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Asymptotic goodness-of-fit tests for the Palm mark distribution of stationary point processes with correlated marks

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Abstract

We consider spatially homogeneous marked point patterns in an unboundedly expanding convex sampling window. Our main objective is to identify the distribution of the typical mark by constructing an asymptotic χ^2-goodness-of-fit test. The corresponding test statistic is based on a natural empirical version of the Palm mark distribution and a smoothed covariance estimator which turns out to be mean-square consistent. Our approach does not require independent marks and allows dependences between the mark field and the point pattern. Instead we impose a suitable β-mixing condition on the underlying stationary marked point process which can be checked for a number of Poisson-based models and, in particular, in the case of geostatistical marking. Our method needs a central limit theorem for β-mixing random fields which is proved by extending Bernstein's blocking technique to non-cubic index sets and seems to be of interest in its own right. By large-scale model-based simulations the performance of our test is studied in dependence of the model parameters which determine the range of spatial correlations.

Keywords : empirical Palm mark distribution, reduced factorial moment measures, β-mixing point process, central limit theorem, Bernstein's blocking technique, smoothed covariance estimation, χ^2-goodness-of-fit test

MSC 2000: Primary 62 G 10, 60 G 55; Secondary 60 F 05, 62 G 20

1 Introduction

Marked point processes (MPPs) are versatile models for the statistical analysis of data recorded at irregularly scattered locations. The most simple marking scenario is independent marking, where marks are given by a sequence of independent and identically distributed random elements, which is also independent of the underlying point pattern of locations. A more complex class of models considers a so-called geostatistical marking, where the marks are determined by the values of a random field at the given locations. Although the random field usually exhibits intrinsic spatial correlations, it is assumed to be independent of the location point process (PP). However, in many real datasets interactions between locations and marks occur. Moreover, many marked point patterns arising in models from stochastic geometry such as edge centers in (anisotropic) Voronoi-tessellations marked by orientation or PPs marked by nearest-neighbour distances do not fit the setting of geostatistical marking. Statistical tests for independence between marks and points are e.g. discussed in [8, 9, 23, 25].

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A frequent approach to investigate dependences in marked point patterns is based on mark variogram and mark covariance functions. Recently, asymptotic normality of empirical versions of these functions with applications to mark correlation analysis has been studied in [10, 11, 14]. The main goal of this paper is to investigate estimators of the Palm mark distribution $P_M^{\boldsymbol{o}}$ in point patterns exhibiting correlations between different marks as well as between marks and locations. The probability measure $P_M^{\boldsymbol{o}}$ can be interpreted as the distribution of the typical mark which denotes the mark of a randomly chosen point of the pattern. For any mark set C we consider the scaled deviations $Z_k(C) = \sqrt{|W_k|}\big((\widehat{P}_M^{\boldsymbol{o}})_k(C) - P_M^{\boldsymbol{o}}(C) \big)$ as measure of the distance between $P_M^{\boldsymbol{o}}$ and an empirical Palm mark distribution $(\widehat{P}_M^{\boldsymbol{o}})_k$. Under appropriate strong mixing conditions we are able to prove asymptotic normality of the scaled deviation vector $\mathbf{Z}_k = (Z_k(C_1), \ldots, Z_k(C_\ell))^T$ when the observation window W_k with volume $|W_k|$ grows unboundedly in all directions as $k \to \infty$. The proof relies on Bernstein's blocking method, see e.g. [4, 21], which so far has been applied only to sequences of cubic or cubelike windows W_k, see e.g. [11, 12]. By means of some convex-geometric arguments it turns out that the blocking method is indeed applicable to any increasing sequence of convex observation windows W_k with unboundedly growing inball radii. In addition we discuss consistent estimators for the covariance matrix of the Gaussian limit of $\mathbf{Z}_k$. This enables us to construct asymptotic χ^2-goodness-of-fit tests for the Palm mark distribution $P_M^{\boldsymbol{o}}$. By means of computer simulations we study the convergence of first and second type errors of the tests for growing observation windows in relation to the range of dependence of the MPP. In this way we demonstrate the practicability of the tests in analysis of real data. A promising field of application of our testing methodology could be the directional analysis of random surfaces. Based on our results one can e.g. consider Cox processes on the boundary of Boolean models, marking them with the local outer normal direction and testing for a hypothetical directional distribution. This allows to identify the rose of directions of the surface process associated with the Boolean model and represents an alternative to a Monte-Carlo test for the rose of direction suggested in [2]. The occurring marked point patterns differ basically from the setting of independent and geostatistical marking, for which functional central limit theorems (CLTs) and corresponding tests have been derived in [16, 22]. Our paper is organized as follows. Section 2 introduces basic notation and definitions. In Section 3 we present our main results, which are proved in Section 4. In Section 5 we briefly discuss some models satisfying the assumptions needed to prove our asymptotic results. In the final Section 6 we study the performance of the proposed tests by large-scale simulations.

2 Stationary marked point processes

An MPP $X_M = \sum_{n\geq 1} \delta_{(X_n, M_n)}$ is a random locally finite counting measure acting on the Borel sets of $\mathbb{R}^d \times \mathbb{M}$ with atoms (X_n, M_n), where the marks M_n belong to some Polish mark space $\mathbb{M}$ endowed with the Borel σ-algebra $\mathcal{B}(\mathbb{M})$. Throughout we assume that X_M is simple, i.e. all locations X_n in $\mathbb{R}^d$ have multiplicity 1 regardless which mark they have. Mathematically spoken, X_M is a measurable mapping $X_M : \Omega \longrightarrow \mathsf{N}_{\mathbb{M}}$ from some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into the set $\mathsf{N}_{\mathbb{M}}$ of counting measures $\varphi(\cdot)$ on $\mathcal{B}(\mathbb{R}^d \times \mathbb{M})$ satisfying $\varphi(B \times \mathbb{M}) < \infty$ for all bounded $B \in \mathcal{B}(\mathbb{R}^d)$, where $\mathsf{N}_{\mathbb{M}}$ is endowed with the smallest σ-algebra $\mathcal{N}_{\mathbb{M}}$ containing all sets of the form $\{\varphi \in \mathsf{N}_{\mathbb{M}} : \varphi(B \times C) = j\}$ for $j \geq 0$, bounded $B \in \mathcal{B}(\mathbb{R}^d)$, and $C \in \mathcal{B}(\mathbb{M})$. In what follows we only consider *stationary* MPPs, which means that the distribution $P_{X_M}(\cdot) = \mathbb{P}(X_M \in (\cdot))$ of X_M on $\mathcal{N}_{\mathbb{M}}$ is invariant under location shifts of the atoms, i.e.,

$$X_M \stackrel{\mathrm{D}}{=} \sum_{n\geq 1} \delta_{(X_n - x, M_n)} \quad \text{for all} \quad x \in \mathbb{R}^d .$$

Provided that X_M is stationary and the *intensity* $\lambda = \mathbb{E}X_M([0,1)^d \times \mathbb{M})$ is finite we have $\mathbb{E}X_M(B \times C) = |B|\, \mathbb{E}X_M([0,1)^d \times C)$ for all bounded $B \in \mathcal{B}(\mathbb{R}^d)$ and $C \in \mathcal{B}(\mathbb{M})$, where $|\cdot|$ denotes d-dimensional Lebesgue measure.

2.1 Palm mark distribution

For a stationary MPP X_M the probability measure $P_M^{\mathbf{o}}$ on $\mathcal{B}(\mathbb{M})$ defined by

$$P_M^{\mathbf{o}}(C) = \frac{1}{\lambda}\, \mathbb{E}X_M([0,1)^d \times C)\,, \quad C \in \mathcal{B}(\mathbb{M})\,, \tag{2.1}$$

is called the *Palm mark distribution* of X_M. It can be interpreted as the conditional distribution of the mark of an atom of X_M located at the origin $\mathbf{o}$. A random element M_0 in $\mathbb{M}$ with distribution $P_M^{\mathbf{o}}$ is called *typical mark* of X_M.

Definition 2.1. An increasing sequence $\{W_k\}$ of convex and compact sets in $\mathbb{R}^d$ such that $\varrho(W_k) = \sup\{r > 0 : B(x,r) \subset W_k \text{ for some } x \in W_k\} \to \infty$ as $k \to \infty$ is called a *convex averaging sequence* (briefly CAS). Here $B(x,r)$ denotes the closed ball (w.r.t. the Euclidean norm $\|\cdot\|$) with midpoint at $x \in \mathbb{R}^d$ and radius $r \geq 0$.

Some results from convex geometry applied to CAS $\{W_k\}$ yield the following inequalities

$$\frac{1}{\varrho(W_k)} \leq \frac{\mathcal{H}_{d-1}(\partial W_k)}{|W_k|} \leq \frac{d}{\varrho(W_k)} \qquad \text{and} \qquad 1 - \frac{|W_k \cap (W_k - x)|}{|W_k|} \leq \frac{d\,\|x\|}{\varrho(W_k)} \tag{2.2}$$

for $\|x\| \leq \varrho(W_k)$, where $\mathcal{H}_{d-1}(\partial W_k)$ is the surface content (i.e. $(d-1)$-dimensional Hausdorff measure) of the boundary ∂W_k, see [3] and [16] for details.

If X_M is ergodic (for a precise definition see [5], pp. 194), the individual ergodic theorem applied to MPPs (see Theorem 12.2.IV and Corollary 12.2.V in [5]) provides the $\mathbb{P}-$a.s. limits

$$\widehat{\lambda}_k = \frac{X_M(W_k \times \mathbb{M})}{|W_k|} \underset{k\to\infty}{\overset{\mathbb{P}-\text{a.s.}}{\longrightarrow}} \lambda \quad \text{and} \quad (\widehat{P}_M^{\mathbf{o}})_k(C) = \frac{X_M(W_k \times C)}{X_M(W_k \times \mathbb{M})} \underset{k\to\infty}{\overset{\mathbb{P}-\text{a.s.}}{\longrightarrow}} P_M^{\mathbf{o}}(C) \tag{2.3}$$

for any $C \in \mathcal{B}(\mathbb{M})$ and an arbitrary CAS $\{W_k\}$.

2.2 Factorial moment measures and the covariance measure

For any integer $m \geq 1$, the mth *factorial moment measure* $\alpha_{X_M}^{(m)}$ of the MPP X_M is defined on $\mathcal{B}((\mathbb{R}^d \times \mathbb{M})^m)$ by

$$\alpha_{X_M}^{(m)}\big(\mathop{\times}_{i=1}^{m}(B_i \times C_i)\big) = \mathbb{E} \sum_{n_1,\dots,n_m \geq 1}^{\neq} \prod_{i=1}^{m} \big(1\!\!1_{B_i}(X_{n_i}) 1\!\!1_{C_i}(M_{n_i}) \big)\,, \tag{2.4}$$

where the sum $\sum^{\neq}_{n_1,\dots,n_m\geq 1}$ runs over all $m-$tuples of pairwise distinct indices $n_1,\dots,n_m \geq 1$ for bounded $B_i \in \mathcal{B}(\mathbb{R}^d)$ and $C_i \in \mathcal{B}(\mathbb{M})$, $i = 1,\dots,m$. We also need the mth factorial moment measure $\alpha_X^{(m)}$ of the *unmarked* PP $X(\cdot) = X_M((\cdot)\times\mathbb{M}) = \sum_{n\ge 1}\delta_{X_n}(\cdot)$ defined on $\mathcal{B}((\mathbb{R}^d)^m)$ by

$$\alpha_X^{(m)}\left(\mathop{\times}_{i=1}^{m} B_i\right) = \alpha_{X_M}^{(m)}\big(\mathop{\times}_{i=1}^{m}(B_i\times\mathbb{M})\big) \quad \text{for bounded} \quad B_1,\dots,B_m\in\mathcal{B}(\mathbb{R}^d)\,.$$

The stationarity of X_M implies that $\alpha_X^{(m)}$ is invariant under diagonal shifts, which allows to define the mth *reduced factorial moment measure* $\alpha_{X,red}^{(m)}$ uniquely determined by the following desintegration formula

$$\alpha_X^{(m)}\left(\mathop{\times}_{i=1}^{m} B_i\right) = \lambda\int_{B_1}\alpha_{X,red}^{(m)}\big(\mathop{\times}_{i=2}^{m}(B_i - x)\big)\,\mathrm{d}x\,. \tag{2.5}$$

We need a condition of weak dependence between parts of X defined over distant Borel sets which can be expressed by the (factorial) *covariance measure* $\gamma_X^{(2)}$ on $\mathcal{B}((\mathbb{R}^d)^2)$ defined by

$$\gamma_X^{(2)}\big(B_1\times B_2\big) = \alpha_X^{(2)}\big(B_1\times B_2\big) - \lambda^2\,|B_1|\,|B_2|\,.$$

The *reduced covariance measure* $\gamma_{X,red}^{(2)} : \mathcal{B}(\mathbb{R}^d)\to[0,\infty]$ is in general a signed measure defined by (2.5) with $\gamma_X^{(2)}$ instead of $\alpha_X^{(2)}$, which means that

$$\gamma_{X,red}^{(2)}(B) = \alpha_{X,red}^{(2)}(B) - \lambda\,|B| \quad \text{for bounded} \quad B\in\mathcal{B}(\mathbb{R}^d)\,.$$

For more details on factorial moment measures and measures related with them we refer to Chapters 8 and 12 in [5].

2.3 m-point Palm mark distribution

For fixed mark sets $C_1,\dots,C_m\in\mathcal{B}(\mathbb{M})$, $m\ge 1$, the mth factorial moment measure $\alpha_{X_M}^{(m)}$ defined by (2.4) can be regarded as a measure on the Borel sets $\mathcal{B}((\mathbb{R}^d)^m)$ depending on $C_1,\dots,C_m$. This new measure is absolutely continuous w.r.t. the mth factorial moment measure $\alpha_X^{(m)}$. Thus, the Radon-Nikodym theorem (cf. [7], p.90) implies the existence of a density $P_M^{x_1,\dots,x_m}(C_1\times\cdots\times C_m)$, which is uniquely determined for $\alpha_X^{(k)}$-almost all $(x_1,\dots,x_m)\in(\mathbb{R}^d)^m$, such that for any $B_1,\dots,B_m\in\mathcal{B}(\mathbb{R}^d)$,

$$\alpha_{X_M}^{(m)}\big(\mathop{\times}_{i=1}^{m}(B_i\times C_i)\big) = \int_{\mathop{\times}_{i=1}^{m} B_i} P_M^{x_1,\dots,x_m}\left(\mathop{\times}_{i=1}^{m} C_i\right)\,\alpha_X^{(m)}(\mathrm{d}(x_1,\dots,x_m)). \tag{2.6}$$

Since the mark space $\mathbb{M}$ is Polish, this Radon-Nikodym density can be extended to a regular conditional distribution of the mark vector $(M_1,\dots,M_m)$ given that the corresponding atoms $X_1,\dots,X_m$ are located at pairwise distinct points $x_1,\dots,x_m$, i.e.,

$$P_M^{x_1,\dots,x_m}(C) = \mathbb{P}((M_1,\dots,M_m)\in C\mid X_1 = x_1,\dots,X_m = x_m) \quad \text{for} \quad C\in\mathcal{B}(\mathbb{M}^m)\,.$$

This means that the mapping $(x_1,\dots,x_m,C)\mapsto P_M^{x_1,\dots,x_m}(C)$ is a stochastic kernel , i.e., $P_M^{x_1,\dots,x_m}(C)$ is $\mathcal{B}((\mathbb{R}^d)^m)$-measurable in $(x_1,\dots,x_m)\in(\mathbb{R}^d)^m$ for fixed $C\in\mathcal{B}(\mathbb{M}^m)$ and a probability measure in $C\in\mathcal{B}(\mathbb{M}^m)$ for fixed $(x_1,\dots,x_m)\in(\mathbb{R}^d)^m$. For details we refer to

[18], p.164. The regular conditional distribution $P_M^{x_1,\ldots,x_m}(C)$ for $C \in \mathcal{B}(\mathbb{M}^m)$ is called the *m-point Palm mark distribution* of X_M . This stochastic kernel is only of interest for $m-$tuples $(x_1,\ldots,x_m)$ of pairwise distinct points $x_i \in \mathbb{R}^d$, $i = 1,...m$. In case of a stationary simple MPP X_M it can be shown that

$$P_M^{x_1,\ldots,x_m}(C) = P_M^{\mathbf{o},x_2-x_1\ldots,x_m-x_1}(C) \quad \text{for} \quad C \in \mathcal{B}(\mathbb{M}^m)\,,\; m \geq 1$$

and any $x_1,\ldots,x_m \in \mathbb{R}^d$ with $x_i \neq x_j$ for $i \neq j$. In this way the Palm mark distribution defined in (2.1) can be considered as one-point Palm mark distribution.

The following result is crucial to prove asymptotic properties of variances estimators of the empirical mark distribution. It generalizes an analoguous result stated for unmarked PPs in [17] to MPPs by involving the notion m-point Palm mark distribution for $m = 2,3,4$. The proof is just a slight extension of the one of Lemma 5 in [17] by using the relation (2.6) for $m = 2,3,4$. The details are left to the reader.

