Multiscale Analysis of Stochastic Partial Differential Equations

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Wael Wagih Elbayoumi Mohammed

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Supervisor
Prof. Dr. Dirk Blömker
Institut für Mathematik
Universität Augsburg
Germany

Erstgutachter:	Prof. Dr. Dirk Blömker	Universität Augsburg, Germany
Zweitgutachter:	Prof. Dr. Martin Hairer	Warwick University, United Kingdom

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Preface

A stochastic partial differential equation (SPDE) is a partial differential equation containing a random (noise) term. The study of SPDEs is an exciting topic which brings together techniques from probability theory, functional analysis, and the theory of partial differential equations.

Stochastic partial differential equations appear in several different applications:

- random evolution of systems with a spatial extension (random interface growth, random evolution of surfaces, fluids subject to random forcing),
- stochastic models where the state variable is infinite dimensional (for example, a curve or surface).

The solution to a stochastic partial differential equations may be viewed in several manners. One can view a solution as a random field (set of random variables indexed by a multidimensional parameter). Alternatively, in the case where the SPDE is an evolution equation, the infinite dimensional point of view consists in viewing the solution at a given time as a random element in a function space and thus view the SPDE as a stochastic evolution equation in an infinite dimensional space. In the pathwise point of view one tries to give a meaning to the solution for (almost) every realization of the noise and then view the solution as a random variable on the set of (infinite dimensional) paths thus defined.

All equations considered are parabolic nonlinear SPDEs perturbed by additive forcing. Near a change of stability, we can use the natural separation of time-scales, in order to derive simpler equations for the evolution of the dominant pattern. As these equations describe the amplitudes of dominant pattern, they are referred to as amplitude equations.

This work is based on one hand on Blömker, Hairer, and Pavliotis [9] and Roberts [40] for the Burgers' equation driven by degenerate noise (i.e. noise does not act directly on the dominant pattern). On the other hand it discusses the observation of Axel Hutt and collaborators [23–25] for the Swift-Hohenberg equation with degenerate noise. Where they established that constant noise in space leads to a deterministic amplitude equation by using a formal argument based on center manifold theory.

The aim of this thesis is to establish rigorously amplitude equations for quite general classes of SPDEs with quadratic or cubic nonlinearities. In the examples we investigate whether additive degenerate noise leads to stabilization of the solutions, or not.

The thesis consists of five chapters:

Chapter 1. This chapter is an introductory chapter. It contains some basic definitions, inequalities and some previously known results without proof for approximation of SPDEs via amplitude equations. These will be used throughout the next chapters and in the main results of approximation for the stochastic partial differential equations via amplitude equations.

Chapter 2. In this chapter we rigorously derive stochastic amplitude equations for SPDEs of the following type

$$du = \left[\mathcal{A}u + \varepsilon^2 \mathcal{L}u + B(u,u)\right] dt + \varepsilon^2 dW,$$

where B is a bilinear map modelling a quadratic nonlinearity. We also show that the solution u of the original SPDE is well approximated by the solution of the amplitude equation of the type

$$da = \left[\mathcal{L}_c a - 2\mathcal{F}(a)\right] dT + d\tilde{W}_c,$$

where $\mathcal{F}(a) = B_c(a, \mathcal{A}_s^{-1} B_s(a,a))$. We give applications to the one-dimensional Burgers' equation

$$\partial_t u = \left(\partial_x^2 + 1\right) u + \varepsilon^2 \nu u + u\partial_x u + \varepsilon^2 \partial_t W,$$

and a model from surface growth

$$\partial_t h = -\triangle^2 h - \mu_\varepsilon \triangle h - \triangle |\nabla h|^2 + \varepsilon^2 \partial_t W(t).$$

Chapter 3. In this chapter we derive rigorously an amplitude equation for

$$du = \left[\mathcal{A}u + \varepsilon^2 \mathcal{L}u + \mathcal{F}(u)\right] dt + \varepsilon dW,$$

where W is a degenerate noise acting on finitely many Fourier modes only. We also show that adding noise will stabilize the dynamics of the dominant modes. We focus on equations with cubic nonlinearity and give applications to the Swift-Hohenberg equation

$$\partial_t u = (\partial_x^2 + 1)^2 u + \nu\varepsilon^2 u - u^3 + \varepsilon\partial_t W(t),$$

the Allen-Cahn equation

$$\partial_t u = (\partial_x^2 + 1)u + \nu\varepsilon^2 u - u^3 + \varepsilon\partial_t W(t),$$

and a model from surface growth

$$\partial_t u = -\triangle^2 u - \mu_\varepsilon \triangle u + \nabla\left(|\nabla u|^2 \nabla u|\right) + \varepsilon\partial_t W(t).$$

Chapter 4. In this chapter we improve the result which we obtained in Chapter 3 in the case of one dimensional kernels of $\mathcal{A}$, by studying higher order corrections. Moreover, we give applications to the Swift-Hohenberg equation and the Allen-Cahn equation.

Chapter 5. The purpose of this chapter is to study the influence of large or unbounded domains, where there is a band of dominant pattern. This leads to a slow modulation of the dominant pattern changing stability. We derive rigorously the Ginzburg-Landau equation

$$\partial_T A = 4\partial_X^2 A + (\nu - \frac{3\sigma^2}{2})A - 3\left|A\right|^2 A\ ,$$

as a modulation equation for the stochastic Swift-Hohenberg equation

$$\partial_t U = \mathcal{A}U + \varepsilon^2 \nu U - U^3 + \varepsilon\sigma\partial_t \beta(t).$$

Here

$$U(t,x) \simeq \varepsilon A(\varepsilon^2 t, \varepsilon x)e^{ix} + \varepsilon \bar{A}(\varepsilon^2 t, \varepsilon x)e^{-ix}.$$

We show that adding noise will stabilize the modulation equation, and thus the dominant pattern.

Chapter 1

Introduction

In this chapter we will collect relevant results and notations from probability theory and functional analysis that we will need later. As most of the results are well known. We either give a proof, if they are short or relevant, or give a reference to the proof in order to keep the presentation short. This chapter is organized as follows. In the next section, we define the fractional Sobolev spaces $\mathcal{H}^\alpha$ and the semigroup $\left\{e^{t\mathcal{A}}\right\}_{t\geq 0}$ generated by the operator $\mathcal{A}$. Also, we state and prove some property for the semigroup $\left\{e^{t\mathcal{A}}\right\}_{t\geq 0}$. In Section 1.2, we recall the definition of stochastic process, Q-Wiener process, stochastic convolution, and martingale. We also summarize some results about the representation of the Q-Wiener process. In Section 1.3, we introduce the solution concept for certain types of stochastic evolution problems and prove existence and uniqueness of their solutions. In Section 1.4 we present some basic inequalities which we will use in our proofs. Finally, in Section 1.5 we recall some previously known results without proof for the approximation of SPDEs via amplitude equations.

1.1 Spaces and Semigroup

Throughout this thesis we will work in some Hilbert space $\mathcal{H}$ equipped with scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. In following definition let us define the fractional Sobolev spaces $\mathcal{H}^\alpha$.

Definition 1.1.1 *For $\alpha \in \mathbb{R}$, we define the space $\mathcal{H}^\alpha$ as*

$$\mathcal{H}^\alpha = \left\{ \sum_{k=1}^{\infty} \gamma_k e_k : \sum_{k=1}^{\infty} \gamma_k^2 k^{2\alpha} < \infty \right\} \quad \text{with norm} \quad \left\| \sum_{k=1}^{\infty} \gamma_k e_k \right\|_\alpha^2 = \sum_{k=1}^{\infty} \gamma_k^2 k^{2\alpha},$$

where $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$ and $\{\gamma_k\}_{k\in\mathbb{N}}$ are real numbers.

Definition 1.1.2 *If we suppose that $\mathcal{A}$ is a non-positive operator on $\mathcal{H}$ with eigenvalues $0 \le \lambda_1 \le .... \le \lambda_k \le ....$ and for some $m > 0$ we assume $\lambda_k \ge Ck^m$ for all sufficiently large k. Suppose we have a complete orthonormal system of eigenvectors $\{e_k\}_{k=1}^\infty$ such that $\mathcal{A}e_k = -\lambda_k e_k$. Then the operator $\mathcal{A}$ generates an analytic semigroup $\left\{e^{t\mathcal{A}}\right\}_{t\ge 0}$ defined by*

$$e^{\mathcal{A}t}\left(\sum_{k=1}^{\infty} \gamma_k e_k\right) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \gamma_k e_k \quad \forall\, t \ge 0.$$

The semigroup $\left\{e^{t\mathcal{A}}\right\}_{t\ge 0}$ has the following property:

Lemma 1.1.3 *For all $t > 0$ and $\beta \le \alpha$, there are constants $M > 0$ and $\tilde{M} = \delta^{-\frac{\alpha-\beta}{m}} M$ for any $\delta \in (0,1)$ such that for all $u \in \mathcal{H}^\beta$*

$$\left\| e^{t\mathcal{A}} u \right\|_\alpha \le M t^{-\frac{\alpha-\beta}{m}} \left\| u \right\|_\beta \,, \tag{1.1}$$

and

$$\left\| e^{t\mathcal{A}} u \right\|_\alpha \le \tilde{M} t^{-\frac{\alpha-\beta}{m}} e^{-\omega t} \left\| u \right\|_\beta \,, \tag{1.2}$$

where $\omega = (1-\delta)\lambda_1$.

Proof. Let $u = \sum_{k=1}^\infty u_k e_k$ be an element in H^β, then

$$\begin{aligned} \left\| e^{\mathcal{A}t} u \right\|_\alpha^2 &= \sum_{k=1}^{\infty} e^{-2t\lambda_k} k^{2\alpha} u_k^2 \\ &= \sum_{k=1}^{\infty} e^{-2t\lambda_k} k^{2(\alpha-\beta)} k^{2\beta} u_k^2 \\ &\le \sup_{k\in\mathbb{N}} \{ e^{-2t\lambda_k} k^{2(\alpha-\beta)} \} \left\| u \right\|_\beta^2 . \end{aligned}$$

Thus,

$$\left\| e^{\mathcal{A}t} u \right\|_\alpha \le \sup_{k\in\mathbb{N}} \{ e^{-t\lambda_k} k^{(\alpha-\beta)} \} \left\| u \right\|_\beta , \tag{1.3}$$

with

$$\begin{aligned}\sup_{k\in\mathbb{N}}\{e^{-t\lambda_k}k^{\alpha-\beta}\} &\leq \sup_{k>0}\{\exp\{-Ctk^m\}(kt^{\frac{1}{m}})^{(\alpha-\beta)}t^{-\frac{\alpha-\beta}{m}}\} \\ &= \underbrace{\sup_{z>0}\{\exp\{-Cz^m\}z^{(\alpha-\beta)}\}}_{:=M<\infty}t^{-\frac{\alpha-\beta}{m}},\end{aligned}$$

where $z = t^{\frac{1}{m}}k$. Thus,

$$\left\|e^{\mathcal{A}t}u\right\|_\alpha \leq Mt^{-\frac{\alpha-\beta}{m}}\left\|u\right\|_\beta .$$

Analogously, from (1.3), consider

$$\begin{aligned}\sup_{k>0}\{e^{-t\lambda_k}k^{\alpha-\beta}\} &= \sup_{k\geq 1}\{e^{-t\lambda_k(1-\delta)}e^{-\delta t\lambda_k}k^{\alpha-\beta}\} \\ &\leq \sup_{k\geq 1}\{e^{-t\lambda_1(1-\delta)}e^{-\delta t\lambda_k}k^{\alpha-\beta}\} \\ &= e^{-t\omega}\sup_{k\geq 1}\{e^{-\delta t\lambda_k}k^{\alpha-\beta}\} \\ &\leq Me^{-t\omega}(\delta t)^{-\frac{\alpha-\beta}{m}}.\end{aligned}$$

Thus,

$$\left\|e^{t\mathcal{A}}u\right\|_\alpha \leq \tilde{M}t^{-\frac{\alpha-\beta}{m}}e^{-\omega t}\left\|u\right\|_\beta .$$

□

Definition 1.1.4 *Let $L(\mathcal{H})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$.*

1. *$Q \in L(\mathcal{H})$ is called symmetric if*

$$\langle Qu, v\rangle = \langle u, Qv\rangle \quad \text{for all } u,\ v \in \mathcal{H}.$$

2. *$Q \in L(\mathcal{H})$ is called positive if $\langle Qu, u\rangle \geq 0$ for all $u \in \mathcal{H}$.*

3. *Let $Q \in L_1(\subset L(\mathcal{H}))$ and e_k be an orthonormal basis of $\mathcal{H}$ for $k \in \mathbb{N}$. The trace of Q is defined as*

$$trQ := \sum_{k=1}^{\infty}\langle Qe_k, e_k\rangle ,$$

if the series is convergent.

Let us recall that the trace is independent of the choice of the basis. For more results form the theory of operators in Hilbert spaces can be found in Chapter 1 [27].

1.2 Q-Wiener Process

In this section, we recall the definitions of stochastic process, Q-Wiener process, stochastic convolution, and martingale. We also summarize some results about the representation of the Q-Wiener process. We follow the presentation in [36] and [37].

Definition 1.2.1 *Let* $(\Omega,\ \mathcal{F},\ \mathbb{P})$ *be a probability space and* $T \subset \mathbb{R}$ *be an interval (possibly infinite). A* $\mathcal{H}$*-valued stochastic process* $\{X(t)\}_{t\in T}$ *is a set of* $\mathcal{H}$*-valued random variables* $X(t)$ *on* $(\Omega,\ \mathcal{F},\ \mathbb{P})$ *where* $t \in T$.

Definition 1.2.2 *(Q-Wiener process) Let* Q *be a symmetric, nonnegative, linear operator with* $trQ < \infty$. *A* $\mathcal{H}$*-valued stochastic process* $W(t)$, $t \geq 0$, *is called* Q*-Wiener process, if*

1. $W(0) = 0$;
2. $\{W(t)\}_{t\geq 0}$ *has continuous paths almost surely. That is, the mapping* $t \mapsto W(t,\omega)$ *is continuous for almost every* $\omega \in \Omega$;
3. $\{W(t)\}_{t\geq 0}$ *has independent increments. That is, for any finite partition* $0 = t_0 \leq t_1 \leq ..... \leq t_n < \infty$ *the random variables* $W(t_1), W(t_2) - W(t_1), ..., W(t_n) - W(t_{n-1})$, *are independent;*
4. $W(t) - W(s)$ *is* $\aleph(0,(t-s)Q)$*-distributed for all* $t > s \geq 0$.

Lemma 1.2.3 *If* $Q \in L(\mathcal{H})$ *is nonnegative and symmetric, with finite trace, then there exists an orthonormal basis* e_k, $k \in \mathbb{N}$, *of* $\mathcal{H}$ *such that*

$$Qe_k = \alpha_k^2 e_k \ \textit{ for } \ \alpha_k \geq 0, \quad k \in \mathbb{N},$$

and 0 *is the only accumulation point of the sequence* $\{\alpha_k\}_{k\in\mathbb{N}}$.

Proof. See Theorem VI.21; Theorem VI.16 (Hilbert-Schmidt theorem) in [39]. □

Remark 1.2.4 *Let* e_k, $k \in \mathbb{N}$, *be an orthonormal basis of* $\mathcal{H}$ *consisting of eigenvectors of* Q *with corresponding eigenvalues* α_k, $k \in \mathbb{N}$, *then*

$$trQ = \sum_{k=1}^{\infty} \alpha_k^2.$$

Proposition 1.2.5 *(Representation of the Q-Wiener process) Let e_k, $k \in \mathbb{N}$, be an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of Q with corresponding eigenvalues α_k, $k \in \mathbb{N}$. Then a $\mathcal{H}$-valued stochastic process $W(t)$, $t \in [0,T]$, is a Q-Wiener process if and only if*

$$W(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) e_k, \qquad t \in [0,T]\,,$$

where $\beta_k(t)$, $k \in \mathbb{N}$, are independent real-valued Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. See Proposition 2.1.10 in [37]. □

In the following definition, we define the stochastic convolution $W_{\mathcal{A}}$ for the operator $\mathcal{A}$ and the Q-Wiener process W.

Definition 1.2.6 *(Stochastic convolution) The stochastic convolution $W_{\mathcal{A}}$ is a stochastic process defined for $t \geq 0$, as*

$$W_{\mathcal{A}}(t) = \int_0^t e^{(t-s)\mathcal{A}} dW(s) = \sum_{k=1}^{\infty} \alpha_k \int_0^t e^{(t-s)\mathcal{A}} d\beta_k(t) e_k, \tag{1.4}$$

where $e^{t\mathcal{A}}$ is the semigroup generated by the operator $\mathcal{A}$.

Theorem 1.2.7 *If*

$$\sum_{k=1}^{\infty} \frac{\alpha_k^2 k^{2\alpha}}{\lambda_k^{1-2\gamma}} < \infty \quad \textit{for} \quad \frac{1}{2} > \gamma > 0,$$

then

$$W_{\mathcal{A}} \in C^0([0,T], \mathcal{H}^{\alpha}) \quad \mathbb{P} - a.s. \quad \forall \;\; T > 0.$$

Proof. See Theorem 5.9 in [36]. □

1.3 Stochastic Evolution Equations

In this section we introduce the solution concept for certain types of stochastic evolution problems and prove existence and uniqueness of their solutions. This is a standard approach based on Banach's fixed argument for mild solutions of SPDEs defined by the variation of constants formula.

The remainder of this section is organised as follows. In Subsection 1.3.1 we give all basic assumptions, while in Subsection 1.3.2 we sketch briefly the results on existence & uniqueness of local solutions. Finally, in Subsection 1.3.3 we give the Burgers' equation as an example. Nevertheless, the result would apply to all models discussed in this thesis.

1.3.1 Setting

We consider the following equation written formally as

$$\begin{aligned} \partial_t u(t) &= \mathcal{A}u(t) + f(u(t)) + \partial_t W(t) \\ u(0) &= u_0, \end{aligned} \tag{1.5}$$

where we make the following assumptions (cf. Definition 1.1.2):

Assumption (A1) $\mathcal{A}$ is a non-positive operator on $\mathcal{H}$ with eigenvalues $0 \leq \lambda_1 \leq .... \leq \lambda_k \leq ....$ such that $\lambda_k \geq Ck^m$ for all sufficiently large k, and a corresponding complete orthonormal system of eigenvectors $\{e_k\}_{k=1}^{\infty}$ such that $\mathcal{A}e_k = -\lambda_k e_k$.

Assumption (A2) The process $W(t)$ for $t \geq 0$ is a Q-Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$W_{\mathcal{A}} \in C^0([0,T], \mathcal{H}^\alpha) \quad \mathbb{P} - a.s. \;\forall\; T.$$

Assumption (A3) Define $f : \mathcal{H}^\alpha \to \mathcal{H}^\beta$ for $\alpha - m < \beta \leq \alpha$ such that f satisfies for some $C \geq 0$ and $p \geq 0$ the local Lipschitz condition

$$\|f(u) - f(v)\|_\beta \leq C \|u - v\|_\alpha (1 + \|u\|_\alpha + \|v\|_\alpha)^p,$$

for all $u, v \in \mathcal{H}^\alpha$.

1.3.2 Existence and Uniqueness of Mild Solutions

Before we state and prove the theorem of the existence and uniqueness of mild solutions, let us define the concept of a mild solution for Equation (1.5).

Definition 1.3.1 *(Mild solution) An $\mathcal{H}$-valued process $\{u(t)\}_{t\in[0,T]}$ is a mild solution of (1.5) if for some random time $\tau > 0$ we have $u \in C^0([0,\tau], \mathcal{H}^\alpha) \quad \mathbb{P} - a.s.$ such that*

$$u(t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}} f(u(s))ds + W_{\mathcal{A}}(t), \quad \forall\, t \in (0,\tau]\,, \mathbb{P} - a.s, \qquad (1.6)$$

where $W_{\mathcal{A}}$ is the stochastic convolution defined in (1.4).

Theorem 1.3.2 *Assume that assumptions (A1), (A2) and (A3) are satisfied. Given $u_0 \in \mathcal{H}^\alpha$. Then, there is a unique mild solution $u \in C^0([0,T], \mathcal{H}^\alpha)$ of (1.5).*

Proof. The proof is based on the classical fixed point theorem for contractions. Fix $\omega \in \Omega$ and define

$$\mathcal{G}(u)(t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}} f(u(s))ds + W_{\mathcal{A}}(t).$$

1. It is easy to check that

   $$\mathcal{G} : C^0([0,T], \mathcal{H}^\alpha) \to C^0([0,T], \mathcal{H}^\alpha) \quad \forall\, T > 0,$$

   as follows:

   - $t \to e^{t\mathcal{A}}u_0 \in C^0([0,T], \mathcal{H}^\alpha)$ as $u_0 \in \mathcal{H}^\alpha$. This follows from the regularity of the semigroup $e^{t\mathcal{A}}$,
   - $t \to W_{\mathcal{A}}(t) \in C^0([0,T], \mathcal{H}^\alpha)$ by Assumption (A2),
   - $f : C^0([0,T], \mathcal{H}^\alpha) \to C^0([0,T], \mathcal{H}^\beta)$ follows from the assumption on $f : \mathcal{H}^\alpha \to \mathcal{H}^\beta$ locally Lipschitz,
   - $\int_0^t e^{(t-s)\mathcal{A}}ds : C^0([0,T], \mathcal{H}^\beta) \to C^0([0,T], \mathcal{H}^\alpha)$ provided $\alpha - m < \beta \le \alpha$.

2. The operator $\mathcal{G}$ is a contraction:

   Define the set Γ as

   $$\Gamma = \{u \in C^0([0,T], \mathcal{H}^\alpha) \text{ such that } \|u(t) - u_0\|_\alpha \le \delta \;\; \forall\, t \in [0,T]\},$$

for some fixed $\delta > 0$. Let

$$|||u||| = \sup_{t\in[0,\tau]} \|u(t)\|_{\alpha}.$$

To show that $\mathcal{G}$ is a contraction on Γ, provided T is sufficiently small, we consider for $u, v \in \Gamma$

$$\|\mathcal{G}(u)(t) - \mathcal{G}(v)(t)\|_{\alpha} = \left\| \int_0^t e^{(t-\tau)\mathcal{A}}[f(u(s)) - f(v(s))]ds \right\|_{\alpha}.$$

Using (1.1), we obtain

$$\begin{aligned}
\|\mathcal{G}(u)(t) - \mathcal{G}(v)(t)\|_{\alpha} &\le M \int_0^t (t-s)^{\frac{\beta-\alpha}{m}} \|f(u(s)) - f(v(s))\|_{\beta} ds \\
&\le CM \int_0^t (t-s)^{\frac{\beta-\alpha}{m}} \|u - v\|_{\alpha} (1 + \|u\|_{\alpha} + \|v\|_{\alpha})^p ds \\
&\le Ct^{1+\frac{\beta-\alpha}{m}} |||u - v||| (2\delta + 2\|u_0\|_{\alpha} + 1)^p.
\end{aligned}$$

Thus, as $(2\delta + 2\|u_0\|_{\alpha} + 1)$ is constant

$$|||\mathcal{G}(u) - \mathcal{G}(v)||| \le CT^{1+\frac{\beta-\alpha}{m}} |||u - v|||.$$

We denoted by C various constant depending only on $u_0, \delta, \alpha, \beta$ and p but not on T. Choose T such that $CT^{1+\frac{\beta-\alpha}{m}} \le \frac{1}{2}$. Then $\mathcal{G} : \Gamma \to \Gamma$ is a contraction.

3. To show that $\mathcal{G} : \Gamma \to \Gamma$ is a self mapping, consider

$$\|\mathcal{G}(u)(t) - u_0\|_{\alpha} \le \|\mathcal{G}(u)(t) - \mathcal{G}(u_0)\|_{\alpha} + \|\mathcal{G}(u_0) - u_0\|_{\alpha}.$$

As $\mathcal{G}$ is a contraction

$$\begin{aligned}
\|\mathcal{G}(u)(t) - u_0\|_{\alpha} &\le \frac{1}{2}\|u - u_0\|_{\alpha} + \|\mathcal{G}(u_0) - u_0\|_{\alpha} \\
&\le \frac{\delta}{2} + \|\mathcal{G}(u_0) - u_0\|_{\alpha}.
\end{aligned}$$

Consider

$$\begin{aligned}
\|\mathcal{G}(u_0) - u_0\|_{\alpha} &\le \left\|e^{t\mathcal{A}}u_0 - u_0\right\|_{\alpha} + \left\| \int_0^t e^{(t-s)\mathcal{A}} f(u_0) ds \right\|_{\alpha} + \|W_{\mathcal{A}}(t)\|_{\alpha} \\
&\le \left\|(e^{t\mathcal{A}} - \mathcal{I})u_0\right\|_{\alpha} + M \int_0^t (t-s)^{\frac{\beta-\alpha}{m}} \|f(u_0)\|_{\beta} ds + \|W_{\mathcal{A}}(t)\|_{\alpha},
\end{aligned}$$

where we used Lemma 1.1.3. Thus,

$$\begin{aligned} \|\mathcal{G}(u_0) - u_0\|_\alpha &\le \sup_{t\in[0,T]} \left\|(e^{t\mathcal{A}} - \mathcal{I})u_0\right\|_\alpha + \sup_{t\in[0,T]} \|W_\mathcal{A}(t)\|_\alpha + CT^{1+\frac{\beta-\alpha}{m}} \|f(u_0)\|_\beta \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

We note that

- $I_1 \to 0$ when $T \to 0$ from the continuity of $e^{t\mathcal{A}}$ in 0,
- $I_2 \to 0$ when $T \to 0$ from the continuity of $W_\mathcal{A}$, as $W_\mathcal{A}(0) = 0$,
- $I_3 \to 0$ when $T \to 0$, obviously.

Hence,

$$\|\mathcal{G}(u_0) - u_0\|_\alpha \le \frac{\delta}{2}.$$

If $T = T(u_0, \omega, \delta)$ is sufficiently small. In the end we obtain for $t \in [0, T]$

$$\|\mathcal{G}(u)(t) - u_0\|_\alpha \le \delta.$$

Thus, $\mathcal{G} : \Gamma \to \Gamma$ is a self mapping.

Therefore, by Banach's fixed point theorem, there is a unique fixed point $u \in \Gamma$, which is the unique mild solution of (1.5) on $[0, T]$.

Thus, for $u_0 \in \mathcal{H}^\alpha$ there is a random time $\tau > 0$ such that there exist a unique solution $u \in C^0([0, \tau], \mathcal{H}^\alpha)$ of $u = \mathcal{G}(u)$.

□

1.3.3 An Example

In this subsection we apply the abstract framework to the stochastic Burgers' equation driven by additive noise. Consider

$$\partial_t u = \partial_x^2 u + \lambda u + \partial_x u^2 + \partial_t W(t), \tag{1.7}$$

on $[0, \pi]$ subject to Dirichlet boundary conditions. Define

$$\mathcal{H} = L^2[0, \pi], \quad e_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx), \quad for \;\; k \in \mathbb{N},$$

and

$$f(u) = \nu u + \partial_x u^2.$$

We consider $\mathcal{A} = \partial_x^2$ as a linear operator on $\mathcal{H}$. It is well known that $\mathcal{A}$ is self-adjoint, positive definite and

$$-\mathcal{A}e_k = \lambda_k e_k,$$

for the orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of $\mathcal{H}$. Moreover, the $\lambda_k = k^2$ are an increasing sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdot\cdot, \ \lambda_k \to \infty \ \text{ as } \ k \to \infty.$$

Let us now verify the local Lipschitz condition. Consider

$$f(u) - f(v) = \nu(u - v) + \partial_x(u^2 - v^2).$$

Thus,

$$\|f(u) - f(v)\|_\beta \leq |\nu| \, \|u - v\|_\beta + \left\|\partial_x(u^2 - v^2)\right\|_\beta . \tag{1.8}$$

Consider

$$\begin{aligned} \left\|\partial_x(u^2 - v^2)\right\|_\beta &= \|\partial_x(u - v)(u + v)\|_\beta \\ &\leq C\,\|(u - v)(u + v)\|_{L^1} \\ &\leq C\,\|u - v\|_{L^2} \left(\|u\|_{L^2} + \|v\|_{L^2}\right), \end{aligned}$$

where we used for $\beta < \frac{-3}{2}$ that

$$\|\partial_x u\|_\beta = \sup_{\|v\|_{-\beta}=1} \int_0^\pi (\partial_x u)\, v dx.$$

Integrating by parts, we obtain

$$\begin{aligned} \|\partial_x u\|_\beta &= \sup_{\|v\|_{-\beta}=1} \int_0^\pi u\,(\partial_x v)\, dx \\ &\leq \|u\|_{L^1} \sup_{\|v\|_{-\beta}=1} \|\partial_x v\|_\infty . \end{aligned}$$

Now, consider $v = \sum_{k=1}^{\infty} \alpha_k e_k \in \mathcal{H}^{-\beta}$, then for $\beta < \frac{-3}{2}$ we obtain

$$\begin{aligned} \|\partial_x v\|_\infty &= \left\|\sum_{k=1}^{\infty} \alpha_k \partial_x e_k\right\|_\infty \leq \sum_{k=1}^{\infty} |\alpha_k| \, \|\partial_x e_k\|_\infty \\ &\leq C \sum_{k=1}^{\infty} |\alpha_k| \, k \leq C \left(\sum_{k=1}^{\infty} \alpha_k^2 k^{-2\beta}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} k^{2+2\beta}\right)^{\frac{1}{2}} \\ &= C\,\|v\|_{-\beta} . \end{aligned}$$

Thus,

$$\|\partial_x u\|_\beta \le \|u\|_{L^1} \cdot \sup_{\|v\|_{-\beta}=1} \|\partial_x v\|_\infty \le C \|u\|_{L^1}.$$

Returning again to (1.8), we obtain

$$\begin{aligned} \|f(u)-f(v)\|_\beta &\le |\nu| \|u-v\|_\beta + C\|u-v\|_{L^2} (\|u\|_{L^2} + \|v\|_{L^2}) \\ &\le C\|u-v\|_\beta (1+\|u\|_{L^2} + \|v\|_{L^2}). \end{aligned}$$

Then according to Theorem 1.3.2 the Equation (1.7) has a unique mild solution $u \in C^0([0,\tau],\mathcal{H})$ given by

$$u(t) = e^{t\mathcal{A}}u_0 + \int_0^t e^{(t-s)\mathcal{A}} f(u(s))ds + W_{\mathcal{A}}(t).$$

1.4 Basic Inequalities

In this section we present some basic results frequently used in the proofs in the following chapters.

