Universität Augsburg

Institut für Mathematik

Lothar Heinrich

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Institut für Mathematik, Universitätsstraße, D-86135 Augsburg http://www.math.uni-augsburg.de/

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Lothar Heinrich
Institut für Mathematik
Universität Augsburg
86135 Augsburg

Some New Results on Second-Order Chord Power Integrals of Convex Quadrangles

LOTHAR HEINRICH

Dedicated to Univ.-Prof. Dr.phil. Dr.h.c.mult. Peter M. Gruber on the occasion of his 70th birthday

Abstract

We study some geometric inequalities for second-order chord power integrals $I_2(K)$ of convex quadrangles K with positive area $A(K)$ and boundary length $L(\partial K)$. Based on different representations of $I_2(K)$ for convex quadrangles K we derive lower and upper bounds and give explicit formulas for $I_2(K)$ in case of parallelograms and rhombs. Further, an elementary proof of the isoperimetric inequality and a Carleman-type inequality for quadrangles is given. At the end of the paper we state two conjectures on sharp upper and lower bounds of $I_2(K)$ for convex polygons and their extensions to parallelotopes in higher dimensions.

Keywords : Poisson strip process, isoperimetric inequality, Carleman-type inequality, Steiner symmetrization, kite, parallelogram, rhomb, parallelotope

MSC 2000 : Primary 52A40 60G05 Secondary 52A07 52A22

1 CLT for Motion-Invariant Poisson Strip Processes and Chord Power Integrals of Star-Shaped Discs

To motivate our study of *chord power integrals* (CPI's) we state a central limit theorem for the total area of the union of stationary and isotropic Poisson strips contained in an expanding planar region ϱK as $\varrho \uparrow \infty$, where K has positive area $A(K)$ and is star-shaped w.r.t. the origin $\mathbf{o}$. Let

$$g(p,\varphi) = \{\, (x,y) \in \mathbb{R}^2 : x\,\cos\varphi + y\,\sin\varphi = p \,\} \;\;,\;\; \varphi \in [0,\pi)\;,\; p \in \mathbb{R}\,,$$

be an unoriented line in a xy-coordinate system with normal unit vector $(\cos\varphi, \sin\varphi)$ directed in the upper half-plane and signed perpendicular

distance $p \in \mathbb{R}$ from $\mathbf{o}$. We define the Poisson strips as dilated Poisson lines and its union set by

$$\Xi_{\lambda,F} = \bigcup_{i\in\mathbb{Z}} g(P_i,\Phi_i) \oplus B_2(R_i)$$

with $B_2(r) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \le r\}$, see Figure 1. Further, $\{P_i, i \in \mathbb{Z}\}$ forms a stationary Poisson process on $\mathbb{R}$ with intensity λ, $\{\Phi_i, i \in \mathbb{Z}\}$ are independent equidistributed angles in $[0,\pi]$, and $\{R_i, i \in \mathbb{Z}\}$ are independent radii with common distribution F such that

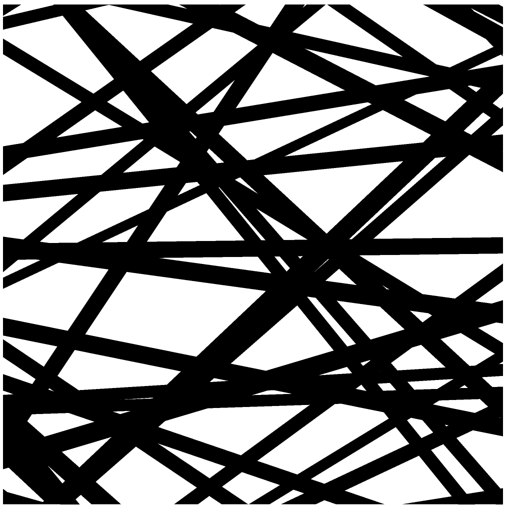

Figure 1: Isotropic Poisson strips

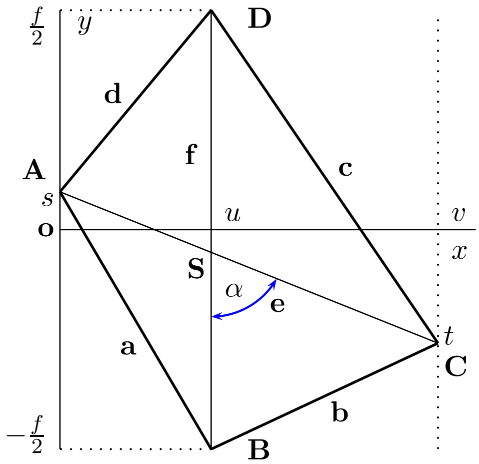

Figure 2: Convex quadrangle

$\mathbb{E}R_0^2 < \infty$. see e.g. [9],[5] for the definition of general stationary (Poisson) cylinder processes.

As special case of a result proved in [5] we get the convergence in distribution

$$\sqrt{\varrho}\,\Big(\frac{A(\Xi_{\lambda,F}\cap\varrho\,K)}{A(\varrho\,K)}-\mu\Big)\xrightarrow[\varrho\to\infty]{\mathrm{d}}\mathcal{N}\Big(0\,,\,\tau^2\,\frac{I_2(K)}{A^2(K)}\Big)\,, \tag{1}$$

where $\mu = 1-\exp\{-2\,\lambda\,\mathbb{E}R_0\,\}$ and $\tau^2 = 4\,\lambda\,\mathbb{E}R_0^2\,\exp\{-4\,\lambda\,\mathbb{E}R_0\,\}/\pi$. Here, $\mathcal{N}(0,\sigma^2)$ is a zero mean Gaussian random variable with variance $\sigma^2>0$ and $I_2(K)$ denotes the second-order CPI of $K\subset\mathbb{R}^2$ defined by

$$I_2(K)=\int_0^{\pi}\int_{\mathbb{R}} L^2(K\cap g(p,\varphi))\,\mathrm{d}p\,\mathrm{d}\varphi=\int_K\int_K\frac{\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|} \tag{2}$$

with length measure $L(\cdot)$, see [3] and [9] for more on CPI's in convex and integral geometry. From the view point of *optimal experimental design* it is

an aim to minimize the ratio $I_2(K)/A^2(K)$ in (1) given the perimeter $L(\partial K)$ of K. In convex geometry, see [1], [8], it is also of interest to maximize $I_2(K)$ when $A(K)$ is fixed, at least within certain families of convex discs.

In Section 2 and 4 we derive integral representations of $I_2(K)$ in case of general quadrangles K and evaluate them expclitly for parallelograms and rhombs. From these expressions we deduce some inequalities between their second-order CPI's. But before doing this, we discuss the comparatively simple case of triangles $\triangle=\triangle(a,b,c)$ with sides a,b,c and half-perimeter $s=(a+b+c)/2$. In [4] we could show that

$$\frac{I_2(\triangle)}{A^2(\triangle)}=\frac{4}{3}\Big[\frac{1}{a}\log\Big(\frac{s}{s-a}\Big)+\frac{1}{b}\log\Big(\frac{s}{s-b}\Big)+\frac{1}{c}\log\Big(\frac{s}{s-c}\Big)\Big]\,, \tag{3}$$

whence together with Heron's formula $A(\triangle)=\sqrt{s(s-a)(s-b)(s-c)}$ and by substituting $x=(s-a)/s$, $y=(s-b)/s$ so that $x+y=c/s$, it follows that the ratio

$I_2(\triangle)/A^{3/2}(\triangle)$ is bounded from above by

$$\begin{aligned}
&\frac{4}{3} \max_{\substack{0\le x,y\le 1\\ x+y\le 1}} \left[\sqrt[4]{x\,y\,(1-x-y)} \left(\frac{\log\left(\frac{1}{x}\right)}{1-x} + \frac{\log\left(\frac{1}{y}\right)}{1-y} + \frac{\log\left(\frac{1}{1-x-y}\right)}{x+y} \right) \right] \\
= &\frac{4}{3} \max_{\substack{0\le x\le a\\ 0\le a\le 1}} \left[\sqrt[4]{x\,(a-x)\,(1-a)} \left(\frac{\log\left(\frac{1}{x}\right)}{1-x} + \frac{\log\left(\frac{1}{a-x}\right)}{1-a+x} + \frac{\log\left(\frac{1}{1-a}\right)}{a} \right) \right] \\
= &\frac{4}{3} \max_{0\le a\le 1} \left[\sqrt[4]{\frac{a^2}{4}\,(1-a)} \left(\frac{4\,\log\frac{2}{a}}{2-a} + \frac{\log\frac{1}{1-a}}{a} \right) \right] = 2\,\sqrt[4]{3}\,\log 3 \approx 2.8917\,,
\end{aligned}$$

where in the second line the maximum is attained at $x = a/2$ and in the third line at $a = 2/3$. This estimate combined with the lower bound of $I_2(\triangle)/A^2(\triangle)$ obtained in [4] yields the inclusion

$$\frac{c_3}{L(\partial\triangle)} \le \frac{I_2(\triangle)}{A^2(\triangle)} \le \frac{\sqrt[4]{3}\,c_3}{6\,\sqrt{A(\triangle)}} \quad \text{with} \quad c_3 = 12\,\log 3 \tag{4}$$

and "=" holds on both sides iff $a = b = c$ (so that $c_3 = 16\,I_2(\triangle(1,1,1))$).

2 Second-Order CPI of Convex Quadrangles

We consider a convex quadrangle $\square ABCD$ embedded in the $xy-$coordinate system as in Figure 2 with diagonal $\mathbf{f} = \overline{BD}$ chosen parallel to the $y-$axis and bisected by the point $(u,0)$ on the $x-$axis with varying $u \in [0,v]$. For the moment let $f = \|\overline{BD}\|$ and $v > 0$ be fixed whereas the vertices $A = (0,s))$ and $C = (v,t)$ may vary on and parallel to the $y-$axis, respectively, as long as the convexity of $\square ABCD$ is not hurt. We shortly write $\square_{f,v}(s,t,u)$ for this quadrangle which has area $A(\square_{f,v}(s,t,u)) = \frac{1}{2}\,f\,v$ not depending on $s,t \in \mathbb{R}$ and $u \in (0,v]$. Another parametrization arises from parallel shifts of both diagonals $\mathbf{f}$ and $\mathbf{e} = \overline{AC}$ (with length $e = \|\overline{AC}\|$) which intersect in $\mathbf{S}$ at a fixed angle $\alpha \in [0,\pi/2]$, see Fig. 2. Put $p = \|\overline{AS}\|/\|\overline{AC}\| \in [0,1]$ and $q = \|\overline{BS}\|/\|\overline{BD}\| \in [0,1]$ and, for fixed e,f,α, let $\square_{e,f,\alpha}(p,q)$ denote the corresponding quadrangle $\square ABCD$. By the obvious relations

$$v = e\,\sin\alpha\,,\ u = p\,e\,\sin\alpha\,,\ s = (q-\frac{1}{2})\,f + p\,e\,\cos\alpha\,,\ t = s - e\,\cos\alpha$$

we get the area $A(\square_{e,f,\alpha}(p,q)) = \frac{1}{2}\,e\,f\,\sin\alpha$ for any $0 \le p,q \le 1$.

We introduce linear functions $f_i(\cdot)$ and $g_i(\cdot)$ which describe the lower line segments $\overline{AB}, \overline{BC}$ (for $i = 1$) and the upper line segments $\overline{AD}, \overline{DC}$ (for $i = 2$). They are given by

$$f_1(x) = -\left(s + \frac{f}{2}\right)\frac{x}{u} + s \quad , \quad f_2(x) = -\left(s - \frac{f}{2}\right)\frac{x}{u} + s\ ,\ \ 0 \le x \le u$$

$$g_1(x) = \left(t + \frac{f}{2}\right)\frac{x-v}{v-u} + t \quad , \quad g_2(x) = \left(t - \frac{f}{2}\right)\frac{x-v}{v-u} + t\ ,\ \ u \le x \le v\,.$$

Making use of the triangles $\triangle_1 = \triangle_1(s,u) = \triangle ABD$ and $\triangle_2 = \triangle_2(t,w) = \triangle BCD$ (with $w := v-u$) we may write $\square := \square_{f,v}(s,t,u) = \triangle_1 \cup \triangle_2$ and, therefore, from (2) it follows the representation

$$I_2(\square) = I_2(\triangle_1) + I_2(\triangle_2) + 2\, I_2(\triangle_1,\triangle_2) \ \text{ with } \ I_2(\triangle_1,\triangle_2) = \int\limits_{\triangle_1}\int\limits_{\triangle_2} \frac{\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|}\,.$$

By means of (3) and $a^2/d^2 = (\frac{f}{2} \pm s)^2 + u^2$, $b^2/c^2 = (\frac{f}{2} \pm s)^2 + w^2$ we get closed-term formulas for $I_2(\triangle_1)$ und $I_2(\triangle_2)$. Next, we rewrite the two-fold integration in $I_2(\triangle_1,\triangle_2)$ over $\mathbf{x} = (x_1,y_1) \in \triangle_1$ and $\mathbf{y} = (x_2,y_2) \in \triangle_2$ leading to the four-fold integral

$$\int\limits_0^u \int\limits_{f_1(x_1)}^{f_2(x_1)} \int\limits_u^v \int\limits_{g_1(x_2)}^{g_2(x_2)} \frac{\mathrm{d}y_2\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}x_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} \;=\; \int\limits_0^u \int\limits_u^v H(x_1,x_2,s,t)\,\mathrm{d}x_2\,\mathrm{d}x_1\,,$$

where

$$H(x_1,x_2,s,t) = \int\limits_{f_1(x_1)}^{f_2(x_1)} \int\limits_{g_1(x_2)}^{g_2(x_2)} \frac{\mathrm{d}y_2\,\mathrm{d}y_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\,.$$

