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Central Limit Theorem for the Volume of Stationary Poisson Cylinder Processes in Expanding Domains

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A stationary Poisson cylinder process in the d-dimensional Euclidean space is composed by a stationary Poisson process of k-flats ($1 \le k \le d-1$) which are dilated by i.i.d. random compact cylinder bases taken from the corresponding $(d-k)$-dimensional orthogonal complement. If the second moment of the $(d-k)$-volume of the typical cylinder base exists, we prove asymptotic normality of the d-volume of the union set of Poisson cylinders that covers an expanding star-shaped domain ϱW as ϱ grows unboundedly. Due to the long-range dependences within the union set of cylinders, the variance of its d-volume in ϱW increases asymptotically proportional to the $(d+k)$th power of ϱ. To obtain the exact asymptotic behaviour of this variance we need a distinction between discrete and continuous directional distributions of the typical k-flat.

keywords: Independently marked Poisson process, truncated typical cylinder, direction space, volume fraction, moment convergence theorem, long-range dependence, asymptotic variance, higher-order (mixed) cumulants

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1 Introduction and Preliminaries

In integral and stochastic geometry, a (poly-) convex *cylinder* in the d-dimensional Euclidean space $\mathbb{R}^d$ is an unbounded set of the form $L \oplus B$ with *direction space* $L \in \mathbb{G}(d,k)$ (= the Grassmannian of k-dimensional subspaces of $\mathbb{R}^d$), $k = 1, \dots, d-1$, and a (poly-) convex, compact subset B of the orthogonal complement $L^\perp$ called *base* of the cylinder, see e.g. [14],[11],[17] for details. The general notion of a stationary point process of poly-convex cylinders (briefly *cylinder process* subsequently abbreviated by CP) has been first considered in [17]. Throughout this paper the orientation of the direction space L is suppressed and the restriction of poly-convexity of B will be dropped. In order to find explicit formulae for numerical characteristics of union sets of CP's such as volume fraction, covariance etc. one needs specific distributional assumptions determining shape, direction and position of the random cylinders. For the description of various real-life random set structures, it is quite natural to assume that the sizes and

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the spatial positions of cylinders are governed by an independently marked Poisson process. Following the concept of Poisson processes defined on the space of cylinders with bases in the convex ring, *Poisson cylinder processes* (briefly PCP's) have been studied in [15] with applications in modelling materials consisting of long thick fibres or thick membranes.

To be precise in describing our problem, we first introduce some notation and give a rigorous definition of a stationary PCP (which slightly differs from that in [15]). For this, let $\{e_1,\dots,e_d\}$ denote the usual orthonormal basis of $\mathbb{R}^d$ defining the orthogonal subspaces $\mathbb{E}_k = span\{e_{d-k+1},\dots,e_d\}$ and $\mathbb{E}_k^{\perp} = span\{e_1,\dots,e_{d-k}\}$, respectively, where $k \in \{1,\dots,d-1\}$ is fixed in what follows. It is well-known that for any given $L \in \mathbb{G}(d,k)$ there exists an equivalence class $\mathbf{O}_L \in \mathbb{SO}_d/\mathbb{S}(\mathbb{O}_{d-k}\times\mathbb{O}_k)$ of orthogonal matrices $O \in \mathbb{R}^{d\times d}$ with $\det(O)=1$ such that $O\,\mathbb{E}_k = L$. In other words two matrices $O,\widehat{O} \in \mathbb{SO}_d$ belong to $\mathbf{O}_L$ iff $O\,\mathbb{E}_k = \widehat{O}\,\mathbb{E}_k = L$ and the product $O^T\,\widehat{O}$ belongs to the set of orthogonal block matrices $\mathbb{S}(\mathbb{O}_{d-k}\times\mathbb{O}_k)$ defined by

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathbb{R}^{(d-k)\times(d-k)}, B \in \mathbb{R}^{k\times k}, A^T = A^{-1}, B^T = B^{-1}, \det(A) = \det(B) \right\}.$$

We identify each class $\mathbf{O}_L$ with a single representative $O_L \in \mathbf{O}_L$ which can be chosen in a canonical (unique) way, e.g. as lexicographically smallest element of the (compact) set $\mathbf{O}_L$. On the other hand, due to the fact from differential geometry that $\dim \mathbb{G}(d,k) = (d-k)\,k$, there always exists a parametric representation of the matrices O_L over some subset of $\mathbb{R}^{(d-k)k}$. In the special cases $d=2, k=1$ and $d=3, k=1$ suitable parameterizations are

$$O_L(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad O_L(\theta_1,\theta_2) = \begin{pmatrix} \sin\theta_1 & \cos\theta_1\,\cos\theta_2 & \cos\theta_1\,\sin\theta_2 \\ -\cos\theta_1 & \sin\theta_1\,\cos\theta_2 & \sin\theta_1\,\sin\theta_2 \\ 0 & -\sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

for $\theta \in [0,\pi)$ resp. $(\theta_1,\theta_2) \in [0,2\pi)\times[0,\frac{\pi}{2})$. In the dual case $d=3, k=2$, the first column of $O_L(\theta_1,\theta_2)$ must be multiplied by -1 and then interchanged with the third column.

Once chosen such a canonical one-to-one correspondence between $L \in \mathbb{G}(d,k)$ and $O_L \in \mathbb{SO}_d$ such that $L = O_L\,E_k$ we denote by $\mathbb{SO}_k^d$ the family of all O_L. In this way, to each random subspace $L \in \mathbb{G}(d,k)$ corresponds a (unique) random matrix $\Theta(L) \in \mathbb{SO}_k^d$. It should be mentioned that instead of $\Theta(L)$ also $\Theta_S(L) = \Theta(L)\,S$ for any fixed $S \in \mathbb{S}(\mathbb{O}_{d-k}\times\mathbb{O}_k)$ can be taken.

Throughout in this paper all random elements are defined on a common probability space $[\Omega,\mathcal{F},\mathbf{P}]$ and $\mathbf{E}$ (resp. $\mathbf{Var}$) denotes expectation (resp. variance) w.r.t. $\mathbf{P}$. In particular, let (Θ_0,Ξ_0) be a measurable mapping from $[\Omega,\mathcal{F},\mathbf{P}]$ into the *mark space* $\mathbb{M}_{d,k} = \mathbb{SO}_k^d \times \mathcal{K}_{d-k}$, where $\mathcal{K}_{d-k}$ denotes the space of compact subsets of $\mathbb{R}^{d-k}$ equipped with the Hausdorff metric. The image measure $Q_{d,k} := \mathbf{P}\circ(\Theta_0,\Xi_0)^{-1}$ acting on the corresponding Borel product σ-field $\mathcal{B}(\mathbb{M}_{d,k})$ determines the joint distribution of the (not necessarily independent) random elements Θ_0 and Ξ_0.

Now we are in a position to introduce a stationary independently marked Poisson process $\Pi_{\lambda,Q_{d,k}} = \sum_{i\geq 1} \delta_{[P_i,(\Theta_i,\Xi_i)]}$ with intensity λ and mark distribution $Q_{d,k}(\cdot)$, i.e. $\Pi_{\lambda,Q_{d,k}}(\cdot)$ is a random locally finite counting measure (shift-invariant in the first component) on the Borel subsets of $\mathbb{R}^{d-k} \times \mathbb{M}_{d,k}$ such that the numbers $\Pi_{\lambda,Q_{d,k}}(B \times M)$ are Poisson distributed with mean $\lambda\,|B|_{d-k}\, Q_{d,k}(M)$ for any bounded $B \in \mathcal{B}(\mathbb{R}^{d-k})$ (with Lebesgue measure $|\cdot|_{d-k}$) and $M \in \mathcal{B}(\mathbb{M}_{d,k})$, see [1] for a standard reference on general (Poisson) point processes. This definition implies that the numbers of atoms of the unmarked Poisson process $\Pi_\lambda = \sum_{i\geq 1} \delta_{P_i}$ located in disjoint subsets of $\mathbb{R}^{d-k}$ are independent and the marks (Θ_i, Ξ_i) associated with the atoms P_i are i.i.d. copies of $(\Theta_0, \Xi_0) \sim Q_{d,k}$ and independent of Π_λ.

Furthermore, we need two important formulae for $\Pi_{\lambda,Q_{d,k}}$, where each of them characterises the distribution of $\Pi_{\lambda,Q_{d,k}}$: The *probability generating functional* of $\Pi_{\lambda,Q_{d,k}}$ takes the form

$$\mathbf{E}\Big[\prod_{i\geq 1} v(P_i,\Theta_i,\Xi_i)\Big] = \exp\Big\{ -\lambda \int\limits_{\mathbb{R}^{d-k}} \int\limits_{\mathbb{M}_{d,k}} \Big(1 - v(x,\theta,\xi)\Big)\, Q_{d,k}(\mathrm{d}(\theta,\xi))\,\mathrm{d}x \Big\} \tag{1}$$

for any measurable function $v : \mathbb{R}^{d-k} \times \mathbb{M}_{d,k} \mapsto [0,1]$ such that $1 - v(\,\cdot\,,\theta,\xi)$ has bounded support for $(\theta,\xi) \in \mathbb{M}_{d,k}$, whereas the nth-order *Campbell formula* reads for any $n \in \mathbb{N}$ as follows:

$$\mathbf{E}\Big(\sum_{i_1,\ldots,i_n \geq 1}^{*} \prod_{j=1}^{n} f_j(P_{i_j},\Theta_{i_j},\Xi_{i_j}) \Big) = \lambda^n \prod_{j=1}^{n} \int\limits_{\mathbb{R}^{d-k}} \int\limits_{\mathbb{M}_{d,k}} f_j(x,\theta,\xi)\, Q_{d,k}(\mathrm{d}(\theta,\xi))\,\mathrm{d}x \tag{2}$$

for non-negative measurable functions $f_1,\ldots,f_n : \mathbb{R}^{d-k} \times \mathbb{M}_{d,k} \mapsto \mathbb{R}^1$, where the sum $\sum^*$ on the left-hand side of (2) runs over all n-tuples of pairwise distinct indices $i_1,\ldots,i_n \geq 1$, see [1] or [16], [14].