Lemma 1.4.1 *(Chebychev's inequality) If $X:\Omega\to\mathbb{R}$ is a random variable such that*

$$\mathbb{E}[|X|^p] < \infty \quad \textit{for some } p \in (0,\infty),$$

then

$$\mathbb{P}[|X| \ge \lambda] \le \frac{1}{\lambda^p}\mathbb{E}[|X|^p] \textit{ for all } \lambda > 0.$$

Theorem 1.4.2 *(Young's inequality (cf. Theorem A.5 in [5])) For p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ there is a constant $C>0$ such that*

$$xy \le C(x^p + y^q) \quad \textit{for all } x,\ y>0.$$

Especially, for all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$xy \le \varepsilon x^p + C_\varepsilon y^q \quad \textit{for all } x,\ y>0.$$

Lemma 1.4.3 *(Itô Isometry) For all Hilbert space valued stochastic processes f adapted to the filtration of the Brownian motion β, there is a constant C such that*

$$\mathbb{E}\left\|\int_0^t f(s)d\beta(s)\right\|^2 = C\mathbb{E}\int_0^t \|f(s)\|^2 ds.$$

Proof. See Lemma 3.1.5 in [28]. □

We also need the celebrated Itô Formula. Here we will state only a simplified version. For the general case see for example [36].

Theorem 1.4.4 *(Itô Formula) Let $\{u(t)\}_{t\geq 0}$ be a stochastic process in $\mathcal{H}$ and let β be a standard real-valued Brownian motion. Suppose that*

$$du = f(u)dt + g(u)d\beta,$$

for some functions $f, g : \mathcal{H} \to \mathcal{H}$. Then for a twice continuously differentiable function $\varphi : \mathcal{H} \to \mathbb{R}$, we have

$$\begin{aligned}\varphi(u(t)) - \varphi(u(0)) &= \int_0^t D\varphi(u(s))\,[f(u(s))]\,ds + \int_0^t D\varphi(u(s))\,[g(u(s))]\,d\beta(s) \\ &\quad + \frac{1}{2}\int_0^t D^2\varphi(u(s))\,[g(u(s)), g(u(s))]\,ds.\end{aligned}$$

Theorem 1.4.5 *(Burkholder-Davis-Gundy (cf. Theorem A.7 in [5])). Let β be a Brownian motion, and f some $\mathcal{H}$-valued stochastic process adapted to β. Then for all $p > 0$, there exists a constant $C = C_p > 0$, depending only on p, such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\int_0^t f(s)d\beta(s)\right\|^p\right) \leq C_p \cdot \mathbb{E}\left(\int_0^T \|f(s)\|^2\,ds\right)^{\frac{p}{2}}.$$

Theorem 1.4.6 *(Doob) Consider f and β as in Theorem 1.4.5. Then for arbitrary $p > 1$*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\int_0^t f(s)d\beta(s)\right\|^p\right) \leq \frac{p}{p-1}\mathbb{E}\left\|\int_0^T f(s)d\beta(s)\right\|^p.$$

Theorem 1.4.7 *(Burkholder-Davis-Gundy (cf. Theorem 1.2.4 in [32])) For arbitrarily given $T > 0$, let $\phi(t,\omega), t \in [0,T]$, be an $\mathcal{F}_t$-adapted, $L_2^0(\mathcal{H})$-valued process such that $\mathbb{E}\int_0^T \|\phi(s,\omega)\|_{L_2^0}^2\,ds < \infty$, where $L_2^0(\mathcal{H})$ is the family of all Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{H}$. Then for arbitrary $p > 0$, there exists a constant $C = C_p > 0$, depending only on p, such that for any $T \geq 0$,*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\int_0^t \phi(s,\omega)dW_s\right\|^p\right) \leq C_p\mathbb{E}\left(\int_0^T \|\phi(s,\omega)\|_{L_2^0}^2\,ds\right)^{\frac{p}{2}},$$

where $\|\cdot\|_{L_2^0}$ denotes the Hilbert-Schmidt norm.

Lemma 1.4.8 *If* $X = \sum_{k=1}^{\infty} X_k e_k$, *for independent real valued Gaussian* X_k *with* $\mathbb{E}X_k = 0$, *then for all* $p > 0$, *there exists a constant* $C_p > 0$ *such that*

$$\mathbb{E}\| \mathbf{X} \|_\alpha^{2p} \leq C_p \left(\mathbb{E}\| \mathbf{X} \|_\alpha^2\right)^p.$$

We give an elementary proof here. Also the result can be found in [36], Corollary 2.17, by using characteristic functions.

Proof. We consider two cases.

First case $p \in \mathbb{N}$. In this case

$$\begin{aligned}
\mathbb{E} \parallel X \parallel_\alpha^{2p} &= \mathbb{E}\| \sum_{k=1} X_k e_k \|_\alpha^{2p} = \mathbb{E}\left(\sum_{k=1} X_k^2 k^{2\alpha}\right)^p \\
&= \sum_{k_1=1} \dots\dots\dots \sum_{k_p=1} k_1^{2\alpha} \dots\dots k_p^{2\alpha} \mathbb{E}\left(X_{k_1}^2 \dots\dots X_{k_p}^2\right) \\
&\leq C_p \sum_{k_1=1} \dots\dots\dots \sum_{k_p=1} k_1^{2\alpha} \dots\dots k_p^{2\alpha} \mathbb{E}X_{k_1}^2 \dots\dots \mathbb{E}X_{k_p}^2 \\
&= C_p \left(\sum_{k=1} \mathbb{E}X_k^2 k^{2\alpha}\right)^p = C_p \left(\mathbb{E}\| \mathbf{X} \|_\alpha^2\right)^p.
\end{aligned}$$

Second case $p \notin \mathbb{N}$. In this case, using Hölder inequality, we obtain for $k > p$, $k \in \mathbb{N}$

$$\mathbb{E}\| \mathbf{X} \|_\alpha^{2p} \leq \left(\mathbb{E}\| \mathbf{X} \|_\alpha^{2k}\right)^{\frac{p}{k}}.$$

We finish the proof by using the first case. □

Let us finally recall Gronwall's lemma as follow:

Lemma 1.4.9 *(Gronwall's lemma (cf. Lemma A.8 in [5])) Let* $u : [0, T] \to \mathbb{R}$ *and* $a : [0, T] \to \mathbb{R}$ *be continuous functions, such that* $a \geq 0$. *Fix* $b \in \mathbb{R}$. *Then*

$$u(t) \leq b + \int_0^t a(s)u(s)ds \quad \textit{for all } t \in [0, T],$$

implies

$$u(t) \leq b \cdot \exp\left\{\int_0^t a(s)ds\right\} \quad \textit{for all } t \in [0, T].$$

1.5 Approximation via Amplitude Equation

Amplitude equations are well known in the physics literature (see, e.g., [20] or [46]). They usually describe some order parameter for the system, which evolves on a much slower time-scale. This separation of time-scale is present in all cases where a change of stability occurs.

The approximation of SPDEs on bounded domians via amplitude equations was first rigorously verified in [10] for a simple Swift-Hohenberg model, and later on extended in [3, 4, 6]. Here the amplitude equation for the dominant modes is given by ODE or SDE.

In contrast to that in the case of unbounded domain or just very large domains the situation is significant different. The amplitude of the dominant modes are subject to a long-range modulation in space, and hence not given by an ODE/SODE, but by some PDE/SPDE instead.

The case of large domain, but still bounded domains, is discussed in [8]. See also [34] for the deterministic equation.

The main difference between small and large domains is the existence of a large spectral gap of order $\mathcal{O}(1)$ in the linearised operator of the PDE. On bounded domains, we have a finite number $e = (e_1, ..., e_n)$ of modes (or eigenfunctions) such that the corresponding eigenvalues change sign at the change of stability. If we are close to the bifurcation, all other eigenvalues are negative and sufficiently far away from 0. Formal arguments show, that the amplitudes $A \in R^n$ of the dominating modes are given by the so called amplitude equation, while the solution u of the SPDE is well approximated by

$$u(t, x) = \varepsilon A(\varepsilon^2 t) \cdot e(x) + \mathcal{O}(\varepsilon^2),$$

where ε^2 is the typical scale for the distance from bifurcation.

On unbounded or just very large domains this picture changes completely. Even very close to the bifurcation a large number of modes are near or already above the threshold of stability, but still small. In this case the amplitude A is also a function in x that is concentrated in Fourier space near the dominant modes. Hence, A is subject to slow modulations in space, taking into account the large number of weakly (un)stable modes. In this case the solution u of the SPDE is approximated by

$$u(t,x) = \varepsilon A(\varepsilon^2 t, \varepsilon x) \cdot e(x) + \mathcal{O}(\varepsilon^2),$$

and A fulfills a (stochastic) PDE, which is called amplitude or modulation equation.

Let us now state some previous results without proof which been used before to approximate the solution of SPDEs with additive noise via the solution of amplitude equation.

in the literature there are numerous examples of SPDEs with additive noise. For instance,

$$\partial_t u = \mathcal{A}u + \varepsilon^2 \mathcal{L}u + B(u,u) + \sigma\xi, \tag{1.9}$$

and

$$\partial_t u = \mathcal{A}u + \varepsilon^2 \mathcal{L}u - \mathcal{F}(u) + \sigma\xi, \tag{1.10}$$

where $\mathcal{A}$ is non-positive operator with finite dimensional kernel, $\varepsilon^2\mathcal{L}u$ is a small deterministic perturbation, B is a symmetric and bilinear operator, $\mathcal{F}$ is a cubic nonlinearity and ξ is a Gaussian noise in space and time.

Blömker [4] established the following theorem for (1.9) with $\sigma = \varepsilon^2$ and noise being the generalised derivative of some Wiener process $\{QW(t)\}_{t\geq 0}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where W is the standard cylindrical Wiener process.

Theorem 1.5.1 *Fix $\delta > 0$, some small $1 \gg \kappa \geq 0$, and some $T_0 > 0$. Let a be the solution of the amplitude equation*

$$a(T) = a(0) + \int_0^T \mathcal{L}_c a(\tau)d\tau - 2\int_0^T B_c(a, \mathcal{A}_s^{-1} B_s(a))d\tau + \tilde{W}_c(T),$$

with initial condition $a(0) = \varepsilon^{-1} P_c u(0)$, and ψ is the solution of

$$\psi(t) = e^{t\mathcal{A}}\psi(0) + \int_0^t e^{(t-\tau)\mathcal{A}} B_s(a(\varepsilon^2\tau), a(\varepsilon^2\tau))d\tau + \int_0^t e^{(t-\tau)\mathcal{A}} dW_s(\tau),$$

with $\psi(0) = \varepsilon^{-2} P_s u(0)$. Then for all mild solutions u of (1.9)

$$\mathbb{P}\left(\sup_{t\in[0,\varepsilon^{-2}T_0]} \left\| u(t) - \varepsilon a(\varepsilon^2 t) - \varepsilon^2 \psi(\varepsilon^2 t) \right\| \leq \ln(\varepsilon^{-1})\varepsilon^{2-2\kappa} \right)$$

$$\geq 1 - \mathbb{P}\left\{ \|u(0) > 2\delta\| \right\} - \mathbb{P}\left\{ \|P_s u(0)\| > \delta\varepsilon^2 \right\} - o_\varepsilon(1).$$

In Chapter 2, we will extend the above result to a fairly large class of noise given by Q-Wiener processes. Moreover, we improve probability estimate significantly.

Blömker, Hairer and Pavliotis [9] gave a rigorous proof for (1.9) with $\sigma = \varepsilon$, degenerate noise, $\mathcal{L} = \nu\mathcal{I}$, and $\ker\mathcal{A} = span\{e_1\}$ by a multiscale analysis. They showed that, although not forced directly, the amplitude equation includes the fluctuations from the fast mode due to the nonlinear interaction.

Theorem 1.5.2 *Let u be a continuous solution of (1.9) with initial condition $u(0)$ such that $\|u(0)\| < C\varepsilon$ for $C > 0$. Furthermore, assume that the covariance of the noise satisfies $q_k = \sigma$ for $k \geq 2$. Then there exists a Brownian motion $B(t)$ such that, if $a(t)$ is a solution of*

$$da(t) = \tilde{\nu}a(t) - \frac{1}{12}a^3(t) + \sqrt{\sigma_b + \sigma_a a^2(t)}dB(t), \quad \varepsilon a(0) = \frac{2}{\pi}\langle u(0), e_1\rangle,$$

where

$$\tilde{\nu} = \nu + \frac{\sigma^2}{36\pi} - \frac{\sigma^2}{4\pi}\sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k(k+1)}\right)\frac{1}{2k^2+2k-1},$$

$$\sigma_a = \frac{\sigma^2}{18\pi},\ \sigma_b = \frac{1}{2\pi^2}\sum_{k=2}^{\infty}\frac{\sigma^4}{(2k^2+2k+1)(k^2-1)(k^2+2k)},$$

and

$$R(t) = \frac{1}{\varepsilon}e^{t\mathcal{A}}P_s u(0) + \int_0^t e^{(t-\tau)\mathcal{A}}dQW(\tau),$$

then for all p, $\kappa > 0$ there is a constant C such that

$$\mathbb{P}\left(\sup_{t\in[0,\varepsilon^{-2}T]}\left\|u(t) - \varepsilon a(\varepsilon^2 t)e_1 - \varepsilon R(t)\right\| \leq C\varepsilon^{\frac{5}{4}-\kappa}\right) \geq 1 - C\varepsilon^p,$$

for all $\varepsilon \in (0,1)$.

For the deterministic equation (1.10), with $\nu = 1$, on unbounded domain. Kirrmann, Mielke and Schneider [26] proved the following approximation result for the deterministic Swift-Hohenberg equation (1.10) through the Ginzburg-Landau equation

$$\partial_T A = 4\partial_x^2 A + A - 3|A|^2 A. \tag{1.11}$$

Theorem 1.5.3 *Let $A = A(T, X) \in C_b^4(\mathbb{R})$, be a solution of (1.11) with initial condition $A(0, X) \in C_b^4(\mathbb{R})$, where $C_b^4(\mathbb{R})$ denotes the space of 4-times differentiable functions, where all derivatives and the function are bonded and uniformly continuous. Define*

$$u_A(t, x) = \varepsilon A(T, X)e^{ix} + \varepsilon \bar{A}(T, X)e^{-ix},$$

where $T = \varepsilon^2 t$ and $X = \varepsilon x$. Then, for each $T_0 > 0$ and $d > 0$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following statement holds. Let $u = u(t, x)$ be a solution of the deterministic equation (1.10) such that $|u(0, x) - u_A(0, x)| \le d\varepsilon^2$ for all x. Then the estimate

$$|u(t, x) - u_A(t, x)| < C\varepsilon^2, \quad \textit{for all } (t, x) \in \left[0, \varepsilon^{-2}T_0\right] \times \mathbb{R},$$

is satisfied.

Blömker, Hairer and Pavliotis [8] considered the SPDEs (1.10), with $\sigma = \varepsilon^{\frac{3}{2}}$, on a large domain $\left[\frac{-L}{\varepsilon}, \frac{L}{\varepsilon}\right]$ near its change of stability and showed that, under appropriate scaling, its solutions can be approximated by the solution of the stochastic Ginzburg-Landau equation

$$\partial_T A = 4\partial_X^2 A + \nu A - 3|A|^2 A + \sqrt{\hat{q}(1)}\eta, \quad X \in [-L, L], \ A(0) = A_0, \tag{1.12}$$

where η is complex space-time white noise and $\hat{q}$ is the Fourier transform of q. The noise strength $\hat{q}(1)$ is derived from the spatial correlation function q of ξ.

Theorem 1.5.4 *Let u be a solution of (1.10) with an admissible initial condition $u_0(x) = 2\varepsilon\Re\left(A_0(\varepsilon x)e^{ix}\right)$ and A be a solution of (1.12) with initial condition A_0. Then, for every $T_0 > 0$, $\kappa > 0$ and $p \ge 1$,*

$$\mathbb{E} \sup_{t \in [0, \varepsilon^{-2}T_0]} \sup_{x \in \left[\frac{-L}{\varepsilon}, \frac{L}{\varepsilon}\right]} \left|u(t, x) - 2\varepsilon\Re\left(A(\varepsilon^2 t, \varepsilon x)e^{ix}\right)\right|^p \le C_{\kappa,p}\varepsilon^{\frac{3}{2}p - p\kappa},$$

for every $\varepsilon \in (0, 1]$.

In the above theorem, we stated the admissible random variable. This mean we can split this random variable into two parts such that the first part is in $\mathcal{H}^1$ space and the second part is a Gaussian in C^0 space, and both parts are of order one.

In Chapter 5, we will consider an intermediate step. Using unbounded domains, but highly degenerate finite dimensional noise, where the amplitude equation will be deterministic.

On bounded domains Blömker and Hairer [6] gave a rigorous formulation of the approximation result for the transient dynamics of (1.10), with $\sigma = \varepsilon^2$, including higher order corrections as follow:

Theorem 1.5.5 *Let*

$$u(t) = e^{t\mathcal{A}}u(0) + \int_0^t e^{(t-\tau)\mathcal{A}} \left(\varepsilon^2 \mathcal{L}u(\tau) + \mathcal{F}(u(\tau))\right) d\tau + \varepsilon^2 \int_0^t e^{(t-\tau)\mathcal{A}} dW(\tau),$$

be the mild solution of (1.10) with initial condition $u(0) = u_0$ satisfying for some family of positive constant $\{C_p,\ p \geq 1\}$,

$$\mathbb{E}\,\|u_0\|^p \leq C_p\varepsilon^p \quad \text{and} \quad \mathbb{E}\,\|P_s u_0\|^p \leq C_p\varepsilon^p.$$

Define ψ as

$$\psi(t) := \varepsilon a(\varepsilon^2 t) + \varepsilon^2 e^{t\mathcal{A}} P_s\psi(0) + \varepsilon^2 \int_0^t e^{(t-\tau)\mathcal{A}} dP_s W(\tau),$$

with initial condition $P_s\psi(0) = \varepsilon^{-2}P_s u(0)$ and a is a solution of

$$\partial_T a(T) = \mathcal{L}_c a(T) + \mathcal{F}_c(a(T)) + \partial_T \beta(T),$$

with initial condition $a(0) = \varepsilon^{-1}P_c u_0$ and Brownian motion $\beta(T) = \varepsilon P_c QW(\varepsilon^{-2}T)$. Then for all $T_0 > 0$, $\kappa > 0$ and $p > 1$ there is a constant $C > 0$ such that the estimate

$$\mathbb{P}\left(\sup_{t\in[0,\varepsilon^{-2}T_0]} \|u(t) - \psi(t)\| \leq \varepsilon^{3-\kappa}\right) \geq 1 - C\varepsilon^p,$$

holds for $\varepsilon > 0$ sufficiently small.

In Chapter 3, we will study a combination of Theorem 1.5.2 and Theorem 1.5.5 by using degenerate noise for an SPDE with cubic nonlinearity. In Chapter 4, we also study higher order corrections. This is related to the work of Wang and Roberts [41]. They considered the SPDEs (1.10) with $\mathcal{A} = (\triangle + 1)$ and $\sigma = \varepsilon$, on bounded domain $(0, \pi)$ to study higher order corrections to the amplitude equation, in order to see the fluctuations induced by the impact of the noise on the dominant pattern as follow:

Theorem 1.5.6 *For any $T > 0$, there is positive constant $C > 0$, such that for any solution $(u^\varepsilon, v^\varepsilon)$ of*

$$\partial_t u^\varepsilon = A_c u^\varepsilon + P_c \mathcal{F}(u^\varepsilon + v^\varepsilon),$$

$$\partial_t v^\varepsilon = A_c v^\varepsilon + P_s \mathcal{F}(u^\varepsilon + v^\varepsilon) + \varepsilon P_s \partial_t W,$$

where P_c is the projection operator, $P_s = \mathcal{I} - P_c$ and A_c is high-pass filter defined by $A_c = (P_s + \varepsilon^2 P_c)\triangle$, there is a N-dimensional Wiener process $\tilde{W}$ such that with high probability

$$\sup_{t\in[0,\varepsilon^{-2}T]} \left\| u^\varepsilon(t) - \varepsilon a(\varepsilon^2 t) - \varepsilon^2 \rho_c(\varepsilon^2 t) \right\| \le C\varepsilon^{2+},$$

where a solves

$$\partial_t a = \triangle a + P_c \tilde{\mathcal{F}}_0(a),$$

and ρ_c solves the following stochastic differential equation

$$\partial_t \rho_c = A_c \rho_c + P_c[\tilde{\mathcal{F}}_0'(a)\rho_c] + \sqrt{B(a)}\partial_t \tilde{W},$$

with $\rho_c(0) = 0$. Here, the average

$$\tilde{\mathcal{F}}_0(a) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathcal{F}_0(a + \psi^*(s))ds,$$

where $\mathcal{F}_0$ is the cubic component of $\mathcal{F}$ and ψ^* is the stationary solution solving the linear stochastic partial differential equation

$$\partial_t \psi = A_c \psi + \varepsilon P_s \partial_t W,$$

and

$$\begin{aligned} B(a) &= 2\int_0^\infty \mathbb{E}\left[P_c\mathcal{F}_0(a+\psi^*(s)) - P_c\tilde{\mathcal{F}}_0(a)\right] \\ &\quad \otimes \left[P_c\mathcal{F}_0(a+\psi^*(0)) - P_c\tilde{\mathcal{F}}_0(a)\right] ds, \end{aligned}$$

where $\otimes$ is the tensor product. Furthermore

$$\mathbb{E}\sup_{t\in[0,\varepsilon^{-2}T]} \|v^\varepsilon(t) - \varepsilon\psi^*(t)\| \le C\varepsilon^3.$$

Chapter 2

Amplitude Equations for SPDEs with Quadratic Nonlinearities

2.1 Introduction

Stochastic partial differentail equations (SPDEs) with quadratic nonlinearities arise in various applications in physics. One example is the stochastic Burgers' equation in the study of closure models for hydrodynamic turbulence [14]. Other examples are the growth of rough amorphous surfaces [38, 45], and the Kuramoto-Sivashinsky model, which originally models a fire front, but it is also used for surface erosion [17, 30]. All these models fit in the abstract framework of this chapter.

Consider the following SPDE in Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$:

$$du = \big[\mathcal{A}u + \varepsilon^2\mathcal{L}u + B(u,u)\big]\,dt + \varepsilon^2 dW. \tag{2.1}$$

We consider (2.1) near a change of stability, where $\varepsilon^2\mathcal{L}u$ measures the distance from bifurcation. The operator $\mathcal{A}$ is assumed to be self-adjoint and non-positive, and we call the kernel of $\mathcal{A}$ the dominant modes. We allow for noise given by a fairly general Q-Wiener process.

Near the bifurcation the equation exhibits two widely separated characteristic time-scales and it is desirable to obtain a simplified equation which governs the evolution of the dominant modes. This is well known on a formal level in many examples in physics (see e.g. [16]). Moreover, for deterministic PDEs on unbounded domains, this method [19, 26, 43, 44] successfully overcomes the gap of a lacking centre manifold theory. This is also useful for SPDEs on bounded domains [7], where no centre manifold theory is available yet.

Moreover, there are numerous variants of this method. However, most of these results are non-rigorous approximations using this type of formal multi-scale analysis. A noteable example is [18].

The purpose of this chapter is to derive rigorously an amplitude equation for a quite general class of SPDEs (cf. (2.1)) with quadratic nonlinearities. This work is based on [9], where degenerate noise in a different scaling was considered, and it improves significantly previously know results of [4], where in a similar situation much more regular noise was considered. A related result can be found in [5], where a simple multiplicative noise was considered, but again with much weaker results.

In this chapter we follow [11] and focus on quadratic nonlinearities only. The case of cubic equations is much simpler, as one can rely on nonlinear stability. This case was already considered in [6], for instance.

As an application of our approximation result of Theorem 2.3.1, we discuss the stochastic Burgers' equation and surface growth model. To illustrate our results consider the Burgers' equation

$$\partial_t u = \left(\partial_x^2 + 1\right) u + \varepsilon^2 \nu u + u \partial_x u + \varepsilon^2 \partial_t W, \tag{2.2}$$

on $[0, \pi]$ subject to Dirichlet boundary conditions.

We show in our main result that near a change of stability on a time-scale of order ε^{-2} the solution of (2.2) is of the type

$$u(t, x) = \varepsilon b(\varepsilon^2 t) \sin(x) + \mathcal{O}(\varepsilon^2),$$

where b is the solution of the amplitude equation on the slow time-scale

$$\partial_T b(T) = \nu b(T) - \frac{1}{12} b^3(T) + \partial_T \beta(T),$$

and β is a Brownian motion with a suitable variance.

This approximating equation is called amplitude equation, as it is rewritten into an SDE for the amplitudes of an expansion of the dominant modes with respect to a basis in $\mathcal{N}$.

For the proofs we rely on a cut-off technique, as in general we cannot control moments of solution and exclude the possibility of a blow up. Therefore all estimates are established only with high probability and not in moments. To be more precise, we use a stopping time, in order to look only at solutions that are not too large. Then we can use moments for time uniformly up to the stopping time. Later we use the amplitude equation itself to verify that the stopping is not small.

As the general strategy we first show that all non-dominant modes are given by an Ornstein-Uhlenbeck process and a quadratic term in the dominant modes. Then we rely on Itô -Formula and some averaging argument, in order to transform the equation for the dominant modes to an amplitude equation with an additional small remainder.

The rest of this chapter is organised as follows. In Section 2.2 we state the assumptions that we make. In Section 2.3 we give a formal derivation of the amplitude equation and state the main results. In Section 2.4 we give the main results. Finally, in Section 2.5 we apply our theory to the stochastic Burgers' equation and surface growth model.

2.2 Main Assumptions and Definitions

This section summarises all assumptions necessary for our results. For the linear operator $\mathcal{A}$ in (2.1) we assume the following:

Assumption 2.2.1 ***(Linear Operator** $\mathcal{A}$**)*** *Suppose $\mathcal{A}$ is a self-adjoint and non-positive operator on $\mathcal{H}$ with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq .... \leq \lambda_k \leq ....$ and $\lambda_k \geq Ck^m$ for all large k and for $m > 0$. The corresponding complete orthonormal system of eigenvectors is $\{e_k\}_{k=1}^{\infty}$ with $\mathcal{A}e_k = -\lambda_k e_k$.*

We use the notation $\mathcal{N} := \ker \mathcal{A}$, $\mathcal{S} = \mathcal{N}^{\perp}$ the orthogonal complement of $\mathcal{N}$ in $\mathcal{H}$, and P_c for the projection $P_c : \mathcal{H} \to \mathcal{N}$. Define, $P_s := \mathcal{I} - P_c$, and suppose that P_c and P_s commute with $\mathcal{A}$. Suppose that $\mathcal{N}$ has finite dimension n with basis $(e_1, ..., e_n)$.

Definition 2.2.2 *Define the operator D^{α} by $D^{\alpha}e_k = k^{\alpha}e_k$, so that $\|u\|_{\alpha} = \|D^{\alpha}u\|$.*

Assumption 2.2.3 ***(Operator** $\mathcal{L}$**)*** *Fix $\alpha \in \mathbb{R}$ and let $\mathcal{L} : \mathcal{H}^\alpha \to \mathcal{H}^{\alpha-\beta}$ for some $\beta \in [0, m)$ be a continuous linear mapping that in general does not commute with P_c and P_s.*

Assumption 2.2.4 ***(Bilinear Operator** B**)*** *With α and β from Assumption 2.2.3 let B be a bounded bilinear mapping from $\mathcal{H}^\alpha \times \mathcal{H}^\alpha$ to $\mathcal{H}^{\alpha-\beta}$. Suppose without loss of generality that B is symmetric, i.e. $B(u, v) = B(v, u)$. Moreover, assume that $P_c B(u, u) = 0$ for $u \in \mathcal{N}$.*

Remark 2.2.5 *If B is not symmetric we can use*

$$\tilde{B}(u, v) := \frac{1}{2}B(u, v) + \frac{1}{2}B(v, u).$$

Denote for shorthand notation $B_s = P_s B$ and $B_c = P_c B$.

For the nonlinearity appearing later in the amplitude equation we define the following.

Definition 2.2.6 *Define $\mathcal{F} : \mathcal{N} \to \mathcal{N}$, for $u \in \mathcal{N}$, as*

$$\mathcal{F}(u, u, u) := B_c(u, \mathcal{A}_s^{-1} B_s(u, u)). \tag{2.3}$$

Assume without loss of generality that $\mathcal{F}$ is given by a symmetric map $\mathcal{F} : \mathcal{N}^3 \to \mathcal{N}$.

By Assumption 2.2.4 the operator $\mathcal{F}$ is already trilinear, continuous and therefore bounded. One standard example being a cubic like u^3.

Remark 2.2.7 *If $\mathcal{F}$ is not symmetric we can always use*

$$\tilde{\mathcal{F}}(u, v, w) := \frac{1}{3}B_c(u, \mathcal{A}_s^{-1} B_s(v, w)) + \frac{1}{3}B_c(w, \mathcal{A}_s^{-1} B_s(u, v)) + \frac{1}{3}B_c(v, \mathcal{A}_s^{-1} B_s(w, u)).$$

Moreover, we assume the following:

Assumption 2.2.8 ***(Stability)*** *Assume that the nonlinearity $\mathcal{F}$ satisfies the following conditions*

$$\langle u, \mathcal{F}(u) \rangle > 0 \quad \forall\, u \in \mathcal{N} - \{0\}, \tag{2.4}$$

and

$$\langle \mathcal{F}(u, u, w), w \rangle > 0 \;\; \forall\, u,\; w \in \mathcal{N} - \{0\}, \tag{2.5}$$

where we define $\mathcal{F}(u) = \mathcal{F}(u, u, u)$ for short.