To calculate $H(x_1,x_2,s,t)$ (for $x_1 < x_2$) we need the integral functions

$$F(z) := \int\limits_0^z \frac{\mathrm{d}y}{\sqrt{1+y^2}} \;=\; \log(z+\sqrt{z^2+1}) = -F(-z)$$

$$G(z) := \int\limits_0^z F(x)\,\mathrm{d}x \;=\; z\,\log(z+\sqrt{z^2+1}) - \sqrt{z^2+1}+1 = G(-z)$$

defined for all $z \in \mathbb{R}$, where the odd function $F(\cdot)$ is strictly increasing (convex for $z \geq 0$) and the even function $G(\cdot)$ is strictly convex since $G''(z) = F'(z) = (z^2+1)^{-1/2} > 0$. From the inner integral of $H(x_1,x_2,s,t)$

$$\int\limits_{g_1(x_2)}^{g_2(x_2)} \frac{\mathrm{d}y_2}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} = F\Big(\frac{y_1-g_1(x_2)}{x_2-x_1}\Big) - F\Big(\frac{y_1-g_2(x_2)}{x_2-x_1}\Big)\,,$$

it follows after a short calculation that

$$\frac{H(x_1,x_2,s,t)}{x_2-x_1} = \sum_{i=1}^{2}(-1)^{i-1}\Big(G\Big(\frac{f_2(x_1)-g_i(x_2)}{x_2-x_1}\Big) - G\Big(\frac{f_1(x_1)-g_i(x_2)}{x_2-x_1}\Big)\Big)\,.$$

In summary after some rearrangements we arrive at

$$I_2(\triangle_1,\triangle_2) = u\,w \int\limits_0^1\int\limits_0^1 (u\,x+w\,y) \sum_{i=0}^{1} \Big[G\Big(\frac{s\,x-t\,y}{u\,x+w\,y} + (-1)^i \frac{f\,(2-x-y)}{2\,(u\,x+w\,y)}\Big)$$
$$-G\Big(\frac{s\,x-t\,y}{u\,x+w\,y} + (-1)^i \frac{f\,(x-y)}{2\,(u\,x+w\,y)}\Big)\Big]\mathrm{d}y\mathrm{d}x$$

and, by setting $t = w = 0$ resp. $s = u = 0$ and integrating over y resp. x,

$$I_2(\triangle_1) = \frac{2\,u^3}{3} \int\limits_0^1 x \sum_{i=0}^{1} \Big[G\Big(\frac{s}{u} + \frac{(-1)^i f}{2\,u}\big(\frac{2}{x} - 1\big)\Big) - G\Big(\frac{s}{u} - \frac{(-1)^i f}{2\,u}\Big)\Big] \mathrm{d}x\,,$$

$$I_2(\triangle_2) = \frac{2\,w^3}{3} \int\limits_0^1 y \sum_{i=0}^{1} \Big[G\Big(\frac{t}{w} + \frac{(-1)^i f}{2\,w}\big(\frac{2}{y} - 1\big)\Big) - G\Big(\frac{t}{w} - \frac{(-1)^i f}{2\,w}\Big)\Big] \mathrm{d}y\,.$$

Now, we are ready to prove two inequalities concerning the behaviour of the CPI $I_2(\Box_{f,v}(s,t,u)$ when s and t are shifted.

Theorem 1. For any fixed $f, v > 0$ and $0 \le u \le v$,

$$I_2(\Box_{f,v}(s,t,u)) \le I_2(\Box_{f,v}(0,0,u) \le I_2(\Box_{f,v}(0,0,v/2) \tag{5}$$

and, if $s \le 0 \le t$ or $t \le 0 \le s$,

$$I_2(\Box_{f,v}(s,t,u)) \le I_2(\Box_{f,v}(-s,t,u) = I_2(\Box_{f,v}(s,-t,u)\,. \tag{6}$$

Corollary. For any $s \in \mathbb{R}$, $I_2(\Box_{f,v}(s,-s,v/2) \le I_2(\Box_{f,v}(s,s,v/2))$ and $I_2(\triangle_1(s,u)) \le I_2(\triangle_1(0,u))$, $I_2(\triangle_2(t,w)) \le I_2(\triangle_2(0,w))$, where the latter inequality means that, among all triangles with fixed altitude and base, the corresponding isosceles triangle has the greatest second-order CPI.

Proof of Theorem 1. We only need to prove (5) and (6) for the mixed second-order CPI $I_2(\triangle_1(s,u), \triangle_2(t,w))$ since $I_2(\triangle_1(s,u)) \le I_2(\triangle_1(0,u))$ and $I_2(\triangle_2(t,w)) \le I_2(\triangle_2(0,w))$ follow by direct computation from (3). Without loss of generality let $s \le 0 \le t$. It suffices to verify the inequalities $H(x_1,x_2,s,t) \le H(x_1,x_2,-s,t) \le H(x_1,x_2,0,t)$ for fixed $x_1 \in [0,u], x_2 \in (u,v]$. For brevity, introduce $a_i(x) = (f_i(x) - g_1(x_2))/(x_2 - x_1)$ and $b_i(x) = (f_i(x) - g_2(x_2))/(x_2 - x_1)$ for $i = 1,2$. Write $a_i^0(x), b_i^0(x)$ resp. $a_i^+(x), b_i^+(x)$ if $f_i(x)$ is defined with s replaced by 0 resp. $-s$. By definition of $f_i(\cdot)$, the relations $f_i(x_1) \le f_i^0(x_1) \le f_i^+(x_1)$ for $i = 1,2$ and $f_2(x_1) - f_1(x_1) = f_2^0(x_1) - f_1^0(x_1) = f_2^+(x_1) - f_1^+(x_1)$ (being valid also for a_i and b_i instead of f_i) hold and imply, by exploiting the convexity of the function $G(\cdot)$, that

$$G(a_2(x_1)) - G(a_1(x_1)) = G\Big(\frac{f_2(x_1) - f_1(x_1)}{x_2 - x_1} + a_1(x_1)\Big) - G(a_1(x_1))$$
$$\le G\Big(\frac{f_2^+(x_1) - f_1^+(x_1)}{x_2 - x_1} + a_1^+(x_1)\Big) - G(a_1^+(x_1)) = G(a_2^+(x_1)) - G(a_1^+(x_1))$$

since $a_1(x_1) \le a_1^+(x_1)$, and, by $f_1(x_1) - f_1^+(x_1) = f_2(x_1) - f_2^+(x_1) \le 0$, we get that

$$G(b_1(x_1)) - G(b_2(x_1)) = G\Big(\frac{f_2(x_1) - f_2^+(x_1)}{x_2 - x_1} + b_1^+(x_1)\Big)$$
$$-G\Big(\frac{f_2(x_1) - f_2^+(x_1)}{x_2 - x_1} + b_2^+(x_1)\Big) \le G(b_1^+(x_1)) - G(b_2^+(x_1))\,.$$

Notice that the convexity of $G(\cdot)$ means that $G(b) - G(a) \le G(b+h) - G(a+h)$ for any $h \ge 0$ and $a \le b$. By addition of the previous inequalities it follows $H(x_1,x_2,s,t) \le$

$H(x_1, x_2, -s, t)$ proving (6). Likewise, together with $G(-x) = G(x)$ for $x \in \mathbb{R}$, we find that $G(a_2^+(x_1)) - G(a_1^+(x_1)) \leq G(a_2^0(x_1)) - G(a_1^0(x_1))$ and $G(b_1^+(x_1)) - G(b_2^+(x_1)) \leq G(b_1^0(x_1)) - G(b_2^0(x_1))$ and , thus, $H(x_1, x_2, -s, t) \leq H(x_1, x_2, 0, t)$ proving the left part of (5). Finally, we rotate the kite $\Box_{f,v}(0, 0, u)$ by a right angle and apply the last step once more showing that the rhomb $\Box_{f,v}(0, 0, v/2)$ has a greater (mixed) second-order CPI. This completes the proof of Theorem 1. □

Remark. The transformation $\Box_{f,v}(s, t, u) \mapsto \Box_{f,v}(0, 0, u)$ coincides with the well-known Steiner-symmetrization, see [3]. It turns out that $(s, t) \mapsto I_2(\triangle_1(s, u), \triangle_2(t, w))$ is a strictly concave function. The mixed second-order CPI of the kite $\Box_{f,e}(0, 0, p\, e)$ for $0 < p < 1$ can be expressed in terms involving the *area sinus hyberbolicus* function $F(\cdot)$.

3 Isoperimetric Inequality for Quadrangles

We return to the quadrangle $\Box_{e,f,\alpha}(p, q)$ as defined in Sect.2 with fixed e, f, acute angle $\angle ASD = \alpha \in [0, \pi/2]$ and varying $p, q \in [0, 1]$. By a little trigonometry we get the following formula for the perimeter $L_\alpha(p, q) = L(\partial\Box_{e,f,\alpha}(p, q)) = P(p, q) + P(1-p, 1-q)$, where $P(p, q) = a + d =$

$$\sqrt{p^2\, e^2 + q^2\, f^2 + 2\, p\, q\, e\, f\, \cos\alpha} + \sqrt{p^2\, e^2 + (1-q)^2\, f^2 - 2\, p\, (1-q)\, e\, f\, \cos\alpha}\,.$$

Lemma. Among all convex quadrangles $\Box_{e,f,\alpha}(p, q)$ the parallelogram $\Box_{e,f,\alpha}(\frac{1}{2}, \frac{1}{2})$ has the least perimeter, i.e.

$$L_\alpha(p, q) \geq L_\alpha(\frac{1}{2}, \frac{1}{2}) = \sum_{i=0}^{1} \sqrt{e^2 + f^2 + (-1)^i\, 2\, e\, f\, \cos\alpha} \tag{7}$$

for all $p, q \in [0, 1]$ with "=" iff $p = q = \frac{1}{2}$.

Proof of Lemma. For fixed $p, q \in [0, 1]$, the function $[0, 1] \ni s \mapsto Q(s) := P(\,(1-s)p + s(1-p)\,,\, (1-s)q + s(1-q)\,)$ turns out to be (strictly) convex. To show this, define $a(s, p) := (1-s)\, p + s\, (1-p) \in [0, 1]$. Since $1 - a(s, q) = a(s, 1-q)$ we may write

$$\begin{aligned} Q(s) &= \sqrt{a^2(s,p)\, e^2 + a^2(s,q)\, f^2 + 2\, a(s,p)\, a(s,q)\, e\, f\, \cos\alpha} \\ &+ \sqrt{a^2(s,p)\, e^2 + a^2(s,1-q)\, f^2 - 2\, a(s,p)\, a(s,1-q)\, e\, f\, \cos\alpha}\,. \end{aligned}$$

A rather lengthy calculation (for details see Appendix) shows that

$$\begin{aligned} Q''(s) &= \frac{e^2\, f^2\, \sin^2\alpha\, [\,(1-2p)\, a(s,q) - (1-2q)\, a(s,p)\,]^2}{[\,a^2(s,p)\, e^2 + a^2(s,q)\, f^2 + 2\, a(s,p)\, a(s,q)\, e\, f\, \cos\alpha\,]^{3/2}} \\ &+ \frac{e^2\, f^2\, \sin^2\alpha\, [\,(1-2p)\, a(s,1-q) + (1-2q)\, a(s,p)\,]^2}{[\,a^2(s,p)\, e^2 + a^2(s,1-q)\, f^2 - 2\, a(s,p)\, a(s,1-q)\, e\, f\, \cos\alpha\,]^{3/2}} > 0 \end{aligned}$$

for $p, q \neq 1/2$. Hence, applying Jensen's inequality reveals that

$$P\big(\frac{1}{2}, \frac{1}{2}\big) = Q\big(\frac{1}{2}\big) \leq \frac{1}{2}\, Q(0) + \frac{1}{2}\, Q(1) = \frac{1}{2}\, P(p, q) + \frac{1}{2}\, P(1-p, 1-q)\,,$$

which coincides with (7). $\square$

From (7) we can easily derive the isoperimetric inequality for quadrangles.

Theorem 2. For any quadrangle $\square_{e,f,\alpha}(p,q)$ with area $A_\alpha(p,q)$ we have

$$\frac{L_\alpha^2(p,q))}{A_\alpha(p,q))} \geq \frac{L_\alpha^2(\frac{1}{2},\frac{1}{2})}{A_\alpha(\frac{1}{2},\frac{1}{2})} \geq \frac{L_{\pi/2}^2(\frac{1}{2},\frac{1}{2})}{A_{\pi/2}(\frac{1}{2},\frac{1}{2})} = \frac{8\,(e^2+f^2)}{e\,f} \geq 16$$

with "=" iff $e=f$ and $\alpha=\pi/2$.