Lemma 2.1. *Let $X_M = \sum_{n\geq 1} \delta_{(X_n,M_n)}$ be an MPP satisfying $\mathbb{E}\big(X_M(B\times\mathbb{M})\big)^4 < \infty$ for all bounded $B \in \mathcal{B}(\mathbb{R}^d)$, and let $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{M}^2 \mapsto \mathbb{R}$ be a Borel-measurable function such that the second moment of $\sum_{p,q\geq 1}^{\neq} |\, f(X_p,X_q,M_p,M_q)\,|$ exists. Then,*

$$\operatorname{Var}\Big(\sum_{p,q\geq 1}^{\neq} f(X_p,X_q,M_p,M_q)\Big) \tag{2.7}$$

$$= \int_{(\mathbb{R}^d)^2}\int_{\mathbb{M}^2} f(x_1,x_2,u_1,u_2)\Big[f(x_1,x_2,u_1,u_2)+f(x_2,x_1,u_2,u_1)\Big]P_M^{x_1,x_2}\big(\mathrm{d}(u_1,u_2)\big)\alpha_X^{(2)}\big(\mathrm{d}(x_1,x_2)\big)$$

$$+ \int_{(\mathbb{R}^d)^3}\int_{\mathbb{M}^3} f(x_1,x_2,u_1,u_2)\Big[f(x_1,x_3,u_1,u_3)+f(x_3,x_1,u_3,u_1)$$

$$+ f(x_2,x_3,u_2,u_3)+f(x_3,x_2,u_3,u_2)\Big]P_M^{x_1,x_2,x_3}\big(\mathrm{d}(u_1,u_2,u_3)\big)\alpha_X^{(3)}\big(\mathrm{d}(x_1,x_2,x_3)\big)$$

$$+ \int_{(\mathbb{R}^d)^4}\int_{\mathbb{M}^4} f(x_1,x_2,u_1,u_2)f(x_3,x_4,u_3,u_4)\Big[P_M^{x_1,x_2,x_3,x_4}\big(\mathrm{d}(u_1,u_2,u_3,u_4)\big)\alpha_X^{(4)}\big(\mathrm{d}(x_1,x_2,x_3,x_4)\big)$$

$$- P_M^{x_1,x_2}\big(\mathrm{d}(u_1,u_2)\big)P_M^{x_3,x_4}\big(\mathrm{d}(u_3,u_4)\big)\alpha_X^{(2)}\big(\mathrm{d}(x_1,x_2)\big)\alpha_X^{(2)}\big(\mathrm{d}(x_3,x_4)\big)\Big].$$

2.4 β-mixing coefficient and covariance inequality

For any $B \in \mathcal{B}(\mathbb{R}^d)$, let $\mathcal{A}_{X_M}(B)$ denote the sub-σ-algebra of $\mathcal{A}$ generated by the restriction of the MPP X_M to the set $B\times\mathbb{M}$. For any $B,B' \in \mathcal{B}(\mathbb{R}^d)$ a natural measure of dependence between $\mathcal{A}_{X_M}(B)$ and $\mathcal{A}_{X_M}(B')$ can be formulated in terms of the $\beta-$*mixing* (or *absolute regularity, respectively weak Bernoulli*) *coefficient*

$$\beta\big(\mathcal{A}_{X_M}(B),\mathcal{A}_{X_M}(B')\big) = \frac{1}{2}\sup_{\{A_i\},\{A'_j\}}\sum_{i,j}\big|\,\mathbb{P}(A_i\cap A'_j) - \mathbb{P}(A_i)\,\mathbb{P}(A'_j)\,\big|\,, \tag{2.8}$$

where the supremum is taken over all finite partitions $\{A_i\}$ and $\{A'_j\}$ of Ω such that $A_i \in \mathcal{A}_{X_M}(B)$ and $A'_j \in \mathcal{A}_{X_M}(B')$ for all i,j , see e.g. [6], [12] or [26]. It should be noticed that the supremum in (2.8) does not change if the sets A_i and A'_j belong to semi-algebras generating $\mathcal{A}_{X_M}(B)$ and $\mathcal{A}_{X_M}(B')$, respectively. To express the degree of dependence of the MPP X_M for disjoint sets $K_a = [-a,a]^d$ and $K^c_{a+b} = \mathbb{R}^d \backslash K_{a+b}$, where $b \geq 0$, we consider non-increasing functions $\beta^*_{X_M},\ \beta^{**}_{X_M} : [\frac{1}{2},\infty) \to [0,\infty)$ such that

$$\beta\big(\mathcal{A}_{X_M}(K_a), \mathcal{A}_{X_M}(K^c_{a+b})\big) \leq \begin{cases} \beta^*_{X_M}(b) & \text{for} \quad \frac{1}{2} \leq a \leq b\,, \\ a^{d-1}\,\beta^{**}_{X_M}(b) & \text{for} \quad \frac{1}{2} \leq b \leq a\,. \end{cases} \tag{2.9}$$

A stationary MPP X_M is called β-*mixing* or *absolutely regular, respectively weak Bernoulli* if both β-*mixing rates* $\beta^*_{X_M}(r)$ and $\beta^{**}_{X_M}(r)$ tend to 0 as $r \to \infty$. By standard measure-theoretic approximation arguments it is easily seen that any stationary β-mixing MPP X_M is mixing in the usual sense and therefore also ergodic, see Lemma 12.3.II and Proposition 12.3.III in [5] Vol. II pp. 206. In order to prove CLTs we need further conditions on the decay of the β-*mixing rates* $\beta^*_{X_M}(r)$ and $\beta^{**}_{X_M}(r)$ on the right-hand side (rhs) of (2.9). For this we formulate

Condition $\beta(\delta)$: There exists some $\delta > 0$ such that $\mathbb{E}\big(X_M(\,[0,1]^d \times \mathbb{M}\,)\big)^{2+\delta} < \infty\,,$

$$\int_1^\infty r^{d-1}\big(\beta^*_{X_M}(r)\big)^{\delta/(2+\delta)}\,\mathrm{d}r < \infty \qquad \text{and} \qquad r^{2d-1}\beta^{**}_{X_M}(r) \underset{r\to\infty}{\longrightarrow} 0\,.$$

The following type of covariance bound in terms of the $\beta-$mixing coefficient (2.8) was first stated in [26].

Lemma 2.2. *Let Y and Y' denote the restrictions of the MPP X_M to $B \times \mathbb{M}$ and $B' \times \mathbb{M}$ for some $B, B' \in \mathcal{B}(\mathbb{R}^d)$, respectively. Furthermore, let $\widetilde{Y}$ and $\widetilde{Y}'$ be independent copies of Y and Y', respectively. Then, for any $\mathcal{N}_{\mathbb{M}} \otimes \mathcal{N}_{\mathbb{M}}-$ measurable function $f : \mathsf{N}_{\mathbb{M}} \times \mathsf{N}_{\mathbb{M}} \to [0,\infty)$ and for any $\eta > 0$*

$$\begin{aligned} \big|\mathbb{E}f(Y,Y') - \mathbb{E}f(\widetilde{Y},\widetilde{Y}')\big| \;&\leq\; 2\,\beta(\mathcal{A}_{X_M}(B), \mathcal{A}_{X_M}(B'))^{\frac{\eta}{1+\eta}} \\ &\times\; \max\Big\{\big(\mathbb{E}f^{1+\eta}(Y,Y')\big)^{\frac{1}{1+\eta}},\, \big(\mathbb{E}f^{1+\eta}(\widetilde{Y},\widetilde{Y}')\big)^{\frac{1}{1+\eta}}\Big\}\,. \end{aligned} \tag{2.10}$$

If f is bounded, then (2.10) remains valid for $\eta = \infty$. In the particular case $f(y,y') = f_1(y)\,f_2(y')$ and $\eta = \delta/2$ for $\delta > 0$, the Cauchy-Schwarz inequality applied to the expectations on the rhs of (2.10) yields

$$\big|\,\mathrm{Cov}\big(\,f_1(Y), f_2(Y')\,\big)\,\big| \leq 2\,\|\,f_1(Y)\,\|_{2+\delta}\,\|\,f_2(Y')\,\|_{2+\delta}\,\big(\,\beta(\mathcal{A}_{X_M}(B), \mathcal{A}_{X_M}(B')\,\big)^{\frac{\delta}{2+\delta}}\,, \tag{2.11}$$

where $\|\,Z\,\|_q = (\mathbb{E}|Z|^q)^{1/q}$ is the L^q-norm ($q \geq 1$) of a random variable Z.

3 Results

3.1 Central limit theorem

We consider a sequence of set-indexed empirical processes $\{\,Y_k(C)\,,\,C \in \mathcal{B}(\mathbb{M})\,\}$ defined by

$$Y_k(C) = \frac{1}{\sqrt{|W_k|}}\sum_{n\geq 1}\mathbb{I}_{W_k}(X_n)\big(\mathbb{I}_C(M_n) - P^{\mathbf{o}}_M(C)\big) = \sqrt{|W_k|}\,\widehat{\lambda}_k\,\big((\widehat{P}^{\mathbf{o}}_M)_k(C) - P^{\mathbf{o}}_M(C)\big)\,, \tag{3.1}$$

where $\{W_k\}$ is a CAS of observation windows in $\mathbb{R}^d$. We will first state a multivariate CLT for the joint distribution of $Y_k(C_1),\dots,Y_k(C_\ell)$. For this, let '$\xrightarrow{\mathrm{D}}$' denote *convergence in distribution* and $\mathcal{N}_\ell(a,\mathbf{\Sigma})$ be an ℓ-dimensional Gaussian vector with expectation vector $a\in\mathbb{R}^\ell$ and covariance matrix $\mathbf{\Sigma}=(\sigma_{ij})_{i,j=1}^{\ell}$.

Theorem 3.1. *Let X_M be a stationary MPP with $\lambda>0$ satisfying Condition $\beta(\delta)$. Then*

$$\mathbf{Y}_k=\big(Y_k(C_1),\dots,Y_k(C_\ell)\big)^\top \xrightarrow[k\to\infty]{\mathrm{D}} \mathcal{N}_\ell(\mathbf{o}_\ell,\mathbf{\Sigma}) \quad \textit{for any} \quad C_1,\dots,C_\ell\in\mathcal{B}(\mathbb{M})\,, \tag{3.2}$$

where $\mathbf{o}_\ell=(0,\dots,0)^\top$ and the asymptotic covariance matrix $\mathbf{\Sigma}=(\sigma_{ij})_{i,j=1}^{\ell}$ is given by the limits

$$\sigma_{ij}=\lim_{k\to\infty}\mathbb{E}Y_k(C_i)Y_k(C_j). \tag{3.3}$$

The above result can also be stated in terms of the empirical set-indexed process $\{Z_k(C),\,C\in\mathcal{B}(\mathbb{M})\}$, where

$$Z_k(C)=(\,\widehat{\lambda}_k\,)^{-1}Y_k(C)=\sqrt{|W_k|}\,\big(\,(\widehat{P}_M^{\mathbf{o}})_k(C)-P_M^{\mathbf{o}}(C)\,\big)\,.$$

In other words, as refinement of the ergodic theorem (2.3), we derive asymptotic normality of a suitably scaled deviation of the ratio-unbiased empirical Palm mark probabilities $(\widehat{P}_M^{\mathbf{o}})_k(C)$ from $P_M^{\mathbf{o}}(C)$ defined by (2.1) for any $C\in\mathcal{B}(\mathbb{M})$. Since Condition $\beta(\delta)$ ensures the ergodicity of X_M , the first limiting relation in (2.3) combined with Slutsky's lemma yields the following result as a corollary of Theorem 3.1.

Corollary 3.2. *The conditions of Theorem* 3.1 *imply the CLT*

$$\mathbf{Z}_k=(Z_k(C_1),\dots,Z_k(C_\ell))^\top \xrightarrow[k\to\infty]{\mathrm{D}} \mathcal{N}_\ell(\mathbf{o}_\ell,\lambda^{-2}\,\mathbf{\Sigma})\,.$$

3.2 β-mixing and integrability conditions

In this subsection we give a condition in terms of the mixing rate $\beta^*_{X_M}(r)$ which implies finite total variation of the reduced covariance measure $\gamma^{(2)}_{X,red}$ and a certain integrability condition (3.5) which expresses weak dependence between any two marks located at far distant sites. Both of these conditions are needed to get the asymptotic unbiasedness resp. L^2-consistency of some estimators for the asymptotic covariances (3.3).

Note that the total variation measure $|\gamma^{(2)}_{X,red}|$ of $\gamma^{(2)}_{X,red}$ is defined as sum of the positive part $\gamma^{(2)+}_{X,red}$ and negative part $\gamma^{(2)-}_{X,red}$ of the Jordan decomposition of $\gamma^{(2)}_{X,red}$, i.e.,

$$\gamma^{(2)}_{X,red}=\gamma^{(2)+}_{X,red}-\gamma^{(2)-}_{X,red} \qquad \text{and} \qquad |\gamma^{(2)}_{X,red}|=\gamma^{(2)+}_{X,red}+\gamma^{(2)-}_{X,red}\,,$$

where the positive measures $\gamma^{(2)+}_{X,red}$ and $\gamma^{(2)-}_{X,red}$ are mutually singular, see [7], p.87.

Lemma 3.1. *Let X_M be a stationary MPP satisfying*

$$\mathbb{E}\big(X_M(\,[0,1]^d\times\mathbb{M}\,)\big)^{2+\delta}<\infty \quad \textit{and} \quad \int_1^\infty r^{d-1}\big(\beta^*_{X_M}(r)\big)^{\delta/(2+\delta)}\,\mathrm{d}r<\infty \quad \textit{for some} \quad \delta>0\,.$$

Then $\gamma^{(2)}_{X,red}$ has finite total variation on $\mathbb{R}^d$, i.e.,

$$|\gamma^{(2)}_{X,red}|(\mathbb{R}^d)<\infty\,. \tag{3.4}$$

Furthermore, for any $C_1,C_2\in\mathcal{B}(\mathbb{M})$

$$\int_{\mathbb{R}^d}\Big|\,P_M^{\mathbf{o},x}(C_1\times C_2)-P_M^{\mathbf{o}}(C_1)\,P_M^{\mathbf{o}}(C_2)\,\Big|\,\alpha^{(2)}_{X,red}(\mathrm{d}x)<\infty\,. \tag{3.5}$$

3.3 Representation of the asymptotic covariance matrix

In Theorem 3.1 we stated conditions for asymptotic normality of the random vector $\mathbf{Y}_k$. Clearly, (2.1) and (3.1) immediately imply that $\mathbb{E}Y_k(C) = 0$ for any $C \in \mathcal{B}(\mathbb{M})$. The following theorem gives a representation formula for the asymptotic covariance matrix $\mathbf{\Sigma}$.

Theorem 3.3. *Let X_M be a stationary MPP satisfying* (3.5) *and let $\{W_k\}$ be a CAS. Then, the limits in* (3.3) *exist and take the form*

$$\begin{aligned} \sigma_{ij} &= \lambda\big(P_M^{\mathbf{o}}(C_i \cap C_j) - P_M^{\mathbf{o}}(C_i)\, P_M^{\mathbf{o}}(C_j) \big) + \lambda \int_{\mathbb{R}^d} \big(P_M^{\mathbf{o},x}(C_i \times C_j) \\ &- P_M^{\mathbf{o},x}(C_i \times \mathbb{M})\, P_M^{\mathbf{o}}(C_j) - P_M^{\mathbf{o},x}(C_j \times \mathbb{M})\, P_M^{\mathbf{o}}(C_i) + P_M^{\mathbf{o}}(C_i)\, P_M^{\mathbf{o}}(C_j) \big)\, \alpha^{(2)}_{X,red}(\mathrm{d}x)\,. \end{aligned} \tag{3.6}$$

In particular, if X_M is an independently MPP, then

$$\sigma_{ij} = \lambda\big(P_M^{\mathbf{o}}(C_i \cap C_j) - P_M^{\mathbf{o}}(C_i)\, P_M^{\mathbf{o}}(C_j) \big). \tag{3.7}$$

3.4 Estimation of the asymptotic covariance matrix

In Section 6 we will exploit the normal convergence (3.2) for statistical inference of the typical mark distribution. More precisely, assuming that the asymptotic covariance matrix $\mathbf{\Sigma}$ is invertible, we consider asymptotic χ^2-goodness-of-fit tests, which are based on the distributional limit

$$\mathbf{Y}_k^\top \widehat{\mathbf{\Sigma}}_k^{-1} \mathbf{Y}_k \xrightarrow[k\to\infty]{\mathrm{D}} \chi^2_\ell.$$

which is an immediate consequence of (3.2) and Slutsky's lemma, given that $\widehat{\mathbf{\Sigma}}_k$ is a consistent estimator for $\mathbf{\Sigma}$. Here we use the notation $\mathbf{Y}_k = \big(Y_k(C_1), \ldots, Y_k(C_\ell)\big)^\top$ (see (3.1)) and the random variable χ^2_ℓ is χ^2-distributed with ℓ degrees of freedom. In the following we will discuss several estimators for $\mathbf{\Sigma}$. Our first observation is that the simple plug-in estimator $\widehat{\mathbf{\Sigma}}_k^{(0)} = \big(Y_k(C_i)Y_k(C_j)\big)_{i,j=1}^\ell$ for $\mathbf{\Sigma}$ is useless, since the determinant of $\widehat{\mathbf{\Sigma}}_k^{(0)}$ vanishes. Instead of $\widehat{\mathbf{\Sigma}}_k^{(0)}$ we take the edge-corrected estimator $\widehat{\mathbf{\Sigma}}_k^{(1)} = \big((\widehat{\sigma}_{ij}^{(1)})_k\big)_{i,j=1}^\ell$ with

$$\begin{aligned} (\widehat{\sigma}_{ij}^{(1)})_k &= \frac{1}{|W_k|} \sum_{p\ge 1} \mathbb{1}_{W_k}(X_p)\big(\mathbb{1}_{C_i\cap C_j}(M_p) - P_M^{\mathbf{o}}(C_i)\, P_M^{\mathbf{o}}(C_j) \big) \\ &+ \sum_{p,q\ge 1}^{\neq} \frac{\mathbb{1}_{W_k}(X_p)\mathbb{1}_{W_k}(X_q)\big(\mathbb{1}_{C_i}(M_p) - P_M^{\mathbf{o}}(C_i) \big)\big(\mathbb{1}_{C_j}(M_q) - P_M^{\mathbf{o}}(C_j)\big)}{|(W_k - X_p) \cap (W_k - X_q)|}\,. \end{aligned} \tag{3.8}$$

As an alternative, which can be implemented in a more efficient way, we neglect the edge correction and consider the naive estimator $\widehat{\mathbf{\Sigma}}_k^{(2)} = \big((\widehat{\sigma}_{ij}^{(2)})_k\big)_{i,j=1}^\ell$ for $\mathbf{\Sigma}$ with

$$\begin{aligned} (\widehat{\sigma}_{ij}^{(2)})_k &= \frac{1}{|W_k|} \sum_{p\ge 1} \mathbb{1}_{W_k}(X_p)\big(\mathbb{1}_{C_i\cap C_j}(M_p) - P_M^{\mathbf{o}}(C_i)\, P_M^{\mathbf{o}}(C_j) \big) \\ &+ \frac{1}{|W_k|} \sum_{p,q\ge 1}^{\neq} \mathbb{1}_{W_k}(X_p)\mathbb{1}_{W_k}(X_q)\big(\mathbb{1}_{C_i}(M_p) - P_M^{\mathbf{o}}(C_i) \big)\, \big(\mathbb{1}_{C_j}(M_q) - P_M^{\mathbf{o}}(C_j) \big)\,. \end{aligned}$$

Theorem 3.4. *Let X_M be a stationary MPP satisfying* (3.5) *and let* $\{W_k\}$ *be a CAS. Then* $(\widehat{\sigma}_{ij}^{(1)})_k$ *is an unbiased estimator, whereas* $(\widehat{\sigma}_{ij}^{(2)})_k$ *is an asymptotically unbiased estimator for* $\sigma_{ij}\,,\; i,j=1,...,\ell\,.$

Remark: In general, neither $(\widehat{\sigma}_{ij}^{(1)})_k$ nor $(\widehat{\sigma}_{ij}^{(2)})_k$ are L^2-consistent estimators for σ_{ij}, even if stronger moment and mixing conditions are supposed.