Definition. *For the independently marked Poisson process $\Pi_{\lambda,Q_{d,k}} = \sum_{i\geq 1} \delta_{[P_i,(\Theta_i,\Xi_i)]}$ satisfying the above assumptions, by a stationary* PCP *we understand a countable family of cylinders*

$$\{\, \Theta_i(\, (\Xi_i' + P_i') \oplus E_k\,)\,,\, i \geq 1 \,\} = \{\, \Theta_i(\, (\Xi_i + P_i) \times \mathbb{R}^k\,)\,,\, i \geq 1 \,\} \tag{3}$$

with $\Xi_i' + P_i' = \{(x + P_i, \mathbf{o}_k)^T : x \in \Xi_i\} \subset E_k^{\perp}$ for $i \geq 1$, where $\oplus$ denotes the Minkowski addition in $\mathbb{R}^d$ and $\mathbf{o}_k$ is the null vector in $\mathbb{R}^k$.

In this paper we are interested in the d-volume measure $\big|\,\Xi_{\lambda,Q_{d,k}} \cap (\cdot)\,\big|_d$ of the stationary random set

$$\Xi_{\lambda,Q_{d,k}} = \bigcup_{i\geq 1} \Theta_i\big(\, (\Xi_i + P_i) \times \mathbb{R}^k\,\big) \tag{4}$$

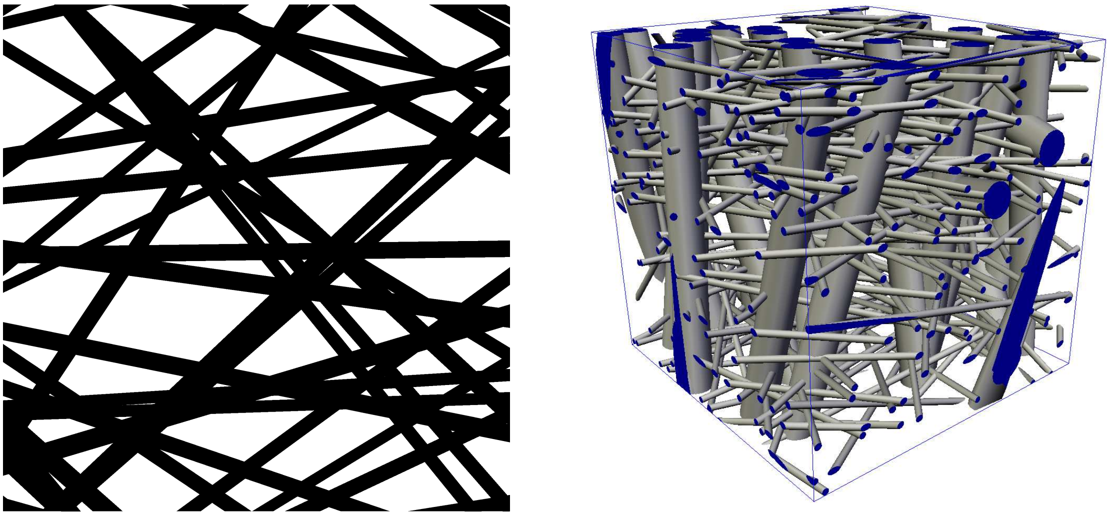

Figure 1: Planar isotropic and spatial anisotropic PCP with one-dimensional direction space

derived from (3) and, in particular, in the asymptotic behaviour of $\left|\,\Xi_{\lambda,Q_{d,k}} \cap \varrho W\,\right|_d$ as $\varrho \to \infty$, where the set $W \in \mathcal{K}_d$ is chosen star-shaped (w.r.t. the origin $\mathbf{o}_d$) such that $B_d(\delta_W) \subseteq W \subseteq B_d(1)$ for some $\delta_W > 0$ and the $(d-k)$-volume $|\Xi_0|_{d-k}$ of the typical cylinder base possesses a second moment. Here $B_d(r)$ is a closed ball in $\mathbb{R}^d$ with radius $r \geq 0$ centred at the origin.

Remark 1. *In the degenerate case $k = 0$ (where $E_0 = \{\mathbf{o}_d\}$ and $\Theta_0 =$ unit matrix) the union set (4) coincides with well-studied Boolean (or Poisson grain, Poisson blob, Swiss cheese) model in $\mathbb{R}^d$ with typical grain Ξ_0, see e.g. [2],[16] for more information.*

Remark 2. *Provided that $\mathbf{E}|\Xi_0|_{d-k} < \infty$ the random union set (4) is ($\mathbf{P}$-a.s.) closed iff $\mathbf{E}|\Xi_0 \oplus B_{d-k}(\varepsilon)|_{d-k} < \infty$ for some $\varepsilon > 0$, see Lemma 4 in [8]. In this case we may derive from (1) with suitable v the hitting functional of $\Xi_{\lambda,Q_{d,k}}$*

$$\mathbf{P}(\Xi_{\lambda,Q_{d,k}} \cap C \neq \emptyset) = 1 - \exp\Big\{ - \lambda\,\mathbf{E}\big|\,\Xi_0 \oplus (-\,\pi_{d-k}(\Theta_0^T C))\,\big|_{d-k} \Big\}$$

for any $C \in \mathcal{K}_d$, see [11], [15], [8]. Here $\pi_{d-k}(y)$ denotes the projection of the vector $y \in \mathbb{R}^d$ on its first $d-k$ components. Note that even $\mathbf{P}(|\Xi_0|_{d-k} \leq 1) = 1$ does in general not imply the ($\mathbf{P}$-a.s.) closedness of (4), see also [8] for a counter-example. Realisations of two sets (4) for $d = 2, k = 1$ resp. $d = 3, k = 1$, are shown in Fig. 1.

In the next section we state the announced central limit theorem (briefly CLT) for the d-volume $\left|\,\Xi_{\lambda,Q_{d,k}} \cap \varrho W\,\right|_d$ and give the exact asymptotic behaviour of its variance as $\varrho \to \infty$.

2 Main Results

For notational ease we will mostly use the abbreviation Ξ instead of $\Xi_{\lambda,Q_{d,k}}$. We first recall the fact that the probability space $[\Omega,\mathcal{F},\mathbf{P}]$ on which the marked Poisson process $\Pi_{\lambda,Q_{d,k}}=\sum_{i\ge 1}\delta_{[P_i,(\Theta_i,\Xi_i)]}$ is defined can be chosen in such a way that the mapping $\mathbb{R}^d\times\Omega\ni(x,\omega)\mapsto \mathbf{1}_{\Xi(\omega)}(x)\in\{0,1\}$ is measurable w.r.t. the product-σ-field $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{F}$, see Appendix in [5]. This enables us to apply Fubini's theorem to the stationary random field of indicator variables $\mathbf{1}_{\Xi}(x)$, $x\in\mathbb{R}^d$, and implies among others that its nth-order mixed moments (also called *n-point probabilities* of Ξ)

$$p_{\Xi}(x_1,\dots,x_n):=\mathbf{E}\Big(\prod_{i=1}^{n}\mathbf{1}_{\Xi}(x_i)\Big)=\mathbf{P}(x_1\in\Xi,\dots,x_n\in\Xi)$$

are $\mathcal{B}(\mathbb{R}^{dn})$-measurable functions of $(x_1,\dots,x_n)$ for any $n\in\mathbb{N}$ and the void probabilities $p_{\Xi^c}(x_1,\dots,x_n)=\mathbf{P}(\Xi\cap\{x_1,\dots,x_n\}=\emptyset)$ take on the explicit form

$$p_{\Xi^c}(x_1,\dots,x_n)=\mathbf{E}\Big(\prod_{i=1}^{n}(1-1_{\Xi}(x_i))\Big)=\exp\Big\{-\lambda\,\mathbf{E}\Big|\bigcup_{i=1}^{n}(\,\Xi_0-\pi_{d-k}(\Theta_0^T x_i)\,)\,\Big|_{d-k}\Big\}\,, \tag{5}$$

which follows from (1) with $v(\,\cdot\,,\theta,\xi)=1$ if $\theta\,(\,(\xi+(\cdot))\times\mathbb{R}^k\,)\cap\{x_1,\dots,x_n\}=\emptyset$, and $v(\,\cdot\,,\theta,\xi)=0$ otherwise. Since the random fields $\mathbf{1}_{\Xi}(\cdot)$ and $1-\mathbf{1}_{\Xi}(\cdot)$ have the same covariance function, it follows from (5) for $n=1,2$ together with the shift-invariance and additivity of the Lebesgue measure $|\cdot|_{d-k}$ that, for any $x_1,x_2\in\mathbb{R}^d$,

$$\mathbf{Cov}\,(\mathbf{1}_{\Xi}(x_1),\mathbf{1}_{\Xi}(x_2))=\exp\Big\{-\lambda\,\mathbf{E}|\,\Xi_0\cup(\Xi_0-\pi_{d-k}(\Theta_0^T\,(x_2-x_1))\,\Big|_{d-k}\Big\}-e^{-2\,\lambda\,M_1}$$

$$=e^{-2\,\lambda\,M_1}\Big(\exp\Big\{\lambda\,\mathbf{E}|\Xi_0\cap(\Xi_0-\pi_{d-k}(\Theta_0^T\,(x_2-x_1))\Big|_{d-k}\Big\}-1\Big).$$

Here and below, let $M_s=\mathbf{E}|\,\Xi_0\,|_{d-k}^{s}$ denote the moment of order $s>0$ of the $(d-k)$-volume of Ξ_0. By multiple use of Fubini's theorem we get for any bounded $B\in\mathcal{B}(\mathbb{R}^d)$ that

$$\mathbf{Var}(|\Xi\cap B|_d)=\int_B\int_B \mathbf{Cov}\,(\mathbf{1}_{\Xi}(x_1),\mathbf{1}_{\Xi}(x_2))\;\mathrm{d}\,x_1\;\mathrm{d}\,x_2 \tag{6}$$

$$=e^{-2\,\lambda\,M_1}\int_{\mathbb{R}^d}|B\cap(B-x)\big|_d\Big(\,\exp\Big\{\lambda\,\mathbf{E}|\,\Xi_0\cap(\Xi_0-\pi_{d-k}(\Theta_0^T\,x)\,\Big|_{d-k}\Big\}-1\Big)\;\mathrm{d}\,x\,.$$

We are now in a position to formulate our main results.