4 Lower and Upper Bounds for Second-Order CPI of Parallelograms

We consider a parallelogram $\square_\alpha(a,b) = \{\lambda_1\,(b,0) + \lambda_2\,(a\,\cos\alpha, h) : 0 \leq \lambda_1, \lambda_2 \leq 1\}$ with altitude $h = a\,\sin\alpha$ and base $b = \|\overline{AB}\|$ ($a = \|\overline{AC}\|$, $\alpha = \angle(CAB) \in (0,\pi/2]$, $A(\square_\alpha(a,b)) = b\,h = a\,b\,\sin\alpha$) which yields

$$I_2(\square_\alpha(a,b)) = \int\limits_{[0,1]^4} \frac{b^2\,h^2\,\mathrm{d}(\lambda_1\,,\,\lambda_2\,,\,\mu_1\,,\,\mu_2)}{\sqrt{[(\lambda_1-\mu_1)b + (\lambda_2-\mu_2)a\,\cos\alpha]^2 + [(\lambda_2-\mu_2)\,h]^2}}$$

$$= \int_0^1\int_0^1\int_0^1\int_0^1 \frac{a^2\,b^2\,\sin^2\alpha\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2\,\mathrm{d}\mu_1\,\mathrm{d}\mu_2}{\sqrt{b^2(\lambda_1-\mu_1)^2 + 2\,a\,b\cos\alpha\,(\lambda_1-\mu_1)(\lambda_2-\mu_2) + a^2(\lambda_2-\mu_2)^2}}$$

$$\begin{aligned}
\frac{I_2(\square_\alpha(a,b))}{A^2(\square_\alpha(a,b))} &= 2\sum_{i=0}^{1}\int_0^1\int_0^1 \frac{(1-x)\,(1-y)\,\mathrm{d}x\,\mathrm{d}y}{\sqrt{a^2\,x^2\, + (-1)^i\,2\,a\,b\,\cos\alpha\,x\,y + b^2\,y^2}} \\
&= \frac{1}{3}\sum_{i=0}^{1}\int_0^1 \frac{(3-z)\,\mathrm{d}z}{\sqrt{a^2 + (-1)^i\,2\,a\,b\,z\,\cos\alpha + b^2\,z^2}} \\
&+ \frac{1}{3}\sum_{i=0}^{1}\int_0^1 \frac{(3-z)\,\mathrm{d}z}{\sqrt{a^2\,z^2 + (-1)^i\,2\,a\,b\,z\,\cos\alpha + b^2}}
\end{aligned} \tag{8}$$

Theorem 3. For any parallelogram $\square_\alpha(a,b)$ with altitude $h = a\,\sin\alpha$ and perimeter $L(\square_\alpha(a,b)) = 2\,(a+b)$ the following inequalities are valid:

$$\frac{c_4}{2\,(a+b)} \leq \frac{I_2(\square_{\pi/2}(a,b))}{A^2(\square_{\pi/2}(a,b))} \leq \frac{I_2(\square_\alpha(a,b))}{A^2(\square_\alpha(a,b))} \leq \frac{I_2(\square_{\pi/2}(h,b))}{A^2(\square_{\pi/2}(h,b))} \leq \frac{c_4}{4\sqrt{a\,b\,\sin\alpha}} \tag{9}$$

with "=" left and right iff $a=b$, $\alpha=\pi/2$, where $c_4 = 4\,I_2(\square_{\pi/2}(1,1))$ and

$$I_2(\square_{\pi/2}(1,1)) = 4\int_0^1\int_0^1 \frac{(1-x)(1-y)}{\sqrt{x^2+y^2}}\mathrm{d}x\mathrm{d}y = \frac{4}{3}\Big[3\log(1+\sqrt{2}) + 1 - \sqrt{2}\Big]\,.$$

Proof of Theorem 3. We only sketch the essential steps. The above integral transformations are more or less straightforward and they are left to the reader. Next, using the elementary inequality $(x+c)^{-1/2}+(x-c)^{-1/2}\geq 2\,x^{-1/2}$ for $x>0$, $-x<c<x$, it follows that

$$\frac{1}{\sqrt{a^2-2\,a\,b\,z\,\cos\alpha+b^2\,z^2}}+\frac{1}{\sqrt{a^2+2\,a\,b\,z\,\cos\alpha+b^2\,z^2}}\geq\frac{2}{\sqrt{a^2+b^2\,z^2}} \tag{10}$$

and, by interchanging a and b and using (8) we get the first lower bound in Theorem 3. To verify the first upper bound it suffices to prove with $c:=a\,\cos\alpha$ and $h:=a\,\sin\alpha$ that

$$\int_0^1\frac{(3-z)\;\mathrm{d}z}{\sqrt{(b\,z+c)^2+h^2}}+\int_0^1\frac{(3-z)\;\mathrm{d}z}{\sqrt{(b\,z-c)^2+h^2}}\leq 2\int_0^1\frac{(3-z)\;\mathrm{d}z}{\sqrt{b^2\,z^2+h^2}} \tag{11}$$

and the corresponding inequality with $(b\,z\pm c)^2+h^2$ and $b^2\,z^2+h^2$ replaced by $(b\pm c\,z)^2+h^2\,z^2$ and $b^2+h^2\,z^2$, respectively. The analytic proof of (11) is somewhat lenghty. The remaining bounds left and right are obtained by direct evaluation of the integrals and determining their extreme values. For details we refer to the Sect.2 of the Appendix. □

The four integrals on the right-hand side of (8) can be taken from a usual table of integrals. Putting $H(z)=F(z)+z\,\sqrt{z^2+1}$ and

$$J_\alpha(a,b)=\sum_{i=0}^{1}\big(3\,b+(-1)^i\,a\,\cos\alpha\big)\,H\Big(\frac{b+(-1)^i a\,\cos\alpha}{a\,\sin\alpha}\Big)-2\,a\,\cos\alpha\,H\big(\cot\alpha\big)$$

we may write together with the diagonals $e/f=\sqrt{a^2+b^2\pm 2\,a\,b\,\cos\alpha}$:

$$\frac{I_2(\square_\alpha(a,b))}{A^2(\square_\alpha(a,b))}=\frac{2\,(\,a^3+b^3\,)-4\,(\,e^3+f^3\,)}{3\,a^2\,b^2\,\sin^2\alpha}+\frac{J_\alpha(a,b)}{3\,b^2}+\frac{J_\alpha(b,a)}{3\,a^2}\,. \tag{12}$$

In the special case of a rhomb $\square_\alpha(a,a)$ with diagonals $e=a\,\sqrt{2\,(1+\cos\alpha)}=2\,a\,\cos\frac{\alpha}{2}$ and $f=a\,\sqrt{2\,(1-\cos\alpha)}=2\,a\,\sin\frac{\alpha}{2}$ we obtain together with $\sin\alpha=2\,\sin\frac{\alpha}{2}\,\cos\frac{\alpha}{2}$ that

$$\frac{4\,a\,I_2(\square_\alpha(a,a))}{A^2(\square_\alpha(a,a))}=\frac{16}{3}\,T(\alpha)\quad\text{and}\quad\frac{I_2(\square_\alpha(a,a))}{(A(\square_\alpha(a,a)))^{3/2}}=\frac{4\,\sqrt{2}}{3}\,T(\alpha)\,\sqrt{\sin\frac{\alpha}{2}\,\cos\frac{\alpha}{2}}\,, \tag{13}$$

where

$$T(\alpha)=1-\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}+\Big(1+\sin^2\frac{\alpha}{2}\Big)\log\Big(1+\frac{1}{\sin\frac{\alpha}{2}}\Big)+\Big(1+\cos^2\frac{\alpha}{2}\Big)\log\Big(1+\frac{1}{\cos\frac{\alpha}{2}}\Big)\,.$$

It is easily verified that the left/right-hand ratio in (13) decreases/increases in $\alpha\in[0,\pi/2]$ with minimum/maximum at $\alpha=\pi/2$. In summary, this result combined with (5) proves

Theorem 4. Among all convex quadrangles $\square ABCD$ with fixed $A=A(\square ABCD)$ the square $\square_{\pi/2}(\sqrt{A},\sqrt{A})$ has the greatest second-order CPI, more precisely:

$$\frac{I_2(\square ABCD)}{(A(\square ABCD))^{3/2}}\leq\frac{4}{3}\,T\Big(\frac{\pi}{2}\Big)=\frac{c_4}{4}\approx 2.97321<\frac{16}{3\,\sqrt{\pi}}\approx 3.00901\,.$$

We remark that Carleman's inequality, see [1], says that the ratio in Theorem 4 attains its maximum (taken over all convex discs) for circles giving the right-hand value, see also [8].

5 Conjectures for Convex Polygons and Parallelotopes in Higher Dimensions

To conclude this paper we formulate two conjectures which have been proved only for special cases so far.

Conjecture I. For any convex n-gon K_n the inequality

$$\frac{c_n}{L(\partial K_n)} \leq \frac{I_2(K_n)}{A^2(K_n)} \leq \frac{c_n}{\sqrt{4\,n\,\tan\left(\frac{\pi}{n}\right)A(K_n)}} \quad \text{with } \; c_n = \frac{16}{n}\tan\left(\frac{\pi}{n}\right)I_2(K_n^*)$$

holds with "=" on both sides iff K_n is a regular n-gon. Here, K_n^* denotes the regular n-gon with edges of length 1, see [4] for explicit values of c_n.

Let $\mathbf{a}_i = (a_i^{(1)}, ..., a_i^{(d)})$, $i = 1, ..., d$, be linearly independent vectors in $\mathbb{R}^d$ which define a so-called $d-$*parallelotope* $P(\mathbf{a}_1, ..., \mathbf{a}_d) = \{\sum_{i=1}^d \lambda_i\,\mathbf{a}_i : 0 \leq \lambda_1, ..., \lambda_d \leq 1\}$. Without loss of generalty, we may assume that $a_i^{(1)}, ..., a_i^{(i-1)} \geq 0$, $a_i^{(i+1)} = ... = a_i^{(d)} = 0$ and put $a_i := a_i^{(i)} > 0$ for $i = 1, ..., d$. Further, let $I_d(K)$ denote the rigth-hand double integral of (2) for some convex body K in $\mathbb{R}^d$ with $d-$volume $V_d(K) > 0$ and *mean breadth* $b_d(K)$, see [3], [9].

Conjecture II. For any positive $a_1, ..., a_d \in \mathbb{R}$ the inequality

$$\frac{d\,I_d([0,1]^d)}{\|\mathbf{a}_1\| + \cdots + \|\mathbf{a}_d\|} \leq \frac{I_d(P(\mathbf{a}_1, ..., \mathbf{a}_d))}{(\,a_1 \cdots a_d\,)^2} \leq \frac{I_d([0,1]^d)}{(\,a_1 \cdots a_d\,)^{1/d}}$$

holds with "=" on each side iff $a_1 = \|\mathbf{a}_1\| = \cdots = a_d = \|\mathbf{a}_d\|$.

Note that the lower bound could be verified by the author, see Sect. 5 in the Appendix. Further, one can show that $V_d(P(\mathbf{a}_1, ..., \mathbf{a}_d)) = a_1 \cdots a_d$, $b_d(P(\mathbf{a}_1, ..., \mathbf{a}_d))\, d\,\kappa_d/2\,\kappa_{d-1}$ $= \|\mathbf{a}_1\| + \cdots + \|\mathbf{a}_d\|$ (where $\kappa_d = d-$volume of the unit ball in $\mathbb{R}^d$) and

$$I_d([0,1]^d) = 2^d \int_0^1 \cdots \int_0^1 \prod_{i=1}^d (1 - x_i) \Big(\sum_{i=1}^d x_i^2\Big)^{-1/2} \mathrm{d}x_1 \cdots \mathrm{d}x_d\,.$$

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Postal address: Universität Augsburg, 86135 Augsburg, Germany
E-mail address: heinrich@math.uni-augsburg.de

6 Appendix

1. Supplements to the proofs in Section 3

At first we calculate in detail the second derivative of the function $Q(\cdot)$ defined in the proof of the Lemma. The second derivative $G''(s)$ is equal to

$$\frac{\partial}{\partial s}\left(\frac{(1-2p)\,a(s,p)\,e^2+(1-2q)\,a(s,q)\,f^2+[(1-2p)\,a(s,q)+(1-2q)\,a(s,p)]\,e\,f\,\cos\alpha}{[\,a^2(s,p)\,e^2+a^2(s,q)\,f^2+2\,a(s,p)\,a(s,q)\,e\,f\,\cos\alpha\,]^{1/2}}\right.$$

$$\left.+\frac{(1-2p)\,a(s,p)\,e^2-(1-2q)\,a(s,1-q)\,f^2-[(1-2p)\,a(s,1-q)-(1-2q)\,a(s,p)]\,e\,f\,\cos\alpha}{[\,a^2(s,p)\,e^2+a^2(s,1-q)\,f^2-2\,a(s,p)\,a(s,1-q)\,e\,f\,\cos\alpha\,]^{1/2}}\right)$$

$$= \frac{[(1-2p)^2\,e^2+(1-2q)^2\,f^2+2\,(1-2p)\,(1-2q)\,e\,f\,\cos\alpha]}{[\,a^2(s,p)\,e^2+a^2(s,q)\,f^2+2\,a(s,p)\,a(s,q)\,e\,f\,\cos\alpha\,]^{1/2}}$$

$$\times\quad\frac{[a^2(s,p)\,e^2+a^2(s,q)\,f^2+2\,a(s,p)\,a(s,q)\,e\,f\,\cos\alpha]}{a^2(s,p)\,e^2+a^2(s,q)\,f^2+2\,a(s,p)\,a(s,q)\,e\,f\,\cos\alpha}$$

$$-\quad\frac{[(1-2p)\,a(s,p)\,e^2+(1-2q)\,a(s,q)\,f^2+((1-2p)\,a(s,q)+(1-2q)\,a(s,p))\,e\,f\,\cos\alpha]^2}{[\,a^2(s,p)\,e^2+a^2(s,q)\,f^2+2\,a(s,p)\,a(s,q)\,e\,f\,\cos\alpha\,]^{3/2}}$$

$$+\quad\frac{[(1-2p)^2\,e^2+(1-2q)^2\,f^2+2\,(1-2p)\,(1-2q)\,e\,f\,\cos\alpha]}{[\,a^2(s,p)\,e^2+a^2(s,1-q)\,f^2-2\,a(s,p)\,a(s,1-q)\,e\,f\,\cos\alpha\,]^{1/2}}$$

$$\times\quad\frac{a^2(s,p)\,e^2+a^2(s,1-q)\,f^2-2\,a(s,p)\,a(s,1-q)\,e\,f\,\cos\alpha}{a^2(s,p)\,e^2+a^2(s,1-q)\,f^2-2\,a(s,p)\,a(s,1-q)\,e\,f\,\cos\alpha}$$

$$-\quad\frac{[(1-2p)\,a(s,p)\,e^2-(1-2q)\,a(s,1-q)\,f^2}{[\,a^2(s,p)\,e^2+a^2(s,1-q)\,f^2-2\,a(s,p)\,a(s,1-q)\,e\,f\,\cos\alpha\,]^{3/2}}$$

$$+\quad\frac{-((1-2p)\,a(s,1-q)-(1-2q)\,a(s,p))\,e\,f\,\cos\alpha]^2}{[\,a^2(s,p)\,e^2+a^2(s,1-q)\,f^2-2\,a(s,p)\,a(s,1-q)\,e\,f\,\cos\alpha\,]^{3/2}}$$

$$= \frac{e^2 f^2 \sin^2\alpha\,[\,(1-2p)\,a(s,q)-(1-2q)\,a(s,p)\,]^2}{[\,a^2(s,p)\,e^2+a^2(s,q)\,f^2+2\,a(s,p)\,a(s,q)\,e\,f\,\cos\alpha\,]^{3/2}}$$

$$+ \frac{e^2 f^2 \sin^2\alpha\,[\,(1-2p)\,a(s,q)-(1-2q)\,a(s,p)\,(p-q)\,]^2}{[\,a^2(s,p)\,e^2+a^2(s,1-q)\,f^2-2\,a(s,p)\,a(s,1-q)\,e\,f\,\cos\alpha\,]^{3/2}}$$

$$= \frac{e^2 f^2 \sin^2\alpha\,(\,q-p\,)^2}{[\,a^2(s,p)\,e^2+a^2(s,q)\,f^2+2\,a(s,p)\,a(s,q)\,e\,f\,\cos\alpha\,]^{3/2}}$$

$$+ \frac{e^2 f^2 \sin^2\alpha\,(\,1-p-q\,)^2}{[\,a^2(s,p)\,e^2+a^2(s,1-q)\,f^2-2\,a(s,p)\,a(s,1-q)\,e\,f\,\cos\alpha\,]^{3/2}} > 0 \text{ für } p,q \neq 1/2\,.$$

