi\Big(\int_Q\Big|R_l(u_m)-R_l(u)\Big|^{\frac{3+\beta}{2+\beta}}|\varepsilon_{ij}(v)|^{\frac{3+\beta}{2+\beta}}\,dx\,dt\Big)^{\frac{2+\beta}{3+\beta}}\\
&\le\xi\Big(\int_Q\Big|R_l(u_m)-R_l(u)\Big|^{\frac{3+\beta}{1+\beta}}\,dx\,dt\Big)^{\frac{1+\beta}{3+\beta}}\Big(\int_Q|\varepsilon_{ij}(v)|^{3+\beta}\,dx\,dt\Big)^{\frac{1}{3+\beta}}.
\end{aligned}\tag{4.83}$$

Therefore, our lemma will be proved, if we argue that

$$R_l(u_m)\to R_l(u)\ \text{ in } L_{\frac{3+\beta}{1+\beta}}(Q).\tag{4.84}$$

Denote

$$(g(w))(x,t)=\frac{d(k(w,x,t),\ r(x,S_1))\rho}{(\varphi(I(w_\gamma)))(x,t)},\qquad w\in L_{3+\beta}(0,T;V).\tag{4.85}$$

(2.1) and (4.85) yield

$$R_l(u_m)=|u_m|g(u_m),\qquad R_l(u)=|u|g(u).\tag{4.86}$$

Since $|\,|u_m|-|u|\,|\le|u_m-u|$, the relation (4.81) implies

$$|u_m|\to|u|\ \text{ in } L_{3+\beta}(Q).\tag{4.87}$$

By (C1), (C2), and (C4), the functions $(x,t)\to(g(u_m))(x,t)$ are bounded in $L_\infty(Q)$. By (4.81), we have $g(u_m)\to g(u)$ a.e. in Q. Therefore,

$$g(u_m)\to g(u)\ \text{ in } L_{\frac{3+\beta}{\beta}}(Q).\tag{4.88}$$

The function $e,h\to eh$ is a bilinear continuous mapping of $L_{3+\beta}(Q)\times L_{\frac{3+\beta}{\beta}}(Q)$ into $L_{\frac{3+\beta}{1+\beta}}(Q)$. Because of this (4.84) follows from (4.86), (4.87), and (4.88). ■

Lemma 4.12. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n=2$ or 3, with a Lipschitz continuous boundary S. Then the operators N_1 and N_4 are bounded mappings of $L_{3+\beta}(0,T;V)$ into $L_{\frac{3+\beta}{2+\beta}}(0,T;V^*)$. The operator N_2 also is a bounded mapping of $L_{3+\beta}(0,T;V)$ into $L_{\frac{3+\beta}{2+\beta}}(0,T;V^*)$ provided that (2.13) holds.*

Proof. Let v and h be elements of $L_{3+\beta}(0,T;V)$. Taking into account (4.60) and applying the Hölder inequality with the numbers $(3+\beta)/(2+\beta)$ and $(3+\beta)$, we obtain

$$\begin{aligned}
|(N_1(v),h)|&\le 2\alpha\int_0^T||v(t)||_V^{2+\beta}\,||h(t)||_V\,dt\\
&\le 2\alpha\,||v||^{2+\beta}_{L_{3+\beta}(0,T;V)}\,||h||_{L_{3+\beta}(0,T;V)}.
\end{aligned}\tag{4.89}$$

(4.71) and the Hölder inequality with the same numbers $(3+\beta)/(2+\beta)$ and $(3+\beta)$ yield

$$\begin{aligned}
|(N_4(v),h)|&\le c\int_0^T||v(t)||_V^2\,||h(t)||_V\,dt\\
&\le c||v||^2_{L_{\frac{2(3+\beta)}{2+\beta}}(0,T;V)}\,||h||_{L_{3+\beta}(0,T;V)}.
\end{aligned}\tag{4.90}$$

By (2.13) and (4.55), we get

$$|(N_2(v),h)| \leq 2a_2 \int_Q (I(v))^{\frac{1}{2}} \, (I(h))^{\frac{1}{2}} \, dx \, dt$$
$$\leq c_1 ||v||_{L_{3+\beta}(0,T;V)} \, ||h||_{L_{3+\beta}(0,T;V)}. \tag{4.91}$$

and our lemma is proved.

Lemma 4.13. *Suppose that the conditions* $(C1)$, $(C2)$, $(C3)$, (2.13) *and* (3.20) *are satisfied. Let the local Reynolds numbers be defined by* (2.1) *where* μ *is either a positive constant, or is given by* (2.20). *Then the operator* N_3 *is a bounded mapping of* $L_{3+\beta}(0,T;V)$ *into* $L_{\frac{3+\beta}{2+\beta}}(0,T;V^*)$.

Proof. Let v and h be elements of $L_{3+\beta}(0,T;V)$. We take in (3.20) $y_1 = (R_l(v))(x,t)$, $(x,t) \in Q$, $y_2 = 0$. Then by (2.12), we receive

$$\varphi_t(R_l(v)) \leq \xi R_l(v). \tag{4.92}$$

(2.1), $(C2)$, and (2.13) imply

$$R_l(v) \leq c|v|. \tag{4.93}$$

(4.56), (4.92), and (4.93) yield

$$|(N_3(v),h)| \leq c_1 \int_Q |v| (I(v))^{\frac{1}{2}} \, (I(h))^{\frac{1}{2}} \, dx \, dt$$
$$\leq c_1 \Big(\int_Q |v|^{\frac{3+\beta}{1+\beta}} \, dx \, dt \Big)^{\frac{1+\beta}{3+\beta}} ||v||_{L_{3+\beta}(0,T;V)} \, ||h||_{L_{3+\beta}(0,T;V)}$$
$$\leq c_2 ||v||^2_{L_{3+\beta}(0,T;V)} \, ||h||_{L_{3+\beta}(0,T;V)}. \tag{4.94}$$

This completes the proof.