Theorem 1. *Let* $\Xi_{\lambda,Q_{d,k}}$ *be the union set* (4) *of a stationary* PCP $\Pi_{\lambda,Q_{d,k}}$ *with compact typical cylinder base* $\Xi_0 \subset \mathbb{R}^{d-k}$ *satisfying* $0 < M_2 < \infty$. *Further, let* $W \subset \mathbb{R}^d$ *be compact and star-shaped w.r.t.* $\mathbf{o}_d$ *satisfying* $B_d(\delta_W) \subseteq W \subseteq B_d(1)$ *for some* $\delta_W \in (0,1]$. *Then*

$$\frac{|\,\Xi_{\lambda,Q_{d,k}} \cap \varrho W\,|_d - \varrho^d\,|W|_d\,(\,1 - e^{-\lambda M_1}\,)}{\sqrt{\mathbf{Var}(\,|\,\Xi_{\lambda,Q_{d,k}} \cap \varrho W\,|_d\,)}} \quad \xrightarrow[\varrho\to\infty]{\mathcal{D}} \quad \mathcal{N}(0,1)\,. \tag{7}$$

Note that $\mathbf{P}(\mathbf{o}_d \in \Xi_{\lambda,Q_{d,k}}) = \mathbf{E}|\,\Xi_{\lambda,Q_{d,k}} \cap [0,1]^d\,|_d = 1 - \exp\{-\lambda M_1\}$ is just the *volume fraction* of the stationary random set (4) which coincides with intensity of the random volume measure $|\,\Xi \cap (\cdot)\,|_d$. Lemma 1 in [8] shows that the variance of $|\,\Xi \cap \varrho W\,|_d$ increases to infinity proportional to the $(d+k)$th power of ϱ (in the sense of Hardy-Littlewood), see below relation (17). More precisely, there exist positive constants c_1, c_2 not depending on $\varrho \geq 1$ such that

$$c_1\,\varrho^{d+k} \quad \leq \quad \mathbf{Var}(\,|\,\Xi_{\lambda,Q_{d,k}} \cap \varrho W\,|_d\,) \quad \leq \quad c_2\,\varrho^{d+k} \quad \text{for all} \quad \varrho \geq 1\,. \tag{8}$$

As our second main result the following Theorem 2 provides the exact asymptotic growth rate of the variances of the d-volume $|\,\Xi \cap \varrho W\,|_d$ in dependence of k, d and W in the cases of purely atomic and diffuse directional distribution $\mathbf{P}_0(\cdot) = Q_{d,k}((\cdot) \times \mathcal{K}_{d-k})$. By the (unique) decomposition of $\mathbf{P}_0(\cdot)$ into an atomic and diffuse part and combining both of the below relations (10) and (11) we are able to guarantee the existence and positivity of the asymptotic variance

$$\sigma^2_{\lambda,Q_{d,k}}(W) = \lim_{\varrho\to\infty} \frac{\mathbf{Var}(\,|\,\Xi_{\lambda,Q_{d,k}} \cap \varrho W\,|_d\,)}{\varrho^{d+k}} \tag{9}$$

for any distribution $Q_{d,k}$ (of (Θ_0, Ξ_0)) on $\mathcal{B}(\mathbb{M}_{d,k})$ such that $0 < M_2 < \infty$.

Theorem 2. *Let the assumptions of Theorem 1 be satisfied. If the marginal distribution* $\mathbf{P}_0(\cdot)$ *is discrete, i.e. it is concentrated on* $\{\theta_i \in \mathbb{SO}^d_k\,,\, i \in I\}$ *for some at most countable index set* I , *then*

$$\sigma^2_{\lambda,Q_{d,k}}(W) = e^{-2\lambda M_1} \sum_{i\in I} \int_{\mathbb{R}^k} \big|W \cap (W - \theta_i(\mathbf{o}_{d-k}, x)^T)\big|_d \,\mathrm{d}x \int_{\mathbb{R}^{d-k}} \Big(e^{\lambda f(y,\theta_i)} - 1\Big)\,\mathrm{d}y\,, \tag{10}$$

where $f(y,\theta_i) = \mathbf{E}(|\Xi_0 \cap (\Xi_0 - y)|_{d-k}\,\mathbf{1}\{\Theta_0 = \theta_i\}) = \mathbf{E}(|\Xi_0 \cap (\Xi_0 - y)|_{d-k}\,|\,\Theta_0 = \theta_i)\;\mathbf{P}_0(\{\theta_i\})$ *for* $i \in I$.

On the other hand, if $\mathbf{P}_0(\cdot)$ *is diffuse, i.e.* $\mathbf{P}_0(\{\theta\}) = 0$ *for all* $\theta \in \mathbb{SO}^d_k$, *we have*

$$\sigma^2_{\lambda,Q_{d,k}}(W) = \lambda\,e^{-2\lambda M_1} \int_{\mathbb{SO}^d_k} M_2(\theta) \int_{\mathbb{R}^k} \big|W \cap (W - \theta(\mathbf{o}_{d-k}, x)^T)\big|_d \,\mathrm{d}x\;\mathbf{P}_0(\mathrm{d}\theta)\,, \tag{11}$$

where $M_2(\theta) = \mathbf{E}(\,|\Xi_0|^2_{d-k}\,|\,\Theta_0 = \theta\,)$ *for* $\theta \in \mathbb{SO}^d_k$.

We mention further that the above theorems can be extended to analogous results for estimators of the *covariance* $C_{\Xi^c}(x)$ of the random set Ξ^c defined by the two-point probability $p_{\Xi^c}(\mathbf{o}_d, x)$ for any $x \in \mathbb{R}^d$, see e.g. [2], [16] and [15]. This is seen from the obvious relation $C_{\Xi^c}(x) = 1 - \mathbf{P}(\mathbf{o}_d \in \Xi \cup (\Xi - x))$ and the fact that the union $\Xi \cup (\Xi - x)$ takes the form (4) with typical base $\Xi_0 \cup (\Xi_0 - \pi_{d-k}(\Theta_0^T x))$. In the same way one can treat the corresponding estimator for the n-point probability $p_{\Xi^c}(\mathbf{o}_d, x_1, \ldots, x_{n-1})$.

The rest of this paper is organised as follows: In the next Section 3, we prove the CLT by using a truncation technique which allows to approximate the union set (4) by a union set $\Xi^{(\tau)}$ of cylinders with truncated cylinder bases. In this way we make use of the estimates of the nth-order cumulants of $|\, \Xi \cap \varrho W \,|_d$ derived in [8] if $M_j < \infty$ for $j = 1, \ldots, n$. In Section 4, we prove the formulas (10) and (11) for the asymptotic variances of the volume $|\, \Xi \cap \varrho W \,|_d$. Furthermore, we show that the limit (9) always exists. In Section 5 we discuss the formulae (10) and (11) for $W = B_d(1)$ and the isotropic case in (11).

3 Central limit theorem for a truncated Poisson cylinder process

We now introduce a truncated version $\Xi^{(\tau)}$ of the PCP (4)

$$\Xi^{(\tau)} = \bigcup_{i \geq 1} \Theta_i \Big((\, \Xi_i^{(\tau)} + P_i \,) \times \mathbb{R}^k \Big) ,$$

where the second component of the typical mark (Θ_0, Ξ_0) in (4) is replaced by the truncated typical grain

$$\Xi_0^{(\tau)} = \begin{cases} \Xi_0 & , \text{ if } \; |\Xi_0|_{d-k} \leq \tau \,, \\ \emptyset & , \text{ if } \; |\Xi_0|_{d-k} > \tau \,. \end{cases} \qquad \text{with} \qquad \tau = \varepsilon\, \varrho^{(d-k)/2} \tag{12}$$

for arbitrarily small $\varepsilon > 0$ and large enough $\varrho > 0$ such that $\tau \geq 1$ just for convenience.

Obviously, by (3) and (4), we have $\Xi^{(\tau)} \subseteq \Xi$ as well as the inclusion

$$\Xi \setminus \Xi^{(\tau)} \subseteq \bigcup_{i \geq 1} \Theta_i \Big[(\, \Xi_i \setminus \Xi_i^{(\tau)} + P_i \,) \times \mathbb{R}^k \Big] =: \widetilde{\Xi}^{(\tau)} ,$$

where $\widetilde{\Xi}^{(\tau)}$ can be regarded as a PCP with typical mark $(\Xi_0 \setminus \Xi_0^{(\tau)}, \Theta_0)$. The latter relation yields

$$\mathbf{E}|\, (\Xi \setminus \Xi^{(\tau)}) \cap \varrho W \,|_d^2 \leq \mathbf{E}|\, \widetilde{\Xi}^{(\tau)} \cap \varrho W \,|_d^2 = \mathbf{Var}(|\, \widetilde{\Xi}^{(\tau)} \cap \varrho W \,|_d) + (\, \mathbf{E}|\, \widetilde{\Xi}^{(\tau)} \cap \varrho W \,|_d)^2 .$$

Next replace in (6) the bounded Borel set B by the star-shaped set ϱW which increases when ϱ does. In view of the relation $\{x \in \mathbb{R}^d : \varrho W \cap (\varrho W - x) \neq \emptyset\} = \varrho\, (\, W \oplus (-W)\,) \subseteq$