Remark 2.2.9 *From Assumption 2.2.8 there exist* $\delta > 0$ *such that*

$$\langle u, \mathcal{F}(u)\rangle \geq \delta \|u\|^4 \quad \forall\, u \in \mathcal{N}\,, \tag{2.6}$$

and

$$\langle \mathcal{F}(u,u,w), w\rangle \geq \delta \|u\|^2 \|w\|^2 \quad \forall\, u,\ w \in \mathcal{N}.$$

For the noise we suppose:

Assumption 2.2.10 ***(Wiener Process*** W***)*** *Let* W *be a Wiener process on an abstract probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with a bounded covariance operator* $Q : \mathcal{H} \to \mathcal{H}$ *defined by* $Qf_k = \alpha_k^2 f_k$ *where* $(\alpha_k)_k$ *is a bounded sequence of real numbers and* $(f_k)_{k\in\mathbb{N}}$ *is an orthonormal basis in* $\mathcal{H}$*. For the orthonormal basis* e_k *from Assumption 2.2.1 we assume*

$$\sum_{k=1}^{n}\sum_{l=n+1}^{\infty} \frac{1}{\lambda_l} k^{\alpha} l^{\alpha} \left|\langle Qe_k, e_l\rangle\right| < \infty \quad \textit{and} \quad \sum_{l=n+1}^{\infty} l^{2\alpha}\lambda_l^{2\gamma-1}\|Q^{\frac{1}{2}}e_l\|^2 < \infty, \tag{2.7}$$

for some $\gamma \in (0, \frac{1}{2})$.

We note that $W(t)$ and $\varepsilon W(\varepsilon^{-2}t)$ are in law the same process due to scaling properties.

Let us discuss two different representations of W. One with the basis e_k and the other one with f_k. For $t \geq 0$, we can write $W(t)$ (cf. Da Prato and Zabczyk [36]) as

$$W(t) = \sum_{k=1}^{\infty}\alpha_k\beta_k(t)f_k = \sum_{l=1}^{\infty}ß_l(t)e_l, \tag{2.8}$$

where $(\beta_k)_k$ are independent, standard Brownian motions in $\mathbb{R}$. Furthermore, the

$$ß_l := \sum_{k=1}^{\infty}\alpha_k\langle f_k, e_l\rangle\beta_k \tag{2.9}$$

are real valued Brownian motions, which are in general not independent.

Moreover, it follows easily from the definition of P_c, P_s and $W(t)$ that

$$P_cW(t) = \sum_{k=1}^{\infty}\alpha_k\beta_k(t)P_cf_k = \sum_{l=1}^{n}ß_l(t)e_l, \tag{2.10}$$

and

$$P_sW(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) P_s f_k = \sum_{l=n+1}^{\infty} ß_l(t) e_l. \tag{2.11}$$

For our result we rely on a cut off argument. We consider only solutions (a, ψ) that are not too large. To be more precise we introduce a cut-off time, after which the solution is too big. Later we will show that this time is large with high probability.

Definition 2.2.11 ***(Stopping Time)*** *For the $\mathcal{N}\times\mathcal{S}$-valued stochastic process (a, ψ) defined later in (2.14) we define, for some small $0 < \kappa < \frac{1}{7}$ and some time $T_0 > 0$, the stopping time τ^* as*

$$\tau^* := T_0 \wedge \inf\left\{T > 0 : \|a(T)\|_\alpha > \varepsilon^{-\kappa} \text{ or } \|\psi(T)\|_\alpha > \varepsilon^{-3\kappa}\right\}. \tag{2.12}$$

Definition 2.2.12 *For a real-valued family of processes $\{X_\varepsilon(t)\}_{t\geq 0}$ we say $X_\varepsilon = \mathcal{O}(f_\varepsilon)$, if for every $p \geq 1$ there exists a constant C_p such that*

$$\mathbb{E} \sup_{t\in[0,\tau^*]} |X_\varepsilon(t)|^p \leq C_p f_\varepsilon^p. \tag{2.13}$$

We use also the analogous notation for time-independent random variables.

Finally note, that we use the letter C for all constants that depend only on other constants like T_0, κ, or α and the data of the equation given by B, Q, $\mathcal{L}$, and $\mathcal{A}$.

2.3 Formal Derivation and the Main Result

Let us first discuss a formal derivation of the Amplitude equation corresponding to Equation (2.1). We split the solution u into

$$u(t) = \varepsilon a(\varepsilon^2 t) + \varepsilon^2 \psi(\varepsilon^2 t)\ , \tag{2.14}$$

with $a \in \mathcal{N}$ and $\psi \in \mathcal{S}$, and rescale to the slow time scale $T = \varepsilon^2 t$, in order to obtain for the dominant modes

$$da = [\mathcal{L}_c a + \varepsilon \mathcal{L}_c \psi + 2B_c(a, \psi) + \varepsilon B_c(\psi, \psi)]\, dT + d\tilde{W}_c. \tag{2.15}$$

For the fast modes we derive

$$d\psi = [\varepsilon^{-2}\mathcal{A}_s\psi + \varepsilon^{-1}\mathcal{L}_s a + \mathcal{L}_s\psi + \varepsilon^{-2}B_s(a,a) + 2\varepsilon^{-1}B_s(a,\psi) \\ + B_s(\psi,\psi)]dT + \varepsilon^{-1}d\tilde{W}_s \ , \tag{2.16}$$

where $\tilde{W}(T) := \varepsilon W(\varepsilon^{-2}T)$ is a rescaled version of the Wiener process. Now we use (2.16) in order to remove ψ from Equation (2.15).

From (2.16) we obtain in lowest order of ε that

$$\mathcal{A}_s\psi \approx -B_s(a,a).$$

As $\mathcal{A}_s$ is invertible on $\mathcal{S}$, we derive

$$\psi \approx -\mathcal{A}_s^{-1}B_s(a,a),$$

which we substitute into (2.15). Neglecting all small terms in ε, yields

$$da \approx [\mathcal{L}_c a - 2\mathcal{F}(a)]\, dT + d\tilde{W}_c.$$

Thus, we consider solutions $b : [0,T_0] \to \mathcal{N}$ of

$$db = [\mathcal{L}_c b - 2\mathcal{F}(b)]\, dT + d\tilde{W}_c. \tag{2.17}$$

This is the amplitude equation that approximates the dynamics of the original SPDE. The main aim of this chapter is to show that the solution of (2.1)

$$u(t) = \varepsilon b(\varepsilon^2 t) + \mathcal{O}(\varepsilon^{2-}) \ .$$

In the following, let us be more precise. Applying Itô's formula to $B_c(a, \mathcal{A}_s^{-1}\psi)$ we obtain

$$\begin{aligned} dB_c(a(T), A_s^{-1}\psi(T)) &= B_c(da, A_s^{-1}\psi) + B_c(a, A_s^{-1}d\psi) + \frac{1}{2}B_c(da, A_s^{-1}d\psi) \\ &= B_c(L_c a, A_s^{-1}\psi)dT + \varepsilon B_c(L_c\psi, A_s^{-1}\psi)dT \\ &\quad + B_c(a, A_s^{-1}L_s\psi)dT + \varepsilon^{-1}B_c(a, A_s^{-1}L_s a)dT \\ &\quad + 2B_c(B_c(a,\psi), A_s^{-1}\psi)dT + \varepsilon B_c(B_c(\psi,\psi), A_s^{-1}\psi)dT \\ &\quad + B_c(d\tilde{W}_c, A_s^{-1}\psi) + \varepsilon^{-2}B_c(a, A_s^{-1}B_s(a,a))dT \\ &\quad + \varepsilon^{-2}B_c(a,\psi)dT + 2\varepsilon^{-1}B_c(a, A_s^{-1}B_s(a,\psi))dT \\ &\quad + B_c(a, A_s^{-1}B_s(\psi,\psi))dT + \varepsilon^{-1}B_c(a, A_s^{-1}d\tilde{W}_s) \\ &\quad - \frac{\varepsilon}{2}B_c(d\tilde{W}_c(T), \mathcal{A}_s^{-1}d\tilde{W}_s(T)). \end{aligned}$$

Integrating from 0 to T, yields

$$\begin{aligned}
\int_0^T B_c(a(\tau),\psi(\tau))d\tau \;&=\; \varepsilon^2 B_c(a(T),A_s^{-1}\psi(T)) - \int_0^T \mathcal{F}(a(\tau))d\tau \\
&-\varepsilon^2\int_0^T B_c(L_c a, A_s^{-1}\psi)d\tau - \varepsilon^3\int_0^T B_c(L_c\psi, A_s^{-1}\psi)d\tau \\
&-\varepsilon^3\int_0^T B_c(B_c(\psi,\psi),A_s^{-1}\psi)d\tau - \varepsilon\int_0^T B_c(a, A_s^{-1}L_s a)d\tau \\
&-2\varepsilon^2\int_0^T B_c(B_c(a,\psi),A_s^{-1}\psi)d\tau - \varepsilon^2\int_0^T B_c(d\tilde{W}_c, A_s^{-1}\psi) \\
&-\varepsilon^2\int_0^T B_c(a, A_s^{-1}L_s\psi)d\tau - 2\varepsilon\int_0^T B_c(a, A_s^{-1}B_s(a,\psi))d\tau \\
&-\varepsilon^2\int_0^T B_c(a, A_s^{-1}B_s(\psi,\psi))d\tau - \varepsilon\int_0^T B_c(a, A_s^{-1}d\tilde{W}_s) \\
&-\frac{\varepsilon}{2}\int_0^T B_c(d\tilde{W}_c(\tau), \mathcal{A}_s^{-1}d\tilde{W}_s(\tau)). \qquad (2.18)
\end{aligned}$$

Integrating (2.15) and using (2.18) we obtain the amplitude equation with remainder

$$a(T) = a(0) + \int_0^T \mathcal{L}_c a(\tau)d\tau - 2\int_0^T \mathcal{F}(a(\tau))d\tau + \tilde{W}_c(T) + R(T), \qquad (2.19)$$

where the remainder R is given by

$$\begin{aligned}
R(T) =&\varepsilon^2 B_c(a(T),\mathcal{A}_s^{-1}\psi(T)) - 2\varepsilon^2\int_0^T B_c(B_c(a(\tau),\psi(\tau)),\mathcal{A}_s^{-1}\psi(\tau))d\tau \\
&-\varepsilon^3\int_0^T B_c(B_c(\psi(\tau),\psi(\tau)),\mathcal{A}_s^{-1}\psi(\tau))d\tau - \varepsilon^2\int_0^T B_c(\mathcal{L}_c a, \mathcal{A}_s^{-1}\psi)d\tau \\
&-2\varepsilon\int_0^T B_c(a(\tau),\mathcal{A}_s^{-1}B_s(a(\tau),\psi(\tau)))d\tau - \varepsilon^3\int_0^T B_c(\mathcal{L}_c\psi, \mathcal{A}_s^{-1}\psi)d\tau \\
&-\varepsilon\int_0^T B_c(a,\mathcal{A}_s^{-1}\mathcal{L}_s a)d\tau - \varepsilon^2\int_0^T B_c(a,\mathcal{A}_s^{-1}\mathcal{L}_s\psi)d\tau \\
&+\varepsilon\int_0^T \mathcal{L}_c\psi(\tau)d\tau - \varepsilon^2\int_0^T B_c(a(\tau),\mathcal{A}_s^{-1}B_s(\psi(\tau),\psi(\tau)))d\tau \\
&+\varepsilon\int_0^T B_c(\psi(\tau),\psi(\tau))d\tau - \varepsilon^2\int_0^T B_c(dW_c(\tau),\mathcal{A}_s^{-1}\psi(\tau)) \\
&-\varepsilon\int_0^T B_c(a(\tau),\mathcal{A}_s^{-1}d\tilde{W}_s(\tau)) - \varepsilon\int_0^T B_c(d\tilde{W}_c(\tau),\mathcal{A}_s^{-1}d\tilde{W}_s(\tau)).
\end{aligned} \qquad (2.20)$$

For our main aim we need to show that the remainder R is of order ε. This involves carefully analysis of all terms using moments of uniform bounds up to the stopping time like $\mathbb{E}\sup_{[0,\tau^*]}\|R\|_\alpha^p$. Later, we need an explicit error estimate to actually remove R from the equation. Finally, we use the nonlinear stability of the amplitude equation to show that $\tau^* = T_0$ with high probability.

To be more precise, the main result is:

Theorem 2.3.1 ***(Approximation)*** *Under Assumptions 2.2.1, 2.2.3, 2.2.4 and 2.2.10, let u be a solution of (2.1) defined in (2.14) with the initial condition $u(0) = \varepsilon a(0) + \varepsilon^2\psi(0)$ where $a(0)$ and $\psi(0)$ are of order one. Suppose that b is a solution of the amplitude equation (2.17). Then for all $p > 1$ and $T_0 > 0$ there exists a constant $C > 0$ such that*

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\|u(t)-\varepsilon b(\varepsilon^2 t)\|_\alpha > \varepsilon^{2-7\kappa}\Big) \le C\varepsilon^p\ . \tag{2.21}$$

2.4 Proof of the Main Result

As a first step of the approximation result, we show that in (2.14) the modes $\psi \in \mathcal{S}$ are essentially an OU-process plus a quadratic term in the modes $a \in \mathcal{N}$. Later we use this to replace ψ in (2.15). After this, we proceed to show that ψ is with high probability not too large.

Lemma 2.4.1 *Under Assumption 2.2.1, 2.2.3, 2.2.4 and 2.2.10 let $z(T)$, $T > 0$ be the $\mathcal{S}$-valued process solving the SDE*

$$dz = \varepsilon^{-2}\mathcal{A}_s z dT + \varepsilon^{-1}d\tilde{W}_s, \quad z(0) = \psi(0). \tag{2.22}$$

Then for $\varepsilon \in (0,1)$ and $T \le \tau^*$

$$\Big\|\psi(T) - z(T) - \varepsilon^{-2}\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}B_s(a(\tau),a(\tau))d\tau\Big\|_\alpha \le C\varepsilon^{1-5\kappa}. \tag{2.23}$$

Proof. The mild formulation of (2.16) is

$$\psi(T) = z(T) + \int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}\left[\mathcal{L}_s\psi + \varepsilon^{-1}\mathcal{L}_s a + \varepsilon^{-2}B_s(a+\varepsilon\psi)\right]d\tau.$$

Thus, we derive

$$\begin{aligned}
&\Big\|\psi(T) - z(T) - \varepsilon^{-2}\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}B_s(a,a)d\tau\Big\|_\alpha \\
\leq\ & \Big\|\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}\mathcal{L}_s\psi(\tau)d\tau\Big\|_\alpha + \varepsilon^{-1}\Big\|\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}\mathcal{L}_s a(\tau)d\tau\Big\|_\alpha \\
& +2\varepsilon^{-1}\Big\|\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}B_s(a(\tau),\psi(\tau))d\tau\Big\|_\alpha \\
& +\Big\|\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}B_s(\psi(\tau),\psi(\tau))d\tau\Big\|_\alpha \\
=:\ & I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We now bound all four terms separately. Using Lemma 1.1.3 with $0 \leq \beta < m$, we obtain for the first term for all $T \leq \tau^*$

$$\begin{aligned}
I_1 &= \Big\|\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}\mathcal{L}_s\psi(\tau)d\tau\Big\|_\alpha \\
&\leq C\varepsilon^{\frac{2\beta}{m}}\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\left\|\psi(\tau)\right\|_\alpha d\tau \\
&\leq C\varepsilon^{2-3\kappa}\ ,
\end{aligned}$$

where we used the definition of τ^* and Assumption 2.2.3. Analogously, to the second term, we obtain for all $T \leq \tau^*$

$$I_2 \leq C\varepsilon^{\frac{2\beta}{m}-1}\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\left\|\mathcal{L}_s a(\tau)\right\|_{\alpha-\beta} d\tau \leq C\varepsilon^{1-\kappa}\ .$$

For the third term we obtain

$$\begin{aligned}
I_3 &\leq C\varepsilon^{\frac{2\beta}{m}-1}\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\|B_s(a(\tau),\psi(\tau))\|_{\alpha-\beta}d\tau \\
&\leq C\varepsilon^{\frac{2\beta}{m}-1}\sup_{\tau\in[0,\tau^*]}\|B_s(a(\tau),\psi(\tau))\|_{\alpha-\beta}\int_0^T e^{-\varepsilon^{-2}\omega\tau}\tau^{-\frac{\beta}{m}}d\tau.
\end{aligned}$$

Using Assumption 2.2.4, yields for $T \leq \tau^*$,

$$I_3 \leq C\varepsilon\sup_{\tau\in[0,\tau^*]}\{\|a(\tau)\|_\alpha\|\psi(\tau)\|_\alpha\}\int_0^{\varepsilon^{-2}\omega T} e^{-\eta}\eta^{-\frac{\beta}{m}}d\eta \leq C\varepsilon^{1-4\kappa}.$$

Analogously, we derive for the fourth term

$$\begin{aligned} I_4 &\leq \varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}} \left\|B_s(\psi(\tau),\psi(\tau))\right\|_{\alpha-\beta} d\tau \\ &\leq C\varepsilon^{\frac{2\beta}{m}} \sup_{\tau\in[0,\tau^*]} \|B_s(\psi(\tau),\psi(\tau))\|_{\alpha-\beta} \int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}} d\tau \\ &\leq C\varepsilon^2 \sup_{\tau\in[0,\tau^*]} \|\psi(\tau)\|_\alpha^2 \int_0^{\varepsilon^{-2}\omega T} e^{-\eta}\eta^{-\frac{\beta}{m}} d\eta \leq C\varepsilon^{2-6\kappa}\,. \end{aligned}$$

Combining all four results yields (2.23). □

In the following we will show that $\psi \ll \mathcal{O}(\varepsilon^{-3\kappa})$. First, the following Lemma provides bounds for the stochastic convolution based on the well know factorisation method. This also implies bounds for the process z defined in (2.22).

Lemma 2.4.2 *Under Assumption 2.2.1 and 2.2.10, let* $\|z(0)\|_\alpha = \mathcal{O}(1)$. *Now for every* $\kappa_0 > 0$, $p > 1$, *and* $T_0 > 0$, *there exists a constant* $C > 0$ *such that*

$$\mathbb{E}\Big(\sup_{T\in[0,T_0]} \|z(T)\|_\alpha^{2p}\Big) \leq C\varepsilon^{-\kappa_0}. \tag{2.24}$$

Proof. The mild solution of equation (2.22) is given by

$$z(T) = e^{\varepsilon^{-2}\mathcal{A}_s T} z(0) + \varepsilon^{-1}\tilde{W}_{\varepsilon^{-2}\mathcal{A}_s}(T). \tag{2.25}$$

The bound on $z(T)$ depends on the bound on $\tilde{W}_{\varepsilon^{-2}\mathcal{A}_s}$. We will use the factorization method introduced in [35] to prove the bound on $\tilde{W}_{\varepsilon^{-2}\mathcal{A}_s}$, which is based on the following elementary identity

$$\int_\sigma^T (T-r)^{\gamma-1}(r-\sigma)^{-\gamma} dr = \frac{\pi}{\sin \pi\gamma} \ \text{ for } \ \sigma \leq r \leq T,\ 0<\gamma<1. \tag{2.26}$$

Fix $\gamma \in \left(0, \frac{1}{2}\right)$. By using identity (2.26), we obtain:

$$\tilde{W}_{\varepsilon^{-2}\mathcal{A}_s}(T) = C_\gamma \int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\sigma)} \left[\int_\sigma^T (T-r)^{\gamma-1}(r-\sigma)^{-\gamma} dr\right] d\tilde{W}_s(\sigma).$$

From the stochastic Fubini theorem, we obtain

$$\begin{aligned} \tilde{W}_{\varepsilon^{-2}\mathcal{A}_s}(T) &= C_\gamma \int_0^T \int_0^s e^{\varepsilon^{-2}\mathcal{A}_s(T-\sigma)}(T-s)^{\gamma-1}(s-\sigma)^{-\gamma} d\tilde{W}_s(\sigma) ds \\ &= C_\gamma \int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-s)}(T-s)^{\gamma-1} \int_0^s e^{\varepsilon^{-2}\mathcal{A}_s(s-\sigma)}(s-\sigma)^{-\gamma} d\tilde{W}_s(\sigma) ds. \end{aligned}$$

Thus,

$$\tilde{W}_{\varepsilon^{-2}\mathcal{A}_s}(T) = C_\gamma \int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-s)}(T-s)^{\gamma-1}y(s)ds, \tag{2.27}$$

with $y(s) := \int_0^s e^{\varepsilon^{-2}\mathcal{A}_s(s-\sigma)}(s-\sigma)^{-\gamma}d\tilde{W}_s(\sigma)$. Hence, by Gaussianity

$$\mathbb{E}\,\|y(s)\|_\alpha^{2p} \leq C_p \left(\mathbb{E}\,\|y(s)\|_\alpha^2\right)^p.$$

Using the series expansion (cf. (2.11)), yields

$$y(s) = \sum_{l=n+1}^{\infty} \int_0^s e^{-\varepsilon^{-2}(s-\sigma)\lambda_l}(s-\sigma)^{-\gamma}d\tilde{ß}_l(\sigma)e_l.$$

Using Itô-Isometry in order to obtain

$$\begin{aligned}\mathbb{E}\,\|y(s)\|_\alpha^{2p} &\leq C_p \left(\sum_{l=n+1}^{\infty} l^{2\alpha}\mathbb{E}\left(\int_0^s e^{-\varepsilon^{-2}(s-\sigma)\lambda_l}(s-\sigma)^{-\gamma}d\tilde{ß}_l(\sigma)\right)^2\right)^p \\ &= C_p\varepsilon^{2p-4p\gamma}\left(\sum_{l=n+1}^{\infty} l^{2\alpha}\left(\lambda_l\right)^{2\gamma-1}\left\|Q^{\frac{1}{2}}e_l\right\|^2 \int_0^{\frac{\varepsilon^2 s}{2\lambda_l}} e^{-\tau}\tau^{-2\gamma}d\tau\right)^p,\end{aligned}$$

where we used

$$(d\tilde{ß}_l(\sigma))^2 = \sum_{k=1}^{\infty} \alpha_k^2\langle f_k, e_l\rangle^2 d\sigma = \|Q^{\frac{1}{2}}e_l\|^2 d\sigma. \tag{2.28}$$

Integrating from 0 to T_0, we obtain

$$\mathbb{E}\int_0^{T_0} \|y(s)\|_\alpha^{2p}\,ds \leq C\varepsilon^{2p-4\gamma p}. \tag{2.29}$$

Taking the $\mathcal{H}^\alpha$ norm in (2.27), yields

$$\|\tilde{W}_{\varepsilon^{-2}\mathcal{A}_s}(T)\|_\alpha^{2p} \leq C\Big(\int_0^T e^{(-\varepsilon^{-2}\omega)(T-s)}(T-s)^{\gamma-1}\|y(s)\|_\alpha ds\Big)^{2p}.$$

Using Hölder inequality with $\frac{1}{2p}+\frac{1}{2q}=1$ for sufficiently large p implies

$$\|\tilde{W}_{\varepsilon^{-2}\mathcal{A}_s}(T)\|_\alpha^{2p} \leq C\varepsilon^{4p\gamma-2}\int_0^T \|y(s)\|_\alpha^{2p}ds.$$

Hence, using (2.29) we obtain

$$\mathbb{E}\sup_{T\in[0,T_0]}\|\tilde{W}_{\varepsilon^{-2}\mathcal{A}_s}(T)\|_\alpha^{2p} \leq C\varepsilon^{4p\gamma-2}\int_0^{T_0}\mathbb{E}\|y(s)\|_\alpha^{2p}ds \leq C\varepsilon^{2p-2}.$$

For the bound on z take the norm of Equation (2.25) to obtain for sufficiently large p

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,T_0]}\|z(T)\|_\alpha^{2p} &\leq C\left[\mathbb{E}\sup_{T\in[0,T_0]}\|e^{\varepsilon^{-2}T\mathcal{A}_s}z(0)\|_\alpha^{2p}+\varepsilon^{-2p}\mathbb{E}\sup_{T\in[0,T_0]}\|\tilde{W}_{\varepsilon^{-2}\mathcal{A}_s}(T)\|_\alpha^{2p}\right]\\ &\leq C\mathbb{E}\sup_{T\in[0,T_0]}e^{-2p\varepsilon^{-2}\omega T}\|z(0)\|_\alpha^{2p}+C\cdot\varepsilon^{-2p}\cdot\varepsilon^{2p-2}\\ &\leq C\varepsilon^{-2}.\end{aligned}$$

Using Hölder inequality we derive for all $p>1$ and sufficiently large $q>\frac{2}{\kappa_0}$

$$\mathbb{E}\sup_{T\in[0,T_0]}\|z(T)\|_\alpha^{2p}\leq\mathbb{E}\Big(\sup_{T\in[0,T_0]}\|z(T)\|_\alpha^{2pq}\Big)^{\frac{1}{q}}\leq C\varepsilon^{-\kappa_0}\ ,$$

where the constant C depends among other things on T, p, and κ_0. □

We now need the following simple estimate.

Lemma 2.4.3 *Under Assumption 2.2.1 and 2.2.4, using τ^* defined in Definition 2.2.11,*

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\left\|\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}B_s(a\,(\tau)\,,a\,(\tau))d\tau\right\|_\alpha^{2p}\leq C\varepsilon^{4p-4p\kappa}, \tag{2.30}$$

for all $\varepsilon\in(0,1)$.

Proof. Using Lemma 1.1.3 and Assumption 2.2.4 we obtain for $T<\tau^*$

$$\begin{aligned}\Big\|\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}B_s(a,a)d\tau\Big\|_\alpha &\leq C\varepsilon^{\frac{2\beta}{m}}\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\|B_s(a,a)\|_{\alpha-\beta}d\tau\\ &\leq C\varepsilon^2\sup_{\tau\in[0,\tau^*]}\|a(\tau)\|_\alpha^2\int_0^{\varepsilon^{-2}\omega T}e^{-\eta}\eta^{-\frac{\beta}{m}}d\eta\\ &\leq C\varepsilon^{2-2\kappa}.\end{aligned}$$

□

Now we can proceed to bound ψ. The following lemma states that $\psi(T)$ is with high probability much smaller than $\varepsilon^{-3\kappa}$, as asserted by the Definition 2.2.11 for $T\leq\tau^*$. Here a key fact is that in the definition of τ^* we have $a=\mathcal{O}(\varepsilon^{-\kappa})$, while $\psi=\mathcal{O}(\varepsilon^{-3\kappa})$, but we already proved that ψ is essentially a quadratic term in a.

Lemma 2.4.4 *Let the assumptions of Lemmas 2.4.1, 2.4.2, and 2.4.3 be true. Then for all* $p \geq 1$ *there is a constant* $C > 0$ *such that*

$$\mathbb{E} \sup_{T \in [0,\tau^*]} \|\psi(T)\|_\alpha^{2p} \leq C\varepsilon^{-4p\kappa}. \tag{2.31}$$

Proof. From (2.23), by triangle inequality and Lemma 2.4.1, we obtain

$$\begin{aligned} \mathbb{E} \sup_{[0,\tau^*]} \|\psi\|_\alpha^{2p} \leq C\varepsilon^{2p-10p\kappa} + C\mathbb{E} \sup_{[0,\tau^*]} \|z\|_\alpha^{2p} \\ + C\varepsilon^{-4p}\mathbb{E} \sup_{[0,\tau^*]} \Big\| \int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)} B_s(a,a) d\tau \Big\|_\alpha^{2p}. \end{aligned}$$

We finish the proof by using Lemma 2.4.2 and 2.4.3. □

Corollary 2.4.5 *Under the assumptions of Lemma* 2.4.4, *there is for every every* $p > 1$ *a constant* $C > 0$ *such that*

$$\mathbb{P}\Big(\sup_{T \in [0,\tau^*]} \|\psi(T)\|_\alpha < \varepsilon^{-3\kappa} \Big) \geq 1 - C\varepsilon^{2p\kappa}. \tag{2.32}$$

Proof. From Chebychev inequality

$$\mathbb{P}\Big(\sup_{[0,\tau^*]} \|\psi\|_\alpha < \varepsilon^{-3\kappa} \Big) \geq 1 - \varepsilon^{6\kappa p} \cdot \mathbb{E} \sup_{[0,\tau^*]} \|\psi\|_\alpha^{2p}.$$

We finish the proof by using (2.31). □

Now the next step is to bound the remainder R defined in (2.20), and use it in order to show the approximation result later.