According to Lemma 3.1, the integrability condition (3.5) in Theorems 3.3 and 3.4 can be replaced by the stronger Condition $\beta(\delta)$. In order to obtain an L^2-consistent estimator, we introduce a smoothed version of the unbiased estimator in (3.8), which is based on some kernel function and a sequence of bandwidths depending on the CAS $\{W_k\}$.

Condition (*wb*): Let $w:\mathbb{R}\mapsto\mathbb{R}$ be a non-negative, symmetric, Borel-measurable *kernel function* satisfying $w(x)\longrightarrow w(0)=1$ as $x\to 0\,.$ In addition, assume that $w(\cdot)$ is bounded by $m_w<\infty$ and vanishes outside $B(\mathbf{o},r_w)$ for some $r_w\in(0,\infty)$. Further, associated with $w(\cdot)$ and some given CAS $\{W_k\}$, let $\{b_k\}$ be a sequence of positive *bandwidths* such that

$$\frac{\varrho(W_k)}{2\,d\,r_w\,|W_k|^{1/d}}\geq b_k\underset{k\to\infty}{\longrightarrow}0\quad,\quad b_k^d\,|W_k|\underset{k\to\infty}{\longrightarrow}\infty\quad\text{and}\quad b_k^{\frac{3}{2}\,d}\,|W_k|\underset{k\to\infty}{\longrightarrow}0\,. \tag{3.9}$$

Theorem 3.5. *Let* $\{W_k\}$ *be an arbitrary CAS and* $w(\cdot)$ *be a kernel function with an associated sequence of bandwidths* $\{b_k\}$ *satisfying Condition* (*wb*). *If the MPP* X_M *satisfies*

$$\mathbb{E}\big(X_M(\,[0,1]^d\times\mathbb{M}\,)\big)^{4+\delta}<\infty\quad\textit{and}\quad\int_1^\infty r^{d-1}\big(\beta^*_{X_M}(r)\big)^{\delta/(4+\delta)}\,\mathrm{d}r<\infty \tag{3.10}$$

for some $\delta>0\,,$ *then*

$$\mathbb{E}\big(\,\sigma_{ij}-(\widehat{\sigma}_{ij}^{(3)})_k\big)^2\underset{k\to\infty}{\longrightarrow}0\,,$$

where $(\widehat{\sigma}_{ij}^{(3)})_k$ *is a smoothed covariance estimator defined by*

$$(\widehat{\sigma}_{ij}^{(3)})_k\;=\;\frac{1}{|W_k|}\sum_{p\geq1}\mathbb{1}_{W_k}(X_p)\,\big(\,\mathbb{1}_{C_i\cap C_j}(M_p)-P_M^{\mathbf{o}}(C_i)\,P_M^{\mathbf{o}}(C_j)\,\big)$$
$$+\sum_{p,q\geq1}^{\neq}\frac{\mathbb{1}_{W_k}(X_p)\,\mathbb{1}_{W_k}(X_q)\,\big(\,\mathbb{1}_{C_i}(M_p)-P_M^{\mathbf{o}}(C_i)\,\big)\big(\,\mathbb{1}_{C_j}(M_q)-P_M^{\mathbf{o}}(C_j)\,\big)}{|(W_k-X_p)\cap(W_k-X_q)|}\,w\Big(\frac{\|X_q-X_p\|}{b_k|W_k|^{1/d}}\Big)\,.$$

Remark: The full strength of condition (3.10) on the β-mixing rate $\beta^*_{X_M}(r)$ introduced in (2.9) is only necessary to prove the consistency result of the preceding Theorem 3.5. However, the β-mixing rate $\beta^*_{X_M}(r)$ in Condition $\beta(\delta)$, which is needed to prove (3.4) and (3.5) as well as Theorem 3.1, can be defined by the slightly smaller non-increasing β-mixing rate function

$$\beta^*_{X_M}(r)=\beta\big(\mathcal{A}_{X_M}(K_a),\mathcal{A}_{X_M}(K^c_{a+r})\big)\quad\text{for}\quad r\geq a=1/2\,. \tag{3.11}$$

Moreover, in order to prove Theorem 3.1, condition $\beta(\delta)$ relying on the β-mixing coefficient considered in (2.8) with $\beta^*_{X_M}(r)$ and $\beta^{**}_{X_M}(r)$ given in (3.11) and (2.9), respectively, can be relaxed by using the slightly smaller α-mixing coefficient

$$\alpha\big(\mathcal{A}_{X_M}(B),\mathcal{A}_{X_M}(B')\big)=\sup\{\,\big|\mathbb{P}(A\cap A')-\mathbb{P}(A)\mathbb{P}(A')\big|\,:A\in\mathcal{A}_{X_M}(B),A'\in\mathcal{A}_{X_M}(B')\}$$

instead of (2.8). The corresponding α-mixing rates $\alpha^{*}_{X_M}(r)$ and $\alpha^{**}_{X_M}(r)$ are then defined in analogy to (3.11) and (2.9), respectively. A covariance inequality for the α-mixing case similar to (2.11) can be found in [6], see [15] for an improved version. Despite of the subtle differences between the discussed mixing conditions, we prefer to present our results under the unified assumptions of Condition $\beta(\delta)$ and (3.10) with $\beta^{*}_{X_M}(r)$ as defined in (2.9). It seems to be difficult to identify models where these differences are relevant.

4 Proofs

4.1 Proof of Theorem 3.1

By the Cramér-Wold technique, the multivariate CLT stated in (3.2) is equivalent to

$$s^{\top}\boldsymbol{Y}_k = s_1\,Y_k(C_1) + \ldots + s_\ell\,Y_k(C_\ell) \xrightarrow[k\to\infty]{\mathrm{D}} \mathcal{N}_1(0\,,\,\sigma^2) \quad \text{with} \quad \sigma^2 = s^{\top}\,\boldsymbol{\Sigma}\,s \tag{4.1}$$

for any $s = (s_1,\ldots,s_\ell)^{\top} \in \mathbb{R}^{\ell} \neq \mathbf{o}_\ell$.

To prove (4.1) we extend Bernstein's classical blocking method for weakly dependent random fields over a cubic index set of $\mathbb{Z}^d$, see e.g. [4], [12] or [21], to $\beta-$mixing fields indexed by elements of $H_k = \{z \in \mathbb{Z}^d : E_z \subset W_k\}$, where $E_z = [-1/2, 1/2)^d + z$ for $z \in \mathbb{Z}^d$ and $\{W_k\}$ is an arbitrary CAS. The proof of (4.1) is divided into four steps.

Step 1. Bounds and asymptotics for the variance of the sum
In view of (3.1) we may write

$$s^{\top}\boldsymbol{Y}_k = \frac{1}{\sqrt{|W_k|}}\big(\,V_k + V'_k\,\big)\,, \quad \text{where} \quad V_k = \sum_{z\in H_k} U_z \quad \text{and} \quad V'_k = \sum_{z\in\partial H_k} U^{(k)}_z$$

with

$$U^{(k)}_z = \sum_{n\geq 1} \mathbb{1}_{E_z\cap W_k}(X_n)\,g(M_n) \quad , \quad U_z = \sum_{n\geq 1} \mathbb{1}_{E_z}(X_n)\,g(M_n) \quad \text{for} \quad z \in \mathbb{Z}^d\,,$$

$\partial H_k = \{z \in \mathbb{Z}^d \setminus H_k : |\,E_z \cap W_k\,| > 0\,\}$ and $g(M_n) = \sum_{i=1}^{\ell} s_i\big(\mathbb{1}_{C_i}(M_n) - P^o_{X_M}(C_i)\big)$. Clearly, $\mathbb{E}U^{(k)}_z = \mathbb{E}U_z = 0$ and $\max\{|U^{(k)}_z|, |U_z|\} \leq c(s)\,X_M(E_z \times \mathbb{M})$ for $z \in \mathbb{Z}^d$, since $|g(M_n)| \leq c(s) = |s_1| + \cdots + |s_\ell|$. Hence, by stationarity of X_M and Condition $\beta(\delta)$,

$$\max\{\|\,U^{(k)}_z\,\|_{2+\delta}\,,\,\|\,U_z\,\|_{2+\delta}\} \leq c(s)\;\|X_M([0,1)^d \times \mathbb{M})\|_{2+\delta} \quad \text{for} \quad z \in \mathbb{Z}^d\,.$$

In the following we use the maximum norm $|z| = \max_{1\leq i\leq d}|z_i|$ to express the distance of $z = (z_1, ..., z_d) \in \mathbb{Z}^d$ to the origin $\mathbf{o}$. By applying the covariance inequality (2.11) together with Condition $\beta(\delta)$, we obtain

$$\begin{aligned}\mathrm{Var}(V'_k) &= \sum_{y,z\in\partial H_k} \mathbb{E}\,U^{(k)}_y\,U^{(k)}_z \leq \sum_{y,z\in\partial H_k} \mathbb{E}\,|U^{(k)}_y\,U^{(k)}_z\,| \leq \#\partial H_k \sum_{z\in\mathbb{Z}^d} \mathbb{E}\,|U^{(k)}_{\boldsymbol{o}}\,U^{(k)}_z\,| \\ &\leq 2\,c(s)^2\,\|X_M([0,1)^d \times \mathbb{M})\|^2_{2+\delta}\,\#\partial H_k \sum_{z\in\mathbb{Z}^d}\Big(\beta\big(\mathcal{A}_{X_M}(E_{\boldsymbol{o}}), \mathcal{A}_{X_M}(E_z)\big)\Big)^{\frac{\delta}{2+\delta}}\end{aligned}$$

$$
\begin{aligned}
&\le 2\,c(s)^2\,\|X_M([0,1)^d\times\mathbb{M})\|_{2+\delta}^2\,\#\partial H_k\Big(1+\sum_{z\neq \boldsymbol{o}}\big(\beta^*_{X_M}(|z|-1)\big)^{\frac{\delta}{2+\delta}}\Big)\\
&\le 2\,c(s)^2\,\|X_M([0,1)^d\times\mathbb{M})\|_{2+\delta}^2\,\#\partial H_k\Big(1+2\,d\sum_{n\ge 0}(2n+3)^{d-1}\,\big(\beta^*_{X_M}(n)\big)^{\frac{\delta}{2+\delta}}\Big) \qquad (4.2)\\
&\le c_1\,\#\partial H_k,
\end{aligned}
$$

for some constant $c_1=c_1(s,d,\delta)>0$ where the relation $\#\{z\in\mathbb{Z}^d:|z|=n\}=(2n+1)^d-(2n-1)^d\le 2\,d\,(2n+1)^{d-1}$ has been used. A simple geometric argument shows that each unit cube E_z hitting the boundary ∂W_k is contained in the annulus $\partial W_k\oplus B(\mathbf{o},\sqrt{d})$ implying that

$$
\#\partial H_k \le |\partial W_k\oplus B(\mathbf{o},\sqrt{d})| \le 2\left(\,|W_k\oplus B(\mathbf{o},\sqrt{d})|-|W_k|\,\right).
$$

Steiner's formula (cf. [24], p. 600) applied to the convex body W_k reveals that the volume $|W_k\oplus B(\mathbf{o},\sqrt{d})|-|W_k|$ does not decrease when W_k is replaced by a larger convex body, e.g. by $d^{3/2}\,R_k$ from relation (4.8) below, where the hyper-rectangle R_k has edge lengths $a_1^{(k)},\ldots,a_d^{(k)}$. Replacing additionally $B(\mathbf{o},\sqrt{d})$ by the cube $[-\sqrt{d},\sqrt{d}]^d$ we get

$$
\begin{aligned}
|W_k\oplus B(\mathbf{o},\sqrt{d})|-|W_k| \;&\le\; |d^{3/2}\,R_k\oplus[-\sqrt{d},\sqrt{d}]^d|-|d^{3/2}\,R_k|\\
&=\; 2\,d^{(3d-2)/2}\sum_{i=1}^{d}a_1^{(k)}\cdots a_{i-1}^{(k)}\Big(a_{i+1}^{(k)}+\frac{2}{d}\Big)\cdots\Big(a_d^{(k)}+\frac{2}{d}\Big)\\
&\le\; 2^{d-1}\,d^{(3d-2)/2}\,\mathcal{H}_{d-1}(\partial R_k), \quad \text{if} \quad \min_{1\le i\le d}a_i^{(k)}\ge\frac{2}{d}.
\end{aligned}
$$

Hence, since (4.8) implies $\mathcal{H}_{d-1}(\partial R_k)\le\mathcal{H}_{d-1}(\partial W_k)$ and $d^{3/2}\,\min_{1\le i\le d}a_i^{(k)}\ge 2\,\varrho(W_k)$, it follows that $\#\partial H_k\le 2^d\,d^{(3d-2)/2}\,\mathcal{H}_{d-1}(\partial W_k)$ if $\varrho(W_k)\ge\sqrt{d}$, which in turn by combining (2.2), (4.2) and the inclusion $\#H_k\le|W_k|\le\#H_k+\#\partial H_k$ implies that

$$
\frac{\mathrm{Var}(V_k')}{|W_k|}\le c_2\,\frac{\mathcal{H}_{d-1}(\partial W_k)}{|W_k|}\le\frac{c_2\,d}{\varrho(W_k)}\underset{k\to\infty}{\longrightarrow}0 \qquad\text{and}\qquad \frac{\#H_k}{|W_k|}\underset{k\to\infty}{\longrightarrow}1 \qquad (4.3)
$$

for any CAS $\{W_k\}$. Thus, by a standard Slutsky argument, (4.1) is equivalent to

$$
\frac{V_k}{\sqrt{\#H_k}}\ \underset{k\to\infty}{\overset{\mathrm{D}}{\longrightarrow}}\ \mathcal{N}_1(0\,,\,\sigma^2)\,. \qquad (4.4)
$$

The technique used above to estimate $\mathrm{Var}(V_k')$ will in the following be applied to show that

$$
\sigma^2\;=\;\lim_{k\to\infty}\mathrm{Var}(s^\top\boldsymbol{Y}_k)\;=\;\lim_{k\to\infty}\frac{\mathrm{Var}(V_k)}{\#H_k}\;=\;\sum_{z\in\mathbb{Z}^d}\mathbb{E}\,(U_{\mathbf{o}}\,U_z)\,. \qquad (4.5)
$$

The series on the rhs of (4.5) converges absolutely as immediate consequence of the estimate

$$
\mathrm{Var}(V_k)\;\le\;\#H_k\sum_{z\in\mathbb{Z}^d}|\,\mathbb{E}\,(U_{\mathbf{o}}\,U_z)\,|\;\le\;c_1\,\#H_k\,,
$$

where the positive constant c_1 is the same as in (4.2). The Cauchy-Schwarz inequality and the previous estimates of $\mathrm{Var}(V_k)$ and $\mathrm{Var}(V_k')$ show that

$$\left| \mathrm{Var}(s^\top \boldsymbol{Y}_k) - \frac{\mathrm{Var}(V_k)}{|W_k|} \right| \le 2\,\frac{|\mathrm{Cov}(V_k, V_k')|}{|W_k|} + \frac{\mathrm{Var}(V_k')}{|W_k|} \le \frac{2c_1\sqrt{\#H_k\,\#\partial H_k}}{|W_k|} + \frac{c_1\#\partial H_k}{|W_k|}$$

proving the second equality in (4.5). To prove the third equality in (4.5) we use the identity

$$\frac{\mathrm{Var}(V_k)}{\#H_k} = \frac{1}{\#H_k}\sum_{y,z\in H_k}\mathbb{E}\,(U_{\mathbf{o}}\,U_{z-y}) = \sum_{z\in\mathbb{Z}^d}\frac{\#(H_k\cap(H_k-z))}{\#H_k}\mathbb{E}\,(U_{\mathbf{o}}\,U_z)$$

and the geometric inequality (following from the very definition of H_k and ∂H_k)

$$\#(\,H_k\cap(H_k-z)\,) \le |\,W_k\cap(W_k-z)\,| \le \#(\,H_k\cap(H_k-z)\,) + \#\partial H_k + \#\partial(H_k-z)$$

for $z\in\mathbb{Z}^d$. This fact combined with (2.2) and (4.3) shows that

$$\frac{\#(H_k\cap(H_k-z))}{\#H_k}\underset{k\to\infty}{\longrightarrow} 1 \quad \text{for any fixed} \quad z\in\mathbb{Z}^d$$

proving the third equality in (4.5) by applying the dominated convergence theorem.

Step 2. Passage to bounded random variables by truncation

For any fixed $a>0$ we define the random field $\{U_z(a)\,,\,z\in H_k\}$ of the truncated (and centered) random variables and the sum $V_k(a)$ by

$$U_z(a) = U_z 1\!\!1_{\{|U_z|\le a\}} - \mathbb{E}\big(U_z 1\!\!1_{\{|U_z|\le a\}}\big) \quad \text{and} \quad V_k(a) = \sum_{z\in H_k} U_z(a) \tag{4.6}$$

so that, for any $z\in H_k$,

$$|\,U_z(a)\,| \le 2\,a \quad \text{and} \quad \big(\,\mathbb{E}|\,U_z - U_z(a)\,|^{2+\delta}\,\big)^{\frac{1}{2+\delta}} = \|\,U_{\mathbf{o}} - U_{\mathbf{o}}(a)\,\|_{2+\delta}\underset{a\to\infty}{\longrightarrow} 0\,.$$

By quite the same arguments as used in Step 1 based on the covariance inequality (2.11) and Condition $\beta(\delta)$, we find that

$$\mathrm{Var}\big(\,V_k - V_k(a)\,\big) \le 2\,\#H_k\,\|\,U_{\mathbf{o}} - U_{\mathbf{o}}(a)\,\|_{2+\delta}^2\left(1 + 2\,d\sum_{n\ge 0}(2n+3)^{d-1}\,\big(\beta^*_{X_M}(n)\big)^{\frac{\delta}{2+\delta}}\right)$$

for $k\ge 1$. Hence, by Slutsky's lemma, the weak limits of $V_k/\sqrt{\#H_k}$ and $V_k(a)/\sqrt{\#H_k}$ as $k\to\infty$ are arbitrarily close whenever $a>0$ is large enough. It therefore remains to prove the CLT in (4.4) for the bounded random variables in (4.6), i.e., for any fixed $a>0$,

$$\frac{V_k(a)}{\sqrt{\#H_k}}\underset{k\to\infty}{\overset{\mathrm{D}}{\longrightarrow}}\mathcal{N}_1\big(\,0\,,\,\sigma^2(a)\,\big) \quad \text{with} \quad \sigma^2(a) = \sum_{z\in\mathbb{Z}^d}\mathbb{E}\,U_{\mathbf{o}}(a)\,U_z(a)\,. \tag{4.7}$$

Step 3. Adaptation of Bernstein's blocking method to non-cubic index sets

We start with some preliminary considerations. A well-known result from convex geometry first proved by F. John, see e.g. [1], asserts that there exists a unique ellipsoid $\mathcal{E}_k$ (called *John ellipsoid*) of maximal volume contained in W_k with midpoint $c(\mathcal{E}_k)$ and semi-axes of lengths $e_1^{(k)}, \ldots, e_d^{(k)}$ such that $\mathcal{E}_k \subseteq W_k \subseteq c(\mathcal{E}_k) + d\,(\,\mathcal{E}_k - c(\mathcal{E}_k)\,)$.