$B_d(2\,\varrho)$ and the inequality $e^y - 1 \le y\, e^y$ for $y \ge 0$ we may write

$$\begin{aligned}
\mathbf{Var}(\,|\,\Xi \cap \varrho W\,|_d\,) &\le \lambda\, e^{-\lambda M_1}\,|\,\varrho W\,|_d \int\limits_{\varrho\,(W\oplus(-W))} \mathbf{E}|\,\Xi_0 \cap (\Xi_0 + \pi_{d-k}(\Theta_0^T\, x))\,|_{d-k}\,\mathrm{d}\,x \\
&\le \lambda\,|\,W\,|_d\, e^{-\lambda M_1}\,\varrho^d\,\mathbf{E} \int\limits_{B_d(2\,\varrho)} \big|\Xi_0 \cap (\Xi_0 + \pi_{d-k}(x))\big|_{d-k}\,\mathrm{d}\,x \\
&\le \lambda\,|\,W\,|_d\, e^{-\lambda M_1}\,\varrho^d\,\mathbf{E} \int\limits_{[-2\,\varrho,2\,\varrho]^k} \int\limits_{\mathbb{R}^{d-k}} \big|\,\Xi_0 \cap (\Xi_0 + y_1)\big|_{d-k}\,\mathrm{d}\,y_1\,\mathrm{d}\,y_2 \\
&= \lambda\,|\,W\,|_d\, e^{-\lambda M_1}\,4^k\,\mathbf{E}|\,\Xi_0\,|_{d-k}^2\,\varrho^{d+k} \quad \text{for any} \quad \varrho > 0\,. \qquad (13)
\end{aligned}$$

Replacing Ξ_0 in (13) by $\Xi_0 \setminus \Xi_0^{(\tau)}$ we obtain that

$$\begin{aligned}
\mathbf{Var}(\big|\,\widetilde{\Xi}^{(\tau)} \cap \varrho W\,\big|_d) &\le \lambda\,|W|_d\,\exp\{-\lambda\,\mathbf{E}|\Xi_0 \setminus \Xi_0^{(\tau)}|_{d-k}\}\,4^k\,\mathbf{E}|\,\Xi_0 \setminus \Xi_0^{(\tau)}\,|_{d-k}^2\,\varrho^{d+k} \\
&\le \lambda\,|W|_d\,4^k\,\mathbf{E}|\,\Xi_0\,\big|_{d-k}^2\,\mathbf{1}(\,|\,\Xi_0\,|_{d-k} > \tau\,)\,\varrho^{d+k}
\end{aligned}$$

and, by $\mathbf{E}|\,\Xi \cap B\,|_d = (\,1 - \exp\{-\lambda\, M_1\}\,)\,|\,B\,|_d \le \lambda\, M_1\,|\,B\,|_d$ for any bounded $B \in \mathcal{B}(\mathbb{R}^d)$, we get the inequality

$$\begin{aligned}
\big(\,\mathbf{E}\,|\,\widetilde{\Xi}^{(\tau)} \cap \varrho W\,|_d\,\big)^2 &\le \lambda^2\,|W|_d^2\,\varrho^{2\,d}\,\big(\,\mathbf{E}\,|\Xi_0 \setminus \Xi_0^{(\tau)}|_{d-k}\,\big)^2 \\
&\le \lambda^2\,|W|_d^2\,\varrho^{d+k}\,\varepsilon^{-2}\,\Big(\,\mathbf{E}\,|\,\Xi_0\,|_{d-k}^2\,\mathbf{1}(\,|\,\Xi_0\,|_{d-k} > \tau\,)\Big)^2\,.
\end{aligned}$$

Setting

$$M_2(\varepsilon,\tau) = \varepsilon^{-2}\,\mathbf{E}\,|\,\Xi_0\,|_{d-k}^2\,\mathbf{1}(|\,\Xi_0\,|_{d-k} > \tau\,)$$

we arrive together with Chebyshev's inequality at

$$\begin{aligned}
\mathbf{P}(\,\varrho^{-(d+k)/2}\,|\,(\Xi \setminus \Xi^{(\tau)}) \cap \varrho W)\,\big|_d \ge \varepsilon\,) &\le \varepsilon^{-2}\,\varrho^{-(d+k)}\,\mathbf{E}|\,(\Xi \setminus \Xi^{(\tau)}) \cap \varrho W\,\big|_d^2 \\
&\le \lambda\,|W|_d\,\big(\,4^k + \lambda\,|W|_d\,M_2(\varepsilon,\tau)\,\big)\,M_2(\varepsilon,\tau) \underset{\varrho\to\infty}{\longrightarrow} 0
\end{aligned}$$

for any $\varepsilon > 0$. By the same arguments,

$$\varrho^{-(d+k)/2}\,\mathbf{E}|\,(\Xi \setminus \Xi^{(\tau)}) \cap \varrho W)\,|_d \le \Big(\,\varrho^{-(d+k)}\,\mathbf{E}|\,\widetilde{\Xi}^{(\tau)} \cap \varrho W)\,|_d^2\Big)^{1/2} \underset{\varrho\to\infty}{\longrightarrow} 0$$

and, together with $\Xi^{(\tau)} \subseteq \Xi$ and Minkowski's inequality, we get that

$$\varrho^{-(d+k)}\,|\,\mathbf{Var}(|\,\Xi\cap\varrho W\,|_d)-\mathbf{Var}(|\,\Xi^{(\tau)}\cap\varrho W\,|_d)\,|\le\varrho^{-(d+k)}\,\big(\mathbf{E}|\,\widetilde{\Xi}^{(\tau)}\cap\varrho W\,|_d^2\big)^{1/2}$$

$$\times\Big(\big(\mathbf{Var}(|\,\Xi^{(\tau)}\cap\varrho W\,|_d)\big)^{1/2}+\big(\mathbf{Var}(|\,\Xi\cap\varrho W\,|_d)\big)^{1/2}\Big)\underset{\varrho\to\infty}{\longrightarrow}0\,.$$

In summary, by applying Slutzky's theorem, to prove the limit (7) in Theorem 1 it suffices to verify the CLT

$$\frac{|\,\Xi^{(\tau)}\cap\varrho W\,|_d-\mathbf{E}\,|\,\Xi^{(\tau)}\cap\varrho W\,|_d}{\sqrt{\mathbf{Var}(\,|\,\Xi^{(\tau)}\cap\varrho W\,|_d|\,)}}\quad\underset{\varrho\to\infty}{\overset{\mathcal{D}}{\longrightarrow}}\quad\mathcal{N}(0,1)\;. \tag{14}$$

for the truncated model $\Xi^{(\tau)}$ instead of Ξ. Notice that, by standard arguments from analysis, $\varepsilon>0$ can be chosen as null sequence $\varepsilon(\varrho)\underset{\varrho\to\infty}{\longrightarrow}0$ such that $\tau(\varrho)=\varepsilon(\varrho)\,\varrho^{(d-k)/2}\underset{\varrho\to\infty}{\longrightarrow}\infty$ and $M_2(\varepsilon(\varrho),\tau(\varrho))\underset{\varrho\to\infty}{\longrightarrow}0$.

To verify (14) it remains the proof of the limits $\mathbf{Cum}_n\big(\,|\,\Xi^{(\tau(\varrho))}\cap\varrho W)\,|_d\,\big)\underset{\varrho\to\infty}{\longrightarrow}0$ for $n\ge 3$. The nth-order cumulants $\mathbf{Cum}_n(\,|\Xi\cap B|_d\,)$ can be expressed in analogy to the nth-order moment of the volume $|\Xi\cap B|_d$ by

$$\mathbf{Cum}_n(\,|\Xi\cap B|_d\,)=\int\limits_{B^n}c_\Xi(x_1,\dots,x_n)\mathrm{d}(x_1,\dots,x_n)\quad\text{for}\quad n\ge 2\,,$$

where the nth-order mixed cumulant $c_\Xi(x_1,\dots,x_n)$ of the $\{0,1\}$-valued random field $\{\,\mathbf{1}_\Xi(x)\,,\,x\in\mathbb{R}^d\,\}$ is defined by

$$\begin{aligned} c_\Xi(x_1,\dots,x_n) &= \frac{\partial^n}{\partial s_1\dots\partial s_n}\log\,\mathbf{E}\exp\Big\{\sum_{j=1}^n s_j\,\mathbf{1}_\Xi(x_j)\Big\}\Big|_{s_1=\dots=s_n=0} \\ &= \sum_{k=1}^n(-1)^{k-1}(k-1)!\sum_{X_1\cup\dots\cup X_k=X}p_\Xi(X_1)\cdots p_\Xi(X_k)\;, \end{aligned}\tag{15}$$

where the inner sum runs over all decompositions of $X=\{x_1,\dots,x_n\}$ into pairwise disjoint, non-empty subsets $X_1,\dots,X_k$. The latter formula gives a representation of the (mixed) cumulant $c_\Xi(X)$ in terms of the (mixed) moment functions $p_\Xi(Y)\,,\,\emptyset\ne Y\subseteq X$ of $\mathbf{1}_\Xi(\cdot)$. If we replace the union set Ξ by its complement Ξ^c, then $c_{\Xi^c}(x_1,\dots,x_n)$ turns out to be the nth-order mixed cumulant of the random field $1-\mathbf{1}_\Xi(\cdot)$. By applying the very definition (15) of mixed cumulants we get the relationship

$$c_\Xi(x_1,\dots,x_n)=(-1)^n\,c_{\Xi^c}(x_1,\dots,x_n)\;,$$

which in turn yields a representation of $c_\Xi(x_1,\dots,x_n)$ in terms of the mixed moment function $p_{\Xi^c}(Y)$ for non-empty subsets Y of $\{x_1,\dots,x_n\}$ given in (5).