Lemma 2.4.6 *We assume that Assumptions 2.2.1, 2.2.3, 2.2.4, and 2.2.10 hold. Then for all* $p > 1$ *there exists a constant* $C > 0$ *such that*

$$\mathbb{E} \sup_{T \in [0,\tau^*]} \|R(T)\|_\alpha^{p} \leq C\varepsilon^{p-6p\kappa}. \tag{2.33}$$

Proof. For the bound on R we bound all terms in (2.20) separately. The estimates rely on Assumption 2.2.4 and the inequality $\|\psi\|_\gamma \leq C\|\psi\|_{\gamma+\delta}$ for all $\gamma \in \mathbb{R}$ and $\delta \geq 0$. Moreover, we use that $B_c(a(\tau), \mathcal{A}_s^{-1}\psi(\tau)) \in \mathcal{N}$ (finite dimensional) and

$\mathcal{A}_s^{-1}$ being a bounded linear operator on $\mathcal{S} \subset \mathcal{H}^\alpha$. Thus, we obtain for all times up to the stopping time τ^* that

$$\begin{aligned}\left\|\varepsilon^2 B_c(a, \mathcal{A}_s^{-1}\psi)\right\|_\alpha &\leq C\varepsilon^2 \left\|B_c(a, \mathcal{A}_s^{-1}\psi)\right\|_{\alpha-\beta} \\ &\leq C\varepsilon^2 \left\|a\right\|_\alpha \left\|\mathcal{A}_s^{-1}\psi\right\|_\alpha \\ &\leq C\varepsilon^2 \left\|a\right\|_\alpha \left\|\psi\right\|_\alpha .\end{aligned}$$

Using the definition of τ^*, we obtain

$$\mathbb{E} \sup_{[0,\tau^*]} \|\varepsilon^2 B_c(a, \mathcal{A}_s^{-1}\psi)\|_\alpha^p \leq C\varepsilon^{2p-4p\kappa}. \tag{2.34}$$

For the second term in (2.20) with $T \leq \tau^* \leq T_0$

$$\begin{aligned}\left\|2\varepsilon^2 \int_0^T B_c(B_c(a,\psi), \mathcal{A}_s^{-1}\psi)d\tau\right\|_\alpha &\leq C\varepsilon^2 \int_0^T \|B_c(B_c(a,\psi), \mathcal{A}_s^{-1}\psi)\|_{\alpha-\beta} d\tau \\ &\leq C\varepsilon^2 T \sup_{[0,\tau^*]} \|B_c(a,\psi)\|_\alpha \|\mathcal{A}_s^{-1}\psi\|_\alpha \\ &\leq C\varepsilon^2 T \sup_{[0,\tau^*]} \|a\|_\alpha \|\psi\|_\alpha^2 \\ &\leq C\varepsilon^{2-7\kappa}.\end{aligned} \tag{2.35}$$

Analogously, for the third term in (2.20)

$$\begin{aligned}\left\|\varepsilon^3 \int_0^T B_c(B_c(\psi,\psi), \mathcal{A}_s^{-1}\psi)d\tau\right\|_\alpha &\leq C\varepsilon^3 \int_0^T \|B_c(B_c(\psi,\psi), \mathcal{A}_s^{-1}\psi)\|_{\alpha-\beta} d\tau \\ &\leq C\varepsilon^3 T \sup_{[0,\tau^*]} \|\psi\|_\alpha^3 \\ &\leq C\varepsilon^{3-9\kappa}.\end{aligned} \tag{2.36}$$

The 4th term in (2.20) is bounded by

$$\begin{aligned}\left\|\varepsilon^2 \int_0^T B_c(\mathcal{L}_c a, \mathcal{A}_s^{-1}\psi)d\tau\right\|_\alpha &\leq C\varepsilon^2 \int_0^T \|B_c(\mathcal{L}_c a, \mathcal{A}_s^{-1}\psi)\|_{\alpha-\beta} d\tau \\ &\leq C\varepsilon^2 \sup_{[0,\tau^*]} \|\mathcal{L}_c a\|_\alpha \|\mathcal{A}_s^{-1}\psi\|_\alpha \\ &\leq C\varepsilon^2 \sup_{[0,\tau^*]} \|a\|_\alpha \|\psi\|_\alpha \\ &\leq C\varepsilon^{2-4\kappa},\end{aligned} \tag{2.37}$$

where we used $\|\mathcal{L}_c a\|_\alpha \leq C\|\mathcal{L}_c a\|_{\alpha-\beta}$, as $\mathcal{N}$ is finite dimensional.

For the 5th term in (2.20)

$$\begin{aligned}\Big\|2\varepsilon\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(a,\psi))d\tau\Big\|_\alpha &\leq C\varepsilon\int_0^T \|B_c(a,\mathcal{A}_s^{-1}B_s(a,\psi))\|_{\alpha-\beta}d\tau\\ &\leq C\varepsilon\sup_{[0,\tau^*]}\|a\|_\alpha\|\mathcal{A}_s^{-1}B_s(a,\psi)\|_\alpha\\ &\leq C\varepsilon\sup_{[0,\tau^*]}\|a\|_\alpha^2\|\psi\|_\alpha\\ &\leq C\varepsilon^{1-5\kappa}. \end{aligned}\tag{2.38}$$

The 6th term in (2.20) is bounded by

$$\begin{aligned}\Big\|\varepsilon^3\int_0^T B_c(\mathcal{L}_c\psi,\mathcal{A}_s^{-1}\psi)d\tau\Big\|_\alpha &\leq C\varepsilon^3\int_0^T \|B_c(\mathcal{L}_c\psi(\tau),\mathcal{A}_s^{-1}\psi(\tau))\|_{\alpha-\beta}d\tau\\ &\leq C\varepsilon^3\sup_{[0,\tau^*]}\|\mathcal{L}_c\psi\|_\alpha\|\mathcal{A}_s^{-1}\psi)\|_\alpha\\ &\leq C\varepsilon^3\sup_{[0,\tau^*]}\|\psi\|_\alpha^2\\ &\leq C\varepsilon^{3-6\kappa}. \end{aligned}\tag{2.39}$$

The 7th term in (2.20) is bounded by

$$\begin{aligned}\Big\|\varepsilon\int_0^T B_c(a,\mathcal{A}_s^{-1}\mathcal{L}_s a)d\tau\Big\|_\alpha &\leq C\varepsilon\int_0^T \|B_c(a,\mathcal{A}_s^{-1}\mathcal{L}_s a)\|_{\alpha-\beta}d\tau\\ &\leq C\varepsilon\sup_{[0,\tau^*]}\|a\|_\alpha\|\mathcal{A}_s^{-1}\mathcal{L}_s a\|_\alpha\\ &\leq C\varepsilon\sup_{[0,\tau^*]}\|a\|_\alpha\|\mathcal{L}_s a\|_{\alpha-m}\\ &\leq C\varepsilon\sup_{[0,\tau^*]}\|a\|_\alpha^2\\ &\leq C\varepsilon^{1-2\kappa}. \end{aligned}\tag{2.40}$$

The 8th term in (2.20) is completely analogous. We have

$$\Big\|\varepsilon^2\int_0^T B_c(a,\mathcal{A}_s^{-1}\mathcal{L}_s\psi)d\tau\Big\|_\alpha \leq C\varepsilon^{2-4\kappa}. \tag{2.41}$$

Moreover for the 9th term in (2.20):

$$\Big\|\varepsilon\int_0^T B_c(\psi,\psi)d\tau\Big\|_\alpha \leq C\varepsilon\int_0^T \|B_c(\psi,\psi)\|_{\alpha-\beta}d\tau \leq C\varepsilon^{1-6\kappa}\ . \tag{2.42}$$

For the 10th term in (2.20)

$$\begin{aligned}\Big\|\varepsilon\int_0^T \mathcal{L}_c\psi d\tau\Big\|_\alpha &\le C\varepsilon\int_0^T \|\mathcal{L}_c\psi\|_\alpha d\tau \\ &\le C\varepsilon\int_0^T \|\mathcal{L}_c\psi\|_{\alpha-\beta} d\tau \\ &\le C\varepsilon \sup_{[0,\tau^*]}\|\psi(\tau)\|_\alpha \\ &\le C\varepsilon^{1-3\kappa}. \end{aligned} \tag{2.43}$$

The 11th term in (2.20) is bounded by

$$\begin{aligned}\Big\|\varepsilon^2\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(\psi,\psi))d\tau\Big\|_\alpha &\le C\varepsilon^2\int_0^T \|B_c(a,\mathcal{A}_s^{-1}B_s(\psi,\psi))\|_{\alpha-\beta} d\tau \\ &\le C\varepsilon^2 \sup_{[0,\tau^*]}\|a\|_\alpha\|\mathcal{A}_s^{-1}B_s(\psi,\psi)\|_\alpha \\ &\le C\varepsilon^2 \sup_{[0,\tau^*]}\|a\|_\alpha\|\psi\|_\alpha^2 \\ &\le C\varepsilon^{2-7\kappa}. \end{aligned} \tag{2.44}$$

For the stochastic integral $\varepsilon^2\int_0^T B_c(d\tilde{W}_c,\mathcal{A}_s^{-1}\psi)$ in (2.20) note that the covariance operator of W_c is $Q_c = P_cQP_c$. Define

$$\pounds(\tau)u := B_c(u(\tau),\mathcal{A}_s^{-1}\psi(\tau)),$$

to obtain

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T B_c(d\tilde{W}_c(\tau),\mathcal{A}_s^{-1}\psi(\tau))\Big\|_\alpha^p = \mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T \pounds(\tau)d\tilde{W}_c(\tau)\Big\|_\alpha^p.$$

By Burkholder-Davis-Gundy (cf. Theorem 1.2.4 in [32]) we derive

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T \pounds d\tilde{W}_c\Big\|_\alpha^p &= \mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T D^\alpha\pounds d\tilde{W}_c\Big\|^p \\ &\le \mathbb{E}\Big(\int_0^{\tau^*}\|D^\alpha\pounds Q_c^{\frac12}\|_{HS}^2 d\tau\Big)^{\frac{p}{2}} \\ &= C\mathbb{E}\Big(\int_0^{\tau^*}\sum_{k=1}^\infty\|D^\alpha\pounds Q_c^{\frac12}g_k\|^2 d\tau\Big)^{\frac{p}{2}}, \end{aligned}$$

where $(g_k)_{k\in\mathbb{N}}$ is any orthonormal basis in $\mathcal{H}$ and D^α was defined in Definition 2.2.2. The space HS is the space of Hilbert-Schmidt operators on $\mathcal{H}$, equipped with the norm $\|\Psi\|_{HS} = \text{Trace}[\Psi\Psi^*]$. Hence,

$$\begin{aligned}
\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T \pounds d\tilde{W}_c\Big\|_\alpha^p &\leq \mathbb{E}\Big(\int_0^{\tau^*}\sum_{k=1}^{\infty}\|D^\alpha B_c(Q_c^{\frac{1}{2}}g_k, \mathcal{A}_s^{-1}\psi)\|^2 d\tau\Big)^{\frac{p}{2}} \\
&= C\mathbb{E}\Big(\int_0^{\tau^*}\sum_{k=1}^{\infty}\|\underbrace{B_c(Q_c^{\frac{1}{2}}g_k, \mathcal{A}_s^{-1}\psi)}_{\in\mathcal{N}}\|_\alpha^2 d\tau\Big)^{\frac{p}{2}} \\
&\leq C\mathbb{E}\Big(\sum_{k=1}^{\infty}\sup_{[0,\tau^*]}\|B_c(Q_c^{\frac{1}{2}}g_k, \mathcal{A}_s^{-1}\psi)\|_{\alpha-\beta}^2\Big)^{\frac{p}{2}} \\
&\leq C\Big(\sum_{k=1}^{\infty}\|Q_c^{\frac{1}{2}}g_k\|_\alpha^2\Big)^{\frac{p}{2}}\mathbb{E}\sup_{[0,\tau^*]}\|\mathcal{A}_s^{-1}\psi(\tau)\|_\alpha^p \\
&\leq C\varepsilon^{-3p\kappa},
\end{aligned}$$

where we used the fact that the norm in HS is invariant under taking the adjoint, and independent of the choice of the basis. To be more precise

$$\begin{aligned}
\sum_{k=1}^{\infty}\Big\|Q_c^{\frac{1}{2}}g_k\Big\|_\alpha^2 &= \sum_{k=1}^{\infty}\Big\|D^\alpha Q_c^{\frac{1}{2}}g_k\Big\|^2 = \Big\|D^\alpha Q_c^{\frac{1}{2}}\Big\|_{HS}^2 \\
&\overset{\text{adjoint}}{=} \Big\|Q_c^{\frac{1}{2}}D^\alpha\Big\|_{HS}^2 \underset{\text{of basis}}{\overset{\text{indep.}}{=}} \sum_{k=1}^{\infty}\Big\|Q_c^{\frac{1}{2}}D^\alpha e_k\Big\|^2 \\
&= \sum_{k=1}^{\infty}\Big\langle Q_c^{\frac{1}{2}}D^\alpha e_k, Q_c^{\frac{1}{2}}D^\alpha e_k\Big\rangle \\
&= \sum_{k=1}^{\infty}\langle Q_c D^\alpha e_k, D^\alpha e_k\rangle = \sum_{k=1}^{\infty}k^{2\alpha}\langle P_cQP_ce_k, e_k\rangle \\
&= \sum_{k=1}^{\infty}k^{2\alpha}\langle QP_ce_k, P_ce_k\rangle = \sum_{k=1}^{n}k^{2\alpha}\langle Qe_k, e_k\rangle \\
&= \sum_{k=1}^{n}k^{2\alpha}\Big\|Q^{\frac{1}{2}}e_k\Big\|^2 \leq C.
\end{aligned}$$

Thus,

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T B_c(d\tilde{W}_c(\tau), \mathcal{A}_s^{-1}\psi(\tau))\Big\|_\alpha^p \leq C\varepsilon^{2p-3p\kappa}. \tag{2.45}$$

For the stochastic integral $\varepsilon\int_0^T B_c(a, \mathcal{A}_s^{-1}d\tilde{W}_s)$ in (2.20), note that the covariance operator of $\tilde{W}_s$ is $Q_s = P_s Q P_s$. Similar to the previous estimate we define

$$\pounds_1(\tau)u := B_c(a(\tau), \mathcal{A}_s^{-1}u).$$

Hence,

$$\begin{aligned}\mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\Big\|\varepsilon\int_0^T B_c(a(\tau), A_S^{-1}d\tilde{W}_s)\Big\|_\alpha^p\Big)\\ &= \mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\Big\|\varepsilon\int_0^T \pounds_1(\tau)d\tilde{W}_s\Big\|_\alpha^p\Big)\\ &= \mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\Big\|\varepsilon\int_0^T D^\alpha\pounds_1(\tau)d\tilde{W}_s\Big\|_{L^2}^p\Big).\end{aligned}$$

By Burkholder-Davis-Gundy (cf. Theorem 1.2.4 in [32]), we obtain

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\varepsilon\int_0^T B_c(a(\tau), A_S^{-1}d\tilde{W}_s)\Big\|_\alpha^p\\ &\le C\varepsilon^p\mathbb{E}\Big(\int_0^{\tau^*}\|D^\alpha\pounds_1(\tau)\|_{L_0^2}^2\,d\tau\Big)^{\frac{p}{2}}\\ &= C\varepsilon^p\mathbb{E}\Big(\int_0^{\tau^*}\Big\|D^\alpha\pounds_1(\tau)Q_s^{\frac{1}{2}}\Big\|_{HS}^2\,d\tau\Big)^{\frac{p}{2}}\\ &= C\varepsilon^p\mathbb{E}\Big(\int_0^{\tau^*}\sum_{k=1}^\infty\Big\|D^\alpha\pounds_1(\tau)Q_s^{\frac{1}{2}}g_k\Big\|_{L^2}^2\,d\tau\Big)^{\frac{p}{2}}\\ &= C\varepsilon^p\mathbb{E}\Big(\int_0^{\tau^*}\sum_{k=1}^\infty\Big\|D^\alpha B_c(a(\tau), A_S^{-1}Q_s^{\frac{1}{2}}g_k)\Big\|_{L^2}^2\,d\tau\Big)^{\frac{p}{2}}\\ &= C\varepsilon^p\mathbb{E}\Big(\int_0^{\tau^*}\sum_{k=1}^\infty\Big\|B_c(a(\tau), A_S^{-1}Q_s^{\frac{1}{2}}g_k)\Big\|_\alpha^2\,d\tau\Big)^{\frac{p}{2}}\\ &\le C\varepsilon^p\mathbb{E}\Big(\sum_{k=1}^\infty\Big(\sup_{[0,\tau^*]}\Big\|B_c(a, A_S^{-1}Q_s^{\frac{1}{2}}g_k)\Big\|_{\alpha-\beta}^2\Big)\cdot\tau^*\Big)^{\frac{p}{2}}\\ &\le C\varepsilon^p\mathbb{E}\Big(\sup_{[0,\tau^*]}\|a\|_\alpha^P\Big)\Big(\sum_{k=1}^\infty\Big\|A_S^{-1}Q_s^{\frac{1}{2}}g_k\Big\|_\alpha^2\Big)^{\frac{p}{2}}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\Big\|\varepsilon\int_0^T B_c(a(\tau),A_S^{-1}d\tilde{W}_s)\Big\|_\alpha^p\Big) &\leq C\varepsilon^{p-p\kappa}\Big(\sum_{k=1}^{\infty}\Big\|A_S^{-1}Q_s^{\frac{1}{2}}g_k\Big\|_\alpha^2\Big)^{\frac{p}{2}}\\ &= C\varepsilon^{p-p\kappa}\Big(\sum_{k=n+1}^{\infty}\frac{k^{2\alpha}}{\lambda_k^2}\Big\|Q^{\frac{1}{2}}e_k\Big\|^2\Big)^{\frac{p}{2}}\\ &\leq C\varepsilon^{p-p\kappa}, \qquad (2.46)\end{aligned}$$

where we used

$$\begin{aligned}\sum_{k=1}^{\infty}\|\mathcal{A}_s^{-1}Q_s^{\frac{1}{2}}g_k\|_\alpha^2 = \|D^\alpha\mathcal{A}_s^{-1}Q_s^{\frac{1}{2}}\|_{HS}^2 &= \|Q_s^{\frac{1}{2}}\mathcal{A}_s^{-1}D^\alpha\|_{HS}^2\\ &= \sum_{k=1}^{\infty}\|Q_s^{\frac{1}{2}}\mathcal{A}_s^{-1}D^\alpha e_k\|^2 = \sum_{k=1}^{\infty}\frac{k^{2\alpha}}{\lambda_k^2}\|Q_s^{\frac{1}{2}}e_k\|^2\\ &= \sum_{k=1}^{\infty}\frac{k^{2\alpha}}{\lambda_k^2}\langle P_sQP_se_k,e_k\rangle\\ &= \sum_{k=n+1}^{\infty}\frac{k^{2\alpha}}{\lambda_k^2}\|Q^{\frac{1}{2}}e_k\|^2\\ &\leq C\ .\end{aligned}$$

The last step follows from Assumption 2.2.10, as $\lambda_k\to\infty$.

Analogously, for the last integral $\varepsilon\int_0^T B_c(d\tilde{W}_c(\tau),\mathcal{A}_s^{-1}d\tilde{W}_s(\tau))$ in (2.20), we obtain

$$\begin{aligned}&\mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\Big\|\varepsilon\int_0^T B_c(d\tilde{W}_c(\tau),\mathcal{A}_s^{-1}d\tilde{W}_s(\tau))\Big\|_\alpha^p\Big)\\ &\quad= \varepsilon^p\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\int_0^T\sum_{k=1}^{n}\sum_{l=n+1}^{\infty}B_c(e_k,-\lambda_l^{-1}e_l)d\beta_k d\beta_l\Big\|_\alpha^p\\ &\quad= \varepsilon^p\Big\|\sum_{k=1}^{n}\sum_{l=n+1}^{\infty}B_c(e_k,\lambda_l^{-1}e_l)\,\langle Qe_k,e_l\rangle\cdot\tau^*\Big\|_\alpha^p\\ &\quad\leq C\varepsilon^p\Big(\sum_{k=1}^{n}\sum_{l=n+1}^{\infty}\big\|B_c(e_k,\lambda_l^{-1}e_l)\big\|_{\alpha-\beta}|\langle Qe_k,e_l\rangle|\Big)^p\end{aligned}$$

Using Assumptions 2.2.4 and 2.2.10, we obtain

$$
\begin{aligned}
\mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\Big\|\varepsilon\int_0^T B_c(d\tilde{W}_c(\tau),\mathcal{A}_s^{-1}d\tilde{W}_s(\tau))\Big\|_\alpha^p\Big)
&\le C\varepsilon^p\Big(\sum_{k=1}^{n}\sum_{l=n+1}^{\infty}\frac{1}{\lambda_l}\|e_k\|_\alpha\|e_l\|_\alpha|\langle Qe_k,e_l\rangle|\Big)^p\\
&\le C\varepsilon^p\Big(\sum_{k=1}^{n}\sum_{l=n+1}^{\infty}\frac{1}{\lambda_l}k^\alpha l^\alpha|\langle Qe_k,e_l\rangle|\Big)^p\\
&\le C\varepsilon^p. \qquad (2.47)
\end{aligned}
$$

As we supposed $\kappa < \frac{1}{7}$ in the definition of τ^*, we can collect all term in the equations from (2.34) until (2.47). This implies the result. □

In order to prove now the approximation result, we first need the following a-priori estimate for solutions of the amplitude equation.

Lemma 2.4.7 *Let Assumptions 2.2.1, 2.2.3, 2.2.8 and 2.2.10 hold. Define the stochastic process $b(T)$ in $\mathcal{N}$ with $\mathbb{E}\|b(0)\| \le C$ as the solution of*

$$b(T) = b(0) + \int_0^T \mathcal{L}_c b(\tau)d\tau - 2\int_0^T \mathcal{F}(b(\tau))d\tau + \tilde{W}_c(T). \qquad (2.48)$$

Then for $T_0 > 0$ there exists a constant $C > 0$ such that

$$\mathbb{E}\sup_{T\in[0,T_0]}\|b(T)\|_\alpha^p \le C\,. \qquad (2.49)$$

We note that all norms in a finite dimensional space are equivalent. Thus for simplicity of notation in the proof we use only the standard Euclidean norm and suppose that $b \in \mathbb{R}^n$.

Proof. The existence and uniqueness of solutions for equation (2.48) is standard. To verify the bound in (2.49) we define X as

$$X(T) = b(T) - \tilde{W}_c(T)\,. \qquad (2.50)$$

Substituting into (2.48), we obtain

$$\partial_T X = \mathcal{L}_c(X + \tilde{W}_c) - 2\mathcal{F}(X + \tilde{W}_c). \qquad (2.51)$$

Taking the scalar product $\langle \cdot, X\rangle$ on both sides of (2.51), yields

$$\frac{1}{2}\partial_T \|X\|^2 = \langle \mathcal{L}_c(X+\tilde{W}_c), X\rangle - 2\langle \mathcal{F}(X+\tilde{W}_c), X\rangle.$$

Using Young and Cauchy-Schwarz inequalities and Assumption 2.2.8, yields

$$\partial_T \|X\|^2 \le C + C\left\|\tilde{W}_c\right\|^4 - \frac{\delta}{2}\|X\|^4.$$

Neglecting the fourth power, integrating from 0 to T, taking $\frac{p}{2}$-th power, and finally the expectation, we obtain

$$\mathbb{E}\sup_{[0,T_0]}\|X\|^p \le CT_0^{\frac{1}{2}p} + CT_0^{\frac{1}{2}p}\mathbb{E}\sup_{[0,T_0]}\left\|\tilde{W}_c\right\|^{2p} \le C.$$

Togather with (2.50), this implies

$$\mathbb{E}\sup_{[0,T_0]}\|b\|^p \le C\mathbb{E}\sup_{[0,T_0]}\|X\|^p + C\mathbb{E}\sup_{[0,T_0]}\left\|\tilde{W}_c\right\|^{p} \le C.$$

□

Definition 2.4.8 *Define the set $\Omega^* \subset \Omega$ such that all these estimates*

$$\sup_{[0,\tau^*]}\|\psi\|_\alpha < C\varepsilon^{-3\kappa}, \tag{2.52}$$

$$\sup_{[0,\tau^*]}\|R\|_\alpha < C\varepsilon^{1-7\kappa}, \tag{2.53}$$

and

$$\sup_{[0,\tau^*]}\|b\|_\alpha < C\varepsilon^{-\frac{\kappa}{2}}, \tag{2.54}$$

hold on Ω^*.

Remark 2.4.9 *Ω^* has probability*

$$\mathbb{P}(\Omega^*) \ge 1-\mathbb{P}(\sup_{[0,\tau^*]}\|\psi\|_\alpha \ge C\varepsilon^{-3\kappa})-\mathbb{P}(\sup_{[0,\tau^*]}\|R\|_\alpha \ge C\varepsilon^{1-7\kappa})-\mathbb{P}(\sup_{[0,\tau^*]}\|b\|_\alpha \ge C\varepsilon^{-\frac{\kappa}{2}}).$$

Using Chebychev inequality and Lemmas 2.4.4, 2.4.6 and 2.4.7, we obtain for sufficiently large $q > 0$

$$\begin{aligned}\mathbb{P}(\Omega^*) &\ge 1 - C[\varepsilon^{q\kappa} + \varepsilon^{q\kappa} + \varepsilon^{\frac{1}{2}q\kappa}] \\ &\ge 1 - C\varepsilon^{\frac{1}{2}q\kappa} \ge 1 - C\varepsilon^p. \end{aligned} \tag{2.55}$$

Theorem 2.4.10 *We assume that Assumption 2.2.1, 2.2.3, 2.2.4, 2.2.8 and 2.2.10 hold. Let b be a solution of (2.48) and a as defined in (2.19) with $\|a(0)\| \leq C$ on Ω^*. If the initial conditions satisfy $a(0) = b(0)$, then, for $\kappa < \frac{1}{7}$, we obtain*

$$\sup_{T\in[0,\tau^*]} \|a(T) - b(T)\|_\alpha \leq C\varepsilon^{1-7\kappa}, \tag{2.56}$$

and

$$\sup_{T\in[0,\tau^*]} \|a(T)\|_\alpha \leq C\varepsilon^{-\frac{\kappa}{2}}, \tag{2.57}$$

on Ω^*.

Proof. Define $\varphi(T)$ as

$$\varphi(T) := a(T) - R(T).$$

From (2.19) we obtain

$$\varphi(T) = a(0) + \int_0^T \mathcal{L}_c\left[\varphi(\tau) + R(\tau)\right] d\tau - 2\int_0^T \mathcal{F}(\varphi(\tau) + R(\tau))d\tau + \tilde{W}_c(T). \tag{2.58}$$

Define now $h(T)$ by

$$h(T) := b(T) - \varphi(T).$$

Subtracting (2.58) from (2.48), we obtain

$$h(T) = \int_0^T \mathcal{L}_c h(\tau)d\tau - \int_0^T \mathcal{L}_c R(\tau)d\tau + 2\int_0^T [\mathcal{F}(b-h+R) - \mathcal{F}(b)](\tau)d\tau.$$

Thus,

$$\partial_T h = \mathcal{L}_c h - \mathcal{L}_c R + 2[\mathcal{F}(b-h+R) - \mathcal{F}(b)]\ . \tag{2.59}$$

Taking the scalar product $\langle\cdot, h\rangle$ on both sides of (2.59), yields

$$\tfrac{1}{2}\partial_T \|h\|^2 = \langle\partial_T h, h\rangle = \langle\mathcal{L}_c h, h\rangle - \langle\mathcal{L}_c R, h\rangle + 2\langle\mathcal{F}(b-h+R) - \mathcal{F}(b), h\rangle\ .$$

Using Young and Cauchy-Schwarz inequalities and (2.5), we obtain the following linear ordinary differential inequality

$$\begin{aligned}\partial_T \|h\|^2 &\leq C[\|h\|^2 + \|h\|^4] + C\|R\|^2\left[1 + \|R\|^2 + \|b\|^2 + \|b\|^4 + \|b\|^2\|R\|^2\right] \\ &\leq C[\|h\|^2 + \|h\|^4] + C\|R\|^2\left[c + c\|R\|^4 + c\|b\|^4\right]\ .\end{aligned}$$

Using (2.53) and (2.54), we obtain

$$\partial_T \|h\|^2 \leq C[\|h\|^2 + \|h\|^4] + C\varepsilon^{2-14\kappa} \quad \text{on } \Omega^*.$$

As long as $\|h\| < 1$, we obtain

$$\partial_T \|h\|^2 \leq 2C \|h\|^2 + C\varepsilon^{2-14\kappa}.$$

Integrating from 0 to T and using Gronwall's lemma, we obtain

$$\|h\|^2 \leq C\varepsilon^{2-14\kappa}.$$

Thus,

$$\sup_{[0,\tau^*]} \|h\| \leq C\varepsilon^{1-7\kappa} \quad \text{on } \Omega^*. \tag{2.60}$$

We finish the first part by using (2.53), (2.60) and

$$\sup_{[0,\tau^*]} \|a - b\| = \sup_{[0,\tau^*]} \|h - R\| \leq \sup_{[0,\tau^*]} \|h\| + \sup_{[0,\tau^*]} \|R\|.$$

For the second part of the theorem we consider

$$\sup_{[0,\tau^*]} \|a\| \leq \sup_{[0,\tau^*]} \|a - b\| + \sup_{[0,\tau^*]} \|b\| .$$

Using the first part and (2.54), we obtain (2.57). □

Finally, we use the results previously obtained to prove the main result of Theorem 2.3.1 for the approximation of the solution of the SPDE (2.1).

Proof of Theorem 2.3.1. For the stopping time we note that

$$\Omega \supset \{\tau^* = T_0\} \supseteq \{ \sup_{T\in[0,\tau^*]} \|a(T)\|_\alpha < \varepsilon^{-\kappa}, \sup_{T\in[0,\tau^*]} \|\psi(T)\|_\alpha < \varepsilon^{-3\kappa}\} \supseteq \Omega^*.$$

Now let us turn to the approximation result. Using (2.14) and triangle inequality, we obtain

$$\sup_{T\in[0,\tau^*]} \|u(\varepsilon^{-2}T) - \varepsilon b(T)\|_\alpha \leq \varepsilon \sup_{[0,\tau^*]} \|a - b\|_\alpha + \varepsilon^2 \sup_{[0,\tau^*]} \|\psi\|_\alpha.$$

From (2.52) and (2.56), we obtain

$$\sup_{t\in[0,\varepsilon^{-2}T_0]} \|u(t) - \varepsilon b(\varepsilon^2 t)\|_\alpha = \sup_{t\in[0,\varepsilon^{-2}\tau^*]} \|u(t) - \varepsilon b(\varepsilon^2 t)\|_\alpha \leq C\varepsilon^{2-7\kappa} \text{ on } \Omega^*.$$

Hence,

$$\mathbb{P}\left(\sup_{t\in[0,\varepsilon^{-2}T_0]} \|u(t) - \varepsilon b(\varepsilon^2 t)\|_\alpha > C\varepsilon^{2-7\kappa} \right) = 1 - \mathbb{P}(\Omega^*) .$$

Using (2.55), yields (2.21). □

2.5 Applications

There are numerous examples in the physics literature of equations with quadratic nonlinearities where our theory applies. Before we give examples, we suppose in all our applications for simplicity that W is a cylindrical Wiener process on $\mathcal{H}$ with a covariance operator Q defined by $Qe_k = \alpha_k^2 e_k$ where $(\alpha_k)_k$ is a bounded sequence of real numbers and e_k are the eigenfunctions of the dominant linear operator.