Further, it is easy to determine a unique hyper-rectangle R_k centered at the origin $\mathbf{o}$ circumscribed by $\mathcal{E}_k - c(\mathcal{E}_k)$ with edge-lengths $a_i^{(k)} = 2\,e_i^{(k)}/\sqrt{d}$ for $i = 1, \ldots, d$ such that $\mathcal{E}_k - c(\mathcal{E}_k) \subseteq \sqrt{d}\, R_k$ and finally

$$R_k \subseteq \mathcal{E}_k - c(\mathcal{E}_k) \subseteq W_k - c(\mathcal{E}_k) \subseteq d\,(\,\mathcal{E}_k - c(\mathcal{E}_k)\,) \subseteq d^{3/2}\, R_k\,. \tag{4.8}$$

Since the MPP X_M observed in the CAS $\{W_k\}$ is stationary, we may assume that $c(\mathcal{E}_k) = \mathbf{o}$ and without loss of generality let the edge lengths of R_k be arranged in ascending order $a_1^{(k)} \le \cdots \le a_d^{(k)}$ (possibly after renumbering of the edges). Note that R_k is not necessarily in a position parallel to the coordinate axes. But there is an orthogonal matrix O_k such that

$$O_k\, R_k = \mathop{\times}_{i=1}^{d} \Big[-\frac{a_i^{(k)}}{2}\,,\,\frac{a_i^{(k)}}{2}\Big]\,. \tag{4.9}$$

Let $\{p_k\}$ and $\{q_k\}$ be two sequences of positive integers (which will be specified later) satisfying $p_k \ge q_k \underset{k\to\infty}{\longrightarrow} \infty$ and $q_k/p_k \underset{k\to\infty}{\longrightarrow} 0$. We define two types of pairwise disjoint cubes

$$P_y^{(k)} = P_{\mathbf{o}}^{(k)} + (2\,p_k + q_k + 1)\,y \quad \text{and} \quad Q_y^{(k)} = Q_{\mathbf{o}}^{(k)} + (2\,p_k + q_k + 1)\,y \quad \text{for} \quad y \in \mathbb{Z}^d\,,$$

where $P_{\mathbf{o}}^{(k)} = \{-p_k, \ldots, 0, \ldots, p_k\}^d$ and $Q_{\mathbf{o}}^{(k)} = \{-p_k, \ldots, 0, \ldots, p_k + q_k\}^d$ for $k \ge 1$.

Now, we describe how to modify *Bernstein's blocking method* in order to prove the CLT stated in (4.7). For the family of *block sums*

$$V_y^{(k)}(a) = \sum_{z \in P_y^{(k)} \cap H_k} U_z(a) \quad \text{for} \quad y \in G_k = \{z \in \mathbb{Z}^d : P_z^{(k)} \cap H_k \ne \emptyset\}$$

we shall show in Step 4 that

$$\frac{1}{\sqrt{\# H_k}} \sum_{y \in G_k} V_y^{(k)}(a) \underset{k\to\infty}{\overset{\mathrm{D}}{\longrightarrow}} \mathcal{N}_1\big(\,0\,,\,\sigma^2(a)\,\big) \tag{4.10}$$

by assuming the mutual independence of the random variables $V_y^{(k)}(a)\,,\; y \in G_k\,$, which can be justified by Condition $\beta(\delta)$ and

$$\frac{1}{\sqrt{\# H_k}} \Big(\, V_k(a) - \sum_{y \in G_k} V_y^{(k)}(a)\,\Big) \underset{k\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} 0\,. \tag{4.11}$$

Next, we specify the choice of p_k and q_k in dependence on the edge lengths of R_k and the supposed decaying rate of $\beta_{X_M}^{**}(r)$. In view of $\varrho(W_k) \longrightarrow \infty$ and (4.8) it follows that $\min\{a_1^{(k)}, \ldots, a_d^{(k)}\} \longrightarrow \infty$ as $k \to \infty$. Note that the choice $p_k = \lfloor \varepsilon_k\, |W_k|^{1/2d} \rfloor$ as in case of a cubic observation window with a certain null sequence $\{\varepsilon_k\}$, see [12], does not always imply (4.10) and (4.11) if at least one of the first $d-1$ ordered edge lengths of R_k increases very slowly to infinity. So one has to choose p_k large enough but much smaller than $a_d^{(k)}$. For this purpose put $r_k(s) = (\,a_{s+1}^{(k)} \cdot \ldots \cdot a_d^{(k)}\,)^{1/(2d-s)}$ for each $s \in \{0,1,\ldots,d-1\}$. Because of $r^{2d-1}\,\beta^{**}_{X_M}(r) \longrightarrow 0$ as $r \to \infty$, there exist non-increasing sequences $\varepsilon_k(s)$ of positive numbers such that

$$\varepsilon_k(s) \underset{k\to\infty}{\longrightarrow} 0, \qquad \varepsilon_k(s)\, r_k(s) \underset{k\to\infty}{\longrightarrow} \infty, \quad \text{and} \quad \frac{(r_k(s))^{2d-1}}{\varepsilon_k(s)}\, \beta^{**}_{X_M}(\varepsilon_k(s)\, r_k(s)) \underset{k\to\infty}{\longrightarrow} 0\,. \tag{4.12}$$

Let $\varepsilon_k = \max\{\varepsilon_k(0),\ldots,\varepsilon_k(d-1)\}$ and $p_k(s) = \varepsilon_k^{1/(2d-s)}\, r_k(s)$ and select s_k to be the smallest number in $\{0,1,...,d-1\}$ such that $a_{s+1}^{(k)} \geq 2\,p_k(s)+1$ for $k \geq k_0$, where k_0 is a sufficiently large positive integer. Thus, we define the integer sequences p_k and q_k by

$$p_k = \lfloor p_k(s_k) \rfloor = \lfloor \varepsilon_k^{1/(2d-s_k)}\, r_k(s_k) \rfloor \quad \text{and} \quad q_k = \lfloor \varepsilon_k\, r_k(s_k) \rfloor \;\text{ for }\; k \geq k_0\,. \tag{4.13}$$

Further, we need lower and upper bounds for the number N_k of cubes $\widetilde{Q}_y^{(k)} = [-p_k - \frac{1}{2}, p_k + q_k + \frac{1}{2})^d + (2\,p_k + q_k + 1)\,y$ hitting H_k, i.e., $N_k = \#\{y \in \mathbb{Z}^d : \widetilde{Q}_y^{(k)} \cap H_k \neq \emptyset\}$. For this put $N_j^{(k)}(c,w) = \#\{y \in \mathbb{Z}^d : \widetilde{Q}_y^{(k)} \cap (L_j^{(k)}(c) + w) \neq \emptyset\}$ for $w \in \mathbb{R}^d$ and some real $c > 0$, where $L_j^{(k)}(c) = O_k^T\,\{(x_1,\ldots,x_d) \in \mathbb{R}^d : -c\,a_j^{(k)}/2 \leq x_j \leq c\,a_j^{(k)}/2\,,\, x_i = 0 \;\text{ for }\; i \neq j\}$. The following rough estimates of $N_j^{(k)}(c,w)$ from below and above can be obtained by elementary geometric arguments:

$$\Big\lfloor \frac{c\,a_j^{(k)}/\sqrt{d}}{2\,p_k + q_k + 1} \Big\rfloor + 1 \;\leq\; N_j^{(k)}(c,w) \;\leq\; d\Big(\Big\lfloor \frac{c\,a_j^{(k)}}{2\,p_k + q_k + 1} \Big\rfloor + 2\Big) \quad \text{for any} \quad w \in \mathbb{R}^d\,.$$

Hence, by (4.8) and (4.9) the minimal number $N_{\min}^{(k)}$ and the maximal number $N_{\max}^{(k)}$ of cubes $\widetilde{Q}_y^{(k)}$ hitting H_k satisfy the inequality

$$\prod_{j=1}^{d}\Big(\Big\lfloor \frac{a_j^{(k)}/(2\sqrt{d})}{2\,p_k + q_k + 1} \Big\rfloor + 1\Big) \;\leq\; N_{\min}^{(k)} \;\leq\; N_k \;\leq\; N_{\max}^{(k)} \;\leq\; d^d \prod_{j=1}^{d}\Big(\Big\lfloor \frac{d^{3/2}\,a_j^{(k)}}{2\,p_k + q_k + 1} \Big\rfloor + 2\Big).$$

In view of the above choice of $s = s_k$ and (4.13), the number N_k allows the estimate

$$c_3\,\frac{a_{s_k+1}^{(k)} \cdots a_d^{(k)}}{p_k^{d-s_k}} \;\leq\; N_k \;\leq\; c_4\,\frac{a_{s_k+1}^{(k)} \cdots a_d^{(k)}}{p_k^{d-s_k}} \quad \text{for all} \quad k \geq k_0 \tag{4.14}$$

with positive constants c_3, c_4 only depending on the dimension d. Combining the obvious fact that $\#\,G_k \leq N_k$ with (4.12), (4.13) and (4.14) (with $p_k \geq 1$ and $\varepsilon_k \leq 1$) we arrive at

$$\#\,G_k\, p_k^{d-1}\, \beta^{**}_{X_M}(q_k) \leq c_4\, \frac{(r_k(s_k))^{2d-1}}{\varepsilon_k^{1/2d}}\, \beta^{**}_{X_M}(q_k) \underset{k\to\infty}{\longrightarrow} 0\,.$$

Likewise, by (4.8) and $a_i^{(k)} \leq 2\,p_k + 3$ for $i = 1,...,s_k$,

$$\frac{p_k^{d-s_k}}{\sqrt{\#\,H_k}} \prod_{j=1}^{s_k} a_j^{(k)} \leq c_5\,\Big(\frac{p_k}{r_k(s_k)}\Big)^{(2d-s_k)/2} \leq c_5\,\sqrt{\varepsilon_k} \underset{k\to\infty}{\longrightarrow} 0\,. \tag{4.15}$$

Finally, we show that

$$\frac{1}{\# H_k} \sum_{y \in G_k} \# (P_y^{(k)} \cap H_k) \underset{k\to\infty}{\longrightarrow} 1 \,, \tag{4.16}$$

which, by the results of Step 1, is equivalent to

$$\frac{1}{|W_k|} \sum_{y \in \mathbb{Z}^d} |\, (\widetilde{Q}_y^{(k)} \setminus \widetilde{P}_y^{(k)}) \cap W_k \,| \underset{k\to\infty}{\longrightarrow} 0 \,, \tag{4.17}$$

where $\widetilde{P}_y^{(k)} = [-p_k - \frac{1}{2}, p_k + \frac{1}{2})^d + (2p_k + q_k + 1)\, y$ for $y \in \mathbb{Z}^d$. To estimate the volume of the space in W_k outside the union of cubes $\widetilde{P}_y^{(k)}$ we introduce equidistant *slices* $S_{ij}^{(k)}$ in $\mathbb{R}^d$ of thickness q_k and distance $2p_k + 1$ defined by

$$S_{ij}^{(k)} = \Big\{ (y_1, \dots, y_d) \in \mathbb{R}^d : (2j+1)\,\big(p_k + \frac{1}{2}\big) + j\, q_k \le y_i < (2j+1)\,\big(p_k + \frac{1}{2}\big) + (j+1)\, q_k \Big\}$$

for $i = 1, ..., d$ and $j \in \mathbb{Z}^1$. By (4.8), (4.9) and the choice of p_k and q_k it might happen that, for at most s_k coordinates $i \in \{1, ..., d\}$, $S_{ij}^{(k)} \cap W_k = \emptyset$ for all intergers j. For the remaining coordinates $i \in \{1, ..., d\}$ there exist sequences of integers $n_k(i)$ (at least one of them tends to infinity as $k \to \infty$) such that $S_{ij}^{(k)} \cap R_k \ne \emptyset$ for $|j| \le n_k(i)$ (and $S_{ij}^{(k)} \cap R_k = \emptyset$ for $|j| > n_k(i)$) and

$$\frac{1}{|R_k|} \sum_{|j| \le n_k(i)} |\, S_{ij}^{(k)} \cap d^{3/2} R_k \,| \le c_6 \frac{q_k}{p_k} \quad \text{for} \quad k \ge k_0 \,,$$

where c_6 depends only on d. This estimate and the evident inequalities $|R_k| \le |W_k|$ and

$$\sum_{y \in \mathbb{Z}^d} |\, (\widetilde{Q}_y^{(k)} \setminus \widetilde{P}_y^{(k)}) \cap W_k \,| \le \sum_{i=1}^{d} \sum_{j \in \mathbb{Z}^1} |\, S_{ij}^{(k)} \cap d^{3/2} R_k \,|$$

show that the lhs of (4.17) is bounded by a constant multiple of q_k/p_k so that (4.16) is finally proved by (4.13).

Step 4. Approximation by sums of independent random variables

For brevity put $P_k = \bigcup_{y \in G_k} (P_y^{(k)} \cap H_k)$. Again by applying the covariance inequality (2.11) and Condition $\beta(\delta)$ to the stationary random field $\{U_z(a), z \in H_k\}$ (with $|U_z(a)| \le 2\,a$ and thus $\delta = \infty$), we find in analogy to (4.2) that

$$\begin{aligned} \frac{1}{\# H_k} \mathbb{E}\Big(V_k(a) - \sum_{y \in G_k} V_y^{(k)}(a) \Big)^2 \;&=\; \frac{1}{\# H_k} \sum_{y,z \in H_k \setminus P_k} \mathbb{E}\big(U_y(a)\, U_z(a)\big) \\ &\le\; 8\,a^2 \Big(1 + 2\,d \sum_{n \ge 0} (2n+3)^{d-1} \beta^*_{X_M}(n) \Big) \frac{\#(H_k \setminus P_k)}{\# H_k} \,. \end{aligned}$$

From (4.16) it is immediately clear that the ratio in the latter line disappears as $k \to \infty$, which confirms (4.11). Thus, in view of Slutsky's lemma, it remains to prove (4.10). We will do this under the assumption of mutual independence of the block sums $V_y^{(k)}(a), y \in G_k$. For this reason we show that the characteristic function $\mathbb{E} \exp\{it\, V_k(a)\}$ differs from the product $\prod_{y \in G_k} \mathbb{E} \exp\{it\, V_y^{(k)}(a)\}$ uniformly in $t \in \mathbb{R}^1$ by certain sequences tending to zero as $k \to \infty$.

Setting $n_k = \#G_k$ we may write

$$\xi_j = \exp\{\mathrm{i}t\, V_{y_j}^{(k)}(a)\} \quad \text{for} \quad y_j \in G_k \quad \text{with} \quad j = 1,\ldots,n_k\,.$$

Using the algebraic identity

$$\mathbb{E}\prod_{j=1}^{n_k}\xi_j - \prod_{j=1}^{n_k}\mathbb{E}\xi_j = \sum_{j=1}^{n_k-1}\mathbb{E}\xi_1\cdots\mathbb{E}\xi_{j-1}\Big(\mathbb{E}\xi_j\xi_{j+1}\cdots\xi_{n_k} - \mathbb{E}\xi_j\,\mathbb{E}\big(\xi_{j+1}\cdots\xi_{n_k}\big)\Big)$$

and $|\xi_j| \le 1$ for $j = 1, ..., n_k$ we get

$$\Big|\mathbb{E}\exp\{\mathrm{i}t\, V_k(a)\} - \prod_{y\in G_k}\mathbb{E}\exp\{\mathrm{i}t\, V_y^{(k)}(a)\}\Big| \le \sum_{j=1}^{n_k-1}\Big|\mathrm{Cov}(\xi_j, \xi_{j+1}\cdots\xi_{n_k})\Big|\,.$$

By the stationarity of X_M we may assume that the real as well as the imaginary part of ξ_j is measurable w.r.t. the σ−algebra $\mathcal{A}_{X_M}(K_{p_k+1/2})$ and the product $\xi_{j+1}\cdots\xi_{n_k}$ is measurable w.r.t. $\mathcal{A}_{X_M}(\mathbb{R}^d \setminus K_{p_k+q_k+1/2})$. By applying the covariance inequality (2.11) with $\delta = \infty$ (to the real and imaginary part of ξ_j resp. $\xi_{j+1}\cdots\xi_{n_k}$) and using (2.9) we find that

$$\begin{aligned}\Big|\mathrm{Cov}(\xi_j, \xi_{j+1}\cdots\xi_{n_k})\Big| &\le 8\,\beta\big(\mathcal{A}_{X_M}(K_{p_k+1/2}), \mathcal{A}_{X_M}(\mathbb{R}^d \setminus K_{p_k+q_k+1/2})\big)\\ &\le 8\,(p_k+1/2)^{d-1}\,\beta_{X_M}^{**}(q_k)\,.\end{aligned}$$

Since $n_k = \#G_k \le N_k$ it follows with (4.14) that

$$\sup_{t\in\mathbb{R}^1}\Big|\mathbb{E}\exp\{\mathrm{i}t\, V_k(a)\} - \prod_{y\in G_k}\mathbb{E}\exp\{\mathrm{i}t\, V_y^{(k)}(a)\}\Big| \le 8\, n_k\,(\,p_k + 1/2\,)^{d-1}\,\beta_{X_M}^{**}(q_k) \underset{k\to\infty}{\longrightarrow} 0\,.$$

The latter relation and the Berry-Esseen bound in the CLT for independent random variables (which can be expressed by the third-order Lyapunov ratio, see e.g. [4], p. 204, and references therein) reveal that (4.10) holds if

$$L_3^{(k)}(a) = \frac{1}{(\sigma_k^2(a))^{3/2}}\sum_{y\in G_k}\mathbb{E}|\,V_y^{(k)}(a)\,|^3 \underset{k\to\infty}{\longrightarrow} 0 \qquad \text{and} \qquad \frac{\sigma_k^2(a)}{\#H_k}\underset{k\to\infty}{\longrightarrow}\sigma^2(a)\,, \tag{4.18}$$

where $\sigma^2(a)$ is defined by (4.7) and $\sigma_k^2(a) = \sum_{y\in G_k}\mathbb{E}(\,V_y^{(k)}(a)\,)^2$ coincides with the variance of $V_k(a)$ in case of independent block sums $V_y^{(k)}(a)$, $y \in G_k$.