5. **Proof of Theorem 3.1.**

5.1. **Approximate solutions.** It is apparent that if a pair u, p is a solution of the problem (3.8), (3.9), then u is a solution of the following problem:

$$u \in V_1, \quad (N(u),h) = (G^1,h), \quad h \in V_1. \tag{5.1}$$

Let $\{V_{1k}\}$ be a sequence of finite dimensional subspaces of V_1 such that

$$\lim_{k\to\infty} \inf_{z \in V_{1k}} ||v - z||_V = 0, \quad v \in V_1, \tag{5.2}$$
$$V_{1k} \subset V_{1k+1}, \quad k \in \mathbb{N}. \tag{5.3}$$

We seek an approximate solution of the problem (5.1) in the form

$$u_k \in V_{1k}, \quad (N(u_k),h) = (G^1,h), \quad h \in V_{1k}. \tag{5.4}$$

Taking into account $(C2)$, $(C3)$, $(C4)$, (3.6), (3.7) and (4.71), we obtain

$$(N(v),v) - (G^1,v) \geq 2\alpha ||v||_V^{3+\beta} + 2a_1 \int_\Omega I(v) \, dx - c||v||_V^3$$
$$-||G^1||_{V^*} \, ||v||_V, \quad v \in V_1. \tag{5.5}$$

Here c is the constant from (4.71).

It follows from (5.5) that there exists a constant $R > 0$ such that

$$(N(v), v) - (G^1, v) \geq 0 \quad \text{if } ||v||_V \geq R. \tag{5.6}$$

Let $j(k)$ be the dimension of V_{1k} and $(w_1, \dots, w_{j(k)})$ be a basis of V_{1k}. The function $J_k : \mathbb{R}^{j(k)} \ni \xi = (\xi_1, \dots, \xi_{j(k)}) \to J_k\xi = \sum_{i=1}^{j(k)} \xi_i w_i$ is an isomorphism of $\mathbb{R}^{j(k)}$ onto V_{1k}. Define an operator P_k that maps $\mathbb{R}^{j(k)}$ into $\mathbb{R}^{j(k)}$ as follows:

$$P_k(\xi) = b = (b_1, \dots, b_{j(k)}), \qquad b_i = (N(J_k\xi) - G^1, w_i). \tag{5.7}$$

It follows from (5.6) and (5.7) that

$$(P_k(\xi), \xi) = (N(J_k\xi) - G^1, J_k\xi) \geq 0 \quad \text{if } |\xi| \geq R_1, \tag{5.8}$$

where $|\xi| = \left(\sum_{i=1}^{j(k)} \xi_i^2 \right)^{\frac{1}{2}}$ and R_1 is a positive constant that depends on k.

(5.8) and the corollary of the Brower fixed point theorem, see [21], Chapter 1, Lemma 4.3, yield the existence of a solution u_k to problem (5.4) for any k.

(5.4) implies

$$(N(u_k), u_k) = (G^1, u_k) \leq ||G^1||_{V^*} \, ||u_k||_V. \tag{5.9}$$

Since $(N(v), v) \geq \alpha ||v||_V^{3+\beta}$ at large $||v||_V$, we deduce from (5.9) that there exists a constant $c_1 > 0$ such that

$$||u_k||_V \leq c_1, \quad k \in \mathbb{N}. \tag{5.10}$$

Therefore,

$$||N_1(u_k)||_{V^*} \leq c_2, \qquad ||N_2(u_k)||_{V^*} \leq c_3, \tag{5.11}$$

and we can extract a subsequence $\{u_m\}$ such that

$$u_m \rightharpoonup u_0, \quad \text{in } V_1, \tag{5.12}$$

$$N_1(u_m) + N_2(u_m) \rightharpoonup \chi \quad \text{in } V^*. \tag{5.13}$$

5.2. **Passage to the limit.** Let m_0 be a fixed positive integer, and let $h \in V_{1m_0}$. Taking Lemmas 4.8, 4.9 and (5.12), (5.13) into account, we pass to the limit in (5.4) with k changed for m; this gives

$$(\chi + N_3(u_0) + N_4(u_0), h) = (G^1, h), \quad h \in V_{1m_0}. \tag{5.14}$$

Since m_0 is an arbitrary positive integer, (5.2) and (5.3) yield

$$(\chi + N_3(u_0) + N_4(u_0), h) = (G^1, h), \quad h \in V_1. \tag{5.15}$$

For an arbitrary fixed $w \in V_1$, we define a mapping $N_w : V_1 \to V_1^*$ as follows:

$$(N_w(v), h) = 2 \int_\Omega \varphi_t(R_l(w)) \varepsilon_{ij}(v) \varepsilon_{ij}(h) \, dx, \quad v, h \in V_1. \tag{5.16}$$