2.5.1 Burgers' Equation

The first example is the Burgers' equation (cf. (2.2)) on the interval $[0, \pi]$, with Dirichlet boundary conditions. We take

$$\mathcal{H} = L^2([0,\pi]), \quad e_k(x) = \sqrt{\tfrac{2}{\pi}}\sin(kx), \quad \text{and } \mathcal{N} = \text{span}\{\sin\}.$$

We note that Assumption 2.2.1 is true, where the eigenvalues of $-\mathcal{A} = -\partial_x^2 - 1$ are $\lambda_k = k^2 - 1$ with $m = 2$ and $\lim_{k\to\infty} \lambda_k = \infty$. If we fix P_c to be the $\mathcal{H}$-orthogonal projection onto $\mathcal{N}$, then both P_c and P_s commute with $\mathcal{A}$.

Moreover, all conditions of Assumption 2.2.4 are satisfied with

$$B(u,v) = \tfrac{1}{2}\partial_x(uv),$$

as follows:

$$P_c B(u,u) = P_c\left[\gamma^2 \sin(x)\cos(x)\right] = 0 \ \text{ for } u = \gamma\sin \in \mathcal{N},$$

and for $\alpha = \frac{1}{4}$ and $\beta = \frac{5}{4} < m$, we obtain

$$\begin{aligned} 2\|B(u,v)\|_{\mathcal{H}^{-1}} &= \|\partial_x(uv)\|_{\mathcal{H}^{-1}} \le \|uv\|_{L^2} \\ &\le C\|u\|_{L^4}\|v\|_{L^4} \le C\|u\|_{\mathcal{H}^{\frac{1}{4}}}\|v\|_{\mathcal{H}^{\frac{1}{4}}}, \end{aligned}$$

where we used Sobolev embedding from $\mathcal{H}^{1/4}$ into L^4. We derive after a straightforward calculation that

$$\mathcal{F}(\gamma_1\sin, \gamma_2\sin, \gamma_3\sin) = \frac{1}{24}\gamma_1\gamma_2\gamma_3\sin.$$

This function is trilinear, continuous, and satisfies the Conditions (2.4) and (2.5) as follows

$$\langle \gamma_1\sin, \mathcal{F}(\gamma_1\sin)\rangle = C\gamma_1^4 > 0\,,$$

and

$$\langle \mathcal{F}(\gamma_1 \sin, \gamma_1 \sin, \gamma_2 \sin), \gamma_2 \sin \rangle = \frac{\pi}{48}\gamma_1^2\gamma_2^2 > 0 \ .$$

Now our main theorem states that

$$u(t) = \epsilon\gamma(\epsilon^2 t) \sin + \mathcal{O}(\epsilon^{2-}) \ ,$$

where

$$\gamma' = \nu\gamma - \tfrac{1}{12}\gamma^3 + \alpha_1\tilde{\beta}' \ ,$$

with a rescaled standard Brownian motion $\tilde{\beta}$.

2.5.2 Surface Growth Model

The second example that falls into the scope of our work is the growth of rough amorphous surfaces. The equation is of the type

$$\partial_t h = -\triangle^2 h - \mu\triangle h - \triangle|\nabla h|^2 + \sigma\partial_t W(t). \tag{2.61}$$

Here $\triangle$ is Laplacian with respect to periodic boundary conditions on $[0, 2\pi]$. Suppose initial condition $h(0) = 0$ corresponding to an initially flat surface.

For this model we consider $\mu = 1 + \varepsilon^2\nu$ and $\sigma = \varepsilon^2$. Hence,

$$\mathcal{A} = -\triangle^2 - \triangle, \ \ \mathcal{L} = -\nu\triangle \ \text{ and } \ B(u, v) = -\triangle(\partial_x u \cdot \partial_x v).$$

We take

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\sin(kx) & \text{if } k > 0, \\ \frac{1}{\sqrt{\pi}}\cos(kx) & \text{if } k < 0, \\ \frac{1}{\sqrt{2\pi}} & \text{if } k = 0, \end{cases}$$

and

$$\mathcal{H} = \{u \in L^2([0, 2\pi]) : \int_0^{2\pi} u dx = 0\} \ \text{ and } \ \mathcal{N} = \text{span}\{\sin, \ \cos\}.$$

The eigenvalues of $-\mathcal{A} = \triangle^2 + \triangle$ are $\lambda_k = k^4 - k^2$ with $m = 4$ and $\lim_{k\to\infty}\lambda_k = \infty$. So, Assumption 2.2.1 is true.

If we define $u(t) := h(t) - h_0(t)e_0$, then we obtain

$$\partial_t u = -\triangle^2 u - \mu\triangle u - \triangle|\nabla u|^2 + \sigma\sum_{k\neq 0}\alpha_k\partial_t\beta_k(t)e_k, \tag{2.62}$$

and

$$h_0 = \sigma\alpha_0\beta_0(t). \tag{2.63}$$

If $u = u_1 \sin + u_{-1} \cos \in \mathcal{N}$, then

$$B(u,u) = 2\left[u_1^2 - u_{-1}^2\right]\cos(2x) - 4u_1u_{-1}\sin(2x),$$

and

$$P_c B(u,u) = 0,$$

and for $\alpha = \frac{5}{4}$ and $\beta = \frac{13}{4} < m$, we obtain

$$\begin{aligned}\|B(u,v)\|_{\mathcal{H}^{-2}} = \|\triangle(\partial_x u \cdot \partial_x v)\|_{\mathcal{H}^{-2}} \le c\|\partial_x u \cdot \partial_x v\|_{L^2} \\ \le c\|u\|_{\mathcal{H}^{\frac{5}{4}}}\|v\|_{\mathcal{H}^{\frac{5}{4}}}.\end{aligned}$$

Hence, all conditions of Assumption 2.2.4 are satisfied. Moreover, it is easy to check that Assumption 2.2.8 also holds true.

For the symmetric version of $\mathcal{F}$ we obtain

$$\begin{aligned}\mathcal{F}(u,u,w) &= \frac{2}{3}B_c(u,\mathcal{A}_s^{-1}B_s(u,w)) + \frac{1}{3}B_c(w,\mathcal{A}_s^{-1}B_s(u,u)) \\ &= \tfrac{1}{18}[(3u_1^2w_1 + w_1u_{-1}^2 + 2u_1w_{-1}u_{-1})\sin \\ &\quad +(u_1^2w_{-1} + 3w_{-1}u_{-1}^2 + 2u_1w_1u_{-1})\cos],\end{aligned}$$

where $w = w_1 \sin + w_{-1} \cos \in \mathcal{N}$. Now

$$\langle \mathcal{F}(u), u\rangle \ge \tfrac{1}{6\pi}\|u\|^4 > 0 \quad \forall\, u \ne 0.$$

If $u \ne 0$ *and* $w \ne 0$, then

$$\langle \mathcal{F}(u,u,w), w\rangle = \frac{\pi}{18}[3(u_1w_1 + w_{-1}u_{-1})^2 + (w_1u_{-1} - u_1w_{-1})^2] > 0\,.$$

The amplitude equation for (2.62) is a system of two stochastic ordinary differential equations:

$$d\gamma_i = [\nu\gamma_i - \tfrac{1}{3}\gamma_i(\gamma_1^2 + \gamma_{-1}^2)]dt + \alpha_i d\tilde{\beta}_i \;\text{ for } i = \pm 1,$$

where $\tilde{\beta}_i(T) = \varepsilon\beta_i(\varepsilon^2 T)$ rescaled Brownian motions.

Now our main theorem states that

$$u(t) = \varepsilon\gamma(\varepsilon^2 t)\cdot\begin{pmatrix}\sin\\ \cos\end{pmatrix} + \mathcal{O}(\varepsilon^{2-})\,.$$

Chapter 3

Amplitude Equations for SPDEs with Cubic Nonlinearities

3.1 Introduction

Stochastic partial differential equations (SPDEs) with cubic nonlinearity appear in several applications, for instance the Swift-Hohenberg equation, which was first used as a toy model for the convective instability in the Rayleigh-Bénard problem (see [16] or [22]). The simplest example is the well known real valued Ginzburg-Landau equation, which depending on the underlying application is also called Allen-Cahn, Chaffee-Infante or nonlinear heat equation. Moreover, we briefly discuss a model from surface growth proposed by Lai and Das-Sama (cf. [29] and see also [31]).

Recently the impact of degenerate noise not acting directly on the dominant pattern was studied for equations of Burgers type formally by Roberts [40] and later rigorously by Blömker, Hairer and Pavliotis [9]. Here noise is transported via nonlinear interaction to the dominant modes.

Our current research was initiated by an observation of Axel Hutt and collaborators [23–25]. Using a formal argument based on centre manifold theory, they showed that noise constant in space leads to a deterministic amplitude equation, which is stabilized by the impact of additive noise. Thus the noise shifts the bifurcation point. The aim of this chapter is to make these results rigorous.

The general prototype of equations under consideration is of the type

$$du(t) = \left[\mathcal{A}u(t) + \varepsilon^2\mathcal{L}u(t) + \mathcal{F}(u(t))\right] dt + \varepsilon dW(t), \tag{3.1}$$

where $\mathcal{A}$ is non-positive self-adjoint operator with finite dimensional kernel, $\varepsilon^2\mathcal{L}u$ is a small deterministic perturbation, $\mathcal{F}$ is a nonlinearity, and W is some finite dimensional Gaussian noise.

Our aim of this chapter is to establish rigorously an amplitude equation for this quite general class of SPDEs with cubic nonlinearities given by (3.1). In the examples we investigate whether additive degenerate noise leads to stabilization of the solutions, or not.

In this chapter we follow [13] and focus on cubic nonlinearities only. The case of quadratic nonlinearities is significantly different. It was already considered in [6].

This chapter is organized as follows. In the next section, we discuss the formal derivation of our results, while giving the precise assumptions and statements of the main results in Section 3.3. Section 3.4 give bounds on the non-dominant modes, while Section 3.5 provides averaging results, in order to remove the impact of the higher modes on the dominant ones. In Section 3.6, we study the approximation via amplitude equations. Finally, in Section 3.7 we apply our theory to the stochastic Swift-Hohenberg equation, Ginzburg-Landau / Allen-Cahn equation and surface growth model.

3.2 Formal Derivation

Here we study the behavior of solutions u of (3.1) on the natural slow time-scale of order ε^{-2}, given by the distance from bifurcation.

So, we consider u on the slow time and split it into the dominant part $a \in \mathcal{N}$ and the orthogonal part $\psi \in S$.

$$u(t) = \varepsilon a(\varepsilon^2 t) + \varepsilon\psi(\varepsilon^2 t) \tag{3.2}$$

Rescaling to the slow time-scale $T = \varepsilon^2 t$, leads to the following system of equations:

$$da = \left[\varepsilon^{-2}\mathcal{A}_c a + \mathcal{L}_c a + \mathcal{L}_c\psi + \mathcal{F}_c(a + \psi)\right] dT + \varepsilon^{-1} d\tilde{W}_c\,, \tag{3.3}$$

and

$$d\psi = \left[\varepsilon^{-2}\mathcal{A}_s\psi + \mathcal{L}_s a + \mathcal{L}_s\psi + \mathcal{F}_s(a+\psi)\right] dT + \varepsilon^{-1} d\tilde{W}_s \,, \tag{3.4}$$

where $\tilde{W}(T) := \varepsilon W(\varepsilon^{-2}T)$ is a rescaled version of the driving Wiener process W. For short-hand notation, we use the subscripts c and s for projection onto $\mathcal{N}$ and $\mathcal{S}$, i.e. $\mathcal{A}_c = P_c\mathcal{A}$ and $\mathcal{A}_s = P_s\mathcal{A}$ for short.

Let us suppose that the projections P_c and P_s commute not only with $\mathcal{A}$, but also with $\mathcal{L}$. Moreover suppose that the noise is degenerate and acts only on S. Then the system (3.3)-(3.4) takes the form

$$da = \left[\mathcal{L}_c a + \mathcal{F}_c(a+\psi)\right] dT, \tag{3.5}$$

and

$$d\psi = \left[\varepsilon^{-2}\mathcal{A}_s\psi + \mathcal{L}_s\psi + \mathcal{F}_s(a+\psi)\right] dT + \varepsilon^{-1} d\tilde{W}_s \,. \tag{3.6}$$

Formally, we immediately see that ψ is a fast Ornstein-Uhlenbeck process (OU, for short) in first approximation. The rigorous statement can be found in Lemma 3.4.1.

Thus, we can eliminate ψ in Equation (3.5) by averaging. In oder to derive error estimates, this procedure will be in the proofs based on the Itô-Formula (see Lemma 3.5.1).

3.2.1 The Impact of Noise

Let us discuss the averaging and the impact of the noise in some more detail here. Consider for simplicity of the argument instead of ψ here some real valued fast OU-process Z given by

$$Z(T) := \alpha\varepsilon^{-1}\int_0^T e^{-\varepsilon^{-2}\lambda(T-\tau)} d\tilde{\beta}(\tau),$$

where $\tilde{\beta}(T) := \varepsilon\beta(\varepsilon^{-2}T)$ denotes a rescaled version of a Brownian motion β on the fast time-scale.

We apply Itô formula to Z and Z^2, in order to obtain

$$Z dT = \frac{\alpha\varepsilon}{\lambda} d\tilde{\beta} - \frac{\varepsilon^2}{\lambda} dZ.$$

and

$$Z^2 dT = \frac{\alpha^2}{2\lambda} dT + \frac{\varepsilon\alpha}{\lambda} Z d\tilde{\beta} - \frac{\varepsilon^2}{2\lambda} dZ^2.$$

Thus, on the slow time-scale T we can suppose that in integrals Z is small due to averaging, and a square of Z can be replaced by a constant. See Lemma 3.5.1 for a rigorous statement. Note that the next order corrections of order ε are always Martingales.

We see in Lemma 3.4.2 that for fast OU-processes $Z = \mathcal{O}(\varepsilon^{-\kappa_0})$ for arbitrarily small $\kappa_0 > 0$. Thus, we obtain formally that Z is a white noise on the slow time scale:

$$Z(T) = \varepsilon \frac{\alpha}{\lambda} \partial_T \tilde{\beta} + \text{error},$$

where this error is small only in the sense of distributions, for example in $\mathcal{H}^{-1}$.

3.2.2 Amplitude Equation

One main result of the chapter is the following approximation by amplitude equations. Suppose for simplicity that the initial condition is sufficiently small, then we obtain for u

$$u(t) \simeq \varepsilon b(\varepsilon^2 t) + \varepsilon \mathcal{Z}(\varepsilon^2 t) + \mathcal{O}(\varepsilon^{2-}), \tag{3.7}$$

where $\mathcal{Z}$ is a fast OU-process and b is the solution of the amplitude equation on the slow time-scale

$$b'(T) = \mathcal{L}_c b(T) + \mathcal{F}_c(b(T)) + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \mathcal{F}_c(b(T), e_k, e_k). \tag{3.8}$$

The exact form of the additional linear terms is discussed later.

To illustrate this approximation result stated later in Theorem 3.3.6, we discuss here the Swift-Hohenberg equation subject to periodic boundary conditions on $[0, 2\pi]$ forced by spatially constant noise:

$$\partial_t u = -(1 + \partial_x^2)^2 u + \nu \varepsilon^2 u - u^3 + \varepsilon \alpha \partial_t \beta. \tag{3.9}$$

Rescaling the solution u of (3.9) to the slow time-scale by $u(t) = \varepsilon v(\varepsilon^2 t)$, our main theorem in this case states that v is of the type

$$v(T) \simeq \gamma_1(T) \sin + \gamma_{-1}(T) \cos + \varepsilon \frac{\alpha}{\sqrt{2\pi}} \partial_T \tilde{\beta}(T) + \mathcal{O}(\varepsilon^{1-}),$$

where γ_1 and γ_{-1} are the solutions of the amplitude equations

$$\partial_T \gamma_i = (\nu - \tfrac{3\alpha^2}{4\pi}) \gamma_i - \tfrac{3}{4} \gamma_i (\gamma_1^2 + \gamma_{-1}^2) \text{ for } i = \pm 1.$$

We note that if α is large compared to ν, then $(\nu - \frac{3\alpha^2}{4\pi})$ is negative. In this case the degenerate additive noise stabilizes the dynamics of the dominant modes.

3.3 Assumptions and Main Results

This section summarizes all assumptions necessary for our results. For the linear operator $\mathcal{A}$ in (3.1) on the Hilbert-space $\mathcal{H}$. We assume that $\mathcal{A}$ satisfies Assumption 2.2.1.

Assumption 3.3.1 *(Operator $\mathcal{L}$) Let $\mathcal{L} : \mathcal{H}^\alpha \to \mathcal{H}^{\alpha-\beta}$ for some $\beta \in [0, m)$ be a linear continuous mapping that commutes with P_c and P_s.*

For the nonlinearity $\mathcal{F}$ we assume that:

Assumption 3.3.2 *Assume that $\mathcal{F} : (\mathcal{H}^\alpha)^3 \to \mathcal{H}^{\alpha-\beta}$ with β as in Assumption 3.3.1 is trilinear, symmetric and satisfies the following conditions, for some $C > 0$,*

$$\|\mathcal{F}(u, v, \omega)\|_{\alpha-\beta} \le C \|u\|_\alpha \|v\|_\alpha \|\omega\|_\alpha \ \ \forall\, u, v, \omega \in \mathcal{H}^\alpha, \tag{3.10}$$

$$\langle \mathcal{F}_c(u), u\rangle \le 0 \ \ \forall\, u \in \mathcal{N}, \tag{3.11}$$

and

$$\langle \mathcal{F}_c(u, u, w), w\rangle \le 0 \ \ \forall \ u,\ w \in \mathcal{N}. \tag{3.12}$$

We use $\mathcal{F}(u) = \mathcal{F}(u, u, u)$ and $\mathcal{F}_c = P_c\mathcal{F}$ for short.

For the noise we suppose:

Assumption 3.3.3 *Let W be a cylindrical Wiener process on $\mathcal{H}$. Suppose for $t \ge 0$,*

$$W(t) = \sum_{k=n+1}^{N} \alpha_k \beta_k(t) e_k \ \text{for}\ N \ge n+1,$$

where the $(\beta_k)_{k\in\{n+1,\dots,N\}}$ are independent, standard Brownian motions in $\mathbb{R}$ and the $(\alpha_k)_{k\in\{n+1,\dots,N\}}$ are real numbers.

We define the fast OU processes $\mathcal{Z}$ and its coefficients $\mathcal{Z}_k(T)$ by

$$\mathcal{Z}_k(T) := \alpha_k \varepsilon^{-1} \int_0^T e^{-\varepsilon^{-2}\lambda_k(T-\tau)} d\tilde{\beta}_k(\tau), \tag{3.13}$$

for $k \in \{n+1, \dots, N\}$ and

$$\mathcal{Z}(T) := \sum_{k=n+1}^{N} \mathcal{Z}_k(T) e_k, \tag{3.14}$$

where $\tilde{\beta}_k(T) := \varepsilon \beta_k(\varepsilon^{-2}T)$ is a rescaled version of the Brownian motion.

Remark 3.3.4 *We take $N < \infty$ in the above assumption for simplicity of presentation. Nevertheless most results are still true for $N = \infty$, using the same method of proof. We only need to control the convergence of various infinite series, which is possible if the noise is not too irregular, which means for α_k decaying sufficiently fast for $k \to \infty$.*

For our result we rely on a cut off argument. We consider only solutions $u = (a, \psi)$ that are not too large, as given by the next definition.

Definition 3.3.5 *For the $\mathcal{N} \times S$-valued stochastic process (a, ψ) defined in (3.2) we define, for some $T_0 > 0$ and $\kappa \in (0, \frac{1}{12})$, the stopping time τ^* as*

$$\tau^* := T_0 \wedge \inf \left\{ T > 0 : \|a(T)\|_\alpha > \varepsilon^{-\kappa} \text{ or } \|\psi(T)\|_\alpha > \varepsilon^{-\kappa} \right\}. \tag{3.15}$$

The main result for our aim is:

Theorem 3.3.6 *(Approximation) Under Assumptions 2.2.1, 3.3.1, 3.3.2 and 3.3.3 let u be a solution of (3.1) defined in (3.2) with the initial conditions $u(0) = \varepsilon a(0) + \varepsilon \psi(0)$ with $\|u(0)\|_\alpha \le \delta_\varepsilon \varepsilon$ for $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$ where $a(0) \in \mathcal{N}$ and $\psi(0) \in S$, and b is a solution of (3.8) with $b(0) = a(0)$. Then for all $p > 1$ and $T_0 > 0$ and all $\kappa \in (0, \frac{1}{12})$, there exists $C > 0$ such that*

$$\mathbb{P}\Big(\sup_{t \in [0, \varepsilon^{-2}T_0]} \Big\| u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon \mathcal{Q}(\varepsilon^2 t) \Big\|_\alpha > \varepsilon^{2 - \frac{38}{3}\kappa} \Big) \le C \varepsilon^p, \tag{3.16}$$

where

$$\mathcal{Q}(T) = e^{\varepsilon^{-2} T \mathcal{A}_s} \psi(0) + \mathcal{Z}(T), \tag{3.17}$$

with $\mathcal{Z}(T)$ defined in (3.14).

The proof will be given in Section 3.6 later. Let us first discuss the additional error term $\mathcal{Q}$ in (3.17). We see that the first part of $\mathcal{Q}$ decays exponentially fast on the fast time-scale $\mathcal{O}(\varepsilon^2)$. The second part is an OU-process $\mathcal{Z}$, which is a small noise term, as discussed in the formal derivation.

Corollary 3.3.7 *Under Assumptions of Theorem 3.3.6 and for arbitrary initial condition $u(0)$ we obtain*

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\Big\|u(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big) \le C\varepsilon^p + \mathbb{P}(\|u(0)\|_\alpha > \varepsilon\delta_\varepsilon). \tag{3.18}$$

3.4 Bounds for the High Modes

In this section, we show that the non-dominant modes ψ are well approximated by a fast OU-process. We also have to include an exponentially fast decaying term depending on the initial conditions $\psi(0)$.

Lemma 3.4.1 *Under Assumption 2.2.1 and 3.3.1, 3.3.2, for $\kappa > 0$ from the definition of τ^* and $p \ge 1$, there is a constant $C > 0$ such that,*

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\Big\|\psi(T)-\mathcal{Q}(T)\Big\|_\alpha^p \le C\varepsilon^{2p-3p\kappa}, \tag{3.19}$$

where $\mathcal{Q}(T)$ is defined in (3.17). (i.e., $\psi = \mathcal{Q} + \mathcal{O}(\varepsilon^{2-3\kappa})$)

Proof. The mild solution of (3.6) is

$$\psi(T) = e^{\varepsilon^{-2}T\mathcal{A}_s}\psi(0) + \int_0^T e^{\varepsilon^{-2}(T-\tau)\mathcal{A}_s}\left[\mathcal{L}_s\psi + \mathcal{F}_s(a+\psi)\right](\tau)\,d\tau + \mathcal{Z}(T).$$

Using triangle inequality

$$\begin{aligned}\Big\|\psi(T)-\mathcal{Q}(T)\Big\|_\alpha &\le \Big\|\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}\mathcal{L}_s\psi(\tau)\,d\tau\Big\|_\alpha \\ &\quad + \Big\|\int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}\mathcal{F}_s(a(\tau)+\psi(\tau))d\tau\Big\|_\alpha \\ &:= I_1 + I_2\,.\end{aligned}$$

We now bound these two terms separately. For the first term, we obtain by using (1.2) for the semigroup

$$\begin{aligned} I_1 &\le C\varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}\nu(T-\tau)}(T-\tau)^{-\frac{\beta}{m}} \|\mathcal{L}_s\psi(\tau)\|_{\alpha-\beta}\, d\tau \\ &\le C\varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}\nu(T-\tau)}(T-\tau)^{-\frac{\beta}{m}} \|\psi(\tau)\|_{\alpha}\, d\tau \\ &\le C\varepsilon^2 \sup_{\tau\in[0,\tau^*]} \|\psi(\tau)\|_\alpha \int_0^{\varepsilon^{-2}\nu T} e^{-\eta}\eta^{-\frac{\beta}{m}} d\eta \\ &\le C\varepsilon^{2-\kappa}, \end{aligned}$$

where we used the definition of τ^*. For the second term, we obtain by using Assumption 3.3.2 for $\mathcal{F}$

$$\begin{aligned} I_2 &\le C\varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}\nu(T-\tau)}(T-\tau)^{-\frac{\beta}{m}} \|\mathcal{F}_s(a(\tau)+\psi(\tau))\|_{\alpha-\beta}\, d\tau \\ &\le C\varepsilon^{\frac{2\beta}{m}} \int_0^T e^{-\varepsilon^{-2}\nu(T-\tau)}(T-\tau)^{-\frac{\beta}{m}} \|a(\tau)+\psi(\tau)\|_{\alpha}^3\, d\tau \\ &\le C\varepsilon^2 \sup_{\tau\in[0,\tau^*]} \|a(\tau)+\psi(\tau)\|_\alpha^3 \int_0^{\varepsilon^{-2}\nu T} e^{-\eta}\eta^{-\frac{\beta}{m}} d\eta \\ &\le C\varepsilon^2\Big(\sup_{[0,\tau^*]}\|a\|_\alpha^3 + \sup_{[0,\tau^*]}\|\psi\|_\alpha^3\Big) \\ &\le C\varepsilon^{2-3\kappa}, \end{aligned}$$

where we used again the definition of τ^*. Combining all results, yields (3.19). □

Let us now provide bounds on $\mathcal{Z}$ and thus later on ψ. These are also used to show that ψ is not too large, even at time τ^*. The following lemma shows that $\mathcal{Z} = \mathcal{O}(\varepsilon^{-\kappa_0})$ for any $\kappa_0 > 0$.

Lemma 3.4.2 *Under Assumption 2.2.1 and 3.3.3, there is a constant $C > 0$, depending on $p > 1,\ \alpha_k,\ \lambda_k,\ \kappa_0 > 0$ and T_0, such that*

$$\mathbb{E}\sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^p \le C\varepsilon^{-\kappa_0},$$

and

$$\mathbb{E}\sup_{T\in[0,T_0]} \|\mathcal{Z}(T)\|_\alpha^p \le C\varepsilon^{-\kappa_0},$$

where $\mathcal{Z}_k(T)$ and $\mathcal{Z}(T)$ are defined in (3.13) and (3.14), respectively.

Proof. In order to prove the first part, we define

$$\delta(T) = e^{-\lambda_\varepsilon T} \ \text{ and } \ \gamma(T) = \int_0^T e^{2\lambda_\varepsilon \tau} d\tau = \frac{1}{2\lambda_\varepsilon}(\delta(T)^{-2} - 1),$$

where $\lambda_\varepsilon = \varepsilon^{-2}\lambda_k$, and

$$Y(T) := \alpha_k \varepsilon^{-1} \delta(T) \cdot \beta\left(\gamma(T)\right).$$

Note that $\mathcal{Z}_k(T)$ and $Y(T)$ are Gaussian stochastic process with

$$\mathbb{E}\mathcal{Z}_k(T) = \mathbb{E}Y(T) = 0,$$

and for $S \leq T$

$$\mathbb{E}\mathcal{Z}_k(T)\mathcal{Z}_k(S) = \mathbb{E}Y(T)Y(S) = \alpha_k^2 \varepsilon^{-2} \delta(T+S)\gamma(S).$$

Thus $\mathcal{Z}_k(T)$ is a version of $Y(T)$, and

$$\begin{aligned}
\mathbb{E}\sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^p &= \mathbb{E}\sup_{T\in[0,T_0]} |Y(T)|^p = \left(\alpha_k\varepsilon^{-1}\right)^p \mathbb{E}\sup_{T\in[0,T_0]} |\delta(T)\cdot\beta\left(\gamma(T)\right)|^p \\
&\leq \left(\alpha_k\varepsilon^{-1}\right)^p \sum_{i=0}^{n-1} \mathbb{E}\sup_{T\in[T_i,T_{i+1}]} |\delta(T)|^p \left|\beta\left(\gamma(T)\right)\right|^p,
\end{aligned}$$

where $(T_i)_{i=0}^n$ is an equidistant decomposition of $[0, T_0]$. Using Doob's theorem, we obtain

$$\begin{aligned}
\mathbb{E}\sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^p &\leq C_{p,\alpha_k}\varepsilon^{-p} \sum_{i=0}^{n-1} \delta(T_i)^p \gamma(T_{i+1})^{\frac{p}{2}} \\
&\leq C_{p,\alpha_k}\varepsilon^{-p}\lambda_\varepsilon^{-p/2} \sum_{i=0}^{n-1} \left[\frac{\delta(T_i)}{\delta(T_{i+1})}\right]^p \\
&= C_{p,\alpha_k}\lambda_k^{-p/2} \sum_{i=0}^{n-1} e^{p\lambda_\varepsilon h} = C_{p,\alpha_k}\lambda_k^{-p/2} \frac{T_0}{h} e^{p\lambda_\varepsilon h},
\end{aligned}$$

where $h = T_{i+1} - T_i$. Taking $h = \frac{1}{\lambda_\varepsilon}$, we obtain

$$\mathbb{E}\sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^p \leq C\varepsilon^{-2}. \tag{3.20}$$

By Hölder inequality we derive for all $p \geq 1$ and sufficiently large $q > \frac{2}{\kappa_0}$

$$\mathbb{E}\sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^p \leq \left(\mathbb{E}\sup_{T\in[0,T_0]} |\mathcal{Z}_k(T)|^{pq}\right)^{1/q} \leq C\varepsilon^{-\kappa_0}.$$

In order to prove the second part, by Gaussianity,

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,T_0]}\|\mathcal{Z}(T)\|_\alpha^p &\leq C_p\Big(\mathbb{E}\sup_{T\in[0,T_0]}\sum_{k=n+1}^{N}k^{2\alpha}\mathcal{Z}_k^2(T)\Big)^{p/2}\\ &\leq C_p\Big(\sum_{k=n+1}^{N}k^{2\alpha}\mathbb{E}\sup_{T\in[0,T_0]}\mathcal{Z}_k^2(T)\Big)^{p/2}.\end{aligned}$$

Using Hölder inequality for all q and (3.20) to obtain

$$\mathbb{E}\sup_{T\in[0,T_0]}\mathcal{Z}_k^2(T)\leq\Big(\mathbb{E}\sup_{T\in[0,T_0]}\mathcal{Z}_k^{2q}(T)\Big)^{1/q}\leq C\varepsilon^{-2/q}.$$

Hence,

$$\mathbb{E}\sup_{T\in[0,T_0]}\|\mathcal{Z}(T)\|_\alpha^p\leq C\varepsilon^{-p/q}\leq C\varepsilon^{-\kappa_0},$$

for q large enough. □

The following corollary states that $\psi(T)$ is with high probability much smaller than $\varepsilon^{-\kappa}$ as asserted by the Definition 3.3.5 for $T\leq\tau^*$. To be more precise, $\psi=\mathcal{O}(\delta_\varepsilon+\varepsilon^{-\kappa_0})$ for any $\kappa_0>0$ and $\delta_\varepsilon\in(0,\varepsilon^{-\frac{1}{3}\kappa})$. We will use this later to show that $\tau^*\geq T_0$ with high probability (cf. Remark 3.6.5 and proof of Theorem 3.3.6).