By the shift-invariance $c_\Xi(\mathbf{o}_d, x_2 - x_1, \ldots, x_{n+1} - x_1) = c_\Xi(x_1, x_2, \ldots, x_{n+1})$ as consequence of the stationarity of Ξ resp. $\mathbf{1}_\Xi(\cdot)$, we may rewrite $\mathbf{Cum}_{n+1}(|\Xi \cap B|_d)$ as

$$\mathbf{Cum}_{n+1}(|\Xi \cap B|_d) = (-1)^{n+1} \int\limits_{(B \oplus (-B))^n} \Big| B \cap \bigcap_{i=1}^{n} (B - x_i) \Big|_d \, c_{\Xi^c}(\mathbf{o}_d, x_1, \ldots, x_n) \, \mathrm{d}(x_1, \ldots, x_n)$$

generalising the variance formula (6). Since $W \oplus (-W) \subseteq B_d(2)$ by our assumptions it follows that, for $n \geq 2$,

$$\Big| \mathbf{Cum}_{n+1}(|\Xi \cap \varrho W|_d) \Big| \leq \varrho^d \, |W|_d \int\limits_{(B_d(2\varrho))^n} |\, c_{\Xi^c}(\mathbf{o}_d, x_1, \ldots, x_n) \,| \, \mathrm{d}(x_1, \ldots, x_n) \,. \quad (16)$$

Lemma 1. *Provided that $M_2 < \infty$ the truncated PCP (12) with $\tau = \varepsilon \, \varrho^{(d-k)/2}$ allows the estimates*

$$\varrho^{-(d+k)\,n/2} \Big| \mathbf{Cum}_n(\,|\,\Xi^{(\tau)} \cap \varrho W\,|_d\,) \Big| \leq \varepsilon^{n-2} \, c_n(\lambda) \, |W|_d \quad \textit{for} \quad n \geq 3 \,,$$

where the constants $c_n(\lambda)$ depend only on λ, n and on the moments M_1 and M_2.

The proof of Lemma 1 relies essentially on the following recursive estimate shown in [8] for a general stationary PCP (4).

Lemma 2. *If $M_{n+1} < \infty$ for fixed $n \geq 2$, then*

$$\int\limits_{(B_d(2\varrho))^n} |\, c_{\Xi^c}(\mathbf{o}_d, x_1, \ldots, x_n) \,| \, \mathrm{d}(x_1, \ldots, x_n) \leq C_{1,n} \, \varrho^{k\,n} \,,$$

where $C_{1,n}$ depends only on λ, n and the moments $M_1, \ldots, M_{n+1}$ and can be calculated successively by means of the double-indexed sequence $C_{m,n}$ defined by $C_{0,n} = 0$ for $n \geq 1$ and, for $m \geq 1$ and $n \geq 1$,

$$C_{m,n} = A_n + \sum_{j=0}^{n-1} \binom{n}{j} A_j \, C_{m-1+j,n-j} \quad \textit{with} \quad C_{m,1} = 4^k \, m \, \lambda \, e^{-\lambda M_1} \, M_2 \,.$$

Here $A_0 = 1$, $A_1 = 4^k \, \lambda \, e^{\lambda M_1} \, M_2$ and, for $n \geq 2$,

$$\begin{aligned} A_n &= A_{n-1} \, A_1 + e^{2\lambda M_1} \sum_{j=0}^{n-2} \binom{n-1}{j} A_j \, B_{n-j} \quad \textit{with} \\ B_n &= 4^{k\,n} \, (n-1)! \sum_{j=1}^{n-1} \frac{\lambda^j}{j!} \sum_{\substack{n_1 + \cdots + n_j = n-1 \\ n_1, \ldots, n_j \geq 1}} \frac{M_{n_1+2}}{n_1!} \prod_{i=2}^{j} \frac{M_{n_i+1}}{n_i!} \,. \end{aligned}$$

Proof of Lemma 1. We replace in Lemma 2 the typical cylinder base Ξ_0 by the truncated cylinder base (12) of the PCP $\Xi^{(\tau)}$. Hence, in B_n the moments M_j are replaced by the truncated moments $M_j^{(\tau)} = \mathbf{E}|\Xi_0^{(\tau)}|_{d-k}^j$ for $j = 2, \dots, n+1$. Since $M_j^{(\tau)} \le \tau^{j-2} M_2$ we are led to

$$B_n \le 4^{k\,n} \sum_{j=1}^{n-1} \frac{(\lambda\, M_2)^j}{j!}\, \tau^{n-j} \sum_{\substack{n_1+\cdots+n_j=n-1 \\ n_1,\dots,n_j \ge 1}} \frac{(n-1)!}{n_1! \,\cdots\, n_j!} \le \tau^{n-1}\, b_n(\lambda)\,,$$

where

$$b_n(\lambda) = 4^{k\,n} \sum_{j=1}^{n-1} \frac{(\lambda\, M_2)^j}{j!} \sum_{\substack{n_1+\cdots+n_j=n-1 \\ n_1,\dots,n_j \ge 1}} \frac{(n-1)!}{n_1! \,\cdots\, n_j!}\,.$$

A simple inductive argument shows that

$$A_n \le \tau^{n-1}\, a_n(\lambda) \quad \text{for} \quad n \ge 1\,,$$

where $a_1(\lambda) = A_1$ and $a_n(\lambda) = a_{n-1}(\lambda)\, a_1(\lambda) + e^{2\,\lambda\, M_1} \big(b_n(\lambda) + \sum_{j=1}^{n-2} \binom{n-1}{j}\, a_j(\lambda)\, b_{n-j}(\lambda)\big)$ for $n \ge 2$. Finally, we put $c_{m,1}(\lambda) = C_{m,1}$ for $m \ge 1$. In view of $C_{m,2} - C_{m-1,2} = A_2 + 2\, A_1\, C_{m,1}$ it is easy to see that $C_{m,2} = m\, A_2 + 2\, A_1 \big(C_{m,1} + \cdots + C_{1,1}\big) \le c_{m,2}\, \tau$ with $c_{m,2} = m\, a_2(\lambda) + 2\, a_1(\lambda) \big(c_{m,1}(\lambda) + \cdots + c_{1,1}(\lambda)\big)$ for any $m \ge 1$. In this way we may proceed for $n = 3, 4, \dots$ and arrive at $C_{m,n} \le c_{m,n}(\lambda)\, \tau^{n-1}$ for all $n \ge 3$ and $m \ge 1$, where the numbers $c_{m,n}(\lambda)$ are defined recursively by $c_{m,n}(\lambda) = c_{m-1,n}(\lambda) + a_n(\lambda) + \sum_{j=1}^{n-1} \binom{n}{j}\, a_j(\lambda)\, c_{m-1+j,n-j}(\lambda)$. Thus, after inserting $\tau = \varepsilon\, \varrho^{(d-k)/2}$, we find that

$$C_{1,n}\, \varrho^{k\,n} \le \varepsilon^{n-1}\; c_{1,n}(\lambda)\, \varrho^{-d+(d+k)(n+1)/2} \quad \text{for} \quad n \ge 2\,.$$

This estimate combined with (16) and the choice of $\varepsilon(\varrho) \underset{\varrho\to\infty}{\longrightarrow} 0$ completes the proof of Lemma 1. $\square$

4 The Asymptotic Variance for Atomic and Diffuse Directional Distribution

We first recall the Hardy-Littlewood equivalence $\mathbf{Var}(|\Xi_{\lambda,Q_{d,k}} \cap \varrho\, W|_d) \asymp \varrho^{d+k}$ as $\varrho \to \infty$ which means that

$$0 \,<\, \liminf_{\varrho\to\infty} \frac{\mathbf{Var}(|\Xi_{\lambda,Q_{d,k}} \cap \varrho\, W|_d)}{\varrho^{d+k}} \;\le\; \limsup_{\varrho\to\infty} \frac{\mathbf{Var}(|\Xi_{\lambda,Q_{d,k}} \cap \varrho\, W|_d)}{\varrho^{d+k}} \,<\, \infty\,. \qquad (17)$$

The asymptotic relation (17) is an obvious consequence of (8) and holds under the assumptions of Theorem 1.

Remark 3. *(17) reveals that the variance of* $|\Xi_{\lambda,Q_{d,k}} \cap \varrho W|_d$ *grows with the power* $|\,\varrho W\,|_d^{1+k/d}$ *of the window volume which expresses long-range dependences within the random set (4). The same effect could be observed in studying the asymptotic behaviour of the total* $(d-k)$*-volume of intersection* $(d-k)$*-flats generated by Poisson hyperplane processes in* $B_d(\varrho)$ *resp.* ϱW *(for convex* W*) as* $\varrho \to \infty$ *, see [6] resp. [7].*

The aim of this section is to prove that both of the limits in (17) coincide. For this we consider the cases of atomic and diffuse (marginal) distribution of Θ_0 separately.

4.1 Diffuse directional distribution

We first prove the second result of Theorem 2 with diffuse distribution $\mathbf{P}_0$ of Θ_0 , i.e. $\mathbf{P}(\Theta_0 = \theta) = 0$ for all $\theta \in \mathbb{SO}_k^d$. The inequality $0 \le e^x - 1 - x \le x^2\, e^x/2$ for $x \ge 0$ leads to

$$\Big|\mathbf{Var}(|\Xi \cap \varrho W|_d) - \lambda e^{-2\lambda M_1} \int\limits_{\mathbb{R}^d} |\varrho W \cap (\varrho W - x)|_d\, \mathbf{E}\,|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T x))|_{d-k}\, \mathrm{d}\,x\Big|$$

$$\le \frac{\lambda^2}{2}\, e^{-\lambda M_1}\, |\,\varrho W\,|_d \int\limits_{\varrho\,(W\oplus(-W))} \big(\mathbf{E}\,|\,\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T x)\,)|_{d-k}\big)^2\, \mathrm{d}\,x$$

$$\le \frac{\lambda^2}{2}\, e^{-\lambda M_1}\, \varrho^d\, |W|_d \int\limits_{B_d(2\,\varrho)} \big(\mathbf{E}\,|\,\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T x)\,)|_{d-k}\big)^2\, \mathrm{d}\,x.$$

We divide both sides of the previous inequality by ϱ^{d+k} and show in the next step that

$$J_\varrho = \varrho^{-k} \int\limits_{B_d(\varrho)} \big(\,\mathbf{E}\,|\,\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T x))\,|_{d-k}\,\big)^2\, \mathrm{d}\,x \underset{\varrho\to\infty}{\longrightarrow} 0\,. \tag{18}$$