It is obvious that $N_v(v) = N_3(v)$. Let

$$Z_m = N_{u_m} + N_1 + N_2. \tag{5.17}$$

Lemmas 4.6 and 4.7 imply

$$(Z_m(u_m) - Z_m(v), u_m - v) \geq 0, \quad v \in V_1, \quad m \in \mathbb{N}. \tag{5.18}$$

It follows from the proof of Lemma 4.8 and (5.12) that

$$\lim(Z_m(v), v) = (Z_0(v), v), \tag{5.19}$$

$$\lim(Z_m(v), u_m) = (Z_0(v), u_0). \tag{5.20}$$

Taking into account Lemma 4.8 and relations (5.12), (5.13), (5.15), and (5.17), we obtain

$$\lim(Z_m(u_m), v) + (N_4(u_0), v) = (G^1, v), \qquad v \in V_1. \tag{5.21}$$

Lemma 4.9, (5.4), and (5.12) yield

$$(Z_m(u_m), u_m) = (G^1, u_m) - (N_4(u_m), u_m) \to (G^1, u_0) - (N_4(u_0), u_0). \tag{5.22}$$

Upon (5.19)–(5.22), we pass to the limit in (5.18). This gives

$$(G^1 - Z_0(v) - N_4(u_0), u_0 - v) \geq 0, \qquad v \in V_1. \tag{5.23}$$

Take here $v = u_0 - \xi h$, $\xi > 0$, $h \in V_1$, and let ξ tends to zero. Then we obtain

$$(G^1 - Z_0(u_0) - N_4(u_0), h) \geq 0, \qquad h \in V_1.$$

Therefore,

$$(N(u_0) - G^1, h) = -(G^1 - Z_0(u_0) - N_4(u_0), h) = 0, \qquad h \in V_1, \tag{5.24}$$

and the function $u = u_0$ is a solution of the problem (5.1).

It follows from (5.24) that $N(u_0) - G^1 \in W$, and by Theorem 4.2 there exists a unique $p \in L_{\frac{3+\beta}{2+\beta}}(\Omega)$ such that

$$(N(u_0) - G^1, h) = (p, \operatorname{div} h), \qquad h \in V.$$

Therefore, the pair $u = u_0$, p is a solution to the problem (3.8), (3.9). ■

The proof of Theorem 3.2 is closely analogous to the proof of Theorem 3.1. Because of this, it is not given.

6. **Approximation of the velocity and pressure for problem** (3.8) (3.9).

We consider a method for simultaneous calculation of approximate velocity and pressure.

Let $\{A_k\}$ and $\{B_k\}$ be sequences of finite-dimensional subspaces of V and $L_{\frac{3+\beta}{2+\beta}}(\Omega)$, such that

$$\lim_{k\to\infty} \inf_{z \in A_k} ||v - z||_V = 0, \qquad v \in V, \tag{6.1}$$

$$\lim_{k\to\infty} \inf_{\chi \in B_k} ||w - \chi||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)} = 0, \qquad w \in L_{\frac{3+\beta}{2+\beta}}(\Omega), \tag{6.2}$$

$$A_k \subset A_{k+1}, \qquad B_k \subset B_{k+1}. \tag{6.3}$$

Define operators $\operatorname{div}_k \in \mathcal{L}(A_k, B_k^*)$ as follows:

$$(\operatorname{div}_k v, \chi) = \int_\Omega \chi \operatorname{div} v \, dx, \qquad v \in A_k, \quad \chi \in B_k. \tag{6.4}$$

The adjoint operator of div_k is given by

$$(\operatorname{div}_k^* \chi, v) = (\operatorname{div}_k v, \chi), \qquad v \in A_k, \quad \chi \in B_k. \tag{6.5}$$

It is evident that $\operatorname{div}_k^* \in \mathcal{L}(B_k, A_k^*)$.

We introduce the following spaces $\overset{\circ}{A}_k$ and W_k

$$\overset{\circ}{A}_k = \{v | v \in A_k,\ (\operatorname{div}_k v, \chi) = 0,\ \chi \in B_k\}, \tag{6.6}$$

$$W_k = \{q | q \in A_k^*,\ (q, v) = 0,\ v \in \overset{\circ}{A}_k\}. \tag{6.7}$$

Lemma 6.1. *Let $\{A_k\}$ and $\{B_k\}$ be sequences of finite-dimensional subspaces of V and $L_{\frac{3+\beta}{2+\beta}}(\Omega)$, respectively. Suppose that there exists a positive constant γ such that*

$$\inf_{\chi \in B_k} \sup_{v \in A_k} \frac{(\operatorname{div}_k v, \chi)}{||v||_V \, ||\chi||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)}} \geq \gamma, \qquad k \in \mathbb{N}. \tag{6.8}$$

Then the operator div_k is an isomorphism of $A_k/\overset{\circ}{A}_k$ onto B_k^*, and the operator div_k^* is an isomorphism of B_k onto W_k, moreover

$$|| \operatorname{div}_k^{-1} ||_{\mathcal{L}(B_k^*,\, A_k/\overset{\circ}{A}_k)} \leq \frac{1}{\gamma}, \qquad k \in \mathbb{N}, \tag{6.9}$$

$$||(\operatorname{div}_k^*)^{-1}||_{\mathcal{L}(W_k,\, B_k)} \leq \frac{1}{\gamma}, \qquad k \in \mathbb{N}. \tag{6.10}$$

Proof. It follows from (6.8) that

$$\sup_{v \in A_k} \frac{(v, \operatorname{div}_k^* \chi)}{||v||_V} \geq \gamma \, ||\chi||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)}, \quad \chi \in B_k.$$

Therefore,

$$|| \operatorname{div}_k^* \chi)||_{A_K^*} \geq \gamma \, ||\chi||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)}, \quad \chi \in B_k. \tag{6.11}$$

and div_k^* is an isomorphism of B_k onto its range $\mathcal{R}(\operatorname{div}_k^*)$.