Corollary 3.4.3 *Under the assumptions of Lemmas 3.4.1 and 3.4.2 with $\kappa<\frac{2}{3}$. For $p\geq 0$ and for $\kappa_0>0$ there exist a constant $C>0$ such that for $\|\psi(0)\|_\alpha\leq\delta_\varepsilon$ one has*

$$\mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\|\psi(T)\|_\alpha^p\Big)\leq C(\delta_\varepsilon+\varepsilon^{-\kappa_0})\,. \tag{3.21}$$

Proof. From (3.19), by triangle inequality and Lemma 3.4.2, we obtain

$$\mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\|\psi(T)\|_\alpha^p\Big)\leq C\delta_\varepsilon+C\varepsilon^{-\kappa_0}+C\varepsilon^{2p-3p\kappa},$$

for $\kappa<\frac{2}{3}$ we obtain (3.21). □

Lemma 3.4.4 *If Assumption 2.2.1 holds, then for $q\geq 1$ there exists a constant $C>0$ such that for $\|\psi(0)\|_\alpha\leq\delta_\varepsilon$ one has*

$$\int_0^T\Big\|e^{\tau\varepsilon^{-2}\mathcal{A}_s}\psi(0)\Big\|_\alpha^q\,d\tau\leq C\delta_\varepsilon^q\varepsilon^2.$$

Proof. Using (1.2) we obtain

$$\int_0^T \left\| e^{\varepsilon^{-2}\mathcal{A}_s\tau}\psi(0)\right\|_\alpha^q d\tau \le c\int_0^T e^{-q\varepsilon^{-2}\omega\tau}\,\|\psi(0)\|_\alpha^q\, d\tau \le \frac{\varepsilon^2}{q\omega}\,\|\psi(0)\|_\alpha^q\,.$$

This easily implies the claim. □

3.5 Averaging over the Fast OU-Process

Let us turn to the averaging result for the OU-process $\mathcal{Z}$. In Lemma 3.5.1, we provide the first order approximation. It states that even powers of a real valued OU-process average to a constant, while odd powers are small of order $\mathcal{O}(\varepsilon)$.

Lemma 3.5.1 *Let X be a real valued stochastic process such that for some $r \ge 0$ we have $X(0) = \mathcal{O}(\varepsilon^{-r})$. Fix any $\kappa_0 > 0$. If $dX = GdT$ with $G = \mathcal{O}(\varepsilon^{-r})$, then*

1. $\int_0^T X\mathcal{Z}_k d\tau = \mathcal{O}(\varepsilon^{1-r-\kappa_0})$,
2. $\int_0^T X\mathcal{Z}_k^2 d\tau = \frac{\alpha_k^2}{2\lambda_k}\int_0^T X d\tau + \mathcal{O}(\varepsilon^{1-r-2\kappa_0})$,
3. $\int_0^T X\mathcal{Z}_k\mathcal{Z}_l d\tau = \mathcal{O}(\varepsilon^{1-r-2\kappa_0})$,
4. $\int_0^T \mathcal{Z}_k\mathcal{Z}_l\mathcal{Z}_j d\tau = \mathcal{O}(\varepsilon^{1-3\kappa_0})$,
5. $\int_0^T \mathcal{Z}_k^2\mathcal{Z}_l d\tau = \mathcal{O}(\varepsilon^{1-3\kappa_0})$,
6. $\int_0^T \mathcal{Z}_k^3 d\tau = \mathcal{O}(\varepsilon^{1-3\kappa_0})$,
7. $\int_0^T \mathcal{Z}_k^2\mathcal{Z}_l\mathcal{Z}_j d\tau = \mathcal{O}(\varepsilon^{1-4\kappa_0})$,
8. $\int_0^T \mathcal{Z}_k^2\mathcal{Z}_l^2 d\tau = \frac{\alpha_k^2\alpha_l^2}{4\lambda_k\lambda_l}\int_0^T d\tau + \mathcal{O}(\varepsilon^{1-4\kappa_0})$,
9. $\int_0^T \mathcal{Z}_k^4 d\tau = \frac{3\alpha_k^4}{4\lambda_k^2}\int_0^T d\tau + \mathcal{O}(\varepsilon^{1-4\kappa_0})$,

where $\mathcal{Z}_k$ is defined in (3.13).

Proof. We note that

$$\mathbb{E}\sup_{[0,T_0]}|X|^p \le C\mathbb{E}\sup_{[0,T_0]}|G|^p \le C\varepsilon^{-pr}.$$

In order to prove the first part, we apply Itô formula to $X\mathcal{Z}_k$

$$\begin{aligned} d(X\mathcal{Z}_k) &= \mathcal{Z}_k dX + X d\mathcal{Z}_k \\ &= G\mathcal{Z}_k dT + \varepsilon^{-1}\alpha_k X d\tilde{\beta}_k - \lambda_k \varepsilon^{-2}\mathcal{Z}_k X dT. \end{aligned}$$

Integrating from 0 to T, we obtain

$$\lambda_k \int_0^T X\mathcal{Z}_k d\tau = -\varepsilon^2 X(T)\mathcal{Z}_k(T) + \varepsilon^2 \int_0^T G\mathcal{Z}_k d\tau + \varepsilon\alpha_k \int_0^T X d\tilde{\beta}_k.$$

Taking the absolute value and using the triangle inequality we obtain, for $p > 0$,

$$\left|\int_0^T X\mathcal{Z}_k d\tau\right|^p \leq c\varepsilon^{2p} |X(T)|^p |\mathcal{Z}_k(T)|^p + c\varepsilon^{2p} \left|\int_0^T G\mathcal{Z}_k d\tau\right|^p + c\varepsilon^p \left|\int_0^T X d\tilde{\beta}_k\right|^p.$$

Taking expectation after supremum on both sides and using Lemma 3.4.2 yields

$$\mathbb{E} \sup_{T\in[0,T_0]} \left|\int_0^T X\mathcal{Z}_k d\tau\right|^p \leq C\varepsilon^{2p-pr-\kappa_0} + C\varepsilon^p \mathbb{E} \sup_{T\in[0,T_0]} \left|\int_0^T X d\tilde{\beta}_k\right|^p.$$

We finish the first part by using the theorem of Burkholder-Davis-Gundy

$$\mathbb{E} \sup_{T\in[0,T_0]} \left|\int_0^T X\mathcal{Z}_k d\tau\right|^p \leq C\varepsilon^{2p-pr-\kappa_0} + C\varepsilon^p \mathbb{E}\Big(\int_0^T X^2(\tau) d\tau\Big)^{\frac{p}{2}} \leq C\varepsilon^{p-pr-\kappa_0}.$$

In order to prove the second part, we apply Itô formula to $X\mathcal{Z}_k^2$

$$\begin{aligned} d\left(X\mathcal{Z}_k^2\right) &= \mathcal{Z}_k^2 dX + 2X\mathcal{Z}_k d\mathcal{Z}_k + X\left(d\mathcal{Z}_k\right)^2 \\ &= G\mathcal{Z}_k^2 dT - 2\lambda_k\varepsilon^{-2} X\mathcal{Z}_k^2 dT + 2\varepsilon^{-1}\alpha_k \mathcal{Z}_k X d\tilde{\beta}_k + \varepsilon^{-2}\alpha_k^2 X dT. \end{aligned}$$

Integrating from 0 to T, we obtain

$$\int_0^T X\left(\mathcal{Z}_k^2 - \frac{\alpha_k^2}{2\lambda_k}\right) d\tau = -\frac{\varepsilon^2}{2\lambda_k} X(T)\mathcal{Z}_k^2(T) + \frac{\varepsilon^2}{2\lambda_k}\int_0^T G\mathcal{Z}_k^2 d\tau + \frac{\alpha_k}{\lambda_k}\varepsilon \int_0^T X\mathcal{Z}_k d\tilde{\beta}_k.$$

Taking the absolute value and using the triangle inequality we obtain, for $p > 0$,

$$\left|\int_0^T X\left(\mathcal{Z}_k^2 - \frac{\alpha_k^2}{2\lambda_k}\right) d\tau\right|^p \leq c\varepsilon^{2p} |X\mathcal{Z}_k|^p + c\varepsilon^{2p} \left|\int_0^T G\mathcal{Z}_k^2 d\tau\right|^p + c\varepsilon^p \left|\int_0^T X\mathcal{Z}_k d\tilde{\beta}_k(\tau)\right|^p.$$

By Burkholder-Davis-Gundy theorem, we obtain

$$\begin{aligned} \mathbb{E} \sup_{T\in[0,T_0]} \left|\int_0^T X\left(\mathcal{Z}_k^2 - \frac{\alpha_k^2}{2\lambda_k}\right) d\tau\right|^p &\leq C\varepsilon^{2p-pr-\kappa_0} + C\varepsilon^p \mathbb{E}\Big(\int_0^{T_0} X^2\mathcal{Z}_k^2 d\tau\Big)^{\frac{p}{2}} \\ &\leq C\varepsilon^{p-pr-2\kappa_0}, \end{aligned}$$

and this finishes the second part. For the third part, we apply Itô formula to $X\mathcal{Z}_k\mathcal{Z}_l$ and integrate from 0 to T

$$\begin{aligned}\int_0^T X\mathcal{Z}_k\mathcal{Z}_l d\tau &= -\frac{\varepsilon^2}{\lambda_k+\lambda_l}X\mathcal{Z}_k\mathcal{Z}_l + \frac{\varepsilon^2}{\lambda_k+\lambda_l}\int_0^T \mathcal{Z}_k\mathcal{Z}_l G d\tau \\ &\quad + \frac{\alpha_l\varepsilon}{\lambda_k+\lambda_l}\int_0^T X\mathcal{Z}_k d\tilde{\beta}_l + \frac{\alpha_k\varepsilon}{\lambda_k+\lambda_l}\int_0^T X\mathcal{Z}_l d\tilde{\beta}_k.\end{aligned}$$

Taking the absolute value and using Burkholder-Davis-Gundy theorem, we obtain for $p > 0$

$$\mathbb{E}\Big(\sup_{T\in[0,T_0]}\Big|\int_0^T X\mathcal{Z}_k\mathcal{Z}_l d\tau\Big|^p\Big) \le C\varepsilon^{p-pr-2\kappa_0}.$$

For the fourth part, we apply Itô formula to $\mathcal{Z}_k\mathcal{Z}_l\mathcal{Z}_j$ and integrating from 0 to T we obtain

$$\begin{aligned}(\lambda_k+\lambda_l+\lambda_j)\int_0^T \mathcal{Z}_k\mathcal{Z}_l\mathcal{Z}_j d\tau &= -\varepsilon^2\mathcal{Z}_k\mathcal{Z}_l\mathcal{Z}_j + \alpha_l\varepsilon\int_0^T \mathcal{Z}_k\mathcal{Z}_j d\tilde{\beta}_l \\ &\quad + \alpha_j\varepsilon\int_0^T \mathcal{Z}_k\mathcal{Z}_l d\tilde{\beta}_j + \alpha_k\varepsilon\int_0^T \mathcal{Z}_l\mathcal{Z}_j d\tilde{\beta}_k.\end{aligned}$$

Taking the absolute value and using Burkholder-Davis-Gundy theorem, we obtain for $p > 0$

$$\mathbb{E}\Big(\sup_{T\in[0,T_0]}\Big|\int_0^T \mathcal{Z}_k\mathcal{Z}_l\mathcal{Z}_j d\tau\Big|^p\Big) \le C\varepsilon^{1-4\kappa_0}.$$

For the fifth part, we apply Itô formula to $\mathcal{Z}_k^2\mathcal{Z}_l$ and integrating from 0 to T

$$\begin{aligned}\int_0^T \mathcal{Z}_k^2\mathcal{Z}_l d\tau &= -\frac{\varepsilon^2}{\lambda_l+2\lambda_k}\mathcal{Z}_k^2\mathcal{Z}_l + \frac{\alpha_l\varepsilon}{\lambda_l+2\lambda_k}\int_0^T \mathcal{Z}_k^2 d\tilde{\beta}_l \\ &\quad + \frac{2\alpha_k\varepsilon}{\lambda_l+2\lambda_k}\int_0^T \mathcal{Z}_k\mathcal{Z}_l d\tilde{\beta}_k + \frac{\alpha_k^2}{\lambda_l+2\lambda_k}\int_0^T \mathcal{Z}_l d\tau. \quad (3.22)\end{aligned}$$

We note that from the first part. If we take $X = 1$ and choose $r = 0$, then we obtain

$$\int_0^T \mathcal{Z}_k(\tau)d\tau = \mathcal{O}(\varepsilon^{1-\kappa_0}). \quad (3.23)$$

Taking the absolute value and using Burkholder-Davis-Gundy theorem and (3.23), we obtain for $p > 0$

$$\mathbb{E}\sup_{T\in[0,T_0]}\Big|\int_0^T \mathcal{Z}_k^2\mathcal{Z}_l d\tau\Big|^p \le C\varepsilon^{1-3\kappa_0}.$$

For the sixth part, we put $l = k$ in (3.22). We obtain

$$\int_0^T \mathcal{Z}_k^3 d\tau = -\frac{\varepsilon^2}{3\lambda_k}\mathcal{Z}_k^3(T) + \frac{\alpha_l}{\lambda_k}\varepsilon\int_0^T \mathcal{Z}_k^2 d\tilde{\beta}_k + \frac{\alpha_k^2}{3\lambda_k}\int_0^T \mathcal{Z}_l d\tau.$$

Analogously, we obtain

$$\mathbb{E}\Big(\sup_{T\in[0,T_0]}\Big|\int_0^T \mathcal{Z}_k^3 d\tau\Big|^p\Big) \le C\varepsilon^{1-3\kappa_0}.$$

For the seventh part, we apply Itô formula to $\mathcal{Z}_k^2\mathcal{Z}_l\mathcal{Z}_j$ and integrating from 0 to T

$$\begin{aligned}(2\lambda_k+\lambda_l+\lambda_j)\int_0^T \mathcal{Z}_k^2\mathcal{Z}_l\mathcal{Z}_j d\tau &= -\varepsilon^2\mathcal{Z}_k^2\mathcal{Z}_l\mathcal{Z}_j + 2\varepsilon\alpha_k\int_0^T \mathcal{Z}_k\mathcal{Z}_l\mathcal{Z}_j d\tilde{\beta}_k \\ &\quad +\varepsilon\alpha_l\int_0^T \mathcal{Z}_k^2\mathcal{Z}_j d\tilde{\beta}_l + \varepsilon\alpha_j\int_0^T \mathcal{Z}_k^2\mathcal{Z}_l d\tilde{\beta}_j \\ &\quad +\alpha_k^2\int_0^T \mathcal{Z}_l\mathcal{Z}_j d\tau.\end{aligned}$$

Using Burkholder-Davis-Gundy theorem and the second part with $X = 1$ in order to obtain the seventh part.

For the eighth part, we apply Itô formula to $\mathcal{Z}_k^2\mathcal{Z}_l^2$ and integrating from 0 to T

$$\begin{aligned}\int_0^T \mathcal{Z}_k^2\mathcal{Z}_l^2 d\tau &= -\frac{\varepsilon^2}{2(\lambda_l+\lambda_k)}\mathcal{Z}_k^2\mathcal{Z}_l^2 + \frac{\alpha_l\varepsilon}{\lambda_l+\lambda_k}\int_0^T \mathcal{Z}_k^2\mathcal{Z}_l d\tilde{\beta}_l \\ &\quad +\frac{\alpha_k\varepsilon}{\lambda_l+\lambda_k}\int_0^T \mathcal{Z}_l^2\mathcal{Z}_k d\tilde{\beta}_k + \frac{\alpha_k^2}{2(\lambda_l+\lambda_k)}\int_0^T \mathcal{Z}_l^2 d\tau \\ &\quad +\frac{\alpha_l^2}{2(\lambda_l+\lambda_k)}\int_0^T \mathcal{Z}_k^2 d\tau.\end{aligned}$$

We finish the proof of the eighth part by using Burkholder-Davis-Gundy theorem and the second part with $X = 1$.

For the ninth part, we apply Itô formula to $\mathcal{Z}_k^4$ and integrating from 0 to T

$$\int_0^T \mathcal{Z}_k^4 d\tau = -\frac{\varepsilon^2}{4\lambda_k}\mathcal{Z}_k^4 + \frac{\alpha_k\varepsilon}{\lambda_k}\int_0^T \mathcal{Z}_k^3 d\tilde{\beta}_k + \frac{3\alpha_k^2}{2\lambda_k}\int_0^T \mathcal{Z}_k^2 d\tau.$$

Using Burkholder-Davis-Gundy theorem and the second part with $X = 1$ to finish this part. □

Remark 3.5.2 *The above Lemma is true even if $X \in \mathcal{N}$ or $\mathbb{C}$.*

3.6 Proof of the Main Result

This section is devoted to the proof of the main result in Theorem 3.3.6 and Corollary 3.3.7 for the approximation (3.7) of the solution of the SPDE (3.1).
Let us first check that, we can apply the averaging lemma to (3.5).

Lemma 3.6.1 *Assume that Assumption 3.3.1 and 3.3.2 hold. Let X be a stochastic process in $\mathcal{N}$ and $dX = GdT$. If $X = \mathcal{F}_c(a, e_k, e_l)$ or $X = \mathcal{F}_c(a, a, e_k)$, then $G = \mathcal{O}(\varepsilon^{-3\kappa})$ or $G = \mathcal{O}(\varepsilon^{-4\kappa})$, respectively.*

Proof. If $X = \mathcal{F}_c(a, e_k, e_k)$, then

$$dX = \mathcal{F}_c(da, e_k, e_l) = \mathcal{F}_c(\mathcal{L}_c a + \mathcal{F}_c(a + \psi), e_k, e_l)dT.$$

Let

$$G = \mathcal{F}_c(\mathcal{L}_c a + \mathcal{F}_c(a + \psi), e_k, e_l).$$

Taking the $\mathcal{H}^\alpha$ norm, using Assumption 3.3.2 and the fact all norms are equivalent on $\mathcal{N}$, to obtain

$$\begin{aligned} \|G\|_\alpha &\leq C\|\mathcal{L}_c a + \mathcal{F}_c(a+\psi)\|_\alpha \leq C\|a\|_\alpha + C\|\mathcal{F}_c(a+\psi)\|_{\alpha-\beta} \\ &\leq C\|a\|_\alpha + C\|a+\psi\|_\alpha^3 \leq C\|a\|_\alpha + C\|a\|_\alpha^3 + C\|\psi\|_\alpha^3 . \end{aligned}$$

Using the definition of τ^*, we obtain for all $p > 0$

$$\mathbb{E}\sup_{[0,\tau^*]} \|G\|_\alpha^p \leq C\varepsilon^{-3p\kappa}.$$

Analogously, if $X = \mathcal{F}_c(a, a, e_k)$, then

$$dX = 2\mathcal{F}_c(da, a, e_k) = 2\mathcal{F}_c(\mathcal{L}_c a + \mathcal{F}_c(a + \psi), a, e_k)dT.$$

Define

$$G := 2\mathcal{F}_c(\mathcal{L}_c a + \mathcal{F}_c(a + \psi), a, e_k),$$

in order to obtain

$$\mathbb{E}\sup_{[0,\tau^*]} \|G\|_\alpha^p \leq C\varepsilon^{-4p\kappa}.$$

□

Lemma 3.6.2 *If Assumptions 2.2.1, 3.3.1, 3.3.2 and 3.3.3 hold and* $\|\psi(0)\|_\alpha \le \delta_\varepsilon$ *for* $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$ *with* $\kappa \in (0, \frac{1}{12})$ *from the definition of* τ^**, then*

$$a(T) = a(0)+\int_0^T \mathcal{L}_c a(\tau)d\tau+\int_0^T \mathcal{F}_c(a)d\tau+\sum_{k=n+1}^{N}\frac{3\alpha_k^2}{2\lambda_k}\int_0^T \mathcal{F}_c(a,e_k,e_k)d\tau+R(T), \tag{3.24}$$

where

$$R = \mathcal{O}(\varepsilon^{1-5\kappa}), \tag{3.25}$$

for $\kappa > 0$ *from the definition of* τ^**.*

Proof. Recall Lemma 3.4.1, which states

$$\psi(T) = y_\varepsilon(T) + \mathcal{Z}(T) + \mathcal{O}(\varepsilon^{2-3\kappa}), \tag{3.26}$$

where

$$y_\varepsilon(T) = e^{\varepsilon^{-2}T\mathcal{A}_s}\psi(0).$$

Substituting from (3.26) into (3.5) and using the bounds on $a = \mathcal{O}(\varepsilon^{-\kappa})$, $\mathcal{Z} = \mathcal{O}(\varepsilon^{-\kappa_0})$, and $y_\varepsilon = \mathcal{O}(\delta_\varepsilon\varepsilon^2)$ we obtain for $\kappa < 2/3$

$$\begin{aligned} da &= [\mathcal{L}_c a + \mathcal{F}_c(a + y_\varepsilon + \mathcal{Z})]\, dT + \mathcal{O}(\varepsilon^{2-5\kappa})dT \\ &= [\mathcal{L}_c a + \mathcal{F}_c(a) + 3\mathcal{F}_c(a,a,\mathcal{Z}) + 3\mathcal{F}_c(a,\mathcal{Z},\mathcal{Z}) + \mathcal{F}_c(\mathcal{Z}) \\ &\quad +3\mathcal{F}_c(a,a,y_\varepsilon) + 6\mathcal{F}_c(a,\mathcal{Z},y_\varepsilon) + 3\mathcal{F}_c(\mathcal{Z},\mathcal{Z},y_\varepsilon) \\ &\quad +3\mathcal{F}_c(a,y_\varepsilon,y_\varepsilon) + 3\mathcal{F}_c(\mathcal{Z},y_\varepsilon,y_\varepsilon) + \mathcal{F}_c(y_\varepsilon)]dT + \mathcal{O}(\varepsilon^{2-5\kappa})dT. \end{aligned}$$

Integrating from 0 to T, yields for $T \le \tau^*$

$$\begin{aligned} a(T) &= a(0) + \int_0^T \mathcal{L}_c a(\tau)d\tau + \int_0^T \mathcal{F}_c(a)d\tau + 3\sum_{k=n+1}^{N}\int_0^T \mathcal{Z}_k\mathcal{F}_c(a,a,e_k)d\tau \\ &\quad +3\sum_{k=n+1}^{N}\int_0^T \mathcal{Z}_k^2\mathcal{F}_c(a,e_k,e_k)d\tau + 3\sum_{k=n+1}^{N}\sum_{l\neq k}^{N}\int_0^T \mathcal{Z}_k\mathcal{Z}_l\mathcal{F}_c(a,e_k,e_l)d\tau \\ &\quad + \sum_{k,l,j=n+1}^{N}\int_0^T \mathcal{F}_c(\mathcal{Z}_k e_k, \mathcal{Z}_l e_l, \mathcal{Z}_j e_j)d\tau + R_1 + \mathcal{O}(\varepsilon^{2-5\kappa}), \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} R_1 &= 3\int_0^T \mathcal{F}_c(a,a,y_\varepsilon)d\tau + 6\int_0^T \mathcal{F}_c(a,\mathcal{Z},y_\varepsilon)d\tau + 3\int_0^T \mathcal{F}_c(a,y_\varepsilon,y_\varepsilon)d\tau \\ &\quad +3\int_0^T \mathcal{F}_c(\mathcal{Z},y_\varepsilon,y_\varepsilon)d\tau + 3\int_0^T \mathcal{F}_c(\mathcal{Z},\mathcal{Z},y_\varepsilon)d\tau + 3\int_0^T \mathcal{F}_c(y_\varepsilon)d\tau \\ &:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \tag{3.28}$$

Now, using Assumption 3.3.1, the definition of τ^* and the equivalence of $\mathcal{H}^\alpha$-norms on $\mathcal{N}$ to bound R_1. We bound all terms in (3.28) separately. For the first term in (3.28) we obtain

$$\|I_1\|_\alpha \le C\int_0^T \|a\|_\alpha^2 \|y_\varepsilon\|_\alpha \, d\tau \le C \sup_{[0,T_0]} \|a\|_\alpha^2 \int_0^T \|y_\varepsilon\|_\alpha \, d\tau.$$

Using Lemma 3.4.4 for $q = 1$, we obtain

$$I_1 = \mathcal{O}(\delta_\varepsilon \varepsilon^{2-2\kappa}).$$

Analogous results hold to all other terms. To be more precise:

$$I_2 = \mathcal{O}(\delta_\varepsilon \varepsilon^{2-\kappa-\kappa_0}),\ I_3 = \mathcal{O}(\delta_\varepsilon^2 \varepsilon^{2-\kappa}),\ I_4 = \mathcal{O}(\delta_\varepsilon^2 \varepsilon^{2-\kappa_0}),$$

$$I_5 = \mathcal{O}(\delta_\varepsilon \varepsilon^{2-2\kappa_0}), \text{and} I_6 = \mathcal{O}(\delta_\varepsilon^3 \varepsilon^2).$$

Collecting all results we obtain for $\kappa_0 \le \kappa$, where $\kappa_0 > 0$ is arbitrary from Lemma 3.4.2,

$$R_1 = \mathcal{O}((1+\delta_\varepsilon^2)\varepsilon^{2-2\kappa}). \tag{3.29}$$

Applying finally Lemmas 3.5.1 and 3.6.1 to (3.27), we obtain (3.24). □

Lemma 3.6.3 *Let Assumptions 2.2.1, 3.3.1 and 3.3.2 hold. Define* $b(t)$ *in* $\mathcal{N}$ *as the solution of (3.8). If the initial condition satisfies* $\mathbb{E}\,|b(0)|^p \le \delta_\varepsilon^p$ *for some* $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$, *then for all* $T_0 > 0$ *and* $p \ge 1$ *there exists a constant* C *such that*

$$\mathbb{E} \sup_{T\in[0,T_0]} |b(T)|^p \le C\delta_\varepsilon^p. \tag{3.30}$$

Proof. Taking the scalar product $\langle\cdot, b\rangle$ on both sides of (3.8), yields

$$\frac{1}{2}\partial_T |b|^2 = \langle \mathcal{L}_c b, b\rangle + \langle \mathcal{F}_c(b), b\rangle + \sum_{k=n+1}^N \frac{3\alpha_k^2}{2\lambda_k} \langle \mathcal{F}_c(b, e_k, e_k), b\rangle .$$

Using Cauchy-Schwarz inequality and Assumption 3.3.2, we obtain

$$\frac{1}{2}\partial_T |b|^2 \le C |b|^2 .$$

We apply now a comparison argument to deduce for all $T \in [0, T_0]$

$$|b(T)| \le |b(0)| e^{CT_0}. \tag{3.31}$$

Taking expectation after supremum on both sides, yields (3.30). □

In the following we are not able to calculate moments of error terms. Thus, we restrict ourselves to a sufficiently large subset of Ω, where our estimates go through.

Definition 3.6.4 *Given $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$ with $\kappa > 0$ from the definition of τ^*. Define the set $\Omega^* \subset \Omega$ such that all these estimates*

$$\sup_{[0,\tau^*]} \|\psi - \mathcal{Q}\|_\alpha < C\varepsilon^{2-4\kappa} , \tag{3.32}$$

$$\sup_{[0,\tau^*]} \|\psi\|_\alpha < \delta_0 + \varepsilon^{-\frac{1}{2}\kappa} , \tag{3.33}$$

$$\sup_{[0,\tau^*]} |R| < \varepsilon^{1-6\kappa} , \tag{3.34}$$

and

$$\sup_{[0,\tau^*]} |b| < \delta_0\varepsilon^{-\frac{1}{2}\kappa} , \tag{3.35}$$

hold on Ω^*.