In the last step we have used the relations

$(1-2p)\,a(s,q)-(1-2q)\,a(s,p) = q-p$ and $(1-2p)\,a(s,1-q)+(1-2q)\,a(s,p) = 1-p-q$

which completes the proof of the Lemma. □

Proof of Theorem 2. By applying the Lemma we obtain the following chain of inequalities:

$$\begin{aligned}
\frac{L^2(\Box_{e,f,\alpha}(p,q))}{A(\Box_{e,f,\alpha}(p,q))} &\geq \frac{\left(\sqrt{e^2+f^2-2\,e\,f\,\cos\alpha}+\sqrt{e^2+f^2+2\,e\,f\,\cos\alpha}\right)^2}{e\,f\,\sin\alpha/2} \\
&= \frac{4\,(\,e^2+f^2\,)+4\,\sqrt{(\,e^2+f^2\,)^2-4\,e^2\,f^2\,\cos^2\alpha}}{e\,f\,\sin\alpha} \\
&= \frac{4\,(\,e^2+f^2\,)+4\,\sqrt{(\,e-f\,)^2+4\,e^2\,f^2\,\sin^2\alpha}}{e\,f\,\sin\alpha} \\
&= \frac{4\,(\,e^2+f^2\,)}{e\,f\,\sin\alpha}+4\,\sqrt{\frac{(\,e-f\,)^2}{e^2\,f^2\,\sin^2\alpha}+4} \\
&\geq \frac{8\,(\,e^2+f^2\,)}{e\,f} \quad \text{mit} \quad \text{“}=\text{”} \quad \text{iff} \quad \alpha=\frac{\pi}{2} \\
&= \frac{8\,(\,e-f\,)^2}{e\,f}+16 \\
&\geq 16 \quad \text{with} \quad \text{“}=\text{”} \quad \text{iff} \quad e=f\,. \quad \Box
\end{aligned}$$

In view of a verification of the lower bounds in Theorem 3 for any convex quadrangle (instead of parallelograms) which is equivalent to the lower bound in Conjecture I for $n = 4$, the following inequality would be the crucial step:

Conjecture III. For all $p, q \in [0,1]$ it holds that

$$L(\partial\Box_{e,f,\alpha}(p,q))\; I_2(\Box_{e,f,\alpha}(p,q)) \geq L(\partial\Box_{e,f,\alpha}(1/2,1/2))\; I_2(\Box_{e,f,\alpha}(1/2,1/2))\,. \tag{14}$$

Remark. There are counter-examples showing that the second-order CPI of a parallelogramm $\Box_{e,f,\alpha}(1/2,1/2)$ does not always maximize $I_2(\Box_{e,f,\alpha}(p,q))$. So far, it is even unproved whether (14) holds for $\alpha = \pi/2$.

2. Supplements to the proof of Theorem 3 in Section 4

Since the function $x \mapsto f(x) = x^{-1/2}$ turns out (strictly) convex for $x > 0$, it is clear that

$$\frac{1}{\sqrt{x+c}} - \frac{1}{\sqrt{x}} \geq \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x-c}} \quad \text{for } x > 0\,,\ 0 \leq c < x \ \text{ and } \ -x < c \leq 0\,.$$

We apply this inequality with $c = 2\,a\,b\,z\,\cos\alpha$ and $x = a^2 + b^2 z^2$ (resp. $x = a^2 z^2 + b^2$) leading immediately to (10). In order to prove (11) and corresponding inequality

$$\int_0^1 \frac{(3-z)\ \mathrm{d}z}{\sqrt{(b^2+c\,z)^2+h^2 z^2}} + \int_0^1 \frac{(3-z)\ \mathrm{d}z}{\sqrt{(b^2-c\,z)^2+h^2 z^2}} \leq 2 \int_0^1 \frac{(3-z)\ \mathrm{d}z}{\sqrt{b^2+h^2 z^2}}\,, \tag{15}$$

where $c = a\,\cos\alpha$ and $h = a\,\sin\alpha$, we put $b = 1$, $p = \cos\alpha$ and $q = \sin\alpha = \sqrt{1-p^2}$ for notational ease and rewrite all the integrals once more as follows:

$$\begin{aligned}
\int_0^1 \frac{(3-z)\ \mathrm{d}z}{\sqrt{(z\pm c)^2+h^2}} &= \int_0^{a^{-1}} \frac{(3-a\,x)\ \mathrm{d}x}{\sqrt{(x\pm\cos\alpha)^2+\sin^2\alpha}} = \int_{\pm p}^{a^{-1}\pm p} \frac{(3\pm a\,p-a\,x)\ \mathrm{d}x}{\sqrt{x^2+q^2}} \\
&= (3\pm a\,p) \int_{\pm p/q}^{(1\pm a\,p)/a\,q} \frac{\mathrm{d}x}{\sqrt{x^2+1}} + a - \sqrt{a^2\pm 2\,a\,p+1}
\end{aligned}$$

and, likewise,

$$\begin{aligned}
\int_0^1 \frac{(3-z)\ \mathrm{d}z}{\sqrt{(1\pm c\,z)^2+h^2 z^2}} &= \frac{1}{a^2}\int_0^{a} \frac{(3\,a-x)\ \mathrm{d}x}{\sqrt{(x\pm\cos\alpha)^2+\sin^2\alpha}} = \frac{1}{a^2}\int_{\pm p}^{a\pm p} \frac{(3\,a\pm p-x)\ \mathrm{d}x}{\sqrt{x^2+q^2}} \\
&= \frac{(3\,a\pm p)}{a^2} \int_{\pm p/q}^{(a\pm p)/q} \frac{\mathrm{d}x}{\sqrt{x^2+1}} + \frac{1-\sqrt{a^2\pm 2\,a\,p+1}}{a^2}\,.
\end{aligned}$$

Using the function $F(z) = \int_0^z (\,x^2+1\,)^{-1/2}\mathrm{d}z = \log(z+\sqrt{z^2+1}) = -F(-z)$ and

$$\begin{aligned}
2\int_0^{a^{-1}} \frac{(3-a\,x)\ \mathrm{d}x}{\sqrt{x^2+q^2}} &= 6\,F(\frac{1}{a\,q}) + 2\,(a\,q-\sqrt{1+a^2 q^2})\,, \\
\frac{2}{a^2}\int_0^{a} \frac{(3\,a-x)\ \mathrm{d}x}{\sqrt{1+q^2 x^2}} &= \frac{6\,F(a\,q)}{a\,q} + \frac{2\,(1-\sqrt{1+a^2 q^2})}{a^2 q^2}\,,
\end{aligned}$$

the inequalities (11) and (15) can be equivalently expressed as follows:

$$F\Big(\frac{1+a\,p}{a\,q}\Big)+F\Big(\frac{1-a\,p}{a\,q}\Big)+\frac{a\,p}{3}\,\Big(F\Big(\frac{1+a\,p}{a\,q}\Big)-F\Big(\frac{1-a\,p}{a\,q}\Big)-2\,F\Big(\frac{p}{q}\Big)\Big)$$
$$+\ \frac{1}{3}\Big(2\,a-\sqrt{a^2+2\,a\,p+1}-\sqrt{a^2-2\,a\,p+1}\Big)\le 2\,F\Big(\frac{1}{a\,q}\Big)+\frac{2}{3}\,\big(a\,q-\sqrt{1+a^2\,q^2}\,\big)$$

and

$$\frac{1}{a}\Big(F\Big(\frac{a+p}{q}\Big)+F\Big(\frac{a-p}{q}\Big)\Big)+\frac{p}{3\,a^2}\Big(F\Big(\frac{a+p}{q}\Big)-F\Big(\frac{a-p}{q}\Big)-2\,F\Big(\frac{p}{q}\Big)\Big)$$
$$+\ \frac{1}{3\,a^2}\Big(2-\sqrt{a^2+2\,a\,p+1}-\sqrt{a^2-2\,a\,p+1}\Big)\le\frac{2\,F(a\,q)}{a\,q}+\frac{2\,(1-\sqrt{1+a^2\,q^2})}{3\,a^2\,q^2}\,,$$

respectively. Both inequalities could be confirmed by direct evaluation for $0 < a \le 1$, $0 < p \le 1$ with Maple 8. The remaining left bound of (9) was already obtained in [4], see Theorem 2. An alternative proof is given in Sect. 5 of this Appendix. The right-hand bound of (9) results from fact that

$$\max\{\, I_2(\Box_{\pi/2}(a,b))\ :\ a,b>0, a\,b=1\}=I_2(\Box_{\pi/2}(1,1))=4\,F(1)+\frac{4}{3}\,(1-\sqrt{2})\,,$$

which is seen by showing that the function

$$I_2\big(\Box_{\pi/2}\big(a,\frac{1}{a}\big)\big)=\frac{2}{a}\,F(a^2)+2\,a\,F(a^{-2})+\frac{2\,a}{3}\,\big(a^2-\sqrt{a^4+1}\big)+\frac{2}{3\,a}\,\big(a^{-2}-\sqrt{1+a^{-4}}\big)$$

is strictly decreasing for $a \ge 1$ and attains its maximum at $a = 1$.

3. Second-order CPI for parallelograms

A parallelogram $\Box_\alpha(a,b)$ having the vertices $A=(0,0)$, $B=(b,0)$, $C=(b+a\,\cos\alpha,h)$ and $D=(a\,\cos\alpha,h)$ (with altitude $h=a\,\sin\alpha$ and base length b) can be described as point set $\{\lambda_1\,(b,0)+\lambda_2\,(a\,\cos\alpha,h)\ :\ 0\le\lambda_1,\lambda_2\le 1\}$, where each pair $x,y\in\Box_\alpha(a,b)$ admits unique representations as linear combinations $x=(x_1,x_2)=(\lambda_1\,b+\lambda_2\,a\,\cos\alpha\,,\ \lambda_2\,h)$ and $y=(y_1,y_2)=(\mu_1\,b+\mu_2\,a\,\cos\alpha\,,\ \mu_2\,h)$ with $0\le\lambda_1,\lambda_2,\mu_1,\mu_2\le 1$. Therefore, by $\|x-y\|^2=(x_1-y_1)^2+(x_2-y_2)^2$,

$$\begin{aligned}\|x-y\| &= \sqrt{\big((\lambda_1-\mu_1)\,b+(\lambda_2-\mu_2)\,a\,\cos\alpha\big)^2+\big((\lambda_2-\mu_2)\,h\big)^2}\\ &= \sqrt{(\lambda_1-\mu_1)^2\,b^2+2\,a\,b\,\cos\alpha\,(\lambda_1-\mu_1)\,(\lambda_2-\mu_2)+(\lambda_2-\mu_2)^2\,a^2}\,,\end{aligned}$$

and together with the Jacobians

$$\frac{\partial(x_1,x_2)}{\partial(\lambda_1,\lambda_2)}=\frac{\partial(x_1,x_2)}{\partial(\lambda_1,\lambda_2)}=\begin{vmatrix} b & 0\\ a\,\cos\alpha & h\end{vmatrix}=b\,h=A(\Box_\alpha(a,b))$$

and the abbreviation $f(\lambda_1,\lambda_2)=\big(\lambda_1^2\,b^2+2\,a\,b\,\cos\alpha\,\lambda_1\,\lambda_2+\lambda_2^2\,a^2\big)^{-1/2}$ we arrive at

$$
\begin{aligned}
\frac{I_2(\Box_\alpha(a,b))}{A^2(\Box_\alpha(a,b))} &= \int_0^1 \int_{-\mu_1}^{1-\mu_1} \int_0^1 \int_{-\mu_2}^{1-\mu_2} f(\lambda_1,\lambda_2)\,\mathrm{d}\lambda_2\,\mathrm{d}\mu_2\,\mathrm{d}\lambda_1\,\mathrm{d}\mu_1 \\
&= \int_0^1 \int_0^1 \Big(\int_0^{1-\mu_1} \int_0^{1-\mu_2} f(\lambda_1,\lambda_2) + \int_0^{\mu_1} \int_0^{\mu_2} f(-\lambda_1,-\lambda_2) \Big) \mathrm{d}\lambda_2\,\mathrm{d}\lambda_1\,\mathrm{d}\mu_2\,\mathrm{d}\mu_1 \\
&+ \int_0^1 \int_0^1 \Big(\int_0^{1-\mu_1} \int_0^{\mu_2} f(\lambda_1,-\lambda_2) + \int_0^{\mu_1} \int_0^{1-\mu_2} f(-\lambda_1,\lambda_2) \Big) \mathrm{d}\lambda_2\,\mathrm{d}\lambda_1\,\mathrm{d}\mu_2\,\mathrm{d}\mu_1 \\
&= \int_0^1 \int_0^1 \Big(f(\lambda_1,\lambda_2) \int_0^{1-\lambda_1} \int_0^{1-\lambda_2} + f(-\lambda_1,-\lambda_2) \int_{\lambda_1}^1 \int_{\lambda_2}^1 \Big) \mathrm{d}\mu_2\,\mathrm{d}\mu_1\,\mathrm{d}\lambda_2\,\mathrm{d}\lambda_1 \\
&+ \int_0^1 \int_0^1 \Big(f(\lambda_1,-\lambda_2) \int_0^{1-\lambda_1} \int_{\lambda_2}^1 + f(-\lambda_1,\lambda_2) \int_{\lambda_1}^1 \int_0^{1-\lambda_2} \Big) \mathrm{d}\mu_2\,\mathrm{d}\mu_1\,\mathrm{d}\lambda_2\,\mathrm{d}\lambda_1 \\
&= 2 \int_0^1 \int_0^1 (1-\lambda_1)(1-\lambda_2) \big(f(\lambda_1,\lambda_2) + f(\lambda_1,-\lambda_2) \big)\,\mathrm{d}\lambda_2\,\mathrm{d}\lambda_1
\end{aligned}
$$