It is obvious that $\mathcal{R}(\operatorname{div}_k^*)$ is a closed subspace of A_k^*. Consequently, $\mathcal{R}(\operatorname{div}_k^*) = W_k$, see [17], Chapter XII, Section 2, and (6.10) follows from (6.11).

Taking into account that $(\operatorname{div}_k^{-1})^* = (\operatorname{div}_k^*)^{-1}$ and

$$|| \operatorname{div}_k^{-1} ||_{\mathcal{L}(B_k^*,\, A_k/\overset{\circ}{A}_k)} = ||(\operatorname{div}_k^{-1})^*||_{\mathcal{L}(W_k,\, B_k)},$$

we obtain (6.9) from (6.10).■

We seek an approximate solution of the problem (3.8), (3.9) in the form

$$(u_k, p_k) \in A_k \times B_k, \tag{6.12}$$

$$\Big(N(u_k), h\Big) - \int_\Omega p_k \operatorname{div}_k h \, dx = \Big(G^1, h\Big), \quad h \in A_k, \tag{6.13}$$

$$(\operatorname{div}_k u_k, q) = 0, \quad q \in B_k. \tag{6.14}$$

Theorem 6.1. *Suppose that the conditions $(C1)$–$(C4)$ and (3.6), (3.7) are satisfied. Let the local Reynolds number be defined by (2.1), where μ is either a positive constant or is given by (2.20). Let also $\{A_k\}$ and $\{B_k\}$ be sequences of finite-dimensional subspaces of V and $L_{\frac{3+\beta}{2+\beta}}(\Omega)$ which satisfy the conditions (6.1), (6.2), (6.3), and (6.8). Then for an arbitrary k, there exists a solution of the problem (6.12)–(6.14), and a subsequence $\{u_m, p_m\}$ can be extracted from the sequence $\{u_k, p_k\}$ such that $u_m \rightharpoonup u$ in V, $u_m \to u$ in $C(\overline{\Omega})^n$, $p_m \rightharpoonup p$ in $L_{\frac{3+\beta}{2+\beta}}(\Omega)$, where u, p is a solution of the problem (3.8), (3.9).*

Proof. It follows from (6.6) and (6.12)–(6.14) that the function u_k is a solution of the problem

$$u_k \in \overset{\circ}{A}_k, \qquad (N(u_k), h) = (G^1, h), \qquad h \in \overset{\circ}{A}_k. \tag{6.15}$$

By analogy with the proof of Theorem 3.1, it is argued that there exists a solution of the problem (6.15) for any k and, in addition,

$$||u_k||_V \leq c, \qquad k \in \mathbb{N}, \tag{6.16}$$

$$||N_i(u_k)||_{V^*} \leq c_1, \qquad i = 1, 2, 3, 4, \qquad k \in \mathbb{N}. \tag{6.17}$$

For an arbitrary $f \in V^*$, we denote by $Y_k f$ the restriction of f to A_k. In this case, $Y_k f \in A_k^*$, $Y_k \in \mathcal{L}(V^*, A_k^*)$.

It follows from (6.7) and (6.15) that $Y_k(N(u_k) - G^1) \in W_k$, and by Lemma 6.1, there exists a unique $p_k \in B_k$ such that

$$\mathrm{div}_k^* \, p_k = Y_k(N(u_k) - G^1). \tag{6.18}$$

Thus, the pair u_k, p_k is a solution of the problem (6.12)–(6.14).

Due to (6.10), (6.17), (6.18), (3.6) and (3.7), we obtain

$$||p_k||_{L_{\frac{3+\beta}{2+\beta}}(\Omega)} \leq c_2. \tag{6.19}$$

By (6.16), (6.17) and (6.19), we can extract a subsequence $\{u_m, p_m\}$ satisfying

$$u_m \rightharpoonup u_0 \text{ in } V \quad \text{and} \quad u_m \to u_0 \text{ in } C(\overline{\Omega})^n, \tag{6.20}$$

$$p_m \rightharpoonup p_0 \text{ in } L_{\frac{3+\beta}{2+\beta}}(\Omega), \tag{6.21}$$

$$N_1(u_m) + N_2(u_m) \rightharpoonup \chi \text{ in } V^*. \tag{6.22}$$

We pass to the limit in (6.13), (6.14) with k changed for m in much the same way as it is carried out in the proof of Theorem 3.1. In so doing, we use (6.20)–(6.22) and lemmas 4.6–4.9. Then we obtain that the pair $u = u_0$, $p = p_0$ is a solution of the problem (3.8), (3.9).