It is easily seen that $|\,V_y^{(k)}(a)\,| \le 2\,a\,\#\,(P_y^{(k)} \cap H_k) \le 2\,a\,(2p_k+1)^{d-s_k}\prod_{i=1}^{s_k}(d^{3/2}\,a_i^{(k)}+1)$ and therefore

$$L_3^{(k)}(a) \le 2\,a\prod_{i=1}^{s_k}\big(d^{3/2}a_i^{(k)}+1\big)\,\frac{(2\,p_k+1)^{d-s_k}}{(\sigma_k^2(a))^{3/2}}\sum_{y\in G_k}\mathbb{E}(\,V_y^{(k)}(a)\,)^2 \le c_7\,\frac{2\,a\,p_k^{d-s_k}}{(\sigma_k^2(a))^{1/2}}\prod_{i=1}^{s_k}a_i^{(k)}$$

with some positive constant c_7 only depending on d. In combination with (4.15) the second relation in (4.18) for $\sigma^2(a) > 0$ yields $L_3^{(k)}(a) \underset{k\to\infty}{\longrightarrow} 0$. Hence, the first part of (4.18) is proved.

To accomplish the proof of (4.18) we remember that $\sigma^2(a)$ is the asymptotic variance (4.5) with $V_k(a)$ from (4.6) instead of V_k. Taking into account (4.16) or (4.17) we may replace

$V_k(a)$ by the reduced sum $\sum_{y\in G_k} V_y^{(k)}(a)$ so that the second part of (4.18) is a consequence of

$$\frac{1}{\#H_k}\Big|\mathbb{E}\Big(\sum_{y\in G_k} V_y^{(k)}(a)\Big)^2 - \sigma_k^2(a)\Big| \le c_8\, a^2\,\frac{\#P_k}{\#H_k}\sum_{n\ge q_k}(2n+3)^{d-1}\,\beta^*_{X_M}(n) \underset{k\to\infty}{\longrightarrow} 0\,.$$

Here we have again used the notation P_k and the standard covariance estimates from the very beginning of Step 4. Summarizing all Steps 1 - 4 completes the proof of Theorem 3.1.

□

4.2 Proof of Lemma 3.1

By definition of the signed measures $\gamma_X^{(2)}$ and $\gamma_{X,red}^{(2)}$ in Section 2.2 and using algebraic induction, for any bounded Borel-measurable function $g:(\mathbb{R}^d)^2\to\mathbb{R}^1$ we obtain the relation

$$\lambda\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x,y)\,\gamma_{X,red}^{(2)}(\mathrm{d}y)\,\mathrm{d}x = \int_{(\mathbb{R}^d)^2} g(x,y-x)\,\gamma_X^{(2)}(\mathrm{d}(x,y)). \tag{4.19}$$

Let H^+,H^- be a Hahn decomposition of $\mathbb{R}^d$ for $\gamma_{X,red}^{(2)}$, i.e.,

$$\gamma_{X,red}^{(2)+}(\cdot) = \gamma_{X,red}^{(2)}(H^+\cap(\cdot)) \qquad\text{and}\qquad \gamma_{X,red}^{(2)-}(\cdot) = -\gamma_{X,red}^{(2)}(H^-\cap(\cdot))\,.$$

We now apply (4.19) for $g(x,y)=1\!\!1_{E_\mathbf{o}}(x)\,1\!\!1_{H^+\cap E_z}(y)$, where $E_z=[-\frac12,\frac12)^d+z$ for $z\in\mathbb{Z}^d$. Combining this with the definition of the (reduced) second factorial moment measures $\alpha_X^{(2)}$ and $\alpha_{X,red}^{(2)}$ of the unmarked PP $X=\sum_{i\ge1}\delta_{X_i}$, see (2.5) for $m=2$, and

$$\gamma_X^{(2)}(A\times B) = \alpha_X^{(2)}(A\times B) - \lambda^2\,|A|\,|B| \quad\text{for all bounded}\quad A,B\in\mathcal{B}(\mathbb{R}^d)\,,$$

leads to

$$\begin{aligned}\lambda\,\gamma_{X,red}^{(2)}(H^+\cap E_z) &= \int_{(\mathbb{R}^d)^2} 1\!\!1_{E_\mathbf{o}}(x)1\!\!1_{H^+\cap E_z}(y-x)\alpha^{(2)}(d(x,y)) - \lambda^2\,|E_\mathbf{o}|\,|H^+\cap E_z|\\ &= \mathbb{E}\sum_{i,j\ge1}^{\neq} 1\!\!1_{E_\mathbf{o}}(X_i)1\!\!1_{H^+\cap E_z}(X_j-X_i) - \mathbb{E}X(E_\mathbf{o})\,\mathbb{E}X(H^+\cap E_z).\end{aligned}$$

Since $\mathbf{o}\notin H^+\cap E_z$ for $z\in\mathbb{Z}^d$ with $|z|\ge2$ we may continue with

$$\begin{aligned}\lambda\,\gamma_{X,red}^{(2)}(H^+\cap E_z) &= \mathbb{E}\sum_{i\ge1}\delta_{X_i}(E_\mathbf{o})\,X\big((H^+\cap E_z)+X_i\big) - \mathbb{E}X(E_\mathbf{o})\,\mathbb{E}X(H^+\cap E_z)\\ &= \mathbb{E}f(Y,Y'_z) - \mathbb{E}f(\widetilde{Y},\widetilde{Y}'_z) \quad\text{for}\quad |z|\ge2\,,\end{aligned} \tag{4.20}$$

where

$$f(Y,Y'_z) = \sum_{i\ge1}\delta_{X_i}(E_\mathbf{o})\,X\big((H^+\cap E_z)+X_i\big) \le X\big(E_\mathbf{o}\big)\,X\big(E_z\oplus E_\mathbf{o}\big) \tag{4.21}$$

with $Y(\cdot)=\sum_{i\ge1}\delta_{X_i}\big((\cdot)\cap E_\mathbf{o}\big)$ resp. $Y'_z(\cdot)=\sum_{j\ge1}\delta_{X_j}\big(((\cdot)\cap E_z)\oplus E_\mathbf{o}\big)$ being restrictions of the stationary PP $X=\sum_{i\ge1}\delta_{X_i}$ to $E_\mathbf{o}$ resp. $E_z\oplus E_\mathbf{o}=[-1,1)^d+z$. Further, let $\widetilde{Y}$ and

$\widetilde{Y}'_z$ denote copies of the PPs Y and Y'_z, respectively, which are assumed to be independent implying that $\mathbb{E}f(\widetilde{Y},\widetilde{Y}'_z) = \mathbb{E}X(E_{\mathbf{o}})\,\mathbb{E}X(H^+\cap E_z)$. Since Y is measurable w.r.t. $\mathcal{A}_X(E_{\mathbf{o}})$, whereas Y' is $\mathcal{A}_X(\mathbb{R}^d\backslash[-(|z|-1),|z|-1]^d)$-measurable, we are in a position to apply Lemma 2.2 with $\beta\big(\mathcal{A}_X(E_{\mathbf{o}}),\mathcal{A}_X(\mathbb{R}^d\backslash[-(|z|-1),|z|-1]^d\big) \le \beta^*_{X_M}(|z|-\frac{3}{2})$ for $|z|\ge 2$. Hence, the estimate (2.10) together with (4.20) and (4.21) yields

$$\big|\lambda\,\gamma^{(2)}_{X,red}(H^+\cap E_z)\big| \le 2\,\Big(\beta^*_{X_M}(|z|-\frac{3}{2})\Big)^{\frac{\eta}{1+\eta}}\,\Big(\max\Big\{\,\mathbb{E}f^{1+\eta}(Y,Y'_z)\,,\,\mathbb{E}f^{1+\eta}(\widetilde{Y},\widetilde{Y}'_z)\,\Big\}\Big)^{\frac{1}{1+\eta}}\,,$$

where the maximum term on the rhs has the finite upper bound $2^{d(1+\eta)}\,\mathbb{E}(X(E_{\mathbf{o}}))^{2+2\eta}$ for $\delta = 2\,\eta > 0$ in accordance with our assumptions. This is seen from (4.21) using the Cauchy-Schwarz inequality and the stationarity of X giving

$$\mathbb{E}f^{1+\eta}(Y,Y'_z) \le \Big(\,\mathbb{E}(X(E_{\mathbf{o}})^{2+2\eta})\,\mathbb{E}(X([-1,1]^d)^{2+2\eta})\,\Big)^{1/2} \le 2^{d(1+\eta)}\,\mathbb{E}(X(E_{\mathbf{o}})^{2+2\eta})$$

and the same upper bound for $\mathbb{E}f^{1+\eta}(\widetilde{Y},\widetilde{Y}'_z)$. By combining all the above estimates with $\lambda\,\gamma^{(2)}_{X,red}(H^+\cap[-\frac{3}{2},\frac{3}{2})^d) \le 3^d\,\mathbb{E}X^2(E_{\mathbf{o}})$ we arrive at

$$\lambda\,\gamma^{(2)}_{X,red}(H^+) \le 3^d\,\mathbb{E}X^2(E_{\mathbf{o}}) + 2^{d+1}\,\Big(\mathbb{E}(X(E_{\mathbf{o}}))^{2+\delta}\Big)^{\frac{2}{2+\delta}}\sum_{z\in\mathbb{Z}^d:|z|\ge 2}\Big(\beta^*_{X_M}(|z|-\frac{3}{2})\Big)^{\frac{\delta}{2+\delta}}.$$

By the assumptions of Lemma 3.1 the moments and the series on the rhs are finite and the same bound can be derived for $-\lambda\,\gamma^{(2)}_{X,red}(H^-)$ which shows the validity of (3.4).

The proof of (3.5) resembles that of (3.4). First we extend the identity (4.19) to the (reduced) second factorial moment measure of the MPP X_M defined by (2.4) and (2.6) for $m=2$ which reads as follows:

$$\begin{aligned}\lambda\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x,y)\,P^{\mathbf{o},x}_M(C_1\times C_2)\,\alpha^{(2)}_{X,red}(\mathrm{d}y)\mathrm{d}x &= \int_{(\mathbb{R}^d)^2} g(x,y-x)\,P^{x,y}_M(C_1\times C_2)\,\alpha^{(2)}_X\big(\mathrm{d}(x,y)\big)\\ &= \mathbb{E}\sum_{i,j\ge 1}^{\neq} g(X_i,X_j-X_i)\mathbb{1}_{C_1}(M_i)\mathbb{1}_{C_2}(M_j)\,.\end{aligned}$$

For the disjoint Borel sets G^+ and G^- defined by

$$G^{+(-)} = \big\{\,x\in\mathbb{R}^d:\;P^{\mathbf{o},x}_M(C_1\times C_2)\;\ge\;(<)\,P^{\mathbf{o}}_M(C_1)\,P^{\mathbf{o}}_M(C_2)\,\big\}$$

we replace $g(x,y)$ in the above relation by $g^{\pm}(x,y) = \mathbb{1}_{E_{\mathbf{o}}}(x)\,\mathbb{1}_{E_z^{\pm}}(y)$, where $E_z^{\pm} = G^{\pm}\cap E_z$ for $|z|\ge 2$, and consider the restricted MPPs $Y_{\mathbf{o}}(\cdot) = X_M\big((\cdot)\cap(E_{\mathbf{o}}\times C_1)\big)$, $Y'_{z,\pm}(\cdot) = X_M\big((\cdot)\cap((E_z^{\pm}\oplus E_{\mathbf{o}})\times C_2)\big)$ and their copies $\widetilde{Y}_{\mathbf{o}}$ and $\widetilde{Y}'_{z,\pm}$, which are assumed to be stochastically independent. Further, in analogy to (4.21), define

$$f(Y_{\mathbf{o}},Y'_{z,\pm}) = \sum_{i\ge 1}\delta_{(X_i,M_i)}(E_{\mathbf{o}}\times C_1)\,X_M\big((E_z^{\pm}+X_i)\times C_2\big) \le X\big(E_{\mathbf{o}}\big)\,X\big(E_z\oplus E_{\mathbf{o}}\big)\,.$$

It is rapidly seen that, for $|z|\ge 2$,

$$\begin{aligned}\mathbb{E}f(Y_{\mathbf{o}},Y'_{z,\pm}) &= \lambda\int_{E_z^{\pm}} P^{\mathbf{o},x}_M(C_1\times C_2)\,\alpha^{(2)}_{X,red}(\mathrm{d}x)\quad\text{and}\\ \mathbb{E}f(\widetilde{Y}_{\mathbf{o}},\widetilde{Y}'_{z,\pm}) &= \mathbb{E}X_M(E_{\mathbf{o}}\times C_1)\,\mathbb{E}X_M\big(E_z^{\pm}\times C_2\big) = \lambda^2\,P^{\mathbf{o}}_M(C_1)\,P^{\mathbf{o}}_M(C_2)\,|E_z^{\pm}|\end{aligned}$$

and in the same way as in the foregoing proof we find that, for $|z| \geq 2$,

$$|\,\mathbb{E}f(Y_{\mathbf{o}}, Y'_{z,\pm}) - \mathbb{E}f(\widetilde{Y}_{\mathbf{o}}, \widetilde{Y}'_{z,\pm})\,| \leq 2^{d+1}\,\big(\mathbb{E}X(E_{\mathbf{o}})^{2+\delta}\big)^{\frac{2}{2+\delta}}\,\big(\,\beta^*_{X_M}(|z| - \frac{3}{2})\,\big)^{\frac{\delta}{2+\delta}}\,.$$

Finally, the decomposition $\alpha^{(2)}_{X,red}(\cdot) = \gamma^{(2)}_{X,red}(\cdot) + \lambda\,|\cdot|$ together with the previous estimate leads to

$$\lambda \int\limits_{E_z} \Big| P^{\mathbf{o},x}_M(C_1 \times C_2) - P^{\mathbf{o}}_M(C_1)\,P^{\mathbf{o}}_M(C_2) \Big|\,\alpha^{(2)}_{X,red}(\mathrm{d}x) = \mathbb{E}f(Y_{\mathbf{o}}, Y'_{z,+}) - \mathbb{E}f(\widetilde{Y}_{\mathbf{o}}, \widetilde{Y}'_{z,+})$$

$$-\big(\,\mathbb{E}f(Y_{\mathbf{o}}, Y'_{z,-}) - \mathbb{E}f(\widetilde{Y}_{\mathbf{o}}, \widetilde{Y}'_{z,-})\,\big) - \lambda P^{\mathbf{o}}_M(C_1)\,P^{\mathbf{o}}_M(C_2)\,\Big(\gamma^{(2)}_{X,red}(\,E^+_z\,) - \gamma^{(2)}_{X,red}(\,E^-_z\,)\Big)$$

$$\leq 2^{d+2}\,\big(\mathbb{E}(X(E_{\mathbf{o}})^{2+\delta})\big)^{\frac{2}{2+\delta}}\,\big(\beta^*_{X_M}(|z| - \frac{3}{2})\big)^{\frac{\delta}{2+\delta}} + \lambda\,|\gamma^{(2)}_{X,red}|(E_z) \quad \text{for} \quad |z| \geq 2\,.$$

Thus, the sum over all $z \in \mathbb{Z}^d$ is finite in view of our assumptions and the above-proved relation (3.4) which completes the proof of Lemma 3.1. □

4.3 Proof of Theorem 3.3

It suffices to show (3.6), since independent marks imply that $P^{\mathbf{o},x}_M(C_1 \times C_2) = P^{\mathbf{o}}_M(C_1)\,P^{\mathbf{o}}_M(C_2)$ for $x \neq \mathbf{o}$ and any $C_1, C_2 \in \mathcal{B}(\mathbb{M})$ so that the integrand on the rhs of (3.6) disappears which yields (3.7) for stationary independently MPPs. By the very definition of $Y_k(C)$ we obtain that

$$\mathrm{Cov}\,\big(Y_k(C_i), Y_k(C_j)\big) = \frac{1}{|W_k|}\,\mathbb{E}\sum_{p\geq 1} 1\!\!1_{W_k}(X_p)\big(1\!\!1_{C_i}(M_p) - P^{\mathbf{o}}_M(C_i)\big)\big(1\!\!1_{C_j}(M_p) - P^{\mathbf{o}}_M(C_j)\big)$$

$$+ \frac{1}{|W_k|}\,\mathbb{E}\sum_{p,q\geq 1}^{\neq} 1\!\!1_{W_k}(X_p) 1\!\!1_{W_k}(X_q)\big(1\!\!1_{C_i}(M_p) - P^{\mathbf{o}}_M(C_i)\big)\big(1\!\!1_{C_j}(M_q) - P^{\mathbf{o}}_M(C_j)\big)\,. \quad (4.22)$$

Expanding the difference terms in the parentheses leads to eight expressions which, up to constant factors, take either the form

$$\mathbb{E}\sum_{p\geq 1} 1\!\!1_{W_k}(X_p) 1\!\!1_C(M_p) = \lambda |W_k|\, P^{\mathbf{o}}_M(C) \quad \text{or} \quad \mathbb{E}\sum_{p,q\geq 1}^{\neq} 1\!\!1_{W_k}(X_p) 1\!\!1_{W_k}(X_q) 1\!\!1_{C_i}(M_p) 1\!\!1_{C_j}(M_q)$$

$$= \int\limits_{(\mathbb{R}^d)^2} 1\!\!1_{W_k}(x) 1\!\!1_{W_k}(y)\, P^{\mathbf{o},y-x}_M(C_i \times C_j)\,\alpha^{(2)}_X(\mathrm{d}(x,y)) = \lambda \int\limits_{\mathbb{R}^d} P^{\mathbf{o},y}_M(C_i \times C_j)\,\gamma_k(y)\,\alpha^{(2)}_{X,red}(\mathrm{d}y)\,,$$

where $y \mapsto \gamma_k(y) = |W_k \cap (W_k - y)|$ denotes the set covariance function of W_k. Summarizing all these terms gives

$$\mathrm{Cov}\,\big(Y_k(C_i), Y_k(C_j)\big) = \lambda\Big(P^{\mathbf{o}}_M(C_i \cap C_j) - P^{\mathbf{o}}_M(C_i) P^{\mathbf{o}}_M(C_j)\Big) + \lambda \int\limits_{\mathbb{R}^d} \frac{\gamma_k(x)}{|W_k|}\,\Big(P^{\mathbf{o},x}_M(C_i \times C_j)$$

$$- P^{\mathbf{o}}_M(C_i)\,P^{\mathbf{o},x}_M(C_j \times \mathbb{M}) - P^{\mathbf{o}}_M(C_j)\,P^{\mathbf{o},x}_M(C_i \times \mathbb{M}) + P^{\mathbf{o}}_M(C_i)\,P^{\mathbf{o}}_M(C_j)\Big)\,\alpha^{(2)}_{X,red}(\mathrm{d}x)\,.$$

The integrand in the latter formula is dominated by the sum

$$\big|P^{\mathbf{o},x}_M(C_i \times C_j) - P^{\mathbf{o}}_M(C_i) P^{\mathbf{o}}_M(C_j)\big| + \big|P^{\mathbf{o},x}_M(C_j \times \mathbb{M}) - P^{\mathbf{o}}_M(C_j)\big| + \big|P^{\mathbf{o},x}_M(C_i \times \mathbb{M}) - P^{\mathbf{o}}_M(C_i)\big|\,,$$

which, by (3.5), is integrable w.r.t. $\alpha^{(2)}_{X,red}$. Hence, (3.6) follows by (2.2) and Lebesgue's dominated convergence theorem. □