The latter double integral we denote by $J_\alpha(a,b)$. Using the transformation rules $f(r, r\,\lambda_2) = r^{-1} f(1,\lambda_2)$ for $r > 0$ and $f(\lambda_1,-\lambda_2) = f(-\lambda_1,\lambda_2)$, we can rewrite $J_\alpha(a,b)$ as single integral in the following way :

$$
\begin{aligned}
J_\alpha(a,b) &= \int_0^1 \int_0^{\lambda_1} (1-\lambda_1)(1-\lambda_2) \big(f(\lambda_1,\lambda_2) + f(\lambda_1,-\lambda_2) \big)\,\mathrm{d}\lambda_2\,\mathrm{d}\lambda_1 \\
&+ \int_0^1 \int_0^{\lambda_2} (1-\lambda_1)(1-\lambda_2) \big(f(\lambda_1,\lambda_2) + f(\lambda_1,-\lambda_2) \big)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2 \\
&= \int_0^1 \int_0^1 (1-\lambda_1)(1-\lambda_1\,\lambda_2) \big(f(1,\lambda_2) + f(1,-\lambda_2) \big)\,\mathrm{d}\lambda_2\,\mathrm{d}\lambda_1 \\
&+ \int_0^1 \int_0^1 (1-\lambda_1\,\lambda_2)(1-\lambda_2) \big(f(\lambda_1,1) + f(\lambda_1,-1) \big)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2
\end{aligned}
$$

$$
= \frac{1}{6} \int_0^1 (3-\lambda_2) \big(f(1,\lambda_2) + f(1,-\lambda_2) \big)\,\mathrm{d}\lambda_2 + \frac{1}{6} \int_0^1 (3-\lambda_1) \big(f(\lambda_1,1) + f(\lambda_1,-1) \big)\,\mathrm{d}\lambda_1 \,,
$$

whence it follows the above formula (8)

$$
\frac{I_2(\Box_\alpha(a,b))}{A^2(\Box_\alpha(a,b))} = \frac{1}{3} \int_0^1 (3-\lambda) \big(f(\lambda,1) + f(\lambda,-1) + f(1,\lambda) + f(-1,\lambda) \big)\,\mathrm{d}\lambda \,.
$$

Further, we are able to calculate each of the four integrals. It suffices to consider the integral over $f(\lambda,1) = \big(\lambda^2 b^2 + 2\,a\,b\,\cos\alpha\,\lambda + a^2\big)^{-1/2}$. We find that

$$\int_0^1 (3-\lambda)\,f(\lambda,\pm 1)\,\mathrm{d}\lambda = \frac{1}{b^2}\int_0^b \frac{(3\,b-x)\,\mathrm{d}x}{\sqrt{x^2 \pm 2\,a\,\cos\alpha\,x + a^2}}$$

$$= \frac{1}{b^2}\int_0^b \frac{(3\,b-x)\,\mathrm{d}x}{\sqrt{(x\pm a\,\cos\alpha)^2 + (a\,\sin\alpha)^2}}\ , \quad \text{Substitution: } x = a\,y\,\sin\alpha \mp a\,\cos\alpha$$

$$= \frac{1}{b^2}\int_{\pm\cot\alpha}^{\frac{b\pm a\,\cos\alpha}{a\,\sin\alpha}} \frac{(3\,b \pm a\,\cos\alpha - y\,a\,\sin\alpha)\,\mathrm{d}y}{\sqrt{y^2+1}}$$

$$= \frac{3\,b\pm a\,\cos\alpha}{b^2}\int_{\pm\cot\alpha}^{\frac{b\pm a\,\cos\alpha}{a\,\sin\alpha}} \frac{\mathrm{d}y}{\sqrt{y^2+1}} - \frac{a\,\sin\alpha}{b^2}\int_{\pm\cot\alpha}^{\frac{b\pm a\,\cos\alpha}{a\,\sin\alpha}} \mathrm{d}\big(\sqrt{y^2+1}\big)$$

$$= \frac{3\,b\pm a\,\cos\alpha}{b^2}\Big(F\Big(\frac{b\pm a\,\cos\alpha}{a\,\sin\alpha}\Big) - F\big(\pm\cot\alpha\big)\Big) - \frac{\sqrt{a^2 \pm 2\,a\,b\,\cos\alpha + b^2} - a}{b^2}\,.$$

Summarizing the above calculations and using the formulae $e/f = \sqrt{a^2+b^2\pm 2\,a\,b\,\cos\alpha}$ for the diagonals of the parallelogram $I_2(\Box_\alpha(a,b))$ we get the final formula

$$\frac{I_2(\Box_\alpha(a,b))}{A^2(\Box_\alpha(a,b))} = R_\alpha(a,b) + R_\alpha(b,a) + \frac{2\,(\,a^3+b^3\,) - (\,a^2+b^2\,)\,(e+f)}{3\,a^2\,b^2}\,, \tag{16}$$

where, by using the the area sinus hyperbolicus function $F(z) = \log(z + \sqrt{z^2+1})$,

$$R_\alpha(a,b) = \sum_{i=0}^{1} \frac{3\,b + (-1)^i\,a\,\cos\alpha}{3\,b^2}\,F\Big(\frac{b+(-1)^i\,a\,\cos\alpha}{a\,\sin\alpha}\Big) - \frac{2\,a\,\cos\alpha}{3\,b^2}\,F\big(\cot\alpha\big)\,.$$

This latter formula coincides with the expression given in (12). For a rectangle having edge lengths a and b we obtain the comparatively simple expression, see [4],

$$I_2(\Box_{\pi/2}(a,b)) = 2\,a^2\,b\,F\Big(\frac{b}{a}\Big) + 2\,a\,b^2\,F\Big(\frac{a}{b}\Big) + \frac{2}{3}\,\Big(a^3+b^3 - \big(\sqrt{a^2+b^2}\big)^3\Big)\,. \tag{17}$$

This formula also yields the closed-term expression of the second-order CPI $I_2(\Box_\alpha(a,a))$ stated at the end of Section 4. In addition, using (17), we can compare the functions

$$\begin{aligned}\frac{I_2(\Box_\alpha(1,1))}{2\,\sin^2\alpha} &= F\Big(\frac{1+\cos\alpha}{\sin\alpha}\Big) + F\Big(\frac{1-\cos\alpha}{\sin\alpha}\Big) + \frac{\cos\alpha}{3}\,\Big(F\Big(\frac{1+\cos\alpha}{\sin\alpha}\Big)\\ &- F\Big(\frac{1-\cos\alpha}{\sin\alpha}\Big) - 2\,F\Big(\frac{\cos\alpha}{\sin\alpha}\Big)\Big) + \frac{2}{3}\Big(1 - \sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\Big) = \frac{2}{3}\,T(\alpha)\end{aligned}$$

and

$$\frac{I_2(\Box_{\pi/2}(1,\sin\alpha))}{2\,\sin^2\alpha} = \frac{F(\sin\alpha)}{\sin\alpha} + F\Big(\frac{1}{\sin\alpha}\Big) - \frac{1+\sin^3\alpha - (\sqrt{1+\sin^2\alpha})^3}{3\,\sin^2\alpha}\,.$$

By numerical evaluation it turns out that the ratio $r(\alpha) = I_2(\Box_\alpha(1,1))/I_2(\Box_{\pi/2}(1,\sin\alpha))$ is strictly increasing over the interval $(0,\pi/2]$ with $\lim_{r\to 0} r(\alpha) = 2/3$ and $r(\pi/2) = 1$.

4. Chord power integrals of parallelotopes in $\mathbb{R}^d$

Let $\mathbf{a}_i = (a_i^{(1)}, ..., a_i^{(d)})$, $i = 1, ..., d$, be linearly independent vectors in $\mathbb{R}^d$ which define the $d-$*parallelotope* $P_d(\mathbf{a}_1, ..., \mathbf{a}_d) = \{\sum_{i=1}^d \lambda_i \mathbf{a}_i : 0 \le \lambda_1, ..., \lambda_d \le 1\}$. For brevity we write P_d instead of $P_d(\mathbf{a}_1, ..., \mathbf{a}_d)$ (if no confusion is possible). It is well-known from analytic geometry that the $d-$volume $V_d(P_d(\mathbf{a}_1, ..., \mathbf{a}_d))$ of our $d-$parallelotope equals the absolute value of the determinant

$$\det\big((a_j^{(i)})_{i,j=1}^d \big) = \begin{vmatrix} a_1^{(1)} & a_2^{(1)} & \cdots & a_d^{(1)} \\ a_1^{(2)} & a_2^{(2)} & \cdots & a_d^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ a_1^{(d)} & a_2^{(d)} & \cdots & a_d^{(d)} \end{vmatrix} .$$

Further, let $I_d(K)$ denote the right-hand double integral of (2) for a convex body K in $\mathbb{R}^d$ with positive $d-$volume which can be regarded - up to some multiplicative constant - as dth-order chord power integral (with resect to $\mu-$random lines in $\mathbb{R}^d$, see [2], [9], [4]). In the special case $d = 3$ the integral $I_3(K)$ coincides with Newton's self-potential of the body $K \subset \mathbb{R}^d$, see e.g. [3].

Since any two distinct points $\mathbf{x} = (x_1, ..., x_d), \mathbf{y} = (y_1, ..., y_d) \in P_d$ can be expressed as linear combination $\mathbf{x} = \lambda_1 \mathbf{a}_1 + \cdots + \lambda_d \mathbf{a}_d$ resp. $\mathbf{y} = \mu_1 \mathbf{a}_1 + \cdots + \mu_d \mathbf{a}_d$ with unique $\lambda_1, \mu_1, ..., \lambda_d, \mu_d \in [0, 1]$, we may apply the integral transformation formula with the Jacobian determinants

$$\det\Big(\Big(\frac{\partial x_i}{\partial \lambda_j}\Big)_{i,j=1}^d \Big) = \det\Big(\Big(\frac{\partial y_i}{\partial \mu_j}\Big)_{i,j=1}^d \Big) = \det\big((a_j^{(i)})_{i,j=1}^d \big) = V_d(P_d) ,$$

which do not depend on the λ_i's and μ_i's. Together with the function

$$[-1,1]^d \ni (z_1, ..., z_d) \mapsto \|z_1 \mathbf{a}_1 + \cdots + z_d \mathbf{a}_d\| = \sqrt{\sum_{i=1}^d z_i^2 \, \|\mathbf{a}_i\|^2 + 2 \sum_{1\le i<j\le d} z_i \, z_j \, \langle \mathbf{a}_i, \mathbf{a}_j \rangle} ,$$

where $\langle \mathbf{a}_i, \mathbf{a}_j \rangle$ denotes the scalar product of $\mathbf{a}_i$ and $\mathbf{a}_j$, we arrive at

$$\begin{aligned}
\frac{1}{V_d^2(P_d)} \int_{P_d}\int_{P_d} \frac{\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|} &= \int_0^1\int_0^1 \cdots \int_0^1\int_0^1 \frac{\mathrm{d}\lambda_d\,\mathrm{d}\mu_d \cdots \mathrm{d}\lambda_1\,\mathrm{d}\mu_1}{\|\,(\lambda_1-\mu_1)\,\mathbf{a}_1 + \cdots + (\lambda_d-\mu_d)\,\mathbf{a}_d\,\|} \\
&= \int_0^1\int_{-\mu_1}^{1-\mu_1} \cdots \int_0^1\int_{-\mu_d}^{1-\mu_d} \frac{\mathrm{d}\lambda_d\,\mathrm{d}\mu_d \cdots \mathrm{d}\lambda_1\,\mathrm{d}\mu_1}{\|\,\lambda_1\,\mathbf{a}_1 + \cdots + \lambda_d\,\mathbf{a}_d\,\|} \\
&= \int_0^1\int_0^{\mu_1} \cdots \int_0^1\int_0^{\mu_d} \sum_{\nu_1,...,\nu_d\in\{0,1\}} \frac{\mathrm{d}\lambda_d\,\mathrm{d}\mu_d \cdots \mathrm{d}\lambda_1\,\mathrm{d}\mu_1}{\|\,(-1)^{\nu_1}\,\lambda_1\,\mathbf{a}_1 + \cdots + (-1)^{\nu_d}\,\lambda_d\,\mathbf{a}_d\,\|} \\
&= \int_0^1\int_{\lambda_1}^1 \cdots \int_0^1\int_{\lambda_d}^1 \sum_{\nu_1,...,\nu_d\in\{0,1\}} \frac{\mathrm{d}\mu_d\,\mathrm{d}\lambda_d \cdots \mathrm{d}\mu_1\,\mathrm{d}\lambda_1}{\|\,(-1)^{\nu_1}\,\lambda_1\,\mathbf{a}_1 + \cdots + (-1)^{\nu_d}\,\lambda_d\,\mathbf{a}_d\,\|} \\
&= \int_0^1 \cdots \int_0^1 \sum_{\nu_1,...,\nu_d\in\{0,1\}} \frac{(1-\lambda_1)\cdots(1-\lambda_d)\,\mathrm{d}\lambda_d \cdots \mathrm{d}\lambda_1}{\|\,(-1)^{\nu_1}\,\lambda_1\,\mathbf{a}_1 + \cdots + (-1)^{\nu_d}\,\lambda_d\,\mathbf{a}_d\,\|} .
\end{aligned}$$