7. **Proof of Theorem 3.3.**

7.1. **Approximate solutions.** It is obvious that if u, p is a solution of problem (3.21)–(3.24), then u is a solution of the following problem:

$$u \in L_{3+\beta}(0, T; V_1), \qquad \frac{du}{dt} \in L_{\frac{3+\beta}{2+\beta}}(0, T; V^*), \tag{7.1}$$

$$\rho(\frac{du}{dt}, h) + (\tilde{N}u, h) + (N(u), h) = (\tilde{G}, h) \text{ in } D'(0, T), \quad h \in V_1, \tag{7.2}$$

$$u(0) = u_0. \tag{7.3}$$

Let $w_1, \dots, w_k, \dots$ be a sequence of functions such that

$$\begin{aligned} &w_i \in V_1, \qquad i \in \mathbb{N}, \\ &w_1, \dots, w_k \text{ are linearly independent}, \quad k \in \mathbb{N}, \\ &\text{linear combinations of } w_i \text{ are dense in } V_1. \end{aligned} \tag{7.4}$$

We seek an approximate solution of the problem (7.1)–(7.3) in the form

$$u_k(t) = \sum_{i=1}^{k} g_{ik}(t) w_i, \tag{7.5}$$

where g_{ik} are defined out of the conditions

$$\rho\,\frac{d}{dt}(u_k(t),w_i)+(\tilde{N}u_k(t),w_i)+(N(u_k(t)),w_i)=(\tilde{G}(t),w_i),\quad i=1,\dots,k, \tag{7.6}$$

$$u_k(0)=u_{0k},\quad u_{0k}=\sum_{i=1}^{k}\alpha_{ik}w_i\to u_0 \quad \text{in} \quad L_2(\Omega)^n. \tag{7.7}$$

The functions u_k are computed from these conditions on some interval $[0,t_k]$, $t_k>0$. We will show that $t_k=T$.

7.2. **A priory estimates.** We multiply the equations (7.6) by the functions g_{ik}, sum over i, and integrate both sides of the sum over t from 0 to t. This gives

$$\begin{aligned}\frac{1}{2}\,\rho\,||u_k(t)||^2_{L_2(\Omega)^n}&+\int_0^t\Big(\tilde{N}u_k(\tau)+N(u_k(\tau)),u_k(\tau)\Big)\,d\tau\\&=\int_0^t\Big(\tilde{G}(\tau),u_k(\tau)\Big)\,d\tau+\frac{1}{2}\,\rho\,||u_k(0)||^2_{L_2(\Omega)^n}.\end{aligned} \tag{7.8}$$

Taking the assumptions $(C3)$, $(C4)$, and Lemma 4.9 into account, we obtain

$$(N(v),v)\geq 2\alpha||v||_V^{3+\beta}-c||v||_V^3,\quad v\in V_1. \tag{7.9}$$

(3.25) and (3.27) yield

$$|(\tilde{N}v,v)|\leq c_1||v||^2_{L_2(\Omega)^n}\leq c_2||v||_V^2,\quad v\in V_1. \tag{7.10}$$

For an arbitrary $z\in C([0,T];V_1)$, bearing in mind (7.9) and (7.10), we get

$$\int_0^t\Big(\tilde{N}z(\tau)+N(z(\tau)),z(\tau)\Big)\,d\tau\geq\alpha\int_0^t||z(\tau)||_V^{3+\beta}\,d\tau+Y(t), \tag{7.11}$$

where

$$Y(t)=\int_0^t(\alpha\,||z(\tau)||_V^{3+\beta}-c\,||z(\tau)||_V^3-c_2\,||z(\tau)||_V^2)\,d\tau\geq -c_3,\quad t\in(0,T], \tag{7.12}$$

$$c_3=c_4T,\quad c_4=-\min_{y\in\mathbb{R}_+}(\alpha\,y^{3+\beta}-c\,y^3-c_2\,y^2)>0. \tag{7.13}$$

Granting (7.11) and (7.12), we obtain from (7.8) that

$$\begin{aligned}&\frac{1}{2}\rho||u_k(t)||^2_{L_2(\Omega)^n}+\alpha\int_0^t||u_k(\tau)||_V^{3+\beta}\,d\tau\\&\leq\int_0^t||\tilde{G}(\tau)||_{V^*}\,||u_k(\tau)||_V\,d\tau+\frac{1}{2}\rho||u_k(0)||^2_{L_2(\Omega)^n}+c_3\\&\leq\frac{\alpha}{2}\int_0^t||u_k(\tau)||_V^{3+\beta}\,d\tau+c_4\int_0^T||\tilde{G}(\tau)||_{V^*}^{\frac{3+\beta}{2+\beta}}\,d\tau+\frac{1}{2}\rho||u_k(0)||^2_{L_2(\Omega)^n}+c_3.\end{aligned} \tag{7.14}$$

Here we used the following Young inequality:

$$ab\leq\frac{1}{p_1}(\varepsilon a)^{p_1}+\frac{1}{p_2}\Big(\frac{b}{\varepsilon}\Big)^{p_2},\quad \frac{1}{p_1}+\frac{1}{p_2}=1,$$

where $a,b\in\mathbb{R}_+$, $\varepsilon>0$, $p_1>1$.