4.4 Proof of Theorem 3.4

We again expand the parentheses in the second term of the estimator $(\widehat{\sigma}_{ij}^{(1)})_k$ defined by (3.8) and express the expectations in terms of $P_M^{\boldsymbol{o},y}$ and $\alpha_{X,red}^{(2)}$. Using the obvious relation $\gamma_k(y) = \int_{\mathbb{R}^d} \mathbb{1}_{W_k}(x)\mathbb{1}_{W_k}(y+x)\,\mathrm{d}x$ we find that, for any $C_i, C_j \in \mathcal{B}(\mathbb{M})$,

$$\mathbb{E}\sum_{p,q\geq 1}^{\neq} \frac{\mathbb{1}_{W_k}(X_p)\mathbb{1}_{W_k}(X_q)\mathbb{1}_{C_i}(M_p)\mathbb{1}_{C_j}(M_q)}{|(W_k - X_p)\cap(W_k - X_q)|} = \int\limits_{(\mathbb{R}^d)^2} \frac{\mathbb{1}_{W_k}(x)\mathbb{1}_{W_k}(y)P_M^{x,y}(C_i\times C_j)}{\gamma_k(y-x)}\,\alpha_X^{(2)}(\mathrm{d}(x,y))$$

$$= \lambda\int\limits_{\mathbb{R}^d} \frac{P_M^{\boldsymbol{o},y}(C_i\times C_j)}{\gamma_k(y)} \int\limits_{\mathbb{R}^d} \mathbb{1}_{W_k}(x)\mathbb{1}_{W_k}(y+x)\,\mathrm{d}x\,\alpha_{X,red}^{(2)}(\mathrm{d}y) = \lambda\int\limits_{\mathbb{R}^d} P_M^{\boldsymbol{o},y}(C_i\times C_j)\,\alpha_{X,red}^{(2)}(\mathrm{d}y)\,.$$

As in the proof of Theorem 3.3 after summarizing all terms we obtain that

$$\begin{aligned}\mathbb{E}(\widehat{\sigma}_{ij}^{(1)})_k &= \lambda\Big(P_M^{\boldsymbol{o}}(C_i\cap C_j) - P_M^{\boldsymbol{o}}(C_i)P_M^{\boldsymbol{o}}(C_j)\Big) + \lambda\int\limits_{\mathbb{R}^d}\Big(P_M^{\boldsymbol{o},x}(C_i\times C_j) \\ &- P_M^{\boldsymbol{o},x}(C_i\times\mathbb{M})P_M^{\boldsymbol{o}}(C_j) - P_M^{\boldsymbol{o},x}(C_j\times\mathbb{M})\,P_M^{\boldsymbol{o}}(C_i) + P_M^{\boldsymbol{o}}(C_i)\,P_M^{\boldsymbol{o}}(C_j)\Big)\,\alpha_{X,red}^{(2)}(\mathrm{d}x)\,,\end{aligned}$$

which, by comparing with (3.6), yields that $\mathbb{E}(\widehat{\sigma}_{ij}^{(1)})_k = \sigma_{ij}$. The asymptotic unbiasedness of $(\widehat{\sigma}_{ij}^{(2)})_k$ is rapidly seen by the equality $\mathbb{E}(\widehat{\sigma}_{ij}^{(2)})_k = \mathrm{Cov}\big(Y_k(C_i), Y_k(C_j)\big) = \mathbb{E}Y_k(C_i)Y_k(C_j)$, which follows directly from (4.22), and (3.3). □

4.5 Proof of Theorem 3.5

Since $\mathbb{E}\big(\sigma_{ij} - (\widehat{\sigma}_{ij}^{(3)})_k\big)^2 = \mathrm{Var}(\widehat{\sigma}_{ij}^{(3)})_k + \big(\sigma_{ij} - \mathbb{E}(\widehat{\sigma}_{ij}^{(3)})_k\big)^2$ we have to show that

$$\mathbb{E}(\widehat{\sigma}_{ij}^{(3)})_k \underset{k\to\infty}{\longrightarrow} \sigma_{ij} \qquad \text{and} \qquad \mathrm{Var}(\widehat{\sigma}_{ij}^{(3)})_k \underset{k\to\infty}{\longrightarrow} 0\,. \tag{4.23}$$

For notational ease, we put $m(u,v) = \big(\mathbb{1}_{C_i}(u) - P_M^{\boldsymbol{o}}(C_i)\big)\big(\mathbb{1}_{C_j}(v) - P_M^{\boldsymbol{o}}(C_j)\big)$, $a_k = b_k|W_k|^{1/d}$,

$$r_k(x,y) = \frac{\mathbb{1}_{W_k}(x)\mathbb{1}_{W_k}(y)}{\gamma_k(y-x)}\,w\left(\frac{\|y-x\|}{a_k}\right) \quad \text{and} \quad \tau_k = \sum_{p,q\geq 1}^{\neq} r_k(X_p,X_q)\,m(M_p,M_q)\,.$$

Hence, together with (2.3) and (3.1) we may rewrite $(\widehat{\sigma}_{ij}^{(3)})_k$ as follows:

$$(\widehat{\sigma}_{ij}^{(3)})_k = \frac{1}{\sqrt{|W_k|}}\,Y_k(C_i\cap C_j) + \widehat{\lambda}_k\Big(P_M^{\boldsymbol{o}}(C_i\cap C_j) - P_M^{\boldsymbol{o}}(C_i)P_M^{\boldsymbol{o}}(C_j)\Big) + \tau_k\,. \tag{4.24}$$

Using the definitions and relations (2.4) - (2.6) and $\int_{\mathbb{R}^d} r_k(x,y+x)\mathrm{d}x = w\big(\|y\|/a_k\big)$ we find that the expectation $\mathbb{E}\,\tau_k$ can be expressed by

$$\int\limits_{(\mathbb{R}^d\times\mathbb{M})^2} r_k(x,y)m(u,v)\alpha_{X_M}^{(2)}\big(\mathrm{d}(x,u,y,v)\big) = \lambda\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{M}^2} m(u,v)P_M^{\boldsymbol{o},y}\big(\mathrm{d}(u,v)\big)w\Big(\frac{\|y\|}{a_k}\Big)\alpha_{X,red}^{(2)}(\mathrm{d}y)\,.$$

The inner integral $\int_{\mathbb{M}^2} m(u,v)\, P_M^{\boldsymbol{o},y}\big(\mathrm{d}(u,v)\big)$ coincides with the integrand occurring in (3.6) and this term is integrable w.r.t. $\alpha^{(2)}_{X,red}$ due to (3.5) which in turn is a consequence of (3.10) as shown in Lemma 3.1. Hence, by Condition (wb) and the dominated convergence theorem, we arrive at

$$\mathbb{E}\,\tau_k \underset{k\to\infty}{\longrightarrow} \lambda \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{M}^2} m(u,v)\, P_M^{\boldsymbol{o},y}\big(\mathrm{d}(u,v)\big)\, \alpha^{(2)}_{X,red}(\mathrm{d}y) = \sigma_{ij} - \lambda\big(P_M^{\boldsymbol{o}}(C_i\cap C_j) - P_M^{\boldsymbol{o}}(C_i)P_M^{\boldsymbol{o}}(C_j)\big)\,.$$

The definitions of $\widehat{\lambda}_k$ and $Y_k(\cdot)$ by (2.3) and (3.1), respectively, reveal that $\mathbb{E}\,\widehat{\lambda}_k = \lambda$ and $\mathbb{E}\,Y_k(C_i\cap C_j) = 0$. This combined with the last limit and (4.24) proves the first relation of (4.23). To verify the second part of (4.23) we apply the Minkowski inequality to the rhs of (4.24) which yields the estimate

$$\big(\mathrm{Var}\,(\widehat{\sigma}^{(3)}_{ij})_k\big)^{1/2} \le |W_k|^{-1/2}\,\big(\mathrm{Var}\,Y_k(C_i\cap C_j)\big)^{1/2} + \big(\mathrm{Var}\,\widehat{\lambda}_k\big)^{1/2} + \big(\mathrm{Var}\,\tau_k\big)^{1/2}\,.$$

The first summand on the rhs tends to 0 as $k\to\infty$ since $\mathbb{E}\,Y_k^2(C)$ has a finite limit for any $C\in\mathcal{B}(\mathbb{M})$ as shown in Theorem 3.3 under condition (3.5). The second summand is easily seen to disappear as $k\to\infty$ if (3.4) is fulfilled, see e.g. [12], [16] or [17]. Condition (3.10) implies both (3.4) and (3.5), see Lemma 3.1. Therefore, it remains to show that $\mathrm{Var}\,\tau_k \longrightarrow 0$ as $k\to\infty$.

For this purpose we employ the variance formula (2.7) stated in Lemma 2.1 in the special case $f(x,y,u,v) = r_k(x,y)\,m(u,v)$. In this way we get the decomposition $\mathrm{Var}\,\tau_k = I_k^{(1)} + I_k^{(2)} + I_k^{(3)}$, where $I_k^{(1)}$, $I_k^{(2)}$ and $I_k^{(3)}$ denote the three multiple integrals on the rhs of (2.7) with $f(x,y,u,v)$ replaced by the product $r_k(x,y)\,m(u,v)$. We will see that the integrals $I_k^{(1)}$ and $I_k^{(2)}$ are easy to estimate only by using (3.4) and (3.5) while in order to show that $I_k^{(3)}$ goes to 0 as $k\to\infty$, the full strength of the mixing condition (3.10) must be exhausted. Among others we use repeatedly the estimate

$$\frac{1}{\gamma_k(a_k y)} \le \frac{2}{|W_k|} \quad \text{for} \quad y\in B(\boldsymbol{o},r_w)\,, \tag{4.25}$$

which follows directly from (2.2) and the choice of $\{b_k\}$ in (3.9). The definition of $I_k^{(1)}$ together with (4.25) and $\alpha^{(2)}_{X,red}(\mathrm{d}x) = \gamma^{(2)}_{X,red}(\mathrm{d}x) + \lambda\,\mathrm{d}x$ yields

$$\begin{aligned} |I_k^{(1)}| &\le 2\int\limits_{(\mathbb{R}^d)^2} \big(r_k(x_1,x_2)\big)^2 \alpha^{(2)}_X\big(\mathrm{d}(x_1,x_2)\big) = 2\lambda\int\limits_{\mathbb{R}^d} \frac{1}{\gamma_k(y)}\, w^2\Big(\frac{\|y\|}{a_k}\Big)\alpha^{(2)}_{X,red}(\mathrm{d}y) \\ &\le \frac{4\,\lambda}{|W_k|}\Big(m_w^2\, |\gamma^{(2)}_{X,red}|(\mathbb{R}^d) + \lambda\, a_k^d \int\limits_{\mathbb{R}^d} w^2(\|y\|)\mathrm{d}y\Big) \underset{k\to\infty}{\longrightarrow} 0\,, \end{aligned}$$

where the convergence results from Condition (wb) and (3.10), which implies $|\gamma^{(2)}_{X,red}|(\mathbb{R}^d) < \infty$ by virtue of Lemma 3.1. Analogously, using besides (4.25) and Condition (wb) the relations

$$w\Big(\frac{\|x\|}{a_k}\Big) \le m_w\,\mathbb{1}_{[-\lceil a_k r_w\rceil,\lceil a_k r_w\rceil]^d}(x) \quad \text{and} \quad W_k \subseteq \bigcup_{z\in\overline{H}_k} E_z \quad \text{with} \quad \overline{H}_k = H_k\cup\partial H_k,$$

with the notation introduced at the beginning of the proof of Theorem 3.1, we obtain that

$$\begin{aligned} |I_k^{(2)}| &\le 4 \int\limits_{(\mathbb{R}^d)^3} r_k(x_1,x_2)\, r_k(x_1,x_3)\, \alpha_X^{(3)}\big(\mathrm{d}(x_1,x_2,x_3)\big) \\ &\le \frac{16\, m_w^2}{|W_k|^2} \sum_{z \in \overline{H}_k} \alpha_X^{(3)}\big((E_z \oplus [-\lceil a_k r_w \rceil, \lceil a_k r_w \rceil]^d) \times (E_z \oplus [-\lceil a_k r_w \rceil, \lceil a_k r_w \rceil]^d) \times E_z\big)\,. \end{aligned}$$

Since the cube $E_z \oplus [-\lceil a_k r_w \rceil, \lceil a_k r_w \rceil]^d$ decomposes into $(2\lceil a_k r_w \rceil + 1)^d$ disjoint unit cubes and $\alpha_X^{(3)}(E_{z_1} \times E_{z_2} \times E_{z_3}) \le \mathbb{E}(X(E_{\boldsymbol{o}}))^3$ by Hölder's inequality, we may proceed with

$$|I_k^{(2)}| \le \frac{16\, m_w^2}{|W_k|^2}\, \#\overline{H}_k\, (2\lceil a_k r_w \rceil + 1)^{2d}\, \mathbb{E}(X(E_{\boldsymbol{o}}))^3 \;\le c_9\, b_k^{2d}\, |W_k| \underset{k\to\infty}{\longrightarrow} 0\,.$$

Here we have used the moment condition in (3.10), (4.3), and the assumptions (3.9) imposed on the sequence $\{b_k\}$.

In order to prove that $I_k^{(3)}$ vanishes as $k \to \infty$ we first evaluate the inner integrals over the product $m(u_1,u_2)\, m(u_3,u_4)$ with $m(u,v) = \big(\mathbb{1}_{C_i}(u) - P_M^{\boldsymbol{o}}(C_i)\big)\big(\mathbb{1}_{C_j}(v) - P_M^{\boldsymbol{o}}(C_j)\big)$ so that $I_k^{(3)}$ can be written as linear combination of 16 integrals taking the form

$$\begin{aligned} J_k &= \int\limits_{(\mathbb{R}^d)^2} \int\limits_{(\mathbb{R}^d)^2} r_k(x_1,x_2)\, r_k(x_3,x_4) \Big[P_M^{x_1,x_2,x_3,x_4}\big(\mathop{\times}_{r=1}^{4} D_r\big)\, \alpha_X^{(4)}\big(\mathrm{d}(x_1,x_2,x_3,x_4)\big) \\ &\qquad - P_M^{x_1,x_2}(D_1 \times D_2)\, P_M^{x_3,x_4}(D_3 \times D_4)\, \alpha_X^{(2)}\big(\mathrm{d}(x_1,x_2)\big)\, \alpha_X^{(2)}\big(\mathrm{d}(x_3,x_4)\big) \Big] \\ &= \int\limits_{\mathop{\times}_{r=1}^{4} (\mathbb{R}^d \times D_r)} r_k(x_1,x_2)\, r_k(x_3,x_4)\, \big(\alpha_{X_M}^{(4)} - \alpha_{X_M}^{(2)} \times \alpha_{X_M}^{(2)}\big)\big(\mathrm{d}(x_1,u_1,\dots,x_4,u_4)\big)\,, \end{aligned}$$

where the mark sets $D_1, D_3 \in \{C_i, \mathbb{M}\}$ and $D_2, D_4 \in \{C_j, \mathbb{M}\}$ are fixed in what follows and the signed measure $\alpha_{X_M}^{(4)} - \alpha_{X_M}^{(2)} \times \alpha_{X_M}^{(2)}$ on $\mathcal{B}((\mathbb{R}^d \times \mathbb{M})^4)$ $\big($and its total variation measure $\big|\alpha_{X_M}^{(4)} - \alpha_{X_M}^{(2)} \times \alpha_{X_M}^{(2)}\big|\big)$ come into play by virtue of the definition (2.6) for the $m-$point Palm mark distribution in case $m = 2$ and $m = 4$.

As $|z_1 - z_2| > \lceil a_k r_w \rceil$ (where, as above, $|z|$ denotes the maximum norm of $z \in \mathbb{Z}^d$) implies $\|x_2 - x_1\| > a_k r_w$ and thus $r_k(x_1,x_2) = 0$ for all $x_1 \in E_{z_1}, x_2 \in E_{z_2}$, we deduce from (4.25) together with Condition (wb) that

$$|J_k| \le \frac{4\, m_w^2}{|W_k|^2} \left(\sum_{n=0}^{2\lceil a_k r_w \rceil} + \sum_{n > 2\lceil a_k r_w \rceil} \right) \sum_{(z_1,z_2) \in S_k}\; \sum_{(z_3,z_4) \in S_{k,n}(z_1)} V_{z_1,z_2,z_3,z_4}\,, \tag{4.26}$$

where $S_k = \{(u,v) \in \overline{H}_k \times \overline{H}_k : |u-v| \le \lceil a_k r_w \rceil\}$, $S_{k,n}(z) = \{(z_1,z_2) \in S_k : \min_{i=1,2} |z_i - z| = n\}$ and $V_{z_1,z_2,z_3,z_4} = \big|\alpha_{X_M}^{(4)} - \alpha_{X_M}^{(2)} \times \alpha_{X_M}^{(2)}\big|\big(\times_{r=1}^4 (E_{z_r} \times D_r)\big)$ for any $z_1,\dots,z_4 \in \mathbb{Z}^d$.