Remark 3.6.5 *The set Ω^* has approximately probability 1 provided $\delta_\varepsilon < \varepsilon^{-\frac{1}{3}\kappa}$, as*

$$\begin{aligned} \mathbb{P}(\Omega^*) &\ge 1 - \mathbb{P}(\sup_{[0,\tau^*]} \|\psi - \mathcal{Q}\|_\alpha \ge \varepsilon^{2-4\kappa}) - \mathbb{P}(\sup_{[0,\tau^*]} \|\psi\|_\alpha \ge \varepsilon^{-\frac{1}{2}\kappa}) \\ &- \mathbb{P}(\sup_{[0,\tau^*]} |b| \ge \varepsilon^{-\frac{5}{6}\kappa}) - \mathbb{P}(\sup_{[0,\tau^*]} |R| \ge \varepsilon^{1-6\kappa}). \end{aligned}$$

Using Chebychev inequality and Lemmas 3.4.1, 3.6.2, 3.6.3 and Corollary 3.4.3 with arbitrarily $\kappa_0 \le \frac{1}{3}\kappa$, we obtain for sufficient large q

$$\mathbb{P}(\Omega^*) \ge 1 - C[\varepsilon^{q\kappa} + \varepsilon^{\frac{1}{2}q\kappa - q\kappa_0} + \varepsilon^{\frac{1}{2}q\kappa} + \varepsilon^{q\kappa}] \ge 1 - C\varepsilon^{\frac{1}{3}q\kappa} \ge 1 - C\varepsilon^p. \tag{3.36}$$

Theorem 3.6.6 *Assume that Assumptions 2.2.1, 3.3.1, 3.3.2 and 3.3.3 hold and suppose* $|a(0)| \leq \delta_\varepsilon$ *and* $\|\psi(0)\|_\alpha \leq \delta_\varepsilon$*. Let* b *be a solution of (3.8) and* a *as defined in (3.2). If the initial condition satisfies* $a(0) = b(0)$*, then*

$$\sup_{T\in[0,\tau^*]} |a(T) - b(T)| \leq C(1 + \delta_\varepsilon^2)\varepsilon^{1-12\kappa}, \tag{3.37}$$

and for $\kappa < \frac{1}{12}$

$$\sup_{T\in[0,\tau^*]} |a(T)| \leq C(1 + \delta_\varepsilon^2), \tag{3.38}$$

on Ω^**.*

Proof. Define $\varphi(T)$ as

$$\varphi(T) := a(T) - R(T),$$

where R is defined in (3.25). From (3.24) we obtain

$$\varphi(T) = a(0) + \int_0^T \mathcal{L}_c\left[\varphi + R\right] d\tau + \int_0^T \mathcal{F}_c(\varphi+R)d\tau + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \int_0^T \mathcal{F}_c(\varphi+R, e_k, e_k)d\tau. \tag{3.39}$$

Subtracting (3.39) from (3.8) and defining $h(T) := b(T) - \varphi(T)$, we obtain

$$\begin{aligned} h(T) &= \int_0^T \mathcal{L}_c h d\tau - \int_0^T \mathcal{L}_c R d\tau + \int_0^T \left[\mathcal{F}_c(b) - \mathcal{F}_c(b - h + R)\right] d\tau \\ &\quad + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \int_0^T \mathcal{F}_c(h - R, e_k, e_k)d\tau. \end{aligned}$$

Thus,

$$\partial_T h = \mathcal{L}_c h - \mathcal{L}_c R + \mathcal{F}_c(b) - \mathcal{F}_c(b - h + R) + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \mathcal{F}_c(h - R, e_k, e_k). \tag{3.40}$$

Taking the scalar product $\langle \cdot, h\rangle$ on both sides of (3.40), we have

$$\begin{aligned} \frac{1}{2}\partial_T |h|^2 &= \langle \mathcal{L}_c h, h\rangle - \langle \mathcal{L}_c R, h\rangle + \langle \mathcal{F}_c(b) - \mathcal{F}_c(b - h + R), h\rangle \\ &\quad + \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \langle \mathcal{F}_c(h, e_k, e_k), h\rangle - \sum_{k=n+1}^{N} \frac{3\alpha_k^2}{2\lambda_k} \left\langle \mathcal{F}_c(R, e_k, e_k), h\right\rangle. \end{aligned}$$

Using Cauchy-Schwarz inequality and Assumption 3.3.2, we obtain the following linear ordinary differential inequality

$$\partial_T |h|^2 \leq C[|h|^2 + |h|^4] + C\left[|R|^4 + |b|^2 |R|^2 + |b|^4 |R|^2 + |b|^2 |R|^4\right].$$

Using (3.34) and (3.35) in the definition of Ω^*, we obtain for $T \leq \tau^*$

$$\partial_T |h|^2 \leq C[|h|^2 + |h|^4] + C(1 + \delta_\varepsilon^4)\varepsilon^{2-24\kappa} \quad \text{on } \Omega^*.$$

As long as $|h| \leq 1$, we obtain

$$\partial_T |h|^2 \leq 2C |h|^2 + C(1 + \delta_\varepsilon^4)\varepsilon^{2-24\kappa}.$$

Using Gronwall's lemma, we obtain for $T \leq \tau^* \leq T_0$

$$|h(T)|^2 \leq C(1 + \delta_\varepsilon^4)\varepsilon^{2-24\kappa} \leq 1,$$

for $\delta_\varepsilon < \varepsilon^{-\frac{1}{3}\kappa}$, $\varepsilon > 0$ sufficiently small, and $\kappa < \frac{3}{38}$. Thus,

$$\sup_{[0,\tau^*]} |h| \leq C(1 + \delta_\varepsilon^2)\varepsilon^{1-12\kappa} \quad \text{on } \Omega^*. \tag{3.41}$$

We finish the first part by using (3.34), (3.41) and

$$\sup_{[0,\tau^*]} |a - b| = \sup_{[0,\tau^*]} |h - R| \leq \sup_{[0,\tau^*]} |h| + \sup_{[0,\tau^*]} |R|.$$

For the second part of the theorem consider

$$\sup_{[0,\tau^*]} |a| \leq \sup_{[0,\tau^*]} |a - b| + \sup_{[0,\tau^*]} |b|.$$

Using the first part and (3.35), we obtain (3.38) as $\kappa < \frac{1}{12}$. □

Now, we can use the results previously obtained to prove the main result of Theorem 3.3.6 for the approximation of the Solution (3.7) of the SPDE (3.1).

Proof of Theorem 3.3.6. For the stopping time, we note that provided $\delta_\varepsilon < \varepsilon^{-\frac{1}{3}\kappa}$

$$\Omega \supset \{\tau^* = T_0\} \supseteq \{ \sup_{T\in[0,T_0]} |a(T)| < \varepsilon^{-\kappa}, \sup_{T\in[0,T_0]} \|\psi(T)\|_\alpha < \varepsilon^{-\kappa}\} \supseteq \Omega^*,$$

where the last inclusion holds due to (3.33) and Theorem 3.6.6. Now let us turn to the approximation result. Using (3.2) and triangle inequality, yields

$$\sup_{T\in[0,\tau^*]} \left\| u(\varepsilon^{-2}T) - \varepsilon b(T) - \varepsilon \mathcal{Q}(T) \right\|_\alpha \leq \varepsilon \sup_{[0,\tau^*]} \|a - b\|_\alpha + \varepsilon \sup_{[0,\tau^*]} \left\| \psi - \mathcal{Q} \right\|_\alpha.$$

From (3.32) and (3.37), we obtain

$$\begin{aligned}\sup_{t\in[0,\varepsilon^{-2}T_0]}\Big\|u(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha &= \sup_{t\in[0,\varepsilon^{-2}\tau^*]}\Big\|u(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha \\ &\le C\varepsilon^{2-\frac{38}{3}\kappa} \qquad \text{on } \Omega^*.\end{aligned}$$

Thus,

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\Big\|u(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big) \le 1-\mathbb{P}(\Omega^*).$$

Using the bound on Ω^* from (3.36), yields the main claim (3.16). □

Proof of Corollary 3.3.7. Define $\Omega_0 \subset \Omega$ as

$$\Omega_0 = \{w : \|u(0)\|_\alpha \le \varepsilon\delta_\varepsilon\},$$

and define

$$\hat{u}(0) = \begin{cases} 0 & \text{on } \Omega_0^c \\ u(0) & \text{on } \Omega_0. \end{cases}$$

Hence,

$$u = \hat{u} \quad \text{on } \Omega_0.$$

Thus,

$$\begin{aligned}&\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\Big\|u(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big) \\ &= \mathbb{P}\Big(\{\sup_{t\in[0,\varepsilon^{-2}T_0]}\Big\|u(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\}\cap\Omega_0\Big) \\ &+\mathbb{P}\Big(\{\sup_{t\in[0,\varepsilon^{-2}T_0]}\Big\|u(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\}\cap\Omega_0^c\Big) \\ &\le \mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\Big\|\hat{u}(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > \varepsilon^{2-\frac{38}{3}\kappa}\Big)+\mathbb{P}\Big(\Omega_0^c\Big) \\ &\le C\varepsilon^p+\mathbb{P}(\|u(0)\|_\alpha > \varepsilon\delta_\varepsilon),\end{aligned}$$

where we used (3.16). □

3.7 Applications

In the literature there are numerous examples of equations with cubic nonlinearities where our theory applies. Examples studied here are Swift-Hohenberg equation, Ginzburg-Landau / Allen-Cahn equation and some Surface growth model. In all these examples we obtain that adding noise has the potential to stabilize the dynamics of the dominant modes. Furthermore, the amplitude equation is always the same type

$$A^{`} = \nu A - C_{\alpha_k} A - CA\,|A|^2 ,$$

where A is the amplitude of the domiant modes in $\mathcal{N}$.

3.7.1 Swift-Hohenberg Equation

The Swift-Hohenberg equation was defined in introduction(cf. (3.9)). It has been used as a toy model for the convective instability in the Rayleigh-Bénard problem (see [16] or [22]). Now it is one of the celebrated models in the theory of pattern formation. For this model we note that

$$\mathcal{A} = -(1+\partial_x^2)^2,\ \ \mathcal{L} = \nu\mathcal{I},\ \mathcal{F}(u) = -u^3.$$

If we take

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\sin(kx) & \text{if } k>0, \\ \frac{1}{\sqrt{2\pi}} & \text{if } k=0, \\ \frac{1}{\sqrt{\pi}}\cos(kx) & \text{if } k<0, \end{cases}$$

and

$$\mathcal{H} = L^2([0,2\pi]) \ \text{ and } \ \mathcal{N} = \text{span}\{\sin,\ \cos\},$$

then the eigenvalues of $-\mathcal{A} = (1+\partial_x^2)^2$ are given by $\lambda_k = (1-k^2)^2$ with $m=4$, $\lambda_0 = 1 > 0$ and $\lim_{k\to\infty}\lambda_k = \infty$. So, the Assumption 2.2.1 is true. If we split $u = u_1\sin + u_{-1}\cos$ and $u = w_1\sin + w_{-1}\cos \in \mathcal{N}$, then the Assumption 3.3.2 is true as follows:

$$\langle \mathcal{F}_c\left(u_1\sin + u_{-1}\cos\right), u_1\sin + u_{-1}\cos\rangle = -\frac{3\pi}{4}\left(u_1^2 + u_{-1}^2\right)^2 \le 0,$$

where

$$\mathcal{F}_c\left(u_1\sin + u_{-1}\cos\right) = -\tfrac{3}{4}\left(u_1^3 + u_1u_{-1}^2\right)\sin - \tfrac{3}{4}\left(u_{-1}^3 + u_1^2u_{-1}\right)\cos .$$

Moreover,

$$\langle \mathcal{F}_c(u,u,w),w\rangle = -\frac{3\pi}{4}\left(u_1^2w_1^2 + w_1^2u_{-1}^2 + w_{-1}^2u_{-1}^2 + w_{-1}^2u_1^2\right) \le 0,$$

and for $\alpha = 1$ and $\beta = 0$ we obtain

$$\|\mathcal{F}(u,v,w)\|_{\mathcal{H}^1} = \|-uvw\|_{\mathcal{H}^1} \le C\,\|u\|_{\mathcal{H}^1}\,\|v\|_{\mathcal{H}^1}\,\|w\|_{\mathcal{H}^1}\,.$$

For Assumption 3.3.3, we consider many cases:

First case. The noise is a constant in the space (i.e., $W(t) = \frac{\alpha_0}{\sqrt{2\pi}}\beta_0(t)$).

In this case our main theorem states that the solution of the (3.9) is of the type

$$u(t,x) = \varepsilon v(\varepsilon^2 t, x),$$

and

$$v(T,x) \simeq \gamma_1(T)\sin(x) + \gamma_{-1}(T)\cos(x) + \varepsilon\frac{\alpha_0}{\sqrt{2\pi}}\partial_T\tilde{\beta}_0(T) + \mathcal{O}(\varepsilon^{1-}),$$

where γ_1 and γ_{-1} are the solution of the amplitude equation

$$\gamma_i^{\backslash} = (\nu - \tfrac{3\alpha_0^2}{4\pi})\gamma_i - \tfrac{3}{4}\gamma_i(\gamma_1^2 + \gamma_{-1}^2) \;\text{ for } i = \pm 1.$$

Second case. If the noise acts on $\sin(kx)$ [or $\cos(kx)$] for one $k \in \{2,3,.....\}$, then the amplitude equations for (3.9) in this case are

$$\gamma_i^{\backslash} = (\nu - \tfrac{3\alpha_k^2}{2\pi(k^2-1)^2})\gamma_i - \tfrac{3}{4}\gamma_i(\gamma_1^2 + \gamma_{-1}^2) \;\text{ for } i = \pm 1,$$

and our main theorem states that the solution of the (3.9) is of the type

$$u(t,x) = \varepsilon v(\varepsilon^2 t, x),$$

and

$$v(T,x) \simeq \gamma_1(T)\sin(x) + \gamma_{-1}(T)\cos(x) + \varepsilon\frac{\alpha_k}{\sqrt{\pi}}\partial_T\tilde{\beta}_k(T)\sin(kx) + \mathcal{O}(\varepsilon^{1-}).$$

Third case. If the noise takes the form $W(t) = \sum_{k=2}^{N}\frac{\alpha_k}{\sqrt{\pi}}\beta_k(t)\sin(kx)$, then the amplitude equations for (3.9) in this case are

$$\gamma_i^{\backslash} = (\nu - \sum_{k=2}^{N}\tfrac{3\alpha_k^2}{2\pi(k^2-1)^2})\gamma_i - \tfrac{3}{4}\gamma_i(\gamma_1^2 + \gamma_{-1}^2) \;\text{ for } i = \pm 1,$$

and our main theorem states that the solution of the (3.9) is of the type

$$u(t,x) = \varepsilon v(\varepsilon^2 t, x),$$

and

$$v(T,x) \simeq \gamma_1(T)\sin(x) + \gamma_{-1}(T)\cos(x) + \varepsilon\sum_{k=2}^{N}\frac{\alpha_k}{\sqrt{\pi}}\partial_T\tilde{\beta}_k(T)\sin(kx) + \mathcal{O}(\varepsilon^{1-}).$$

3.7.2 Ginzburg-Landau / Allen-Cahn Equation

The second example is the Ginzburg-Landau / Allen-Cahn equation

$$\partial_t u = (\partial_x^2 + r)u + \nu\varepsilon^2 u - u^3 + \varepsilon\partial_t W_k(t). \tag{3.42}$$

We consider two cases depending on the boundary conditions. For Neumann boundary conditions we need $r = 0$, while for Dirichlet boundary conditions we need $r = 1$.

First case $r = 0$. In this case, we consider (3.42) subject to Neumann boundary conditions on the interval $[0, \pi]$. We note that

$$\mathcal{A} = \partial_x^2, \;\; \mathcal{L} = \nu\mathcal{I}, \text{ and} \mathcal{F}(u) = -u^3.$$

If we take

$$\mathcal{H} = L^2([0,\pi]), \quad \mathcal{N} = \text{span}\{1\},$$

and

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}} & \text{if } k = 0, \\ \sqrt{\frac{2}{\pi}}\cos(kx) & \text{if } k > 0, \end{cases}$$

then the Assumption 2.2.1 is true. For this we easily see that the eigenvalues of $-\mathcal{A} = -\partial_x^2$ are $\lambda_k = k^2$ with $m = 2$ and $\lim_{k\to\infty}\lambda_k = \infty$. The condition (3.10) is satisfied for $\alpha = 1$ and $\beta = 0$. Furthermore, for $u = \gamma_1$ and $w = \gamma_2 \in \mathcal{N}$ the Condition (3.11) is satisfied as follows:

$$\langle \mathcal{F}_c(u), u\rangle = -\gamma_1^4 \le 0,$$

where

$$\mathcal{F}_c(u) = -\gamma_1^3,$$

and

$$\langle \mathcal{F}_c\,(u, u, w)\,, w\rangle = -\gamma_1^2\gamma_2^2 \leq 0.$$

For Assumption 3.3.3, we consider two cases:

First case. The noise acts only on $\cos(x)$ (i.e., $N = 1$).

In this case the amplitude equation (Landau equation) of (3.42) takes the form

$$\gamma' = \Big(\nu - \frac{3\alpha_1^2}{2\pi}\Big)\gamma - \gamma^3. \tag{3.43}$$

Second case. The noise acts on $\cos(x)$, $\cos(2x)$,...., $\cos(Nx)$ (i.e., $N \geq 1$).

In this case the amplitude equation of (3.42) takes the form

$$\gamma' = \Big(\nu - \frac{3}{2\pi}\sum_{k=1}^{N}\frac{\alpha_k^2}{k^2}\Big)\gamma - \gamma^3, \tag{3.44}$$

where $\mathcal{F}_c\,(u, e_k, e_k) = -\frac{1}{\pi}u$.

The main theorem states that the solution of (3.42) takes the form

$$u(t) = \varepsilon v(\varepsilon^2 t),$$

and

$$v(T) \simeq \gamma(T) + \varepsilon\sum_{k=1}^{N}\frac{\alpha_k}{k^2}\partial_T\tilde{\beta}_k(T)\cos(kx) + \mathcal{O}(\varepsilon^{1-}),$$

where γ is the solution of the amplitude equation (3.43) or (3.44).

Second case $r = 1$. In this case, we consider (3.42) subject to Dirichlet boundary conditions on the interval $[0, \pi]$. If we take

$$e_k(x) = \sqrt{\tfrac{2}{\pi}}\sin(kx) = \delta\sin(kx) \quad \text{and } \mathcal{N} = \text{span}\{\sin\},$$

then the Assumption 2.2.1 is true, where the eigenvalues of $-\mathcal{A} = -\partial_x^2 - 1$ are $\lambda_k = k^2 - 1$ with $m = 2$ and $\lim_{k\to\infty}\lambda_k = \infty$. Furthermore, for $u = \gamma_1 \sin$ and $w = \gamma_2 \sin \in \mathcal{N}$ the condition (3.11) is satisfied as follows:

$$\langle \mathcal{F}_c\,(u)\,, u\rangle = -\frac{3\pi}{8}\gamma_1^4 \leq 0,$$

where

$$\mathcal{F}_c\,(u) = -\frac{3}{4}\gamma_1^3 \sin,$$

and

$$\langle \mathcal{F}_c(u,u,w), w\rangle = -\frac{3\pi}{8}\gamma_1^2\gamma_2^2 \leq 0.$$

For Assumption 3.3.3, we consider two cases:

First case. The noise acts only on $\sin(2x)$.

In this case the amplitude equation (Landau equation) of (3.42) takes the form

$$\gamma` = \Big(\nu - \frac{\sigma^2}{4}\Big)\gamma - \frac{3}{4}\gamma^3. \tag{3.45}$$

Second case. The noise acts on $\sin(2x)$, $\sin(3x)$,......, $\sin(Nx)$.

In this case the amplitude equation of (3.42) takes the form

$$\gamma` = \Big(\nu - \frac{3}{4}\sum_{k=2}^{N}\frac{\sigma_k^2}{k^2-1}\Big)\gamma - \frac{3}{4}\gamma^3, \tag{3.46}$$

If we assume that $\sigma_2 = \sigma_3 = ... = \sigma_N = \sigma$, then the amplitude equation for (3.42) in this case takes the form

$$\gamma` = \Big(\nu - \frac{9\sigma^2}{16} + \frac{3\sigma^2(2N+1)}{8N(N+1)}\Big)\gamma - \frac{3}{4}\gamma^3,$$

where we used that $\mathcal{F}_c(u, e_k, e_k) = -\frac{1}{2}\delta^2 u$, $\sigma_k = \delta\alpha_k$ and $\delta = \sqrt{\frac{2}{\pi}}$.

The main theorem states that the solution of (3.42) takes the form

$$u(t) = \varepsilon v(\varepsilon^2 t),$$

and

$$v(T) \simeq \gamma(T)\sin + \varepsilon\sum_{k=2}^{N}\frac{\sigma_k}{k^2-1}\partial_T\tilde{\beta}_k(T)\sin(kx) + \mathcal{O}(\varepsilon^{1-}),$$

where γ is the solution of the amplitude equation (3.45) or (3.46).

3.7.3 Surface Growth Model

Another example arising in the theory of surface growth is

$$\partial_t u = -\triangle^2 u - \mu\triangle u + \nabla\left(|\nabla u|^2 \nabla u|\right) + \varepsilon\sum_{k=2}^{N}\alpha_k\partial_t\beta_k(t)e_k. \tag{3.47}$$

subject to periodic boundary conditions on the interval $[0, 2\pi]$. Here we can consider $\mu = 1 + \varepsilon^2\nu$, hence

$$\mathcal{A} = -\triangle^2 - \triangle, \;\; \mathcal{L} = -\nu\triangle \;\text{ and }\; \mathcal{F}(u) = \nabla\left(|\nabla u|^2\nabla u|\right).$$

If we take

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\sin(kx) & \text{if } k > 0, \\ \frac{1}{\sqrt{2\pi}} & \text{if } k = 0, \\ \frac{1}{\sqrt{\pi}}\cos(kx) & \text{if } k < 0, \end{cases}$$

and

$$\mathcal{H} = L^2([0, 2\pi]) \;\text{ and }\; \mathcal{N} = \text{span}\{1, \;\sin, \;\cos\},$$

then the eigenvalues of $-\mathcal{A} = \triangle^2 + \triangle$ are $\lambda_k = k^4 - k^2$ with $m = 4$ and $\lim_{k\to\infty}\lambda_k = \infty$. So, the Assumption 2.2.1 is true. Moreover, if $u = \gamma_0 + \gamma_1\sin + \gamma_{-1}\cos \in \mathcal{N}$, then all conditions of Assumption 3.3.2 are satisfied as follows:

$$\langle \mathcal{F}_c(u), u\rangle = -\frac{3\pi}{4}\left(\gamma_1^2 + \gamma_{-1}^2\right)^2 \le 0\,,$$

where

$$\mathcal{F}_c(u) = -\frac{3}{4}\left(\gamma_1^3 + \gamma_{-1}^2\gamma_1\right)\sin - \frac{3}{4}\left(\gamma_{-1}^3 + \gamma_1^2\gamma_{-1}\right)\cos.$$

Moreover, for $\alpha = \beta = 2$ we obtain

$$\|\mathcal{F}(u)\|_{L^2} = \left\|\partial_x\left(\partial_x u\right)^3\right\|_{L^2} \le \left\|\left(\partial_x u\right)^3\right\|_{\mathcal{H}^1} \le C\left\|\partial_x u\right\|^3_{\mathcal{H}^1} \le C\left\|u\right\|^3_{\mathcal{H}^2}.$$

For Assumption 3.3.3, we consider two cases:

First case. The noise acts only on $\sin(2x)$.

In this case the amplitude equation for (3.47) is a system of ordinary differential equations:

$$\gamma_0^{\backprime} = 0,$$

$$\gamma_i^{\backprime} = \left(\nu - \frac{\alpha_2^2}{4\pi}\right)\gamma_i - \frac{3}{4}\gamma_i\left(\gamma_1^2 + \gamma_{-1}^2\right) \quad \text{for } i = \pm 1.$$

Second case. The noise acts on $\sin(2x)$, $\sin(3x)$,..........., $\sin(Nx)$.

In this case the amplitude equation for (3.47) is a system of ordinary differential equations:

$$\gamma_0^{\backprime} = 0,$$

$$\gamma_i^{\backslash} = \Big(\nu - \frac{3}{4}\sum_{k=2}^{N}\frac{\sigma_k^2}{k^2-1}\Big)\gamma_i - \frac{3}{4}\gamma_i\left(\gamma_1^2+\gamma_{-1}^2\right) \quad \text{for } i=\pm 1,$$

where we supposed $\sigma_k = \delta\alpha_k$ for $k \in \{2, 3, ..., N\}$ and $\delta = \frac{1}{\sqrt{\pi}}$.

If we assume additionally that $\sigma_2 = \sigma_3 = ... = \sigma_N = \sigma$, then the amplitude equation for (3.47) in this case takes the form

$$\gamma_0^{\backslash} = 0,$$

$$\gamma_i^{\backslash} = \Big(\nu - \frac{9\sigma^2}{16} + \frac{3\sigma^2(2N+1)}{8N(N+1)}\Big)\gamma_i - \frac{3}{4}\gamma_i\left(\gamma_1^2+\gamma_{-1}^2\right) \quad \text{for } i=\pm 1.$$

where $\mathcal{F}_c\left(\gamma_0 + \gamma_1 \sin + \gamma_{-1}\cos, e_k, e_k\right) = -\frac{k^2}{2}\delta^2\left(\gamma_1 \sin + \gamma_{-1}\cos\right)$.

Chapter 4

Higher Order Correction for the Solution of SPDEs with Cubic Nonlinearities

4.1 Introduction

This chapter is devoted to study the higher order correction for the solution of equation (3.1). If we consider higher order corrections to (3.8), we obtain from the Itô-formula argument martingale terms of order ε. To get an equation for the higher order corrections we need to approximate this martingale term in order to have explicit error bounds. This approach relies on Lemma 6.1 from an article by Blömker, Hairer, and Pavioltis [9], which is based on the martingale representation theorem. Thus, we are limited in the final argument to $\dim \mathcal{N} = 1$.

Moreover, we want to study higher order corrections to the amplitude equation, in oder to see the fluctuations induced by the impact of the noise on the dominant pattern. In this chapter we follow our work [13]. Related results in this direction are discussed by Roberts & Wang [41].

So, our aim of this chapter is to improve the approximation of (3.1) from

$$u(t) \simeq \varepsilon b_1(\varepsilon^2 t) + \varepsilon \mathcal{Z}(\varepsilon^2 t) + \mathcal{O}(\varepsilon^{2-}),$$

to

$$u(t) \simeq \varepsilon b_1(\varepsilon^2 t) + \varepsilon^2 b_2(\varepsilon^2 t) + \varepsilon \mathcal{Z}(\varepsilon^2 t) + \mathcal{O}(\varepsilon^{3-}), \tag{4.1}$$

where b_1 is again the solution of the amplitude equation

$$db_1 = [\mathcal{L}_c b_1 + \mathcal{F}_c(b_1) + \sum_{k=2}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(b_1, e_k, e_k)]dT. \tag{4.2}$$

The higher order correction b_2 is the solution of

$$db_2 = [\mathcal{L}_c b_2 + 3\mathcal{F}_c(b_2, b_1, b_1) + \sum_{k=2}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(b_2, e_k, e_k)]dT + d\tilde{M}_{b_1}, \tag{4.3}$$

where $\tilde{M}_{b_1}(T)$ is a martingale, which is defined by

$$\tilde{M}_{b_1}(T) = \int_0^T \Big(\sum_{k=2}^{N} g_k(b_1) \Big)^{\frac{1}{2}} dB(s). \tag{4.4}$$

The integration is against a one-dimensional Brownian motion B arising from a martingale representation argument (cf. Lemma 4.4.7) and the g_k's are polynomials of degree 4 in b_1 given later in (4.31).

As an application of our approximation result of Theorem 4.2.2, we discuss the stochastic Swift-Hohenberg equation and the Ginzburg-Landau equation. To illustrate our results consider the stochastic Swift-Hohenberg equation

$$\partial_t u = -(1 + \partial_x^2)^2 u + \nu\varepsilon^2 u - u^3 + \varepsilon\sigma\partial_t\beta. \tag{4.5}$$

with respect to Neumann boundary conditions on the interval $[0, \pi]$. Our main theorem states that the solution of (4.5) is

$$u(t, x) \simeq \varepsilon\gamma_1(\varepsilon^2 t)\cos(x) + \varepsilon^2\gamma_2(\varepsilon^2 t)\cos(x) + \varepsilon\mathcal{Z}_0(\varepsilon^2 t) + \mathcal{O}(\varepsilon^{3-}),$$

where γ_1 and γ_2 are the solution of

$$\gamma_1^{\backprime} = \Big(\nu - \frac{3\sigma^2}{2}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = [\Big(\nu - \frac{3\sigma^2}{2}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2]dT + \frac{3\sigma^2}{\sqrt{2}}\gamma_1 dB.$$

This chapter is organized as follows. In the next section, we give assumptions and statements of the main results. Section 4.3 we recall the averaging results and dive higher order corrections, while Section 4.4, we study the approximation with higher order correction via amplitude equations. Finally, in Section 4.5 we apply our theory to the stochastic Swift-Hohenberg equation, and the Ginzburg-Landau / Allen-Cahn equation.

4.2 Assumptions and Main Result

In this chapter we work in the some setting as before, and assume that all assumptions of Chapter 3 hold.

For simplicity here we assume that in Assumption 3.3.3 one has $\alpha_k = \sigma \;\; \forall\, k$. this means the noise takes the form

$$W(t) = \sum_{k=n+1}^{N} \sigma\beta_k(t)e_k \;\text{ for } N \geq n+1. \tag{4.6}$$

This assumption is only for simplicity of presentation. The proofs can easily be modified to the general case.

In next definition we modify the stopping time as follows:

Definition 4.2.1 *For the $\mathcal{N} \times S$-valued stochastic process (a, ψ) defined in (3.2) we split a into $a = a_1 + \varepsilon a_2$ with a_1 being a solution of the amplitude equation (4.2). We define, for some $T_0 > 0$ and $\kappa \in (0, \frac{1}{7})$, the stopping time τ^{**} as*

$$\tau^{**} := T_0 \wedge \inf\left\{T > 0 : \|a_1(T)\|_\alpha > \varepsilon^{-\kappa} \text{ or } \|a_2(T)\|_\alpha > \varepsilon^{-\kappa} \text{ or } \|\psi(T)\|_\alpha > \varepsilon^{-\kappa}\right\}. \tag{4.7}$$

The main result of this chapter is the following approximation result.