It follows from (7.14) and (7.7) that $t_k = T$ and

$$u_k \text{ are bounded in } L_{3+\beta}(0,T;V_1)\cap L_\infty(0,T;L_2(\Omega)^n). \tag{7.15}$$

(2.13), (7.15), and Lemmas 4.12, and 4.13 imply

$$N_i(u_k) \text{ are bounded in } L_{\frac{3+\beta}{2+\beta}}(0,T;V^*), \quad i=1,2,3,4. \tag{7.16}$$

Define functions q_k, $k=1,2,3,...$ by the following relations:

$$\begin{aligned} &e_1 = w_1, \quad q_1 = \frac{e_1}{||e_1||_{L_2(\Omega)^n}},\\ &e_k = w_k - \sum_{i=1}^{k-1}(w_k,q_i)q_i, \quad q_k = \frac{e_k}{||e_k||_{L_2(\Omega)^n}}, \quad k>1. \end{aligned} \tag{7.17}$$

The functions q_k are orthonormal with respect to the scalar product in $L_2(\Omega)^n$, and q_k is a linear combination of $w_1,\dots,w_k$.

We denote the span of the functions $w_1,\dots,w_k$ by V_{1k}; the subspace V_{1k} also is the span of $q_1,\dots,q_k$.

Define a projection operator P_k that maps V^* onto V_{1k}^* as follows:

$$h\in V^*, \quad P_k h = \sum_{i=1}^{k}\alpha_i q_i, \qquad \alpha_i = (h,q_i). \tag{7.18}$$

Relation (7.6) can be represented in the form

$$\rho\frac{du_k}{dt} + P_k(\tilde{N}u_k + N(u_k)) = P_k\tilde{G}. \tag{7.19}$$

Since the sequence $\{P_k\}$ converges weakly to the identity operator in V_1^*, there exists a constant $c>0$ such that $||P_k||_{\mathcal{L}(V^*,V_{1k}^*)} \le c$ for all k. The relations (3.25), (3.27), (7.15), (7.16), and (7.19) yield

$$\frac{du_k}{dt} \text{ are bounded in } L_{\frac{3+\beta}{2+\beta}}(0,T;V^*). \tag{7.20}$$

7.3. **Passage to the limit.** By (7.15) and (7.20), we can extract a subsequence $\{u_m\}$ such that

$$u_m \rightharpoonup u \text{ in } L_{3+\beta}(0,T;V_1), \tag{7.21}$$

$$u_m \overset{*}{\rightharpoonup} u \text{ in } L_\infty(0,T;L_2(\Omega)^n), \tag{7.22}$$

$$u_m(T) \rightharpoonup \xi \text{ in } L_2(\Omega)^n, \tag{7.23}$$

$$\frac{du_m}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L_{\frac{3+\beta}{2+\beta}}(0,T;V^*), \tag{7.24}$$

where $\overset{*}{\rightharpoonup}$ is the sign of $*$ - weak convergence.

We apply Lemma 4.10. Take

$$B_0 = V_1, \quad B = L_{3+\beta}(\Omega)^n, \quad B_1 = V^*, \quad q_0 = 3+\beta, \quad q_1 = \frac{3+\beta}{2+\beta}.$$

Then (7.21) and (7.24) imply

$$u_m \to u \text{ in } L_{3+\beta}(Q)^n. \tag{7.25}$$

We can also consider that the sequence $\{u_m\}$ converges to u almost everywhere in Q, and so, Lemma 4.11 gives

$$\varphi_t(R_l(u_m))\varepsilon_{ij}(v) \to \varphi_t(R_l(u))\varepsilon_{ij}(v) \text{ in } \quad L_{\frac{3+\beta}{2+\beta}}(Q), \quad v \in L_{3+\beta}(0,T;V_1),$$
$$i,j = 1,\dots,n. \tag{7.26}$$

It follows from (7.25) that

$$u_{mj}\,u_{mi} \to u_j\,u_i \text{ in } \quad L_{\frac{3+\beta}{2}}(Q), \tag{7.27}$$

and for an arbitrary $h \in L_{3+\beta}(0,T;V)$, we have

$$u_{mj}\,h_i \to u_j\,h_i \text{ in } \quad L_{\frac{3+\beta}{2}}(Q). \tag{7.28}$$

(4.57), (7.21), (7.27), and (7.28) imply

$$\lim_{m\to\infty}(N_4(u_m),u_m) = (N_4(u),u), \tag{7.29}$$
$$N_4(u_m) \rightharpoonup N_4(u) \text{ in } \quad L_{\frac{3+\beta}{2+\beta}}(0,T;V^*). \tag{7.30}$$

(3.25), (3.27), and (7.25) yield

$$\tilde{N}u_m \to \tilde{N}u \text{ in } \quad L_{\frac{3+\beta}{2+\beta}}(0,T;V^*). \tag{7.31}$$

By (7.16), we can consider that

$$N_1(u_m) + N_2(u_m) + N_3(u_m) \rightharpoonup \chi \text{ in } \quad L_{\frac{3+\beta}{2+\beta}}(0,T;V^*). \tag{7.32}$$

Multiplying the equation (7.6) for $k = m$ by a function $\theta \in C^1([0,T])$, integrating the result from 0 to T, and using the integration by parts formula, we obtain

$$\int_0^T \Big[-\rho\,(u_m(t),w_i)\frac{d\theta}{dt}(t) + \Big((\tilde{N}u_m(t),w_i) + \sum_{i=1}^4 (N_i(u_m(t)),w_i)\Big)\theta(t)\Big]\,dt$$
$$= \int_0^T (\tilde{G}(t),w_i)\theta(t)\,dt + \rho\,(u_m(0),w_i)\theta(0) - \rho(u_m(T),w_i)\theta(T),$$
$$i = 1,2,\dots,m. \tag{7.33}$$