Obviously, for any fixed $z \in \overline{H}_k$, at most $2\,(2\,\lceil a_k r_w \rceil + 1)^d\, (4\,\lceil a_k r_w \rceil + 1)^d$ pairs (z_3,z_4) belong to $\bigcup_{n=0}^{2\lceil a_k r_w \rceil} S_{k,n}(z)$ and the number of pairs (z_1,z_2) in S_k does not exceed the product $\#\overline{H}_k\, (2\,\lceil a_k r_w \rceil + 1)^d$. Finally, remembering that $a_k = b_k\, |W_k|^{1/d}$ and using the evident estimate $V_{z_1,z_2,z_3,z_4} \le 2\,\mathbb{E}(X(E_{\boldsymbol{o}}))^4$ together with (4.3) and Condition (wb), we arrive at

$$\frac{4\,m_w^2}{|W_k|^2} \sum_{(z_1,z_2)\in S_k} \sum_{n=0}^{2\lceil a_k r_w \rceil} \sum_{(z_3,z_4)\in S_{k,n}(z_1)} V_{z_1,z_2,z_3,z_4} \le c_{10}\,\frac{\#\overline{H}_k}{|W_k|^2}\,\big(b_k^d\,|W_k|\big)^3 \underset{k\to\infty}{\longrightarrow} 0\,.$$

It remains to estimate the sums on the rhs of (4.26) running over $n > 2\lceil a_k r_w \rceil$. For the signed measure $\alpha^{(4)}_{X_M} - \alpha^{(2)}_{X_M} \times \alpha^{(2)}_{X_M}$ we consider the Hahn decomposition $H^+, H^- \in \mathcal{B}((\mathbb{R}^d\times\mathbb{M})^4)$ yielding positive (negative) values on subsets of $H^+(H^-)$. Recall that $K_a = [-a,a]^d$. For fixed $z_1 \in \overline{H}_k$, $z_2 \in \overline{H}_k \cap (K_{\lceil a_k r_w \rceil} + z_1)$ and $(z_3,z_4) \in S_{k,n}(z_1)$, we now consider the decompsition $V_{z_1,z_2,z_3,z_4} = V^+_{z_1,z_2,z_3,z_4} + V^-_{z_1,z_2,z_3,z_4}$ with

$$V^{\pm}_{z_1,z_2,z_3,z_4} = \pm\big(\alpha^{(4)}_{X_M} - \alpha^{(2)}_{X_M}\times\alpha^{(2)}_{X_M}\big)\big(H^{\pm} \cap \mathop{\times}_{r=1}^{4} (E_{z_r}\times D_r)\big).$$

Since $(z_3,z_4) \in S_{k,n}(z_1)$ means that $z_3 \in \overline{H}_k \cap \big(K_n^c + z_1\big)$, where $K_a^c = \mathbb{R}^d \setminus K_a$, and $z_4 \in \overline{H}_k \cap \big(K_{\lceil a_k r_w\rceil} + z_3\big) \cap \big(K_n^c + z_1\big)$, we define MPPs Y_k and Y'_n as the restrictions of X_M to $(K_{\lceil a_k r_w\rceil+1/2} + z_1)\times\mathbb{M}$ and $(K^c_{n-1/2} + z_1)\times\mathbb{M}$, respectively. Let furthermore $\widetilde{Y}_k$ and $\widetilde{Y}'_n$ be copies of Y_k and Y'_n which are independent.

Next we define functions $f^+(Y_k,Y'_n)$ and $f^-(Y_k,Y'_n)$ by

$$f^{\pm}(Y_k,Y'_n) = \sum_{p,q\ge1}^{\neq} \sum_{s,t\ge1}^{\neq} 1\!\!1_{\pm}(X_p,M_p,X_q,M_q,X'_s,M'_s,X'_t,M'_t)\,,$$

where $1\!\!1_{\pm}(\cdots)$ denote the indicator functions of the sets $H^{\pm} \cap \mathop{\times}_{r=1}^{4}(E_{z_r}\times D_r)$ so that we get

$$V^{\pm}_{z_1,z_2,z_3,z_4} = \mathbb{E}f^{\pm}(Y_k,Y'_n) - \mathbb{E}f^{\pm}(\widetilde{Y}_k,\widetilde{Y}'_n) \quad \text{for} \quad (z_1,z_2)\in S_k\,,\,(z_3,z_4)\in S_{k,n}(z_1)\,.$$

Hence, having in mind the stationarity of X_M, we are in a position to apply the covariance inequality (2.10), which provides for $\eta > 0$ and $n > 2\lceil a_k r_w\rceil$ that

$$\begin{aligned} V^{\pm}_{z_1,z_2,z_3,z_4} &\le 2\Big(\beta\big(\mathcal{A}(K_{\lceil a_k r_w\rceil+1/2}+z_1),\mathcal{A}(K^c_{n-1/2}+z_1)\big)\Big)^{\frac{\eta}{1+\eta}} \\ &\times \Big(\mathbb{E}\big(\prod_{r=1}^{2} X_M(E_{z_r}\times D_r)\big)^{2+2\eta}\,\mathbb{E}\big(\prod_{r=3}^{4} X_M(E_{z_r}\times D_r)\big)^{2+2\eta}\Big)^{\frac{1}{2+2\eta}} \\ &\le 2\big(\beta^*_{X_M}(n-\lceil a_k r_w\rceil - 1)\big)^{\frac{\eta}{1+\eta}}\,\big(\mathbb{E}(X(E_{\boldsymbol{o}}))^{4+4\eta}\big)^{\frac{1}{1+\eta}}\,. \end{aligned} \tag{4.27}$$

In the last step we have used the Cauchy-Schwarz inequality and the definition (2.9) of the β-mixing rate $\beta^*_{X_M}$. Finally, setting $\eta = \delta/4$ with $\delta > 0$ from (3.10) the estimate (4.27) enables us to derive the following bound of that part on the rhs of (4.26) connected with the series over $n > 2\lceil a_k r_w\rceil$:

$$c_{11}\,\frac{\#\overline{H}_k}{|W_k|^2}\,(2\lceil a_k r_w\rceil+1)^{2d} \sum_{n>2\lceil a_k r_w\rceil} \big((2n+1)^d - (2n-1)^d\big)\big(\beta^*_{X_M}(n-\lceil a_k r_w\rceil-1)\big)^{\frac{\delta}{4+\delta}}\,.$$

Combining $a_k = b_k|W_k|^{1/d}$ and (4.3) with condition (3.10) and the choice of $\{b_k\}$ in (3.9), it is easily checked that the latter expression and thus J_k tend to 0 as $k\to\infty$. This completes the proof of Theorem 3.5. □

5 Examples

5.1 m-dependent marked point processes

A stationary MPP X_M is called *m-dependent* if, for any $B, B' \in \mathcal{B}(\mathbb{R}^d)$, the $\sigma-$algebras $\mathcal{A}_{X_M}(B)$ and $\mathcal{A}_{X_M}(B')$ are stochastically independent if $\inf\{|x-y| : x \in B,\, y \in B'\} > m$ or, equivalently,

$$\beta\big(\mathcal{A}_{X_M}(K_a), \mathcal{A}_{X_M}(K_{a+b}^c)\big) = 0 \quad \text{for} \quad b > m \ \text{ and } \ a > 0\,.$$

In terms of the corresponding mixing rates this means that $\beta^*_{X_M}(r) = \beta^{**}_{X_M}(r) = 0$ if $r > m$. For m-dependent MPPs X_M it is evident that Condition $\beta(\delta)$ in Theorem 3.1 is only meaningful for $\delta = 0$, that is, $\mathbb{E}(X([0,1]^d))^2 < \infty$. This condition also implies (3.4) and (3.5). Likewise, the assumption (3.10) of Theorem 3.5 reduces to $\mathbb{E}(X([0,1]^d))^4 < \infty$ which suffices to prove the L^2-consistency of the empirical covariance matrix $\widehat{\mathbf{\Sigma}}_k^{(3)}$.

5.2 Geostatistically marked point processes

Let $X = \sum_{n\geq 1} \delta_{X_n}$ be an unmarked simple PP on $\mathbb{R}^d$ and $M = \{M(x),\, x \in \mathbb{R}^d\}$ be a measurable random field on $\mathbb{R}^d$ taking values in the Polish mark space $\mathbb{M}$. Further assume that X and M are stochastically independent over a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. An MPP $X_M = \sum_{n\geq 1} \delta_{(X_n, M_n)}$ with atoms X_n of X and marks $M_n = M(X_n)$ is called *geostatistically marked*. Equivalently, the random counting measure $X_M \in \mathsf{N}_{\mathbb{M}}$ can be represented by means of the Borel sets $M^{-1}(C) = \{x \in \mathbb{R}^d : M(x) \in C\}$ (if $C \in \mathcal{B}(\mathbb{M})$) by

$$X_M(B \times C) = X(B \cap M^{-1}(C)) \quad \text{for} \quad B \times C \in \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{M})\,. \tag{5.1}$$

Obviously, if both the PP X and the mark field M are stationary then so is X_M and vice versa. Furthermore, the $m-$dimensional distributions of M coincide the $m-$point Palm mark distributions of X_M. The following Lemma allows to estimate the $\beta-$mixing coefficient (2.8) by the sum of the corresponding coefficients of the PP X and the mark field M.

Lemma 5.1. *Let the MPP X_M be defined by* (5.1) *with an unmarked PP and a random mark field M being stochastically independent of each other. Then, for any $B, B' \in \mathcal{B}(\mathbb{R}^d)$,*

$$\beta\big(\mathcal{A}_{X_M}(B), \mathcal{A}_{X_M}(B')\big) \leq \beta\big(\mathcal{A}_X(B), \mathcal{A}_X(B')\big) + \beta\big(\mathcal{A}_M(B), \mathcal{A}_M(B')\big)\,, \tag{5.2}$$

where the $\sigma-$algebras $\mathcal{A}_X(B), \mathcal{A}_X(B')$ and $\mathcal{A}_M(B), \mathcal{A}_M(B')$ are generated by the restriction of X and M, respectively, to the sets B, B'.

To sketch a proof for (5.2), we regard the differences $\Delta(A_i, A'_j) = \mathbb{P}(A_i \cap A'_j) - \mathbb{P}(A_i)\,\mathbb{P}(A'_j)$ for two finite partitions $\{A_i\}$ and $\{A'_j\}$ of Ω consisting of events of the form

$$A_i = \bigcap_{p=1}^{k} \{X_M(B_p \times C_p) \in \Gamma_{p,i}\}\,,\ A'_j = \bigcap_{q=1}^{\ell} \{X_M(B'_q \times C'_q) \in \Gamma'_{q,j}\} \quad \text{with} \quad \Gamma_{p,i}, \Gamma'_{q,j} \subseteq \mathbb{Z}_+^d\,,$$

with pairwise disjoint bounded Borel sets $B_1, ..., B_k \subseteq B$ and $B'_1, ..., B'_\ell \subseteq B'$. Making use of (5.1) combined with the independence assumption yields the identity

$$\begin{aligned}\Delta(A_i, A'_j) \;&=\; \int_\Omega\int_\Omega \Big(\mathbb{P}_{\mathcal{A}_X(B)\otimes\mathcal{A}_X(B')} - \mathbb{P}_{\mathcal{A}_X(B)} \times \mathbb{P}_{\mathcal{A}_X(B')}\Big)(A_i \cap A'_j)\, \mathrm{d}\mathbb{P}_{\mathcal{A}_M(B)\otimes\mathcal{A}_M(B')} \\ &+\; \int_\Omega\int_\Omega \mathbb{P}_{\mathcal{A}_X(B)}(A_i)\,\mathbb{P}_{\mathcal{A}_X(B')}(A'_j)\, \mathrm{d}\Big(\mathbb{P}_{\mathcal{A}_M(B)\otimes\mathcal{A}_M(B')} - \mathbb{P}_{\mathcal{A}_M(B)} \times \mathbb{P}_{\mathcal{A}_M(B')}\Big)\,,\end{aligned}$$

which by (2.8) and the integral form of the total variation confirms (5.2).

5.3 Cox processes on the boundary of Boolean models

Let $\Xi = \bigcup_{n\geq 1}(\Xi_n + Y_n)$ be a *Boolean model*, see e.g. [20] , governed by some stationary Poisson process $\sum_{n\geq 1}\delta_{Y_n}$ in $\mathbb{R}^d$ with intensity $\lambda > 0$ and a sequence $\{\Xi_n\}_{n\geq 1}$ of independent copies of some random convex, compact set Ξ_0 called *typical grain* (where we may assume that $\boldsymbol{o} \in \Xi_0$). With the radius functional $\|\Xi_0\| = \sup\{\|x\| : x \in \Xi_0\}$, the condition $\mathbb{E}\|\Xi_0\|^d < \infty$ ensures that Ξ is a well-defined random closed set. We consider a marked Cox process X_M, where the unmarked Cox process $X = \sum_{n\geq 1}\delta_{X_n}$ is concentrated on the boundary $\partial\Xi$ of Ξ with random intensity measure being proportional to the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}_{d-1}$ on $\partial\Xi$. As marks M_n we take the outer unit normal vectors at the points $X_n \in \partial\Xi$, which are (a.s.) well-defined for $n \geq 1$ due to the assumed convexity of Ξ_0. This example with marks given by the orientation of outer normals in random boundary points may occur rather specific. However, this way our asymptotic results may be used to construct asymptotic tests for the fit of a Boolean model to a given dataset w.r.t. its rose of directions. For instance, if the typical grain Ξ_0 is rotation-invariant (implying the isotropy of Ξ), then the Palm mark distribution $P_M^{\boldsymbol{o}}$ of the stationary MPP $X_M = \sum_{n\geq 1}\delta_{(X_n,M_n)}$ is the uniform distribution on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. We will now discuss assumptions ensuring that Condition $\beta(\delta)$ and (3.10) hold, which are required for our CLT (3.2) and the consistent estimation of the covariances (3.3), respectively. Using slight modifications of the proofs for Lemmas 5.1 and 5.2 in [13] one can show that for $a, b > 0$

$$\beta\big(\mathcal{A}_{X_M}(K_a), \mathcal{A}_{X_M}(K_{a+b}^c)\big) \leq \lambda\, 2^{d+2}\left(3 + \frac{4a}{b}\right)^{d-1} \mathbb{E}\big(\|\Xi_0\|^d \mathbb{I}\{\|\Xi_0\| \geq b/4\}\big)\,.$$

According to (2.9) we may thus define the β-mixing rates $\beta^*_{X_M}(r)$ and $\beta^{**}_{X_M}(r)$ for $r \geq \frac{1}{2}$ by

$$\begin{aligned}
\beta^*_{X_M}(r) &= c_{12}\,\mathbb{E}\big(\|\Xi_0\|^d \mathbb{I}\{\|\Xi_0\| \geq r/4\}\big) \geq \sup_{1/2\leq a\leq r} \beta\big(\mathcal{A}_{X_M}(K_a), \mathcal{A}_{X_M}(K_{a+r}^c)\big)\,, \\
\beta^{**}_{X_M}(r) &= \frac{c_{12}}{r^{d-1}}\,\mathbb{E}\big(\|\Xi_0\|^d \mathbb{I}\{\|\Xi_0\| \geq r/4\}\big) \geq \sup_{a\geq r} a^{-(d-1)}\,\beta\big(\mathcal{A}_{X_M}(K_a), \mathcal{A}_{X_M}(K_{a+r}^c)\big)\,,
\end{aligned}$$

where $c_{12} = \lambda 2^{d+2} 7^{d-1}$.

It is easily seen that $\mathbb{E}\|\Xi_0\|^{2d} < \infty$ implies $r^{2d-1}\beta^{**}_{X_M}(r) \underset{r\to\infty}{\longrightarrow} 0$. Moreover, $\mathbb{E}\|\Xi_0\|^{2d(p+\delta)/\delta+\varepsilon} < \infty$ for some $\varepsilon > 0$ ensures $\int_1^\infty r^{d-1}\big(\beta^*_{X_M}(r)\big)^{\delta/(2p+\delta)}\mathrm{d}r < \infty$ for $p \geq 0$. Since the random intensity measure of X on $E_{\boldsymbol{o}}$ and thus also $X(E_{\boldsymbol{o}})$ has moments of any order by virtue of $\mathbb{E}\|\Xi_0\|^d < \infty$, the parameter $\delta > 0$ in Condition $\beta(\delta)$ and (3.10) can be chosen arbitrarily large. This results in the following lemma.

Lemma 5.2. *For the above-defined stationary marked Cox process X_M on the boundary of a Boolean model Ξ with typical grain Ξ_0 the assumptions of the Theorems* 3.1 *and* 3.5 *are satisfied whenever*

$$\mathbb{E}\|\Xi_0\|^{2d+\varepsilon} < \infty \quad \textit{for some} \quad \varepsilon > 0\,. \tag{5.3}$$

Remark: The marked Cox process X_M is $m-$dependent if $\|\Xi_0\|$ is bounded by some constant. By using approximation techniques with truncated grains as suggested in [13], pp. 299-302, it can be shown that (5.3) is just needed for $\varepsilon = 0$. Moreover, the statistical analysis of roses of directions via marked Cox processes applies also in case of non-Boolean $\beta-$mixing fibre processes, see e.g. [12] for Voronoi tessellations.

6 Simulation study

6.1 Moving average model in $\mathbb{R}^2$

In this section we introduce an m-dependent MPP model, which was used for our simulations since it allows to control the range of spatial dependence for a fixed Palm mark distribution. The locations of this MPP are given by a homogeneous Poisson process $\sum_{n\geq 1}\delta_{X_n}$ in $\mathbb{R}^2$. Each point is marked by a direction in the upper half $\mathbb{S}^1_+$ of the unit circle. In order to construct the Palm mark distribution, we consider the projected normal distribution $\mathrm{PN}_2(a, \boldsymbol{\kappa})$ on $\mathbb{S}^1$. By definition, $Y \sim \mathrm{PN}_2(a, \boldsymbol{\kappa})$ means that $Y = \frac{Z}{\|Z\|}$ for some Gaussian random vector $Z \sim \mathcal{N}_2(\boldsymbol{a}, \boldsymbol{\kappa})$ in $\mathbb{R}^2$ with an invertible covariance matrix $\boldsymbol{\kappa}$. Note that $\mathrm{PN}_2(\boldsymbol{o}, \sigma^2 I_2)$ is the uniform distribution on $\mathbb{S}^1$ for all $\sigma^2 > 0$, where I_2 is the identity matrix. Formulas for the density of a projected normal distribution can be found in [19]. Let $\{M_n^{(1)}\}_{n\geq 1}$ be iid $\mathrm{N}_2(\boldsymbol{o}, \boldsymbol{\kappa})$-distributed random vectors. The stability of the normal distribution w.r.t. convolution yields

$$M_n^{(2)} = \frac{\sum_{i=1}^{\infty} M_i^{(1)} \mathbb{1}_{\{\|X_i - X_n\| \leq \rho\}}}{\left\| \sum_{i=1}^{\infty} M_i^{(1)} \mathbb{1}_{\{\|X_i - X_n\| \leq \rho\}} \right\|} \sim \mathrm{PN}_2(\boldsymbol{o}, \boldsymbol{\kappa}),$$

for any $\rho \geq 0$ controlling the range of dependence. The marks of our model are finally defined as the axial versions $M_n = M_n^{(2)} \mathbb{1}_{\mathbb{S}_1^+}(M_n^{(2)}) - M_n^{(2)} \mathbb{1}_{\mathbb{S}_1^-}(M_n^{(2)})$ of the averages $M_n^{(2)}$, i.e., points on the lower half-circle $\mathbb{S}^1_-$ are rotated by π. Due to the moving average approach defining the preliminary marks $\{M_n^{(2)}\}$, we call the MPP $X_M = \sum_{n\geq 1}\delta_{(X_n, M_n)}$ the *moving average model* (MAM). The MAM is clearly m-dependent, where the range of dependence is controlled by the averaging parameter ρ.