Taking an independent copy $(\widetilde{\Theta}_0, \widetilde{\Xi}_0)$ of the mark $(\Theta_0, \Xi_0) \sim Q_{d,k}$, applying Fubini's theorem and substituting $x = \Theta_0\, y$ we may rewrite J_ϱ with the total expectation formula in the following way:

$$\begin{aligned} J_\varrho &= \varrho^{-k}\, \mathbf{E}\Big[\int\limits_{B_d(\varrho)} |\,\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T x)\,)|_{d-k}\, |\,\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\Theta}_0^T x)\,)|_{d-k}\, \mathrm{d}\,x\Big] \\ &= \varrho^{-k}\, \mathbf{E}\Big[\int\limits_{B_d(\varrho)} |\,\Xi_0 \cap (\Xi_0 - \pi_{d-k}(y)\,)|_{d-k}\, |\,\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\Theta}_0^T \Theta_0\, y)\,)|_{d-k}\, \mathrm{d}\,y\Big] \\ &= \varrho^{-k} \int\limits_{\mathbb{SO}_k^d} \int\limits_{\mathbb{SO}_k^d} \mathbf{E}\Big[\int\limits_{B_d(\varrho)} |\,\Xi_0 \cap (\Xi_0 - \pi_{d-k}(y)\,)|_{d-k} \\ &\qquad \times\, |\,\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\theta}^T \theta\, y)\,)|_{d-k}\, \mathrm{d}\,y \Big| \Theta_0 = \theta, \widetilde{\Theta}_0 = \widetilde{\theta}\Big]\, \mathbf{P}_0(\mathrm{d}\,\widetilde{\theta})\, \mathbf{P}_0(\mathrm{d}\,\theta). \end{aligned}$$

Since $\mathbf{P}_0$ is diffuse and Θ_0 and $\widetilde{\Theta}_0$ are stochastically independent, it follows that $\mathbf{P}(\Theta_0 = \widetilde{\Theta}_0) = 0$. Thus, it suffices to show that the inner integral disappears as $\varrho \to \infty$ for any pair $(\theta, \widetilde{\theta}) \in \mathbb{SO}_k^d \times \mathbb{SO}_k^d$ with $\theta \neq \widetilde{\theta}$. For this purpose, we consider the subspace $E = (\theta^T \widetilde{\theta} E_k) \cap E_k$ with dimension $\dim E =: l \in \{0, \dots, k-1\}$ depending on the choice of the distinct orthogonal matrices θ and $\widetilde{\theta}$. We note that $\dim E = k$ would imply $\theta^T \widetilde{\theta} E_k = E_k$ and this gives $\theta = \widetilde{\theta}$ by the very definition of $\mathbb{SO}_k^d$. Furthermore, let $\vartheta \in \mathbb{SO}_d$ be chosen such that $E = \vartheta E_l$ and $\vartheta E_k = E_k$ (such ϑ always exists). Now, setting $y = (y_1, y_2)^T$ with $y_1 \in \mathbb{R}^{d-l}$ and $y_2 \in \mathbb{R}^l$ we can continue to estimate the above inner integral over $B_d(\varrho)$ as follows:

$$
\begin{aligned}
&\varrho^{-k} \int\limits_{B_d(\varrho)} \big|\, \Xi_0 \cap (\Xi_0 - \pi_{d-k}(y)\,)\big|_{d-k} \big|\, \widetilde{\Xi}_0 \cap (\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\theta}^T \theta\, y)\,)\big|_{d-k}\, \mathrm{d}\, y \qquad (19)\\
&\le\ \varrho^{-k} \int\limits_{B_l(\varrho)} \int\limits_{B_{d-l}(\varrho)} \big|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\vartheta\,(y_1, y_2)^T))\big|_{d-k} \\
&\qquad\qquad \times \big|\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta\,(y_1, y_2)^T))\big|_{d-k}\, \mathrm{d}\, y_1\, \mathrm{d}\, y_2 \\
&\le\ \varrho^{-k} \int\limits_{B_l(\varrho)} \int\limits_{B_{d-l}(\varrho)} \big|\, \Xi_0 \cap (\Xi_0 - \pi_{d-k}(\vartheta\,(y_1, \mathbf{o}_l)^T)\,)\big|_{d-k} \\
&\qquad\qquad \times \big|\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta\,(y_1, \mathbf{o}_l)^T))\big|_{d-k}\, \mathrm{d}\, y_1\, \mathrm{d}\, y_2\,,
\end{aligned}
$$

where we have used that $\widetilde{\theta}^T \theta\, \vartheta\, E_l$ and and $\vartheta\, E_l$ are subspaces of E_k with dimension less than k. This means that $\pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta\, y) = \pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta\,(y_1, \mathbf{o}_l))$ and $\pi_{d-k}(\vartheta\, y) = \pi_{d-k}(\vartheta\,(y_1, \mathbf{o}_l))$, i.e. the integrand does not depend on y_2 and we can take $y_2 = \mathbf{o}_l$ and evaluate the integral over $y_2 \in B_l(\varrho)$.

Further, by setting $y_1 = (z_1, z_2)^T$ with $z_1 \in \mathbb{R}^{d-k}$ and $z_2 \in \mathbb{R}^{k-l}$ we get the following upper bound of term (19):

$$
\begin{aligned}
&\varrho^{-(k-l)}\, \omega_l \int\limits_{B_{k-l}(\varrho)} \int\limits_{\mathbb{R}^{d-k}} \big|\, \Xi_0 \cap (\Xi_0 - \pi_{d-k}(\vartheta\,(z_1, z_2, \mathbf{o}_l)^T)\,)\,\big|_{d-k} \\
&\qquad\qquad \times \big|\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta\,(z_1, z_2, \mathbf{o}_l)^T)\,)\big|_{d-k}\, \mathrm{d}\, z_1\, \mathrm{d}\, z_2 \\
&= \omega_l \int\limits_{B_{k-l}(1)} \int\limits_{\mathbb{R}^{d-k}} \big|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\vartheta(z_1, \mathbf{o}_k)^T)\,)\big|_{d-k} \\
&\qquad\qquad \times \big|\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_0 - \pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta(z_1, \varrho\, z_2, \mathbf{o}_l)^T)\,)\,\big|_{d-k}\, \mathrm{d}\, z_1\, \mathrm{d}\, z_2 \underset{\varrho \to \infty}{\longrightarrow} 0\,,
\end{aligned}
$$

where we have used the relations $\pi_{d-k}(\vartheta\,(z_1, z_2, \mathbf{o}_l)^T) = \pi_{d-k}(\vartheta\,(z_1, \mathbf{o}_k)^T)$ and $\|\pi_{d-k}(\widetilde{\theta}^T \theta\, \vartheta(z_1, \varrho\, z_2, \mathbf{o}_l)^T)\| \underset{\varrho \to \infty}{\longrightarrow} \infty$ for $z_2 \neq \mathbf{o}_{k-l}$ and any $z_1 \in \mathbb{R}^{d-k}$, and ω_l denotes the volume of the l-dimensional unit ball. Finally, applying the dominated convergence theorem completes the proof of (18).

Turning back at the beginning of Subsection 4.1 we see that in case of diffuse $\mathbf{P}_0$ the limit (9) is obtained in the following way

$$\frac{\lambda e^{-2\lambda M_1}}{\varrho^{d+k}} \int\limits_{\varrho(W\oplus(-W))} \big|\varrho W\cap(\varrho W-x)\big|_d \,\mathbf{E}\,\big|\,\Xi_0\cap(\Xi_0-\pi_{d-k}(\Theta_0^T x)\,\big|_{d-k}\,\mathrm{d}\,x$$

$$= \frac{\lambda e^{-2\lambda M_1}}{\varrho^{d+k}}\,\mathbf{E}\Big(\int\limits_{\mathbb{R}^{d-k}}\int\limits_{\mathbb{R}^k}\big|\varrho W\cap(\varrho W-\Theta_0(x_1,x_2)^T)\big|_d\,\big|\Xi_0\cap(\Xi_0-x_1)\big|_{d-k}\,\mathrm{d}\,x_2\,\mathrm{d}\,x_1\Big)$$

$$= \lambda e^{-2\lambda M_1}\,\mathbf{E}\Big(\int\limits_{\mathbb{R}^{d-k}}\int\limits_{\mathbb{R}^k}\big|W\cap(W-\Theta_0(\frac{x_1}{\varrho},x_2)^T)\big|_d\,\big|\,\Xi_0\cap(\Xi_0-x_1)\,\big|_{d-k}\,\mathrm{d}\,x_2\,\mathrm{d}\,x_1\Big)$$

$$\underset{\varrho\to\infty}{\longrightarrow}\lambda\,e^{-2\,\lambda\,M_1}\int\limits_{\mathbb{SO}_k^d}\mathbf{E}\big(\,|\Xi_0|_{d-k}^2\,\big|\,\Theta_0=\theta\,\big)\int\limits_{\mathbb{R}^k}\big|W\cap(W-\theta(\mathbf{o}_{d-k},x)^T)\big|_d\,\mathrm{d}\,x\;\mathbf{P}_0(\mathrm{d}\,\theta)\,.$$

This finishes the proof of (11).

4.2 Discrete directional distribution

Let $\mathbf{P}_0$ be an atomic distribution, i.e. its support is some finite or countably infinite set $\{\theta_i\in\mathbb{SO}_k^d\,,\,i\in I\}$ of distinct matrices in $\mathbb{SO}_k^d$; for convenience let $I=\mathbb{N}$. With the notation of Theorem 2 we have $f(y,\theta_i)=\mathbf{E}\big(\,|\Xi_0\cap(\Xi_0-y)|_{d-k}\,|\,\Theta_0=\theta_i\,\big)\;\mathbf{P}_0(\{\theta_i\})$ for $i\in\mathbb{N}$ and $y\in\mathbb{R}^{d-k}$.