We next derive a lower bound for the $d-$fold integral in the last line. We again employ the elementary inequality

$$\frac{1}{\sqrt{x+c}} + \frac{1}{\sqrt{x-c}} \geq \frac{2}{\sqrt{x}} \quad \text{for} \quad x > 0 \quad , \quad -x \leq c \leq x$$

for $x = \| \sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \mathbf{a}_i \|^2 + \lambda_d^2 \|\mathbf{a}_d\|^2$ and $c = 2 \langle \lambda_d \mathbf{a}_d , \sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \mathbf{a}_i \rangle$ and obtain that

$$\begin{aligned}
& \sum_{\nu_d \in \{0,1\}} \frac{1}{\| (-1)^{\nu_1} \lambda_1 \mathbf{a}_1 + \cdots + (-1)^{\nu_d} \lambda_d \mathbf{a}_d \|} \\
= & \sum_{\nu_d \in \{0,1\}} \frac{1}{\sqrt{\| \sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \mathbf{a}_i \|^2 + \lambda_d^2 \|\mathbf{a}_d\|^2 + 2 (-1)^{\nu_d} \lambda_d \sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \langle \mathbf{a}_i, \mathbf{a}_d \rangle}} \\
\geq & \frac{2}{\sqrt{\| \sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \mathbf{a}_i \|^2 + \lambda_d^2 \|\mathbf{a}_d\|^2}} \, .
\end{aligned}$$

In the same way we find that

$$\begin{aligned}
& \sum_{\nu_k \in \{0,1\}} \frac{1}{\sqrt{\| (-1)^{\nu_1} \lambda_1 \mathbf{a}_1 + \cdots + (-1)^{\nu_k} \lambda_k \mathbf{a}_k \|^2 + \lambda_{k+1}^2 \|\mathbf{a}_{k+1}\|^2 + \cdots + \lambda_d^2 \|\mathbf{a}_d\|^2}} \\
\geq & \frac{2}{\sqrt{\| \sum_{i=1}^{k-1} (-1)^{\nu_i} \lambda_i \mathbf{a}_i \|^2 + \lambda_k^2 \|\mathbf{a}_k\|^2 + \cdots + \lambda_d^2 \|\mathbf{a}_d\|^2}}
\end{aligned}$$

for $k = d-1, ..., 2$. Summing up all these inequalities yields

$$\sum_{\nu_1, ..., \nu_d \in \{0,1\}} \frac{1}{\| (-1)^{\nu_1} \lambda_1 \mathbf{a}_1 + \cdots + (-1)^{\nu_d} \lambda_d \mathbf{a}_d \|} \geq \frac{2^d}{\sqrt{\lambda_1^2 \|\mathbf{a}_1\|^2 + \cdots + \lambda_d^2 \|\mathbf{a}_d\|^2}}$$

whence it follows the inequality

$$\frac{I_d\big(\times_{i=1}^d [0, \|\mathbf{a}_i\|]\big)}{\|\mathbf{a}_1\|^2 \cdots \|\mathbf{a}_d\|^2} = 2^d \int_0^1 \cdots \int_0^1 \frac{\prod_{i=1}^d (1-\lambda_i) \, \mathrm{d}\lambda_d \cdots \mathrm{d}\lambda_1}{\sqrt{\sum_{i=1}^d \lambda_i^2 \|\mathbf{a}_i\|^2}} \leq \frac{I_d(P_d(\mathbf{a}_1, ..., \mathbf{a}_d))}{V_d^2(P_d(\mathbf{a}_1, ..., \mathbf{a}_d))} \tag{18}$$

comparing the dth-order CPIs of $d-$dimensional hyper-rectangles and $d-$parallelotopes with edge lengths $\|\mathbf{a}_1\|, ..., \|\mathbf{a}_d\|$. In the next Section 5 we give a lower bound of the dth-order CPI of $d-$dimensional hyper-rectangles in terms of their $m$ean widths and the dth-order CPI of the unit cube $[0,1]^d$.

It should be mentioned the remarkable fact that the mean width $b_d(P(\mathbf{a}_1, ..., \mathbf{a}_d))$ of the $d-$parallelotope $P(\mathbf{a}_1, ..., \mathbf{a}_d)$ only depends on the edge lengths $\|\mathbf{a}_1\|, ..., \|\mathbf{a}_d\|$, but not on the angles between the edges. More precisely, we have

$$b_d\big(P(\mathbf{a}_1, ..., \mathbf{a}_d) \big) = \frac{2\,\kappa_{d-1}}{d\,\kappa_d} \big(\|\mathbf{a}_1\| + \cdots + \|\mathbf{a}_d\| \big) = b_d\big([0, \|\mathbf{a}_1\|] \times \cdots \times [0, \|\mathbf{a}_d\|] \big) \, ,$$

where κ_d denotes the $d-$volume of the unit ball in $\mathbb{R}^d$.

5. Schur-convexity and the lower bound in Conjecture II

In connection with the calculation and estimation of the dth-order CPI of an $d-$dimensional hyper-rectangle with edge lengths $a_1, ..., a_d > 0$ we are faced with the parameter integral

$$I_d(a_1, ..., a_d) = \int_0^1 \cdots \int_0^1 \frac{(1-x_1)\,\cdots\,(1-x_d)}{\sqrt{a_1^2\,x_1^2 + \cdots + a_d^2\,x_d^2}}\,\mathrm{d}x_d\,\cdots\,\mathrm{d}x_1\,. \tag{19}$$

Theorem 5. The mapping $(a_1, ..., a_d) \mapsto I_d(a_1, ..., a_d)$ is Schur-convex for $a_1, ..., a_d > 0$, i.e., for any *doubly stochastic* $d \times d$*-matrix* $\mathbf{P}$ and each column vector $\mathbf{a} = (a_1, ..., a_d)$ the inequality $I_d(\mathbf{a}\,\mathbf{P}) \leq I_d(\mathbf{a})$ holds, see [7] or [10].

Corollary. The parameter integral (19) obeys the inequality

$$I_d(a_1, ..., a_d) \geq \frac{d}{a_1 + \cdots + a_d}\,I_d(1, ..., 1)\,.$$

*P*roof of the Corollary. Clearly, we have

$$I_d(a_1, ..., a_d) = \frac{I_d(p_1, ..., p_d)}{a_1 + \cdots + a_d} \quad \text{with} \quad p_i = a_i/(a_1 + \cdots + a_d)\ \ ,\ \ i = 1, ..., d\,,$$

and together with $I_d(\mathbf{p}) \geq I_d(\mathbf{p}\,\mathbf{P}_d) = I_d\big(\frac{1}{d}, ..., \frac{1}{d}\big) = d\,I_d(1, ..., 1)$ for $\mathbf{p} = (p_1, ..., p_d)$ and matrix $\mathbf{P}_d$ having identical entries equal to $\frac{1}{d}$.

*P*roof of Theorem 5. We shall apply a famous criterion going back to I. Schur , see [7],[10], which for the symmetric function $I_d(a_1, ..., a_d)$ reads as follows:

$$(a_1 - a_2)\Big(\frac{\partial I_n}{\partial a_1} - \frac{\partial I_n}{\partial a_2}\Big) \geq 0 \quad \text{für beliebige} \quad a_1, a_2, ..., a_n > 0\,.$$

This means, for $a_1 \geq a_2 > 0$ and any fixed $a_3, ..., a_d > 0$ we have to show that $\frac{\partial I_d}{\partial a_1} \geq \frac{\partial I_d}{\partial a_2}$.

After differentiation and partial integration we arrive at

$$\begin{aligned}
\frac{\partial I_d}{\partial a_1} &= -\int_0^1\int_0^1 \cdots \int_0^1 \frac{a_1\,x_1^2\,(1-x_1)\,(1-x_2)\cdots(1-x_d)}{(\,a_1^2\,x_1^2 + a_2^2\,x_2^2 + \cdots + a_d^2\,x_d^2\,)^{3/2}}\,\mathrm{d}x_d\,\cdots\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&= \frac{1}{a_1}\int_0^1\int_0^1 \cdots \int_0^1 x_1 \prod_{i=1}^{d}(1-x_i)\,\mathrm{d}_{x_1}\big((\,a_1^2\,x_1^2 + a_2^2\,x_2^2 + \cdots + a_d^2\,x_d^2\,)^{-1/2}\big)\,\mathrm{d}x_d\,\cdots\,\mathrm{d}x_2 \\
&= \frac{1}{a_1}\int_0^1\int_0^1 \cdots \int_0^1 \frac{(2\,x_1 - 1)\,(1-x_2)\cdots(1-x_d)}{\sqrt{a_1^2\,x_1^2 + a_2^2\,x_2^2 + \cdots + a_d^2\,x_d^2}}\,\mathrm{d}x_d\,\cdots\,\mathrm{d}x_2\,\mathrm{d}x_1
\end{aligned}$$

and, likewise,

$$
\begin{aligned}
\frac{\partial I_d}{\partial a_2} &= \frac{1}{a_2} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{(2\,x_2 - 1)\,(1 - x_1) \cdots (1 - x_d)}{\sqrt{a_1^2\,x_1^2 + a_2^2\,x_2^2 + \cdots + a_d^2\,x_d^2}}\,\mathrm{d}x_d \cdots \mathrm{d}x_2\,\mathrm{d}x_1 \\
&= \frac{1}{a_2} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{(2\,x_1 - 1)\,(1 - x_2) \cdots (1 - x_d)}{\sqrt{a_2^2\,x_1^2 + a_1^2\,x_2^2 + \cdots + a_n^2\,x_d^2}}\,\mathrm{d}x_d \cdots \mathrm{d}x_2\,\mathrm{d}x_1
\end{aligned}
$$

Let us introduce the abbreviations $Q := a_1^2/a_2^2 \geq 1$ and $P := a_2^{-2}\,(a_3^2\,x_3^2 + \cdots + a_d^2\,x_d^2)$ with fixed $x_3, ..., x_d \in (0,1]$. Now, we prove that

$$
I(Q,P) := \int_0^1 \int_0^1 \frac{(2\,x_1 - 1)\,(1 - x_2)}{\sqrt{Q\,x_1^2 + x_2^2 + P}}\,\mathrm{d}x_2\,\mathrm{d}x_1 \geq \int_0^1 \int_0^1 \frac{(2\,x_1 - 1)\,(1 - x_2)}{\sqrt{\frac{1}{Q}\,x_1^2 + x_2^2 + \frac{P}{Q}}}\,\mathrm{d}x_2\,\mathrm{d}x_1 = I(Q^{-1}, P\,Q^{-1})\,.
$$

For this purpose it suffices to show that the function $I(Q,P)$ is non-decreasing in Q as well as in P. This means we have to show that $\frac{\partial I(Q,P)}{\partial P} \geq 0$ and $\frac{\partial I(Q,P)}{\partial Q} \geq 0$.

$$
\begin{aligned}
\frac{\partial I(Q,P)}{\partial P} &= -\frac{1}{2} \int_0^1 \int_0^1 \frac{(2\,x_1 - 1)\,(1 - x_2)}{(\,Q\,x_1^2 + x_2^2 + P\,)^{3/2}}\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&= \frac{1}{4} \int_0^1 \int_0^1 \left[\frac{x_1\,(1 - x_2)}{\left(\frac{Q}{4}(1 - x_1)^2 + x_2^2 + P\right)^{3/2}} - \frac{x_1\,(1 - x_2)}{\left(\frac{Q}{4}(1 + x_1)^2 + x_2^2 + P\right)^{3/2}} \right] \mathrm{d}x_2\,\mathrm{d}x_1 \geq 0\,.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\frac{\partial I(Q,P)}{\partial Q} &= -\frac{1}{2} \Big(\int_0^{1/2} + \int_{1/2}^1 \Big) \int_0^1 \frac{(2\,x_1 - 1)\,x_1^2\,(1 - x_2)}{(\,Q\,x_1^2 + x_2^2 + P\,)^{3/2}}\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&= \frac{1}{2\,Q} \int_0^1 \int_0^1 (2\,x_1 - 1)\,x_1\,(1 - x_2)\,\mathrm{d}_{x_1}\big((\,Q\,x_1^2 + x_2^2 + P\,)^{-1/2} \big)\,\mathrm{d}x_2 \\
&= \frac{1}{2\,Q} \left[\int_0^1 \frac{1 - x_2}{\sqrt{Q + x_2^2 + P}}\,\mathrm{d}x_2 - \int_0^1 \int_0^1 \frac{(4\,x_1 - 1)\,(1 - x_2)}{\sqrt{Q\,x_1^2 + x_2^2 + P}}\,\mathrm{d}x_2\,\mathrm{d}x_1 \right] \\
&= \frac{1}{2\,Q} \left[\int_0^1 \frac{1 - x_2}{\sqrt{Q + x_2^2 + P}}\,\mathrm{d}x_2 + \int_0^1 \int_0^1 \frac{1 - x_2}{\sqrt{Q\,x_1^2 + x_2^2 + P}}\,\mathrm{d}x_2\mathrm{d}x_1 \right. \\
&\qquad \left. - 2 \int_0^1 \int_0^1 \frac{1 - x_2}{\sqrt{Q\,x_1 + x_2^2 + P}}\,\mathrm{d}x_2\mathrm{d}x_1 \right]
\end{aligned}
$$

$$\geq \frac{1}{Q}\left[\int_0^1 \frac{1-x_2}{\sqrt{Q+x_2^2+P}}\,\mathrm{d}x_2 - \int_0^1\int_0^1 \frac{1-x_2}{\sqrt{Q\,x_1+x_2^2+P}}\,\mathrm{d}x_2\mathrm{d}x_1\right]$$

$$= \frac{1}{Q}\left[\int_0^1 \frac{1-x_2}{\sqrt{Q+x_2^2+P}}\,\mathrm{d}x_2 - \frac{1}{Q}\int_0^1 (1-x_2)\Big(\sqrt{Q+x_2^2+P}-\sqrt{x_2^2+P}\Big)\,\mathrm{d}x_2\right]$$

$$= \frac{1}{Q}\left[\int_0^1 \frac{1-x_2}{\sqrt{Q+x_2^2+P}}\,\mathrm{d}x_2 - \int_0^1 \frac{1-x_2}{\sqrt{Q+x_2^2+P}+\sqrt{x_2^2+P}}\,\mathrm{d}x_2\right] \geq 0\,.$$