6.2 Tests

By simulations of the MAM we investigated the performance of the asymptotic χ^2-goodness-of-fit test, which is based on the test statistic

$$T_k = \mathbf{Y}_k^\top \widehat{\mathbf{\Sigma}}_k^{-1} \mathbf{Y}_k \xrightarrow[k\to\infty]{\mathrm{D}} \chi^2_\ell .$$

If $(\widehat{\mathbf{\Sigma}})_k$ is chosen as the L^2-consistent estimator $(\widehat{\sigma}_{ij}^{(3)})_k$, and $(P_M^{\boldsymbol{o}})_0$ denotes a hypothetical Palm mark distribution, the hypothesis $H_0 : P_M^{\boldsymbol{o}} = (P_M^{\boldsymbol{o}})_0$ is rejected, if $T_k > \chi^2_{\ell,1-\alpha}$, where α is the level of significance, and $\chi^2_{\ell,1-\alpha}$ denotes the $1-\alpha$-quantile of the χ^2_ℓ-distribution. This test will be referred to as *'test for the typical mark distribution'* (TMD). The construction of $(\widehat{\sigma}_{ij}^{(3)})_k$ involves the sequence of bandwidths $\{b_k\}$. By setting

$$b_k = c|W_k|^{-\frac{3}{4d}} \quad \text{for some constant } c > 0, \tag{6.1}$$

condition (wb), for the L^2-consistency of $(\widehat{\sigma}_{ij}^{(3)})_k$, is clearly satisfied. The constant c is crucial for test performance, as discussed below. This choice of c can be avoided if $\mathbf{\Sigma}$ is not estimated from the data to be tested but incorporated into H_0. This means, we specify an MPP as null model, such that $\mathbf{\Sigma}_0$ is either theoretically known or otherwise can be approximated by Monte-Carlo simulation. By means of the combined null hypothesis $H_0 : P_M^{\boldsymbol{o}} = (P_M^{\boldsymbol{o}})_0$ and

$\mathbf{\Sigma} = \mathbf{\Sigma}_0$, the test exploits not only information on the distribution of the typical mark but additionally considers asymptotic effects of spatial dependence. The test can thus be used to investigate if a given point pattern differs from the MPP null model w.r.t. the Palm mark distribution. We will therefore refer to it as *'test for mark-oriented goodness of model fit'* (MGM). By the strong law of large numbers and the asymptotic unbiasedness of $(\widehat{\sigma}_{ij}^{(2)})_k$, a strongly consistent Monte-Carlo estimator for $\mathbf{\Sigma}_0$ in an MPP model X_M is given by

$$\widehat{\mathbf{\Sigma}}_{k,n} = \frac{1}{n}\sum_{j=1}^{n} (\widehat{\sigma}_{ij}^{(2)})_k (X_M^{(j)}),$$

where $X_M^{(1)}, \dots, X_M^{(n)}$ are independent realizations of X_M. Thus, for large k and n the test statistic $T_{k,n} = \mathbf{Y}_k^\top \widehat{\mathbf{\Sigma}}_{k,n}^{-1} \mathbf{Y}_k$ has an approximate χ^2_ℓ distribution. If α is the level of significance, the MGM test rejects H_0, if $T_{k,n} > \chi^2_{\ell,1-\alpha}$. The estimator $\widehat{\mathbf{\Sigma}}_{k,n}$ can also be used to construct a test for the typical mark distribution if independent replications of a point patterns are to be tested. In that case $X_M^{(1)}, \dots, X_M^{(n)}$ are the replications. Note that for replicated point patterns, H_0 does not incorporate an assumption on $\mathbf{\Sigma}$ and hence the corresponding test differs from the MGM test. The edge-corrected unbiased estimator $(\widehat{\sigma}_{ij}^{(1)})_k$ was not used for the Monte-Carlo estimates in our simulation study, since $(\widehat{\sigma}_{ij}^{(2)})_k$ can be computed more efficiently.

6.3 Model parameters

The MAM was simulated on the observation window $W_{1500} = [-1500, 1500]^2$. The expected number of points was set to $\mathbb{E}X(W_{1500}) = 3125$. The asymptotic behavior of the test was studied by considering smaller observation windows corresponding to an expected number of 300, $600, \dots, 3000$ points. Spatial stochastic dependence of marks was varied by the parameter $\rho \in \{0, 50, \dots, 300\}$. In the MAM, marks of points with distance no larger than 2ρ in general exhibit stochastic dependence. If, on the contrary, two points are separated by more than 2ρ, their marks are independent. Thus, $\rho = 0$ corresponds to independent marking. Deviations of the projected normal distribution from the uniform distribution on $\mathbb{S}^1_+$ were controlled by varying $\kappa_{21} \in \{0, 0.1, 0.2, 0.4, 0.8\}$, where $\kappa_{12} = 0$ represents the uniformly distributed case. The parameter $\kappa_{11} = \kappa_{22} = 1$ was kept constant. The bins $C_1, \dots, C_\ell \in \mathcal{B}(\mathbb{S}^1_+)$ for the χ^2-goodness-of-fit test were chosen as

$$C_i = \left\{ (\cos\theta, \sin\theta)^T : \ \theta \in \left[(i-1)\frac{\pi}{\ell+1}, i\frac{\pi}{\ell+1} \right) \right\}, \ i = 1, \dots, \ell.$$

We will discuss the case $\ell = 8$, where the bins had a width of 20°. Simulations for $\ell = 17$ did not reveal different general effects.

6.4 Simulation results

All simulation results are based on 10000 model realizations per scenario. Type II errors were computed for realizations where $\kappa_{12} \neq 0$, which means that the mark distribution was not uniform on $\mathbb{S}^1_+$, whereas $H_0 : P_M^{\boldsymbol{o}} = U(\mathbb{S}^1_+)$ hypothesized a uniform Palm mark distribution on $\mathbb{S}^1_+$ (corresponding to $\kappa_{12} = 0$).

The performance of the MGM test is visualized in Tab. 1. Empirical type I errors of the MGM test were close to the theoretical levels of significance for $\alpha = 0.025, 0.05$, and 0.1 with maximum deviations of around 0.015. They were hardly affected by the dependence parameter ρ. Type II errors increased with ρ. Under independent marking ($\rho = 0$) as well as for $\rho = 50$, error levels were close to 0 for $\kappa_{12} \in \{0.2, 0.4, 0.8\}$. However, for an extreme range of dependence ($\rho = 300$) even for a strong deviation of the data from a uniform Palm mark distribution ($\kappa_{12} = 0.4$), rejection rates were only between 30 and 40%. For $\rho = 300$ the range of dependence corresponds to 20% of the sidelength of W.
Experiments with the TMD test revealed that the choice of the bandwidth parameter c in (6.1) is critical for test performance (Tab. 3). Whereas large values of c result in small type I errors, they decrease the power of the test. On the other hand, small values for c lead to superior power but increase type I errors (Tab. 3). The empirical errors in Fig. 2 were computed for $c = 50$ which yielded a reasonable compromise with respect to the two error types. In comparison to the MGM test the TMD test exhibits a higher sensitivity of empirical type I errors for varying values of κ_{12}, i.e., w.r.t. deviations from the uniform distribution on $\mathbb{S}^1_+$. Moreover, type II errors of the TMD test were up to 20% higher than for the MGM test.
Tab. 3 and Fig. 1 illustrate test performance w.r.t. the mean number of points in W and the dependence parameter ρ. The simulation experiments were conducted for $\alpha = 0.05$. For power analysis, the tested data was simulated for $\kappa_{12} = 0.4$, and thus the Palm mark distribution strongly differed from the uniform distribution on $\mathbb{S}^1$ of H_0. At a mean number of 3000 observed points, H_0 was reliably rejected by the TMD test once $\rho \leq 150$ (for $c = 50$). For $\rho \leq 100$ already 2000 expected points were sufficient to reject H_0 for almost all realizations. The MGM test required around 500 points less than the TMD test in order to achieve comparable rejection rates (Fig. 1).
In summary, our simulation results indicate that the MGM test outperforms the TMD test especially with respect to power. This result is plausible since the additional information incorporated into H_0 by specification of a model covariance matrix can be expected to result in a more specific test. It seems difficult to derive a general rule of thumb relating the required size of the observation window to the dependence structure of the data and the intensity of the point pattern. However, Fig. 1 and Tab. 3 provide an idea on the practical requirements for asymptotic testing.

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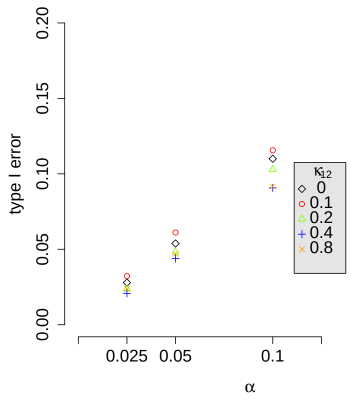
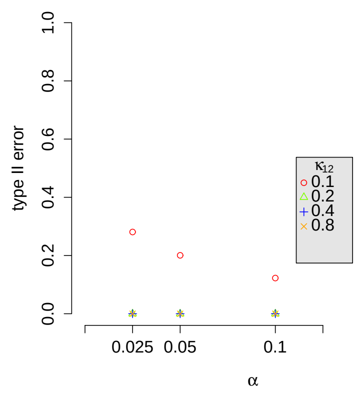
<tr><td>$\rho = 0$</td><td>
</td><td>
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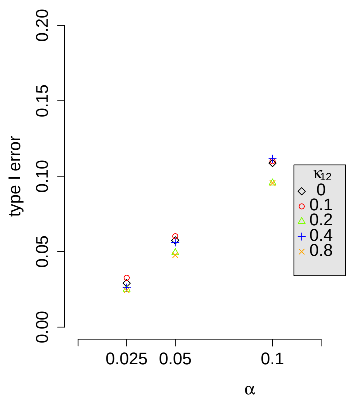
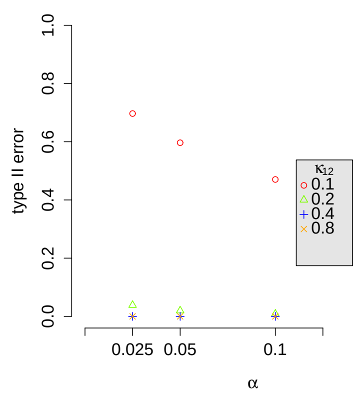
<tr><td>$\rho = 50$</td><td>
</td><td>
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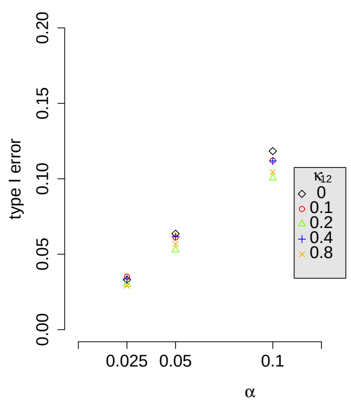
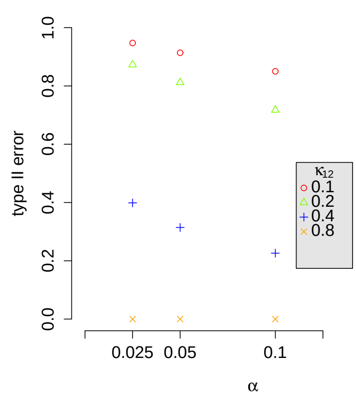
<tr><td>$\rho = 300$</td><td>
</td><td>
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Table 1: Empirical errors of types I and II for the MGM test.

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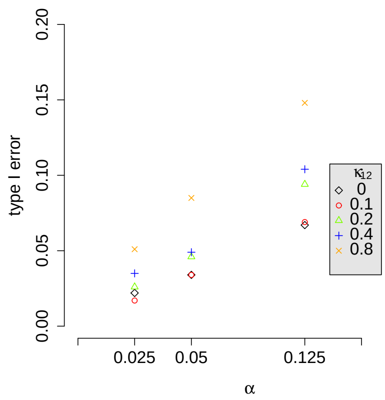
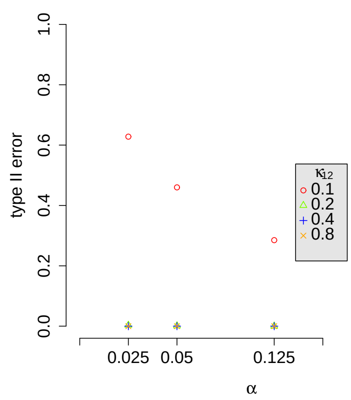
<tr><td>$\rho = 0$</td><td>
</td><td>
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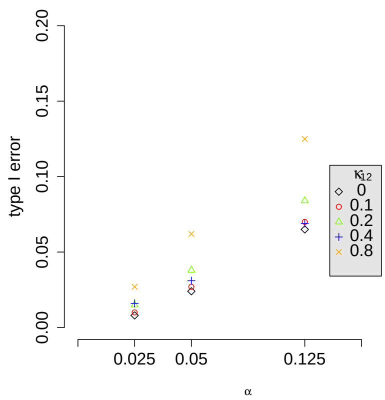
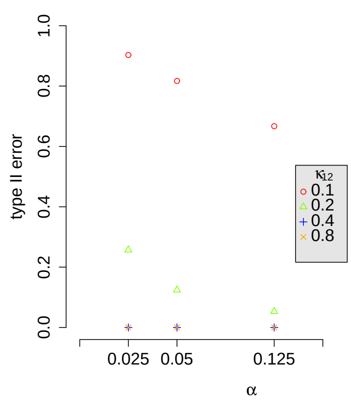
<tr><td>$\rho = 50$</td><td>
</td><td>
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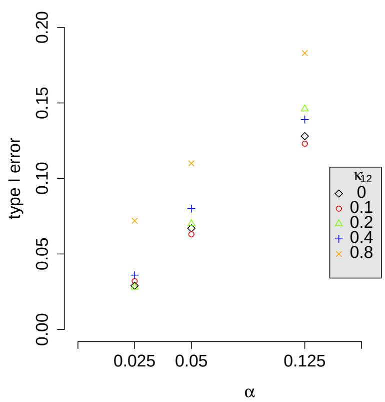
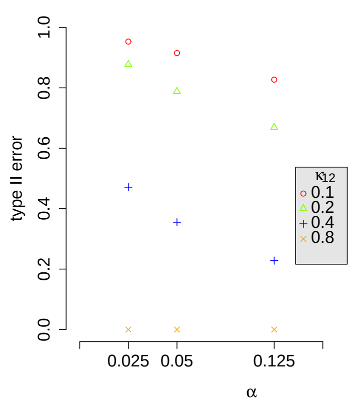
<tr><td>$\rho = 300$</td><td>
</td><td>
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Table 2: Empirical errors of types I and II for the TMD test ($c = 50$).

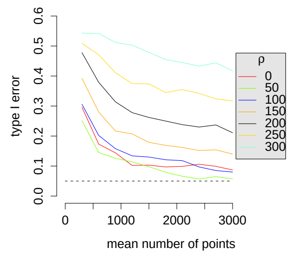
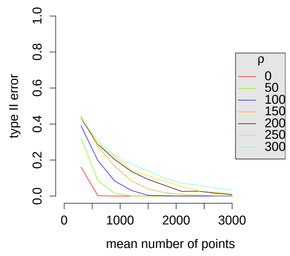
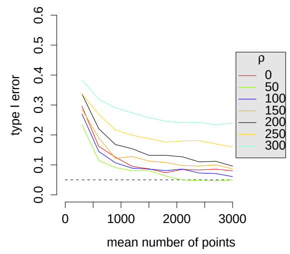
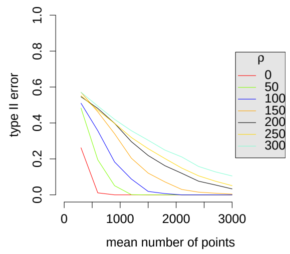
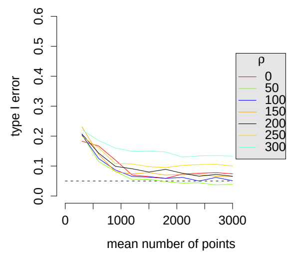
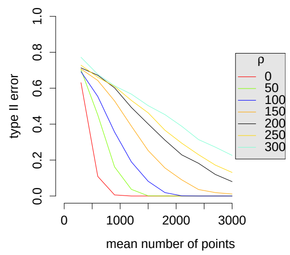
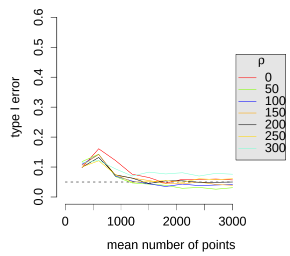
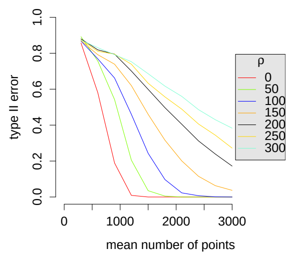
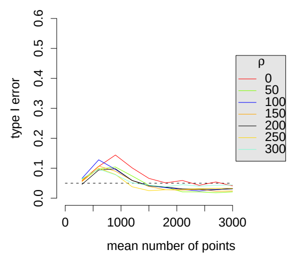
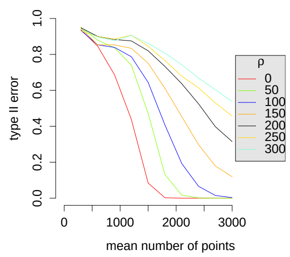
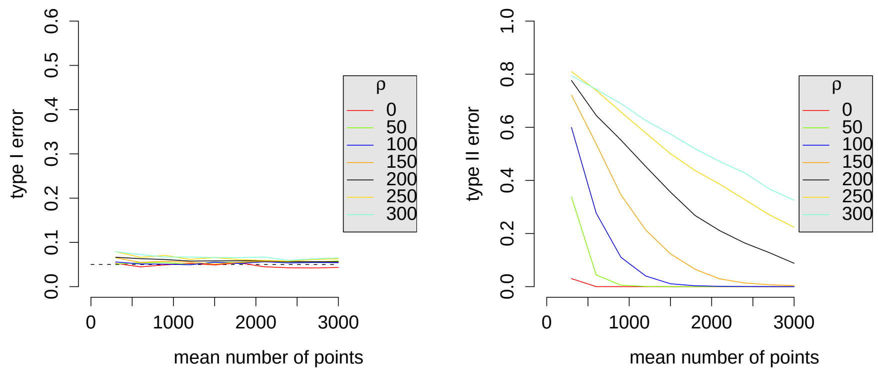
	Type I error	Type II error
$c = 20$		
$c = 30$		
$c = 40$		
$c = 50$		
$c = 60$		

Table 3: Empirical errors of types I and II for the TMD test plotted against the mean number of points in the observation window ($\kappa_{12} = 0.4$, $\ell = 8$, and $\alpha = 0.05$). Different colors correspond to different values of the dependence parameter ρ.



(a) Type I error

(b) Type II error

Figure 1: Empirical errors of types I and II for the MGM test plotted against the mean number of points in the observation window ($\kappa_{12} = 0.4$, $\alpha = 0.5$). Different colors correspond to different values of the dependence parameter ρ.