To begin with we state the elementary inequality

$$e^{x_1+\cdots+x_n}-1-\sum_{i=1}^{n}(e^{x_i}-1)\le\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(e^{x_i}-1)(e^{x_j}-1)e^{x_1+\cdots+x_n}$$

for $x_1,\ldots,x_n\ge 0$, which can be verified by induction on $n\in\mathbb{N}$ and remains valid also in the limit $n\to\infty$.

Applying the previous inequality to the points $x_i=\lambda\,f(\pi_{d-k}(\theta_i^T x),\theta_i)$ for $i\in\mathbb{N}$ and $x\in\mathbb{R}^d$ we are led to the estimate

$$\Big|\,\mathbf{Var}(|\Xi\cap\varrho\,W|_d)\\ -e^{-2\,\lambda\,M_1}\int\limits_{\mathbb{R}^d}\big|\,\varrho\,W\cap(\varrho\,W-x)\,\big|_d\sum_{i=1}^{\infty}\Big(\exp\{\lambda\,f(\pi_{d-k}(\theta_i^T x),\theta_i)\}-1\Big)\,\mathrm{d}\,x\Big|$$

$$\le\lambda^2\,|W|_d\,\varrho^d\int\limits_{B_d(2\,\varrho)}\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}f(\pi_{d-k}(\theta_i^T x),\theta_i)\,f(\pi_{d-k}(\theta_j^T x),\theta_j)\,\mathrm{d}\,x\,,$$

where the simple relations $x_i+x_j+\sum_{k=1}^{\infty}x_k\le 2\,\lambda\,M_1$ for all $i<j$ and $e^{x_i}-1\le x_i\,e^{x_i}$ have been used.

In analogy to (18) we divide both sides of the previous inequality by ϱ^{d+k} and prove that

$$I_\varrho = \varrho^{-k} \int\limits_{B_d(\varrho)} \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} f(\pi_{d-k}(\theta_i^T x), \theta_i)\, f(\pi_{d-k}(\theta_j^T x), \theta_j)\, \mathrm{d}\, x \underset{\varrho\to\infty}{\longrightarrow} 0\,.$$

For any $\varepsilon > 0$ there exists an integer $n = n(\varepsilon) \geq 1$ such that $\sum_{i=n+1}^{\infty} f(\mathbf{o}_{d-k}, \theta_i) \leq \varepsilon$ and this yields the estimate

$$\begin{aligned} I_\varrho \leq \varepsilon\, \varrho^{-k} \sum_{i=1}^{\infty} \int\limits_{B_d(\varrho)} & f(\pi_{d-k}(\theta_i^T x), \theta_i)\, \mathrm{d}\, x \\ & + \varrho^{-k} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \int\limits_{B_d(\varrho)} f(\pi_{d-k}(\theta_i^T x), \theta_i)\, f(\pi_{d-k}(\theta_j^T x), \theta_j)\, \mathrm{d}\, x\,. \end{aligned} \tag{20}$$

By setting $x = (x_1, x_2)^T$ with $x_1 \in \mathbb{R}^{d-k}$ and $x_2 \in \mathbb{R}^k$ it is easily seen that the first summand in (20) is equal to

$$\begin{aligned} & \varepsilon\, \varrho^{-k} \sum_{i=1}^{\infty} \int\limits_{B_d(\varrho)} \mathbf{E}\left[\left|\, \Xi_0 \cap (\Xi_0 - \pi_{d-k}(\theta_i^T (x_1, x_2)^T))\, \right|_{d-k} \mathbf{1}\{\Theta_0 = \theta_i\} \right] \mathrm{d}(x_1, x_2) \\ & = \varepsilon\, \varrho^{-k} \int\limits_{B_d(\varrho)} \mathbf{E} \left| \Xi_0 \cap (\Xi_0 - \pi_{d-k}((x_1, x_2)^T)) \right|_{d-k} \mathrm{d}(x_1, x_2) \\ & \leq \varepsilon\, \varrho^{-k} \int\limits_{B_k(\varrho)} \int\limits_{\mathbb{R}^{d-k}} \mathbf{E} \left| \Xi_0 \cap (\Xi_0 - x_1) \right|_{d-k} \mathrm{d}\, x_1\, \mathrm{d}\, x_2 = \varepsilon\, \omega_k\, M_2\,. \end{aligned}$$

In order to treat the finite double sum in (20) it suffices to consider the integral

$$\begin{aligned} & \varrho^{-k} \int\limits_{B_d(\varrho)} f(\pi_{d-k}(\theta_i^T x), \theta_i)\, f(\pi_{d-k}(\theta_j^T x), \theta_j)\, \mathrm{d}\, x \\ & = \varrho^{-k} \int\limits_{B_d(\varrho)} f(\pi_{d-k}(\theta_i^T \theta_j y), \theta_i)\, f(\pi_{d-k}(y), \theta_j)\, \mathrm{d}\, y \\ & = \varrho^{-k} \int\limits_{B_d(\varrho)} \mathbf{E}\big(|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\theta_i^T \theta_j y))|_{d-k} \,\big|\, \Theta_0 = \theta_i \,\big)\, \mathbf{P}_0(\{\theta_i\}) \\ & \qquad\qquad \times \mathbf{E}\big(|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(y))|_{d-k} \,\big|\, \Theta_0 = \theta_j \,\big)\, \mathbf{P}_0(\{\theta_j\})\, \mathrm{d}\, y\,. \end{aligned}$$

for a single pair $i < j$. This integral can be shown to converge to 0 as $\varrho \to \infty$ by repeating quite the same steps carried out to show that the integral (19) disappears as $\varrho \to \infty$. Thus, the total sum in (20) can be made arbitrarily small. This means that the existence and the explicit form of the limit (9) in case of atomic $\mathbf{P}_0$ is proved by finding the limit (as $\varrho \to \infty$) of

$$\frac{e^{-2\lambda M_1}}{\varrho^{d+k}} \int\limits_{\varrho(W\oplus(-W))} |\,\varrho W \cap (\varrho W - x)\,|_d \sum_{i=1}^{\infty}\big(\exp\{\lambda f(\pi_{d-k}(\theta_i^T x), \theta_i)\} - 1\,\big)\,\mathrm{d}\,x\,.$$

Making use of the monotone convergence theorem we first interchange integration and summation and then we pass to the limit for each term of the above sum:

$$\begin{aligned}
&\frac{1}{\varrho^{d+k}} \int\limits_{\varrho(W\oplus(-W))} |\,\varrho W \cap (\varrho W - x)\,|_d \Big(\exp\{\lambda f(\pi_{d-k}(\theta_i^T x), \theta_i)\} - 1\Big)\,\mathrm{d}\,x \\
&= \frac{1}{\varrho^k} \int\limits_{\mathbb{R}^{d-k}} \int\limits_{\mathbb{R}^k} \Big|\,W \cap \Big(W - \theta_i\Big(\frac{x_1}{\varrho}, \frac{x_2}{\varrho}\Big)^T\Big)\,\Big|_d \,\mathrm{d}\,x_2\,\big(e^{\lambda f(x_1,\theta_i)} - 1\big)\,\mathrm{d}\,x_1 \\
&= \int\limits_{\mathbb{R}^{d-k}} \int\limits_{\mathbb{R}^k} \Big|\,W \cap \Big(W - \theta_i\Big(\frac{x_1}{\varrho}, x_2\Big)^T\Big)\,\Big|_d \,\mathrm{d}\,x_2\,\big(e^{\lambda f(x_1,\theta_i)} - 1\big)\,\mathrm{d}\,x_1 \\
&\underset{\varrho\to\infty}{\longrightarrow} \int\limits_{\mathbb{R}^k} \Big|\,W \cap \Big(W - \theta_i(\mathbf{o}_{d-k}, x_2)^T\Big)\,\Big|_d \,\mathrm{d}\,x_2 \int\limits_{\mathbb{R}^{d-k}} \big(e^{\lambda f(x_1,\theta_i)} - 1\big)\,\mathrm{d}\,x_1\,.
\end{aligned}$$

The last step is justified by the dominated convergence theorem. Thus, the proof of (10) is finished and Theorem 2 is completely proved.

In general, each directional distribution $\mathbf{P}_0$ allows a unique decomposition $\mathbf{P}_0 = \alpha\,\mathbf{P}_0^{\mathrm{a}} + (1-\alpha)\,\mathbf{P}_0^{\mathrm{c}}$ (implying a decomposition of the mark distribution $Q_{d,k} = \alpha\,Q_{d,k}^{\mathrm{a}} + (1-\alpha)\,Q_{d,k}^{\mathrm{c}}$ on $\mathbb{M}_{d,k}$) in an atomic distribution $\mathbf{P}_0^{\mathrm{a}}$ and a diffuse distribution $\mathbf{P}_0^{\mathrm{c}}$ on $\mathbb{SO}_k^d$. Then the limit (9) exists and admits the decomposition

$$\sigma^2_{\lambda,Q_{d,k}}(W) = \sigma^2_{\lambda,Q_{d,k}^{\mathrm{a}},\alpha}(W) + (1-\alpha)\,\sigma^2_{\lambda,Q_{d,k}^{\mathrm{c}}}(W)\,, \tag{21}$$

where $\sigma^2_{\lambda,Q_{d,k}^{\mathrm{a}},\alpha}(W)$ resp. $\sigma^2_{\lambda,Q_{d,k}^{\mathrm{c}}}(W)$ is defined as in (10) resp. (11) with $\mathbf{P}_0$ replaced by $\alpha\,\mathbf{P}_0^{\mathrm{a}}$ (in $f(y,\theta_i)$) resp. by $\mathbf{P}_0^{\mathrm{c}}$.