For fixed w_i, we pass to the limit in (7.33). By applying (7.7), (7.21), (7.23), (7.30), (7.31), and (7.32), we get

$$\int_0^T \Big[-\rho\,(u(t),w_i)\frac{d\theta}{dt}(t) + \Big((\tilde{N}u(t),w_i) + (\chi(t),w_i) + (N_4(u(t)),w_i)\Big)\theta(t)\Big]\,dt$$
$$= \int_0^T (\tilde{G}(t),w_i)\theta(t)\,dt + \rho\,(u_0,w_i)\theta(0) - \rho(\xi,w_i)\theta(T),$$
$$i = 1,2,\dots,m. \tag{7.34}$$

By virtue of (7.4), the function w_i in (7.34) can be changed for an arbitrary function from V_1. Since $D((0,T)) \subset C^1([0,T])$, (7.34) yields

$$\Big(\rho\frac{du}{dt} + \tilde{N}u + \chi + N_4(u), h\Big) = (\tilde{G},h) \text{ in } D'(0,T), \quad h \in V_1. \tag{7.35}$$

We integrate the first term in the left-hand side of (7.34) by parts. Taking a function $\theta \in C^1([0,T])$, $\theta(0) = 0$, we obtain from (7.34) and (7.35) that

$$u(T) = \xi, \tag{7.36}$$

and taking θ such that $\theta(T) = 0$, we get

$$u(0) = u_0. \tag{7.37}$$

If we show that

$$\chi = N_1(u) + N_2(u) + N_3(u), \tag{7.38}$$

the function u will be a solution of the problem (7.1), (7.2), and (7.3).

7.4. **Proof of the equality** (7.38)**.** Take the notation.

$$\begin{gathered}\Phi_m(v) = 2\int_0^T \int_\Omega \big\{\big[\alpha\big(I(u_m)\big)^{\frac{1+\beta}{2}} + \varphi(I(u_m)) + \varphi_t(R_l(u_m))\big]\varepsilon_{ij}(u_m) \\ -\big[\alpha\big(I(v)\big)^{\frac{1+\beta}{2}} + \varphi(I(v)) + \varphi_t(R_l(u_m))\big]\varepsilon_{ij}(v)\big\}\varepsilon_{ij}(u_m - v)\, dx\, dt, \\ v \in L_{3+\beta}(0,T;V_1).\end{gathered} \tag{7.39}$$

Lemmas 4.6 and 4.7 imply

$$\Phi_m(v) \geq 0, \qquad v \in L_{3+\beta}(0,T;V_1), \qquad m \in \mathbb{N}. \tag{7.40}$$

Taking $t = T$ and $k = m$ in (7.8), we obtain

$$\begin{gathered}2\int_0^T \int_\Omega \big[\alpha\big(I(u_m)\big)^{\frac{1+\beta}{2}} + \varphi(I(u_m)) + \varphi_t(R_l(u_m))\big] I(u_m)\, dx\, dt \\ = \int_0^T \big(\tilde{G} - \tilde{N}u_m - N_4(u_m), u_m\big)\, dt + \frac{1}{2}\,\rho\,||u_m(0)||^2_{L_2(\Omega)^n} - \frac{1}{2}\,\rho\,||u_m(T)||^2_{L_2(\Omega)^n}.\end{gathered} \tag{7.41}$$

(7.39) and (7.41) imply

$$\begin{gathered}\Phi_m(v) = \int_0^T (\tilde{G} - \tilde{N}u_m - N_4(u_m), u_m)\, dt + \frac{1}{2}\,\rho\,||u_m(0)||^2_{L_2(\Omega)^n} - \frac{1}{2}\,\rho\,||u_m(T)||^2_{L_2(\Omega)^n}. \\ -2\int_0^T \int_\Omega \big[\alpha\big(I(u_m)\big)^{\frac{1+\beta}{2}} + \varphi(I(u_m)) + \varphi_t(R_l(u_m))\big]\varepsilon_{ij}(u_m)\varepsilon_{ij}(v)\, dx\, dt \\ -2\int_0^T \int_\Omega \big[\alpha\big(I(v)\big)^{\frac{1+\beta}{2}} + \varphi(I(v)) + \varphi_t(R_l(u_m))\big]\varepsilon_{ij}(v)\varepsilon_{ij}(u_m - v)\, dx\, dt.\end{gathered} \tag{7.42}$$

(7.23) and (7.36) yield

$$\lim_{m\to\infty}\ \inf ||u_m(T)||_{L_2(\Omega)^n} \geq ||u(T)||_{L_2(\Omega)^n}. \tag{7.43}$$

By using (7.7), (7.21), (7.26), (7.29), (7.31), (7.32), (7.37), and (7.43), we obtain from (7.40) and (7.42) that

$$\begin{gathered}0 \leq \limsup_{m\to\infty}\ \Phi_m(v) \leq \int_0^T (\tilde{G} - \tilde{N}u - N_4(u), u)\, dt + \frac{1}{2}\,\rho\,||u(0)||^2_{L_2(\Omega)^n} \\ -\frac{1}{2}\,\rho\,||u(T)||^2_{L_2(\Omega)^n} - \int_0^T (\chi, v)\, dt - 2\int_0^T \int_\Omega \big[\alpha\big(I(v)\big)^{\frac{1+\beta}{2}} + \varphi(I(v)) \\ +\varphi_t(R_l(u))\big]\varepsilon_{ij}(v)\varepsilon_{ij}(u - v)\, dx\, dt, \quad v \in L_{3+\beta}(0,T;V_1).\end{gathered} \tag{7.44}$$