Therefore, in view of $Q \geq 1$, we get the inequality

$$I(Q,P) \geq I(Q^{-1},P) \geq I(Q^{-1},P\,Q^{-1})\,,$$

which reveals that Schur's criterion (20) is satisfied. □

In the particular case $d=2$ the foregoing proof becomes much simpler since $P=0$ can be assumed. This allows the following rearrangements:

$$\begin{aligned}
I(Q,0) &= \int_0^1\int_0^1 \frac{(2\,x_1-1)\,(1-x_2)}{(\,Q\,x_1^2+x_2^2\,)^{1/2}}\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&= \int_0^1\int_0^{x_1} \frac{(2\,x_1-1)\,(1-x_2)}{(\,Q\,x_1^2+x_2^2\,)^{1/2}}\,\mathrm{d}x_2\,\mathrm{d}x_1 + \int_0^1\int_0^{x_2} \frac{(2\,x_1-1)\,(1-x_2)}{(\,Q\,x_1^2+x_2^2\,)^{1/2}}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \int_0^1\int_0^1 \frac{(2\,x_1-1)\,(1-x_1\,y_2)\,x_1}{(\,Q\,x_1^2+x_1^2\,y_2^2\,)^{1/2}}\,\mathrm{d}y_2\,\mathrm{d}x_1 + \int_0^1\int_0^1 \frac{(2\,x_2\,y_1-1)\,(1-x_2)\,x_2}{(\,Q\,x_2^2\,y_1^2+x_2^2\,)^{1/2}}\,\mathrm{d}y_1\,\mathrm{d}x_2 \\
&= -\frac{1}{2}\int_0^1 \frac{y}{(\,Q\,+\,y^2\,)^{1/2}}\,\mathrm{d}y - \frac{1}{6}\int_0^1 \frac{3\,-2\,y}{(\,Q\,y^2\,+\,1\,)^{1/2}}\,\mathrm{d}y\,.
\end{aligned}$$

Hence, for $Q > 0$,

$$\frac{\partial I(Q,0)}{\partial Q} = \frac{1}{2}\int_0^1 \frac{y^2}{(\,Q\,+\,y^2\,)^{3/2}}\,\mathrm{d}y + \frac{Q}{6}\int_0^1 \frac{(3\,-2\,y)\,y}{(\,Q\,y^2\,+\,1\,)^{3/2}}\,\mathrm{d}y \,>\, 0$$

providing that $I(Q,0) > I(Q^{-1},0)$ for $Q > 1$.

6. An explicit formula for the third-order CPI of a cuboid in $\mathbb{R}^3$

To begin with we make an expansion of the parameter integral (19) for any $d \geq 2$, which is based on the simple formula $(1-x_1)\,\cdots\,(1-x_d) = 1+\sum_{k=1}^{d}(-1)^k \sum_{1\leq i_1<\cdots<i_k\leq d} x_{i_1}\cdots x_{i_k}$ leading to

$$I_d(a_1,...,a_d) = \int_0^1 \cdots \int_0^1 \frac{\mathrm{d}x_d\,\cdots\,\mathrm{d}x_1}{\sqrt{a_1^2\,x_1^2+\cdots+a_d^2\,x_d^2}} + \sum_{k=1}^{d}(-1)^k \sum_{1\leq i_1<\cdots<i_k\leq d} I^{(d)}_{i_1,...,i_k}(a_1,...,a_d)$$

with the $d-$fold integrals

$$I^{(d)}_{i_1,...,i_k}(a_1,...,a_d) = \int_0^1\int_0^1\cdots\int_0^1 \frac{x_{i_1}\,\cdots\,x_{i_k}\,\mathrm{d}x_d\,\cdots\,\mathrm{d}x_2\,\mathrm{d}x_1}{\sqrt{a_1^2\,x_1^2+a_2^2\,x_2^2+\cdots+a_d^2\,x_d^2}}$$

for $1\leq i_1<\cdots<i_k\leq d$ and $k=1,...,d-1$. The first integral in this expansion can be expressed as sum

$$\int_0^1\cdots\int_0^1 \frac{\mathrm{d}x_d\,\cdots\,\mathrm{d}x_1}{\sqrt{a_1^2\,x_1^2+\cdots+a_d^2\,x_d^2}} = \sum_{\pi_d}\int_0^1\int_0^{x_1}\cdots\int_0^{x_{d-1}} \frac{\mathrm{d}x_d\,\cdots\,\mathrm{d}x_2\,\mathrm{d}x_1}{\sqrt{a^2_{\pi(1)}\,x_1^2+a^2_{\pi(2)}\,x_2^2+\cdots+a^2_{\pi(d)}\,x_d^2}}\,,$$

where the sum $\sum_{\pi_d}$ runs over all $d!$ permutations of the $d-$tuples $(1,2,...,d)$. This expansion is justified by the total symmetry of the left-hand integral in $a_1,...,a_d$ and the fact that the unit cube $C_d=[0,1]^d$ can be decomposed into $d!$ simplices $\{(x_1,...,x_d)\in C_d : x_{\pi(1)}\leq\cdots\leq x_{\pi(d)}\}$ each of them having the volume $(d!)^{-1}$.

Fixing a permutation $\pi_d=(\pi(1),...,\pi(d))$ and setting $b_i=a_{\pi(i)}$ for $i=1,2,...,d$ we can simplify the integrals in the following way:

$$\begin{aligned}
I^{(d)}(b_1,...,b_d) &= \int_0^1\int_0^{x_1}\cdots\int_0^{x_{d-2}}\int_0^{x_{d-1}} \frac{\mathrm{d}x_d\,\cdots\,\mathrm{d}x_2\,\mathrm{d}x_1}{\sqrt{b_1^2\,x_1^2+b_2^2\,x_2^2+\cdots+b_{d-1}^2\,x_{d-1}^2+b_d^2\,x_d^2}}\\
&= \int_0^1\int_0^1\cdots\int_0^1\int_0^1 \frac{\mathrm{d}y_d\,y_{d-1}\,\mathrm{d}y_{d-1}\,\cdots\,y_2^{d-2}\,\mathrm{d}y_2\,y_1^{d-1}\,\mathrm{d}y_1}{\sqrt{b_1^2\,y_1^2+b_2^2\,y_1^2\,y_2^2+\cdots+b_d^2\,y_1^2\,y_2^2\,\cdots\,y_d^2}}\\
&= \frac{1}{d-1}\int_0^1\cdots\int_0^1\int_0^1 \frac{\mathrm{d}y_d\,y_{d-1}\,\mathrm{d}y_{d-1}\,\cdots\,y_2^{d-2}\,\mathrm{d}y_2}{\sqrt{b_1^2+b_2^2\,y_2^2+\cdots+b_d^2\,y_2^2\,\cdots\,y_d^2}}\,.
\end{aligned}$$

In what follows we calculate all the above integrals for $d=3$ starting with

$$\begin{aligned}
I^{(3)}(b_1,b_2,b_3) &= \frac{1}{2}\int_0^1\int_0^1 \frac{\mathrm{d}y_3\,y_2\,\mathrm{d}y_2}{\sqrt{b_1^2+b_2^2\,y_2^2+b_3^2\,y_2^2\,y_3^2}} = \frac{1}{4}\int_0^1\int_0^1 \frac{\mathrm{d}y_3\,\mathrm{d}z}{\sqrt{b_1^2+b_2^2\,z+b_3^2\,z\,y_3^2}}\\
&= \frac{1}{4}\int_0^1 \frac{1}{b_2^2+b_3^2\,y^2}\int_0^{b_2^2+b_3^2\,y^2} \frac{\mathrm{d}z\,\mathrm{d}y}{\sqrt{b_1^2+z}} = \frac{1}{2}\int_0^1 \frac{(\sqrt{b_1^2+b_2^2+b_3^2\,y^2}-b_1)\,\mathrm{d}y}{b_2^2+b_3^2\,y^2}\\
&= \frac{1}{2}\int_0^1 \frac{\mathrm{d}y}{b_1+\sqrt{b_1^2+b_2^2+b_3^2\,y^2}}\,. \qquad (20)
\end{aligned}$$

Substituting $y = z\sqrt{b_1^2+b_2^2}/b_3$ and setting $0 < a = b_1 \le b = \sqrt{b_1^2+b_2^2}$, $c = b_3/b$ yield

$$I^{(3)}(b_1,b_2,b_3) = \frac{1}{2\,c}\int_0^c \frac{\mathrm{d}z}{a+b\sqrt{1+z^2}} = \frac{1}{2\,b\,c}\int_0^c \frac{\mathrm{d}z}{\sqrt{1+z^2}} - \frac{a}{2\,b\,c}\int_1^{\sqrt{1+c^2}} \frac{\mathrm{d}z}{(a+b\,z)\sqrt{z^2-1}}\,.$$

The first integral has been regarded in Sect. 2 and is equal to $F(c) = \log(c+\sqrt{c^2+1})$. In the second integral we substitute $a + b\,z = 1/t$ giving

$$\int_1^{\sqrt{1+c^2}} \frac{\mathrm{d}z}{(a+b\,z)\sqrt{z^2-1}} = \int_{(a+b\sqrt{1+c^2})^{-1}}^{(a+b)^{-1}} \frac{\mathrm{d}t}{\sqrt{1-2\,a\,t-(b^2-a^2)\,t^2}}$$

$$\text{Put}\quad t = \frac{b\,s-a}{b^2-a^2}\ : \qquad = \frac{\sqrt{b^2-a^2}}{b}\int_{(a+b\sqrt{1+c^2})^{-1}}^{(a+b)^{-1}} \frac{\mathrm{d}t}{\sqrt{1-b^{-2}\,(\,t\,(b^2-a^2)+a\,)^2}}$$

$$= \frac{1}{\sqrt{b^2-a^2}}\int_{\frac{b+a\sqrt{1+c^2}}{a+b\sqrt{1+c^2}}}^{1} \frac{\mathrm{d}s}{\sqrt{1-s^2}}$$

$$= \frac{1}{\sqrt{b^2-a^2}}\left(\frac{\pi}{2} - \arcsin\left(\frac{b+a\sqrt{1+c^2}}{a+b\sqrt{1+c^2}}\right)\right)$$

In summary we have

$$I^{(3)}(b_1,b_2,b_3) = \frac{1}{4\,b_3}\left[\,\log\left(\frac{A+b_3}{A-b_3}\right) + \frac{b_1}{b_2}\left(2\,\arcsin\left(\frac{A^2-b_3^2+A\,b_1}{(\,A+b_1\,)\,\sqrt{A^2-b_3^2}}\right) - \pi\right)\right],$$

where we have used the abbreviation $A = \sqrt{b_1^2+b_2^2+b_3^2} = \sqrt{a_1^2+a_2^2+a_3^2}$.