We only sketch the crucial idea leading to (21). We split the exponential term in the representation formula (6) of the variance which gives

$$\begin{aligned}
\exp\{\lambda\,\mathbf{E}|\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T x))|_{d-k}\} - 1 &= \exp\{\lambda T^{\mathrm{a}}(x)\} - 1 + \exp\{\lambda T^{\mathrm{c}}(x)\} - 1 \\
&\quad + (\exp\{\lambda T^{\mathrm{a}}(x)\} - 1)\,(\exp\{\lambda T^{\mathrm{c}}(x)\} - 1)\,,
\end{aligned}$$

where $T^{\mathrm{a}}(x)$ (resp. $T^{\mathrm{c}}(x)$) denotes the atomic (resp. diffuse) part of the expectation term $T(x) = \mathbf{E}\,|\,\Xi_0 \cap (\Xi_0 - \pi_{d-k}(\Theta_0^T x))\,|_{d-k}$. Now, we have to repeat the procedures of Subsections 4.1 and 4.2 with $T(x)$ replaced by $T^{\mathrm{a}}(x)$ and $T^{\mathrm{c}}(x)$, respectively. In view of the inequality

$$(\exp\{\lambda T^{\mathrm{a}}(x)\} - 1)\,(\exp\{\lambda T^{\mathrm{c}}(x)\} - 1) \le \lambda^2\,e^{\lambda M_1}\,T^{\mathrm{a}}(x)\,T^{\mathrm{c}}(x)$$

the additional third term can be shown to disappear as $\varrho \to \infty$ using (19).

5 Some special cases and concluding remarks

5.1 Spherical sampling window

For $W = B_d(1)$ the formulae (10) and (11) can be substantially simplified. This relies on the formula

$$\int_0^2 |\, B_d(1) \cap (B_d(1) + s\, e_1)\,|_d \; s^{k-1} \,\mathrm{d}\, s = 2\,\omega_{d-1} \int_0^2 \int_0^{s/2} \left(\sqrt{1-y^2}\right)^{d-1} \mathrm{d}\, y\, s^{k-1} \,\mathrm{d}\, s$$

$$= \frac{2^k \omega_{d-1}}{k} \int_0^1 z^{\frac{k+1}{2}-1} \,(1-z)^{\frac{d+1}{2}-1} \,\mathrm{d}\, z = \frac{2^k\, \omega_{k+d}}{\pi\, k\, \omega_{k-1}},$$

which, together with $2\,\pi\,\omega_{k-1} = (k+1)\,\omega_{k+1}$, yields

$$\int_{\mathbb{R}^k} \left|W \cap (W - \theta_i(\mathbf{o}_{d-k}, x)^T)\right|_d \,\mathrm{d}\, x = \int_{\mathbb{R}^k} \left|B_d(1) \cap \left(B_d(1) - (\mathbf{o}_{d-k}, x)^T\right)\right|_d \,\mathrm{d}\, x$$

$$= \frac{2^{k+1}\, \omega_k\, \omega_{k+d}}{(k+1)\,\omega_{k+1}}.$$

Thus, we obtain in the discrete case

$$\sigma^2_{\lambda,Q_{d,k}}(B_d(1)) = e^{-2\,\lambda\, M_1}\, \frac{2^{k+1}\, \omega_k\, \omega_{k+d}}{(k+1)\,\omega_{k+1}} \sum_{i\in I} \int_{\mathbb{R}^{d-k}} \left(\, e^{\lambda\, f(x,\theta_i)} - 1\,\right) \mathrm{d}\, x,$$

and analogously in the diffuse case

$$\sigma^2_{\lambda,Q_{d,k}}(B_d(1)) = \lambda\, e^{-2\,\lambda\, M_1}\, \frac{2^{k+1}\, \omega_k\, \omega_{k+d}}{(k+1)\,\omega_{k+1}}\, M_2.$$

5.2 The case of motion-invariant union sets $\Xi_{\lambda,Q_{d,k}}$

Another important special case arises when the stationary random set (4) is additionally isotropic, i.e. $\mathbf{P}_0$ is the uniform distribution on $\mathbb{SO}^d_k$ induced by the normalised Haar measure on the Grassmannian $\mathbb{G}(d,k)$. If the conditional second moment $M_2(\theta)$ does not depend on $\theta \in \mathbb{SO}^d_k$ (e.g. Θ_0 and Ξ_0 are independent) we obtain

$$\int_{\mathbb{SO}_k^d} M_2(\theta) \int_{\mathbb{R}^k} |\, W \cap (W - \theta\,(\mathbf{o}_{d-k}, x)^T)\,|_d \,\mathrm{d}\,x\; \mathbf{P}_0(\mathrm{d}\,\theta)$$

$$= M_2 \int_{\partial B_k(1)} \int_0^\infty \int_{\mathbb{SO}_k^d} |\, W \cap (W - r\,\theta\,(\mathbf{o}_{d-k}, u)^T)\,|_d \;\mathbf{P}_0(\mathrm{d}\,\theta)\, r^{k-1}\;\mathrm{d}\,r\,\mathcal{H}_{k-1}(\mathrm{d}\,u)$$

$$= \frac{k\,\omega_k}{d\,\omega_d}\, M_2 \int_{\partial B_d(1)} \int_0^\infty |\, W \cap (W - r\,v)\,|_d\, r^{k-1}\;\mathrm{d}\,r\,\mathcal{H}_{d-1}(\mathrm{d}\,v)$$

$$= \frac{k\,\omega_k}{d\,\omega_d}\, M_2 \int_{\mathbb{R}^d} \frac{|\, W \cap (W - x)\,|_d}{\|x\|^{d-k}}\;\mathrm{d}\,x \;=\; M_2\, I_{k+1}(W)\,,$$

where $\mathcal{H}_k(\cdot)$ denotes the k-dimensional Hausdorff measure in $\mathbb{R}^d$ and the functional

$$I_{k+1}(W) = \frac{k\,\omega_k}{d\,\omega_d} \int_W \int_W \frac{\mathrm{d}\,y\;\mathrm{d}\,x}{\|y - x\|^{d-k}}$$

is known as $(k+1)$st-order *chord power integral* of W (up to occasionally other multiplicative constants).

5.3 Other expressions for the asymptotic variance in case of isotropy

Applying Blaschke-Petkantschin-type formulae for convex bodies W in $\mathbb{R}^d$ lead to the identities, see [14], pp. 362-364,

$$I_{k+1}(W) = \frac{\omega_k}{k+1} \int_{\mathbb{A}(d,1)} \big(\, V_1(W \cap E)\,\big)^{k+1} \mu_1(\mathrm{d}\,E) = \int_{\mathbb{A}(d,k)} \big(\, V_k(W \cap E)\,\big)^2 \mu_k(\mathrm{d}\,E)\,,$$

where $V_k(\cdot)$ denotes the kth intrinsic volume and $\mathbb{A}(d,k)$ is the space of affine k-flats in $\mathbb{R}^d$ which carries the motion-invariant k-flat measure μ_k satisfying $\mu_k(\{E \in \mathbb{A}(d,k) : E \cap B_d(1) \neq \emptyset\}) = \omega_{d-k}$, see [14] for precise definitions and more details. By virtue of Carleman's inequality we get the estimate

$$I_{k+1}(W) \le \frac{2^{k+1}\,\omega_k\,\omega_{k+d}}{(k+1)\,d\,\omega_{k+1}} \Big(\frac{|W|_d}{\omega_d}\Big)^{(d+k)/d} \quad , \quad k = 1, \dots, d-1\,,$$

for convex W in $\mathbb{R}^d$ with "=" iff $W = B_d(r)$. Hence, for given volume of W, the variance of the volume of the motion-invariant set (4) is maximal in case of a spherical window.

5.4 CLT for stationary Poisson k-flat processes

If we choose $\Xi_0 = B_{d-k}(\varepsilon)$ in (4) with small $\varepsilon > 0$ then the approximative equation

$$|\, \Xi_{\lambda,Q_{d,k}} \cap \varrho\, W\,|_d = \omega_{d-k}\, \varepsilon^{d-k}\, \mathcal{H}_k(\Xi_{\lambda,P_0} \cap \varrho\, W\,) + \mathcal{O}(\varepsilon^{d-k+1}) \quad \text{as} \quad \varepsilon \downarrow 0$$

can be derived for fixed $\varrho \geq 1$, where $\Xi_{\lambda,P_0} = \bigcup_{i\geq 1} \Theta_i \left(P_i \times \mathbb{R}^k \right)$ is a union set of the Poisson k-flats. On the other hand, since $M_1 = \omega_{d-k}\, \varepsilon^{d-k}$ and $M_2(\theta) = (\omega_{d-k}\, \varepsilon^{d-k})^2$ for any θ, it is rapidly seen that

$$\lim_{\varepsilon\to 0} \frac{\mathbf{E}\,|\,\Xi_{\lambda,Q_{d,k}} \cap \varrho\, W\,|_d}{\omega_{d-k}\,\varepsilon^{d-k}} = \lambda\,\varrho^d\,|W|_d \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{\sigma^2_{\lambda,Q_{d,k}}(W)}{(\omega_{d-k}\,\varepsilon^{d-k})^2} = \lambda\,\sigma_0^2(W)$$

with $\sigma_0^2(W) = \int\limits_{\mathbb{R}^k} \mathbf{E}\big|\, W \cap (W - \Theta_0(\mathbf{o}_{d-k}, x)^T)\,\big|_d \,\mathrm{d}\,x$ regardless whether the directional distribution $\mathbf{P}_0$ is atomic or diffuse. These arguments can be made rigorous by the fact that $\mathcal{H}_k(\Xi_{\lambda,P_0} \cap \varrho\, W\,) = \sum_{i\geq 1} \mathcal{H}_k(\,\Theta_i\,(P_i \times \mathbb{R}^k) \cap \varrho\, W\,)$ is a stationary Poisson shot noise process on $\mathbb{R}^{d-k}$ which allows to deduce the CLT

$$\varrho^{-(d+k)/2}\Big(\,\mathcal{H}_k(\Xi_{\lambda,P_0} \cap \varrho\, W\,) - \lambda\,\varrho^d\,|W|_d\Big) \quad \xrightarrow[\varrho\to\infty]{\mathcal{D}} \quad \mathcal{N}(0, \lambda\,\sigma_0^2(W)) \tag{22}$$

for $k = 1,\ldots,d-1$, see e.g. [3]. Note that the special case $k = d-1$ of (22) is a by-product of Theorem 4.1 in [7].

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