Take $h = u$ in (7.35) and integrate the result in t from 0 to T. This gives

$$\int_0^T (\tilde{G} - \tilde{N}u - N_4(u), u)\, dt + \frac{1}{2}\,\rho\,||u(0)||^2_{L_2(\Omega)^n} - \frac{1}{2}\,\rho\,||u(T)||^2_{L_2(\Omega)^n} = \int_0^T (\chi, u)\, dt. \tag{7.45}$$

Upon (7.44) and (7.45)

$$\int_0^T (\chi, u-v)\,dt - 2\int_0^T \int_\Omega \left[\alpha\big(I(v)\big)^{\frac{1+\beta}{2}} + \varphi(I(v)) + \varphi_t(R_l(u))\right]\varepsilon_{ij}(v) \times\varepsilon_{ij}(u-v)\,dx\,dt \geq 0, \quad v \in L_{3+\beta}(0,T;V_1). \tag{7.46}$$

We take here $v = u - \lambda w$, where $\lambda > 0$ and w is an arbitrary element of $C([0,T];V_1)$. Applying the Lebesgue theorem, we pass to the limit as $\lambda \to 0$. This gives

$$\int_0^T (\chi, w)\,dt - 2\int_0^T \int_\Omega \left[\alpha\big(I(u)\big)^{\frac{1+\beta}{2}} + \varphi(I(u)) + \varphi_t(R_l(u))\right] \times\varepsilon_{ij}(u)\varepsilon_{ij}(w)\,dx\,dt \geq 0. \tag{7.47}$$

Since $C([0,T];V_1)$ is dense in $L_{3+\beta}(0,T;V_1)$, we obtain (7.38) from (4.54)–(4.56), and (7.47).

7.5. **Calculation of the pressure.** We denote

$$\Lambda = \rho\,\frac{du}{dt} + \tilde{N}u + N(u) - \tilde{G}. \tag{7.48}$$

If follows from (7.24), (7.30)–(7.32), and (7.2) that

$$\Lambda \in L_{\frac{3+\beta}{2+\beta}}(0,T;V^*), \quad (\Lambda(t),h) = 0 \quad \text{a.e. in} \quad (0,T), \quad h \in V_1, \tag{7.49}$$

i.e. $\Lambda(t) \in W = (V/V_1)^*$.

By Theorem 4.2 there exists a unique function $p = L_{\frac{3+\beta}{2+\beta}}(Q)$ such that $\Lambda(t) = \operatorname{div}^* p(t)$ a.e. in $(0,T)$. Therefore, the pair u, p is a solution of the problem (3.21)–(3.24). ■

The proof of Theorem 3.4 is closely analogous to the proof of Theorem 3.3. Because of this, it is not given.

8. Approximation of the velocity and pressure for problem (3.21)–(3.24).

Let $\{A_k\}$ and $\{B_k\}$ be sequences of finite-dimensional subspaces of V and $L_{\frac{3+\beta}{2+\beta}}(\Omega)$, which satisfy the conditions (6.1), (6.2), (6.3), and (6.8). Consider the problem: Find functions $t \to u_k(t) \in A_k$, $t \to p_k(t) \in B_k$ satisfying

$$\rho\Big(\frac{du_k}{dt}(t), h\Big) + (\tilde{N}u_k(t), h) + (N(u_k(t)), h) - (p_k, \operatorname{div}_k h) = (\tilde{G}(t), h), \quad t \in (0,T), \quad h \in A_k, \tag{8.1}$$

$$(\operatorname{div}_k u_k(t), q) = 0, \qquad q \in B_k, \tag{8.2}$$

$$u_k(0) = u_{0k} \in \overset{\circ}{A}_k, \quad u_{0k} \to u_0 \quad \text{in} \quad L_2(\Omega)^n. \tag{8.3}$$

Theorem 8.1. *Suppose that the conditions* $(C1)$*–*$(C4)$ *and* (3.16)*–*(3.18)*,* (3.20)*,* (3.27) *are satisfied. Let the local Reynolds number be defined by* (2.1)*, where* μ *is either a positive constant or is given by* (2.20)*. Let also* $\{A_k\}$*,* $\{B_k\}$ *be sequences of finite-dimensional subspaces of* V *and* $L_{\frac{3+\beta}{2+\beta}}(\Omega)$ *which meet the conditions* (6.1)*,* (6.2)*,* (6.3)*, and* (6.8)*. Then, for an arbitrary* k*, there exists a solution of the problem* (8.1)*,* (8.2)*,* (8.3)*. From the sequence* $\{u_k, p_k\}$*, one can extract a subsequence* $\{u_m, p_m\}$ *such that* $u_m \rightharpoonup u$ *in* $L_{3+\beta}(0,T;V)$*,* $p_m \rightharpoonup p$ *in* $L_{\frac{3+\beta}{2+\beta}}(Q)$*, where* u*,* p *is a solution of the problem* (3.21)*–*(3.24)*.*

Theorem 8.1 is proved by using the arguments of the proof of Theorems 3.3 and 6.1.

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