We next treat the other integrals. It is easily checked that $I^{(3)}_{1,2,3}(a_1,a_2,a_3)$ equals

$$\frac{1}{8\,a_1^2\,a_2^2\,a_3^2}\int_0^{a_1^2}\int_0^{a_2^2}\int_0^{a_3^2} \frac{\mathrm{d}y_3\,\mathrm{d}y_2\,\mathrm{d}y_1}{\sqrt{y_1+y_2+y_3}} = \int_0^{a_1^2}\int_0^{a_2^2} \frac{\left(\sqrt{y_1+y_2+a_3^2}-\sqrt{y_1+y_2}\right)}{4\,a_1^2\,a_2^2\,a_3^2}\,\mathrm{d}y_2\,\mathrm{d}y_1$$

$$= \frac{1}{6\,a_1^2\,a_2^2\,a_3^2}\int_0^{a_1^2}\left(\left(\sqrt{y_1+a_2^2+a_3^2}\right)^3 - \left(\sqrt{y_1+a_3^2}\right)^3 - \left(\sqrt{y_1+a_2^2}\right)^3 + y_1^{3/2}\right)\mathrm{d}y_1$$

leading to the formula

$$I^{(3)}_{1,2,3}(a_1,a_2,a_3) = \frac{A^5 - \left(\sqrt{A^2-a_1^2}\right)^5 - \left(\sqrt{A^2-a_2^2}\right)^5 - \left(\sqrt{A^2-a_3^2}\right)^5 + a_1^5 + a_2^5 + a_3^5}{15\,a_1^2\,a_2^2\,a_3^2}$$

Similarly, we see that

$$\begin{aligned}
I^{(3)}_{1,2}(a_1,a_2,a_3) &= \int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{x_1\,x_2\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1}{\sqrt{a_1^2\,x_1^2+a_2^2\,x_2^2+a_3^2\,x_3^2}} = \frac{1}{4\,a_1^2\,a_2^2\,a_3}\int_0^{a_3}\!\!\int_0^{a_2^2}\!\!\int_0^{a_1^2}\frac{\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3}{\sqrt{y_1+y_2+y_3^2}}\\
&= \int_0^{a_3}\frac{\big(\sqrt{a_1^2+a_2^2+y_3^2}\big)^3-\big(\sqrt{a_1^2+y_3^2}\big)^3-\big(\sqrt{a_2^2+y_3^2}\big)^3+y_3^3}{3\,a_1^2\,a_2^2\,a_3}\,\mathrm{d}y_3\,.
\end{aligned}$$

By using the integral function

$$\int_0^a(\sqrt{x^2+b^2})^3\mathrm{d}x = \frac{1}{4}\Big[\,\frac{5\,a}{2}\,\big(\sqrt{a^2+b^2}\big)^3-\frac{3\,a^3}{2}\sqrt{a^2+b^2}+\frac{3\,b^4}{4}\,\log\Big(\frac{\sqrt{a^2+b^2}+a}{\sqrt{a^2+b^2}-a}\Big)\Big]\,,$$

which is the result of two-fold partial integration, we obtain the formula

$$\begin{aligned}
I^{(3)}_{1,2}(a_1,a_2,a_3) &= \frac{1}{12\,a_1^2\,a_2^2\,a_3}\Big[\Big(\frac{5}{2}\,a_3\,A^3-\frac{3}{2}\,a_3^3\,A+\frac{3}{4}\,(A^2-a_3^2)^2\,\log\Big(\frac{A+a_3}{A-a_3}\Big)+a_3^4\Big)\\
&- \Big(\frac{5}{2}\,a_3\,\big(\sqrt{A^2-a_2^2}\,\big)^3-\frac{3}{2}\,a_3^3\,\sqrt{A^2-a_2^2}+\frac{3}{4}\,a_1^4\,\log\Big(\frac{\sqrt{A^2-a_2^2}+a_3}{\sqrt{A^2-a_2^2}-a_3}\Big)\Big)\\
&- \Big(\frac{5}{2}\,a_3\,\big(\sqrt{A^2-a_1^2}\,\big)^3-\frac{3}{2}\,a_3^3\,\sqrt{A^2-a_1^2}+\frac{3}{4}\,a_2^4\,\log\Big(\frac{\sqrt{A^2-a_1^2}+a_3}{\sqrt{A^2-a_1^2}-a_3}\Big)\Big)\Big]\,.
\end{aligned}$$

The integrals $I^{(3)}_{1,3}(a_1,a_2,a_3)$ and $I^{(3)}_{2,3}(a_1,a_2,a_3)$ follow from the latter formula by shifting the indices cyclically. Next, we calculate the remaining type of integrals

$$\begin{aligned}
I^{(3)}_{1}(a_1,a_2,a_3) &= \int_0^1\!\!\int_0^1\!\!\int_0^1 \frac{x_1\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1}{\sqrt{a_1^2\,x_1^2+a_2^2\,x_2^2+a_3^2\,x_3^2}} = \frac{1}{2\,a_1^2}\int_0^1\!\!\int_0^1\!\!\int_0^{a_1^2}\frac{\mathrm{d}y_1\,\mathrm{d}x_2\,\mathrm{d}x_3}{\sqrt{y_1+a_2^2\,x_2^2+a_3^2\,x_3^2}}\\
&= \frac{1}{a_1^2}\int_0^1\!\!\int_0^1\big(\sqrt{a_1^2+a_2^2\,x_2^2+a_3^2\,x_3^2}-\sqrt{a_2^2\,x_2^2+a_3^2\,x_3^2}\,\big)\,\mathrm{d}x_2\,\mathrm{d}x_3\,,
\end{aligned}$$

where the first of the two double integrals can be written as

$$\begin{aligned}
&\int_0^1\!\!\int_0^{x_3}\sqrt{a_1^2+a_2^2\,x_2^2+a_3^2\,x_3^2}\,\mathrm{d}x_2\,\mathrm{d}x_3+\int_0^1\!\!\int_0^{x_2}\sqrt{a_1^2+a_2^2\,x_2^2+a_3^2\,x_3^2}\,\mathrm{d}x_3\,\mathrm{d}x_2\\
&= \int_0^1\!\!\int_0^1\sqrt{a_1^2+a_2^2\,x_3^2\,y_2^2+a_3^2\,x_3^2}\;x_3\;\mathrm{d}y_2\,\mathrm{d}x_3+\int_0^1\!\!\int_0^1\sqrt{a_1^2+a_2^2\,x_2^2+a_3^2\,x_2^2\;y_3^2}\;x_2\;\mathrm{d}y_3\,\mathrm{d}x_2\\
&= \frac{1}{2}\int_0^1\!\!\int_0^1\sqrt{a_1^2+(a_2^2\,y_2^2+a_3^2)\,y_3}\;\mathrm{d}y_3\,\mathrm{d}y_2+\frac{1}{2}\int_0^1\!\!\int_0^1\sqrt{a_1^2+(a_2^2+a_3^2\,y_3^2)\,y_2}\;\mathrm{d}y_2\,\mathrm{d}y_3\\
&= \frac{1}{3}\int_0^1\frac{\big(\sqrt{a_1^2+a_2^2\,y_2^2+a_3^2}\big)^3-a_1^3}{a_2^2\,y_2^2+a_3^2}\,\mathrm{d}y_2+\frac{1}{3}\int_0^1\frac{\big(\sqrt{a_1^2+a_2^2+a_3^2\,y_3^2}\big)^3-a_1^3}{a_2^2+a_3^2\,y_3^2}\,\mathrm{d}y_3\,.
\end{aligned}$$

Making use of the identity

$$\frac{v^3 - u^3}{v^2 - u^2} = \frac{v^2 + u\,v + u^2}{u + v} = v + \frac{u^2}{u + v}$$

with $u = a_1$, $v = \sqrt{a_1^2 + a_2^2\,y_2^2 + a_3^2}$ resp. $v = \sqrt{a_1^2 + a_2^2 + a_3^2\,y_3^2}$ and using (20) we can show that

$$\begin{aligned}\int_0^1\!\!\int_0^1 \sqrt{a_1^2 + a_2^2\,x_2^2 + a_3^2\,x_3^2}\,\mathrm{d}x_2\,\mathrm{d}x_3 \;&=\; \frac{1}{3}\int_0^1 \sqrt{a_1^2 + a_2^2\,y_2^2 + a_3^2}\,\mathrm{d}y_2 + \frac{1}{3}\int_0^1 \sqrt{a_1^2 + a_2^2 + a_3^2\,y_3^2}\,\mathrm{d}y_3 \\ &+\; \frac{2\,a_1^2}{3}\left(I^{(3)}(a_1, a_3, a_2) + I^{(3)}(a_1, a_2, a_3) \right).\end{aligned}$$

The latter relation combined with

$$\int_0^1 \sqrt{a^2\,x^2 + b^2}\,\mathrm{d}x = \frac{\sqrt{a^2 + b^2}}{2} + \frac{b^2}{2\,a}\,F\big(\frac{a}{b}\big)$$

leads to

$$I_1^{(3)}(a_1, a_2, a_3) = \frac{2}{3}\left(I^{(3)}(a_1, a_3, a_2) + I^{(3)}(a_1, a_2, a_3) \right) + \widehat{I}_1^{(3)}(a_1, a_2, a_3)\,,$$

where the term $\widehat{I}_1^{(3)}(a_1, a_2, a_3)$ takes on the form

$$\begin{aligned}\widehat{I}_1^{(3)}(a_1, a_2, a_3) \;&=\; \frac{1}{3\,a_1^2}\Big[A + \frac{A^2 - a_2^2}{2\,a_2}\,F\big(\frac{a_2}{\sqrt{A^2 - a_2^2}}\big) + \frac{A^2 - a_3^2}{2\,a_3}\,F\big(\frac{a_3}{\sqrt{A^2 - a_3^2}}\big) \\ &\quad -\sqrt{A^2 - a_1^2} - \frac{a_3^2}{2\,a_2}\,F\big(\frac{a_2}{a_3}\big) - \frac{a_2^2}{2\,a_3}\,F\big(\frac{a_3}{a_2}\big) \Big].\end{aligned}$$

The corresponding integrals $I_2^{(3)}(a_1, a_2, a_3)$ and $I_3^{(3)}(a_1, a_2, a_3)$ can be easily obtained by cyclic shifting of the indices at a_1, a_2, a_3 .

Finally, after collecting the above integrals we obtain the third-order CPI of $\times_{i=1}^{3}[0, a_i]$:

$$\begin{aligned}I_3(a_1, a_2, a_3) = \frac{1}{3}\sum_{\pi_3} I^{(3)}(a_{\pi(1)}, a_{\pi(2)}, a_{\pi(3)}) - \widehat{I}_1^{(3)}(a_1, a_2, a_3) - \widehat{I}_2^{(3)}(a_1, a_2, a_3) - \widehat{I}_3^{(3)}(a_1, a_2, a_3) \\ + I_{1,2}^{(3)}(a_1, a_2, a_3) + I_{1,3}^{(3)}(a_1, a_2, a_3) + I_{2,3}^{(3)}(a_1, a_2, a_3) - I_{1,2,3}^{(3)}(a_1, a_2, a_3)\end{aligned}$$

Hence, in the particular case of the unit cube in $\mathbb{R}^3$, we have

$$I_3(1,1,1) \;=\; 2\,I^{(3)}(1,1,1) \;-\; 3\,\widehat{I}_1^{(3)}(1,1,1) \;+\; 3\,I_{1,2}^{(3)}(1,1,1) \;-\; I_{1,2,3}^{(3)}(1,1,1) \qquad (21)$$

with explicit values on the right-hand side given by

$$
\begin{aligned}
I^{(3)}(1,1,1) &= \frac{1}{4}\Big(\log\big(2+\sqrt{3}\big) + 2\,\arcsin\big(\frac{1+\sqrt{3}}{2\sqrt{2}}\big) - \pi\Big) \approx 0.19833978 \\
\widehat{I}_1^{(3)}(1,1,1) &= \frac{1}{3}\Big(\sqrt{3}-\sqrt{2}+\log\big(2+\sqrt{3}\big) - \log\big(1+\sqrt{2}\big)\Big) \approx 0.251140518 \\
I_{1,2}^{(3)}(1,1,1) &= \frac{1}{12}\Big(6\sqrt{3}-7\sqrt{2}+1+3\,\log\big(2+\sqrt{3}\big) - 3\,\log\big(1+\sqrt{2}\big)\Big) \approx 0.233296903 \\
I_{1,2,3}^{(3)}(1,1,1) &= \frac{1}{5}\Big(3\sqrt{3}-4\sqrt{2}+1\Big) \approx 0.107859634\,.
\end{aligned}
$$

For the sake of completeness, we give once more the connections between these expression and the above-defined integrals :

$$
\begin{aligned}
\int_0^1\int_0^1\int_0^1 \frac{\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1}{\sqrt{x_1^2+x_2^2+x_3^2}} &= 6\,I^{(3)}(1,1,1) \\
&= \frac{3}{2}\Big(\log\big(2+\sqrt{3}\big) + 2\,\arcsin\big(\frac{1+\sqrt{3}}{2\sqrt{2}}\big) - \pi\Big) \approx 1.19003868 \\
\int_0^1\int_0^1\int_0^1 \frac{x_1\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1}{\sqrt{x_1^2+x_2^2+x_3^2}} &= I_1^{(3)}(1,1,1) = \frac{4}{3}\,I^{(3)}(1,1,1) + \widehat{I}_1^{(3)}(1,1,1) = \frac{1}{3}\Big(2\,\arcsin\big(\frac{1+\sqrt{3}}{2\sqrt{2}}\big) \\
&- \pi + \sqrt{3}-\sqrt{2}+2\,\log\big(2+\sqrt{3}\big) - \log\big(1+\sqrt{2}\big)\Big) \approx 0.515593558
\end{aligned}
$$

and

$$
I_{1,2}^{(3)}(1,1,1) = \int_0^1\int_0^1\int_0^1 \frac{x_1\,x_2\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1}{\sqrt{x_1^2+x_2^2+x_3^2}} \quad , \quad I_{1,2,3}^{(3)}(1,1,1) = \int_0^1\int_0^1\int_0^1 \frac{x_1\,x_2\,x_3\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}\,.
$$

According to the relation (21) we arrive at our final formula

$$
\begin{aligned}
I_3(1,1,1) &= \int_0^1\int_0^1\int_0^1 \frac{(1-x_1)\,(1-x_2)\,(1-x_3)}{\sqrt{x_1^2+x_2^2+x_3^2}}\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&= \arcsin\big(\frac{1+\sqrt{3}}{2\sqrt{2}}\big) - \frac{\pi}{2} + \frac{\log\big((2+\sqrt{3})(1+\sqrt{2})\big)}{4} + \frac{1+\sqrt{2}-2\sqrt{3}}{20} \\
&\approx 0.235289081
\end{aligned}
$$

and for the third-order CPI of a cube $[0,a]^3$ with edge-length $a > 0$ it follows together with (2) and (18) that

$$
I_2([0,a]^3) = \int_{[0,a]^3}\int_{[0,a]^3} \frac{\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|} = 8\,a^5\,I_3(1,1,1) \approx 1.88231265\,a^5\,.
$$