Theorem 4.2.2 *(Approximation) Under Assumptions 2.2.1, 3.3.1, 3.3.2 and 3.3.3 with all $\alpha_k = \sigma\ \forall\ k$, $n = 1$, let u be a solution of (3.1) defined in (3.2) with the initial condition $u(0) = \varepsilon a(0) + \varepsilon\psi(0)$ with $\|u(0)\|_\alpha \leq \varepsilon\delta_\varepsilon$ for some $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$, $a(0) \in \mathcal{N}$ and $\psi(0) \in S$. Suppose b_1 and b_2 are solutions of (4.2) and (4.3), respectively, with $b_1(0) = a(0)$ and $b_2(0) = 0$. Then for all $p > 1$ and $T_0 > 0$ and all $\kappa \in (0, \frac{1}{7})$, there exists $C > 0$ such that*

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \Big\|u(t) - \varepsilon b_1(\varepsilon^2 t) - \varepsilon^2 b_2(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > \varepsilon^{\frac{7}{3}-7\kappa}\Big) \leq C\varepsilon^p, \tag{4.8}$$

for all $\varepsilon > 0$ sufficiently small.

Corollary 4.2.3 *Under the Assumptions of Theorem 3.3.6 and for arbitrary initial condition $u(0)$ we obtain*

$$\begin{aligned} &\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \big\|u(t) - \varepsilon b_1(\varepsilon^2 t) - \varepsilon^2 b_2(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\big\|_\alpha > \varepsilon^{\frac{7}{3}-7\kappa}\Big) \\ \leq\; &\mathbb{P}(\|u(0)\|_\alpha > \delta_0\varepsilon) + C\varepsilon^p. \end{aligned} \tag{4.9}$$

4.3 Averaging over the Fast OU-Process

Let us expand the averaging result of Lemma 3.5.1 in order to have higher order corrections. These turn out to be all martingle term.

Lemma 4.3.1 *Let X be a real valued stochastic process and $X(0) = \mathcal{O}(\varepsilon^{-r})$ for $r \geq 0$. If $dX = GdT$ with $G = \mathcal{O}(\varepsilon^{-r})$, then, for all k, l and j, all different,*

1. $\int_0^T X\mathcal{Z}_k d\tau = \varepsilon\frac{\sigma}{\lambda_k}\int_0^T Xd\tilde{\beta}_k + \mathcal{O}(\varepsilon^{2-r-\kappa_0})$,
2. $\int_0^T X\mathcal{Z}_k^2 d\tau = \frac{\sigma^2}{2\lambda_k}\int_0^T Xd\tau + \frac{\sigma}{\lambda_k}\varepsilon\int_0^T X\mathcal{Z}_k d\tilde{\beta}_k + \mathcal{O}(\varepsilon^{2-r-2\kappa_0})$,
3. $\int_0^T X\mathcal{Z}_k\mathcal{Z}_l d\tau = \frac{2\varepsilon\sigma}{\lambda_k+\lambda_l}\int_0^T X\mathcal{Z}_k d\tilde{\beta}_l + \mathcal{O}(\varepsilon^{2-r-2\kappa_0})$,
4. $\int_0^T \mathcal{Z}_k\mathcal{Z}_l\mathcal{Z}_j d\tau = \frac{3\varepsilon\sigma}{(\lambda_k+\lambda_l+\lambda_j)}\int_0^T \mathcal{Z}_l\mathcal{Z}_j d\tilde{\beta}_k + \mathcal{O}(\varepsilon^{2-3\kappa_0})$,
5. $\int_0^T \mathcal{Z}_k^2\mathcal{Z}_l d\tau = \frac{\varepsilon\sigma^3}{\lambda_l(\lambda_l+2\lambda_k)}\tilde{\beta}_l(T) + \frac{\varepsilon\sigma}{(\lambda_l+2\lambda_k)}\left\{\int_0^T \mathcal{Z}_k^2 d\tilde{\beta}_l + 2\int_0^T \mathcal{Z}_k\mathcal{Z}_l d\tilde{\beta}_k\right\} + \mathcal{O}(\varepsilon^{2-3\kappa_0})$,
6. $\int_0^T \mathcal{Z}_k^3 d\tau = \varepsilon\frac{\sigma^3}{\lambda_k^2}\tilde{\beta}_k(T) + \frac{\varepsilon\sigma}{\lambda_k}\int_0^T \mathcal{Z}_k^2 d\tilde{\beta}_k + \mathcal{O}(\varepsilon^{2-3\kappa_0})$,

where $\mathcal{Z}_k$ is defined in (3.13).

Proof. We note first that

$$\mathbb{E}\sup_{[0,T_0]}|X|^p \leq C\mathbb{E}\sup_{[0,T_0]}|G|^p \leq C\varepsilon^{-pr}.$$

In order to prove the first part, we apply Itô formula to $X\mathcal{Z}_k$

$$\begin{aligned} d(X\mathcal{Z}_k) &= \mathcal{Z}_k dX + Xd\mathcal{Z}_k \\ &= G\mathcal{Z}_k dT + \varepsilon^{-1}\sigma Xd\tilde{\beta}_k - \lambda_k\varepsilon^{-2}\mathcal{Z}_k XdT. \end{aligned}$$

Integrating from 0 to T, we obtain

$$\begin{aligned} \int_0^T X\mathcal{Z}_k d\tau &= -\frac{\varepsilon^2}{\lambda_k}X(T)\mathcal{Z}_k(T) + \frac{\varepsilon^2}{\lambda_k}\int_0^T G\mathcal{Z}_k d\tau + \varepsilon\frac{\sigma}{\lambda_k}\int_0^T Xd\tilde{\beta}_k \\ &= \varepsilon\frac{\sigma}{\lambda_k}\int_0^T Xd\tilde{\beta}_k + \mathcal{O}(\varepsilon^{2-r-\kappa_0}). \end{aligned}$$

In order to prove the second part, we apply Itô formula to $X\mathcal{Z}_k^2$

$$\begin{aligned} d\left(X\mathcal{Z}_k^2\right) &= \mathcal{Z}_k^2 dX + 2X\mathcal{Z}_k d\mathcal{Z}_k + X\left(d\mathcal{Z}_k\right)^2 \\ &= G\mathcal{Z}_k^2 dT - 2\lambda_k\varepsilon^{-2}X\mathcal{Z}_k^2 dT + 2\varepsilon^{-1}\sigma\mathcal{Z}_k X d\tilde{\beta}_k + \varepsilon^{-2}\sigma^2 X dT. \end{aligned}$$

Integrating from 0 to T, we obtain

$$\begin{aligned} \int_0^T X\mathcal{Z}_k^2 d\tau &= -\frac{\varepsilon^2}{2\lambda_k}X(T)\mathcal{Z}_k^2(T) + \frac{\varepsilon^2}{2\lambda_k}\int_0^T G\mathcal{Z}_k^2 d\tau \\ &\quad + \frac{\sigma}{\lambda_k}\varepsilon\int_0^T X\mathcal{Z}_k d\tilde{\beta}_k + \frac{\sigma^2}{2\lambda_k}\int_0^T X d\tau \\ &= \frac{\sigma^2}{2\lambda_k}\int_0^T X d\tau + \frac{\sigma}{\lambda_k}\varepsilon\int_0^T X\mathcal{Z}_k d\tilde{\beta}_k + \mathcal{O}(\varepsilon^{2-r-2\kappa_0}). \end{aligned}$$

For the third part, we apply Itô formula to $X\mathcal{Z}_k\mathcal{Z}_l$ and integrate from 0 to T

$$\begin{aligned} \int_0^T X\mathcal{Z}_k\mathcal{Z}_l d\tau &= -\frac{\varepsilon^2}{\lambda_k+\lambda_l}X\mathcal{Z}_k(T)\mathcal{Z}_l(T) \\ &\quad + \frac{\varepsilon^2}{\lambda_k+\lambda_l}\int_0^T \mathcal{Z}_k\mathcal{Z}_l G d\tau + \frac{2\varepsilon\sigma}{\lambda_k+\lambda_l}\int_0^T X\mathcal{Z}_l d\tilde{\beta}_k \\ &= \frac{2\varepsilon\sigma}{\lambda_k+\lambda_l}\int_0^T X\mathcal{Z}_l d\tilde{\beta}_k + \mathcal{O}(\varepsilon^{2-r-2\kappa_0}). \end{aligned}$$

For the fourth part, we apply Itô formula to $\mathcal{Z}_k\mathcal{Z}_l\mathcal{Z}_j$ and integrate from 0 to T in order to obtain

$$\begin{aligned} \int_0^T \mathcal{Z}_k\mathcal{Z}_l\mathcal{Z}_j d\tau &= -\frac{\varepsilon^2}{\lambda_k+\lambda_l+\lambda_j}\mathcal{Z}_k\mathcal{Z}_l\mathcal{Z}_j + \frac{3\varepsilon\sigma}{\lambda_k+\lambda_l+\lambda_j}\int_0^T \mathcal{Z}_l\mathcal{Z}_j d\tilde{\beta}_k \\ &= \frac{3\varepsilon\sigma}{\lambda_k+\lambda_l+\lambda_j}\int_0^T \mathcal{Z}_l\mathcal{Z}_j d\tilde{\beta}_k + \mathcal{O}(\varepsilon^{2-3\kappa_0}). \end{aligned}$$

For the fifth part, we apply Itô formula to $\mathcal{Z}_k^2\mathcal{Z}_l$ and integrate from 0 to T

$$\begin{aligned} \int_0^T \mathcal{Z}_k^2\mathcal{Z}_l d\tau &= -\frac{\varepsilon^2}{\lambda_l+2\lambda_k}\mathcal{Z}_k^2\mathcal{Z}_l + \frac{\varepsilon\sigma}{\lambda_l+2\lambda_k}\int_0^T \mathcal{Z}_k^2 d\tilde{\beta}_l \\ &\quad + \frac{2\varepsilon\sigma}{\lambda_l+2\lambda_k}\int_0^T \mathcal{Z}_k\mathcal{Z}_l d\tilde{\beta}_k + \frac{\sigma^2}{\lambda_l+2\lambda_k}\int_0^T \mathcal{Z}_l d\tau \\ &= \frac{\sigma^2}{\lambda_l+2\lambda_k}\int_0^T \mathcal{Z}_l d\tau + \frac{2\sigma\varepsilon}{\lambda_l+2\lambda_k}\int_0^T \mathcal{Z}_k\mathcal{Z}_l d\tilde{\beta}_k \\ &\quad + \frac{\varepsilon\sigma}{\lambda_l+2\lambda_k}\int_0^T \mathcal{Z}_k^2 d\tilde{\beta}_l + \mathcal{O}(\varepsilon^{2-3\kappa_0}), \end{aligned}$$

we finish the proof of the fifth part by using the first part for $X = 1$.

For the sixth part, we apply Itô formula to $\mathcal{Z}_k^3$ and integrate from 0 to T in order to obtain

$$\begin{aligned}\int_0^T \mathcal{Z}_k^3 d\tau &= -\frac{\varepsilon^2}{3\lambda_k}\mathcal{Z}_k^3(T) + \frac{\sigma}{\lambda_k}\varepsilon\int_0^T \mathcal{Z}_k^2 d\tilde{\beta}_k + \frac{\sigma^2}{\lambda_k}\int_0^T \mathcal{Z}_k d\tau \\ &= \frac{\sigma^2}{\lambda_k}\int_0^T \mathcal{Z}_k d\tau + \frac{\sigma}{\lambda_k}\varepsilon\int_0^T \mathcal{Z}_k^2 d\tilde{\beta}_k + \mathcal{O}(\varepsilon^{2-3\kappa_0}).\end{aligned}$$

Using the first part for $X = 1$, we obtain

$$\int_0^T \mathcal{Z}_k^3 d\tau = \varepsilon\frac{\sigma^3}{\lambda_k^2}\int_0^T d\tilde{\beta}_k + \frac{\varepsilon\sigma}{\lambda_k}\int_0^T \mathcal{Z}_k^2 d\tilde{\beta}_k + \mathcal{O}(\varepsilon^{2-3\kappa_0}).$$

□

Now, let us give some bounds on stochastic integrals containing $\mathcal{Z}_k$'s. These bounds are improved bounds, using similar arguments of the previous lemma.

Lemma 4.3.2 *Let X be as in Lemma 4.3.1, then for $\kappa_0 < \frac{1}{4}$ we obtain*

$$\mathbb{E}\sup_{T\in[0,T_0]}\Big|\int_0^T X\mathcal{Z}_k d\tilde{\beta}_l\Big|^p \le C\varepsilon^{-pr}\,, \tag{4.10}$$

and

$$\mathbb{E}\sup_{T\in[0,T_0]}\Big|\int_0^T \mathcal{Z}_k\mathcal{Z}_l d\tilde{\beta}_j\Big|^p \le C\,. \tag{4.11}$$

Proof. In order to prove (4.10). We first use Burkholder-Davis-Gundy theorem to derive

$$\mathbb{E}\sup_{T\in[0,T_0]}\Big|\int_0^T X\mathcal{Z}_k d\tilde{\beta}_l\Big|^p \le C_p\mathbb{E}\Big(\int_0^{T_0} X^2\mathcal{Z}_k^2 d\tau\Big)^{\frac{p}{2}}.$$

Using Lemma 3.5.1 and Hölder inequality, yields

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,T_0]}\Big|\int_0^T X\mathcal{Z}_k d\tilde{\beta}_l\Big|^p &\le C_p\mathbb{E}\Big(\frac{\alpha_k^2}{2}\int_0^{T_0} X^2 d\tau + C\varepsilon^{1-2r-2\kappa_0}\Big)^{\frac{p}{2}} \\ &\le C\mathbb{E}\int_0^{T_0} X^p d\tau + C\varepsilon^{\frac{p}{2}-pr-p\kappa_0} \\ &\le C\varepsilon^{-pr} + C\varepsilon^{\frac{p}{2}-pr-p\kappa_0} \\ &\le C\varepsilon^{-pr}.\end{aligned}$$

For (4.11). We obtain by using Burkholder-Davis-Gundy theorem again,

$$\mathbb{E}\sup_{T\in[0,T_0]}\Big|\int_0^T \mathcal{Z}_k\mathcal{Z}_l d\tilde{\beta}_j\Big|^p \le C_p\mathbb{E}\Big(\int_0^{T_0}\mathcal{Z}_k^2\mathcal{Z}_l^2 d\tau\Big)^{\frac{p}{2}}. \tag{4.12}$$

Applying Itô formula to $\mathcal{Z}_k^2\mathcal{Z}_l^2$ and integrating from 0 to T_0

$$\begin{aligned}\int_0^{T_0}\mathcal{Z}_k^2\mathcal{Z}_l^2 d\tau &= \frac{1}{(\lambda_k+\lambda_l)}\{-\frac{1}{2}\varepsilon^2\mathcal{Z}_k^2\mathcal{Z}_l^2 + 2\varepsilon\sigma\int_0^{T_0}\mathcal{Z}_l^2\mathcal{Z}_k d\tilde{\beta}_k \\ &\quad +\sigma^2\int_0^{T_0}\mathcal{Z}_k^2 d\tau + 2\delta_{k,l}\sigma^2\int_0^{T_0}\mathcal{Z}_k\mathcal{Z}_l d\tau\},\end{aligned}$$

where $\delta_{k,l} = \begin{cases} 0 & \text{if } l\neq k \\ 1 & \text{if } l = k\end{cases}$. Using Lemma 3.5.1 with $X=1$, we get

$$\begin{aligned}\int_0^{T_0}\mathcal{Z}_k^2\mathcal{Z}_l^2 d\tau &= \frac{1}{(\lambda_k+\lambda_l)}\{-\frac{1}{2}\varepsilon^2\mathcal{Z}_k^2\mathcal{Z}_l^2 + 2\varepsilon\sigma\int_0^{T_0}\mathcal{Z}_l^2\mathcal{Z}_k d\tilde{\beta}_k\} + C + \mathcal{O}(\varepsilon^{1-2\kappa_0}) \\ &= \frac{1}{(\lambda_k+\lambda_l)}\{-\frac{1}{2}\varepsilon^2\mathcal{Z}_k^2\mathcal{Z}_l^2 + 2\varepsilon\sigma\int_0^{T_0}\mathcal{Z}_l^2\mathcal{Z}_k d\tilde{\beta}_k\} + C,\end{aligned}$$

for $\kappa_0 < \frac{1}{2}$. Taking $|\cdot|^{\frac{p}{2}}$ on both sides before expectation and using Burkholder-Davis-Gundy theorem we obtain for $p>1$,

$$\begin{aligned}\mathbb{E}\Big|\int_0^{T_0}\mathcal{Z}_k^2\mathcal{Z}_l^2 d\tau\Big|^{\frac{p}{2}} &\le C\varepsilon^p\mathbb{E}\,|\mathcal{Z}_k\mathcal{Z}_l|^p + C\varepsilon^{\frac{p}{2}}\mathbb{E}\Big(\int_0^{T_0}\mathcal{Z}_l^4\mathcal{Z}_k^2 d\tau\Big)^{\frac{p}{4}} + C \\ &\le C\varepsilon^{p-2\kappa_0} + C\varepsilon^{\frac{p}{2}-2\kappa_0} + C \\ &\le C, \quad \text{for } \kappa_0 < \frac{p}{4}.\end{aligned} \tag{4.13}$$

From (4.12) and (4.13), we obtain

$$\begin{aligned}\mathbb{E}\sup_{T\in[0,T_0]}\Big|\int_0^T \mathcal{Z}_k\mathcal{Z}_l d\tilde{\beta}_j\Big|^p &\le C_p\mathbb{E}\Big(\int_0^{T_0}\mathcal{Z}_k^2\mathcal{Z}_l^2 d\tau\Big)^{\frac{p}{2}} \\ &\le C_p\mathbb{E}\Big|\int_0^{T_0}\mathcal{Z}_k^2\mathcal{Z}_l^2 d\tau\Big|^{\frac{p}{2}} \le C.\end{aligned}$$

□

4.4 Proof of the Main Result

First we prove a technical lemma on ordinary differential equations.

Lemma 4.4.1 *Let X and R_δ be continuous functions from $[0,\tau]$ to $\mathcal{N}$ with $X(0) = R_\delta(0)$. If X is a solution of*

$$X(T) = \int_0^T Q_a(X)ds + \int_0^T Q_b(X)ds + R_\delta,$$

where Q_a and Q_b are linear and bounded operators on $\mathcal{N}$ such that

$$|Q_a(X)| \leq C_a |X|, \quad |Q_b(X)| \leq C_b |X|, \tag{4.14}$$

and

$$\langle Q_b(X), X\rangle \leq 0, \tag{4.15}$$

then

$$\sup_{[0,\tau]} |X|^2 \leq [2 + C_0(C_a^2 + C_b^2)] \sup_{[0,\tau]} |R_\delta|^2, \tag{4.16}$$

where $C_0 = \frac{1}{C_a+1} e^{2[C_a+1]T_0}$.

We note that in the application of this lemma the constant C_b grows with ε while C_a is independent of ε. Therefore the condition (4.15) is important in order to have no C_b in the exponent.

Proof. Define $Y = X - R_\delta$, hence

$$Y' = Q_a(Y) + Q_a(R_\delta) + Q_b(Y) + Q_b(R_\delta).$$

Taking the scalar product $\langle\cdot, Y\rangle$ on both sides, we obtain

$$\frac{1}{2}\partial_T |Y|^2 = \langle Q_a(Y), Y\rangle + \langle Q_b(Y), Y\rangle + \langle Q_a(R_\delta), Y\rangle + \langle Q_b(R_\delta), Y\rangle .$$

Using Cauchy-Schwarz and Young inequalities and (4.15), yields

$$\partial_T |Y|^2 \leq 2[C_a + 1] |Y|^2 + [C_a^2 + C_b^2] |R_\delta|^2 .$$

Applying Gronwall's lemma, yields for all $T \leq \tau$

$$\begin{aligned} |Y(T)|^2 &\leq [C_a^2 + C_b^2] \int_0^T |R_\delta|^2 e^{2[C_a+1](T-s)} ds \\ &\leq C_0[C_a^2 + C_b^2] \sup_{[0,\tau]} |R_\delta|^2 . \end{aligned} \tag{4.17}$$

To prove (4.16) we use

$$|X|^2 = |Y + R_\delta|^2 \leq 2|Y|^2 + 2|R_\delta|^2 ,$$

and (4.17). □

Let us recall Lemma 3.6.2 and look closer at the terms of order ε.

Lemma 4.4.2 *Under Assumptions 2.2.1, 3.3.1, 3.3.2 and 3.3.3 with all $\alpha_k = \sigma$ for $k \in \{n+1, ..., N\}$, we obtain*

$$\begin{aligned} a(T) &= a(0) + \int_0^T \mathcal{L}_c a(\tau) d\tau + \int_0^T \mathcal{F}_c(a) d\tau + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k} \int_0^T \mathcal{F}_c(a, e_k, e_k) d\tau \\ &\quad + \varepsilon M_a(T) + \tilde{R}(T), \end{aligned} \tag{4.18}$$

where $M_a(T)$ is a martingale and it is defined by

$$M_a(T) = \int_0^T \sum_{k=n+1}^{N} \ominus_k(a) d\tilde{\beta}_k(s), \tag{4.19}$$

where all sums are from $n+1$ to N, if it is not explicity stated otherwise

$$\begin{aligned} \ominus_k(a) &= \frac{3\sigma}{\lambda_k} \mathcal{F}_c(a, a, e_k) + \sum_{l=n+1} \frac{6\sigma \mathcal{F}_c(a, e_k, e_l)}{\lambda_k + \lambda_l} \mathcal{Z}_l \\ &\quad + \sum_{l=n+1} \frac{3\sigma^3 \mathcal{F}_c(e_k, e_l, e_l)}{\lambda_k(\lambda_k + 2\lambda_l)} + \sum_{l \neq k} \frac{6\sigma \mathcal{F}_c(e_k, e_k, e_l)}{\lambda_l + 2\lambda_k} \mathcal{Z}_k \mathcal{Z}_l \\ &\quad + \sum_{l=n+1} \sum_{j=n+1} \frac{3\sigma \mathcal{F}_c(e_k, e_l, e_j)}{\lambda_k + \lambda_l + \lambda_j} \mathcal{Z}_l \mathcal{Z}_j \ , \end{aligned} \tag{4.20}$$

and

$$\tilde{R} = R_1 + \mathcal{O}(\varepsilon^{2-5\kappa}),$$

where $R_1 = \mathcal{O}(\delta_\varepsilon^2 \varepsilon^{2-2\kappa})$ is defined in (3.29).

Proof. Using (3.27) and Lemmas 4.3.1 and 4.3.2 in order to obtain (4.18). □

Lemma 4.4.3 *Under Assumptions 2.2.1, 3.3.1, 3.3.2 and 3.3.3 with all $\alpha_k = \sigma \ \forall \ k$, consider some stochastic process $\xi = \mathcal{O}(\varepsilon^{-r})$ for $r \geq 0$. Then for all $p > 0$ there exists $C > 0$ such that*

$$\mathbb{E}\Big(\sup_{T \in [0, \tau^{**}]} |M_\xi(T)|^p \Big) \leq C \varepsilon^{-2pr}, \tag{4.21}$$

where M_ξ is defined in (4.19). If ξ is bounded upto T_0, then (4.21) holds with T_0 instead of τ^{**}.

Proof. In order to prove (4.21), we take $|\cdot|^p$ on both sides for (4.20) in order to obtain

$$\begin{aligned}
\Big|M_\xi(T)\Big|^p &= \Big|\int_0^T \sum_{k=2}^N \ominus_k(a_1)d\tilde{\beta}_k(s)\Big|^p \\
&\leq C\sum_{k=n+1}\frac{1}{\lambda_k^p}\Big|\int_0^T \mathcal{F}_c(\xi,\xi,e_k)d\tilde{\beta}_k\Big|^p \\
&\quad +C\sum_{k=n+1}\sum_{l=n+1}\frac{1}{(\lambda_k+\lambda_l)^p}\Big|\int_0^T \mathcal{F}_c(\xi,e_k,e_l)\mathcal{Z}_l d\tilde{\beta}_k\Big|^p \\
&\quad +C\sum_{k=n+1}\sum_{l=n+1}\frac{\mathcal{F}_c(e_k,e_k,e_l)^p}{\lambda_k^p(\lambda_k+2\lambda_l)^p}\Big|\int_0^T d\tilde{\beta}_k(\tau)\Big|^p \\
&\quad +C\sum_{k=n+1}\sum_{l\neq k}\frac{\mathcal{F}_c(e_k,e_k,e_l)^p}{(\lambda_l+2\lambda_k)^p}\Big|\int_0^T \mathcal{Z}_k\mathcal{Z}_l d\tilde{\beta}_k\Big|^p \\
&\quad +C\sum_{k=n+1}\sum_{l=n+1}\sum_{j=n+1}\frac{\mathcal{F}_c(e_k,e_l,e_j)^p}{(\lambda_k+\lambda_l+\lambda_j)^p}\Big|\int_0^T \mathcal{Z}_l\mathcal{Z}_j d\tilde{\beta}_k\Big|^p.
\end{aligned}$$

Taking expectation after supremum on both sides and using Assumptions 3.3.2, Lemma 4.3.2 and Burkholder-Davis-Gundy theorem, yields (4.21). □

Lemma 4.4.4 *Under Assumptions 2.2.1, 3.3.1, 3.3.2 and 3.3.3 with all* $\alpha_k = \sigma\ \forall\ k$*. If we define* a *as* $a = a_1 + \varepsilon a_2$ *such that* a_1 *is a solution of the amplitude equation*

$$da_1 = [\mathcal{L}_c a_1 + \mathcal{F}_c(a_1) + \sum_{k=n+1}^N \frac{3\sigma^2}{2\lambda_k}\mathcal{F}_c(a_1,e_k,e_k)]dT, \tag{4.22}$$

then a_2 *is a solution of*

$$da_2 = [\mathcal{L}_c a_2 + 3\mathcal{F}_c(a_1,a_1,a_2) + \sum_{k=n+1}^N \frac{3\sigma^2}{2\lambda_k}\mathcal{F}_c(a_2,e_k,e_k)]dT + dM_{a_1} + dR_2, \tag{4.23}$$

where

$$\begin{aligned}
R_2 &= \varepsilon^{-1}\tilde{R} + 3\varepsilon\int_0^T \mathcal{F}_c(a_1,a_2,a_2)d\tau + \varepsilon^2\int_0^T \mathcal{F}_c(a_2)d\tau \\
&\quad +\varepsilon\sum_{k=n+1}^N \frac{6\sigma}{\lambda_k}\int_0^T \mathcal{F}_c(a_1,a_2,e_k)d\tilde{\beta}_k + \varepsilon^2\sum_{k=n+1}^N \frac{3\sigma}{\lambda_k}\int_0^T \mathcal{F}_c(a_2,a_2,e_k)d\tilde{\beta}_k \\
&\quad +\varepsilon\sum_{k=n+1}^N\sum_{l=n+1}^N \frac{6\sigma}{\lambda_k+\lambda_l}\int_0^T \mathcal{F}_c(a_2,e_k,e_l)\mathcal{Z}_l d\tilde{\beta}_k,
\end{aligned} \tag{4.24}$$

with

$$R_2 = \mathcal{O}(\varepsilon^{1-5\kappa}). \tag{4.25}$$

Proof. The equation for a_2 is a straight forward calculation using (4.18) and (4.22). To bound R_2, we take $\|\cdot\|_\alpha^p$ on both sides of (4.24) in order to obtain

$$\begin{aligned}
\Big\|R_2(T)\Big\|_\alpha^p \leq\ & C\varepsilon^{-p}\Big\|\tilde{R(T)}\Big\|_\alpha^p + C\varepsilon^p\Big\|\int_0^T \mathcal{F}_c(a_1,a_2,a_2)d\tau\Big\|_\alpha^p \\
&+C\varepsilon^{2p}\Big\|\int_0^T \mathcal{F}_c(a_2)d\tau\Big\|_\alpha^p + C\varepsilon^p\sum_{k=n+1}\frac{1}{\lambda_k^p}\Big\|\int_0^T \mathcal{F}_c(a_1,a_2,e_k)d\tilde{\beta}_k\Big\|_\alpha^p \\
&+C\varepsilon^{2p}\sum_{k=n+1}\frac{1}{\lambda_k^p}\Big\|\int_0^T \mathcal{F}_c(a_2,a_2,e_k)d\tilde{\beta}_k\Big\|_\alpha^p \\
&+C\varepsilon^p\sum_{k=n+1}\frac{1}{\lambda_k^p}\Big\|\int_0^T \mathcal{F}_c(a_2,e_k,e_k)\mathcal{Z}_k d\tilde{\beta}_k\Big\|_\alpha^p \\
&+C\varepsilon^p\sum_{k=n+1}\sum_{l\neq k}\frac{1}{(\lambda_k+\lambda_l)^p}\Big\|\int_0^T \mathcal{F}_c(a_2,e_k,e_l)\mathcal{Z}_l d\tilde{\beta}_k\Big\|_\alpha^p.
\end{aligned}$$

Taking expectation after supremum on both sides and using Assumption 3.3.2, Lemma 4.3.2, Burkholder-Davis-Gundy inequality and the definition of τ^{**} (cf. (4.7)) in order to obtain (4.25). □

Lemma 4.4.5 *Under assumptions of Lemma 4.4.4. Let a_1 be a solution of (4.22) with initial condition $a_1(0) = \frac{1}{\varepsilon}P_c u(0)$. Define ζ in $\mathcal{N}$ with $\zeta(0) = 0$ as the solution of*

$$d\zeta = [\mathcal{L}_c\zeta + 3\mathcal{F}_c(a_1,a_1,\zeta) + \sum_{k=n+1}^{N}\frac{3\sigma^2}{2\lambda_k}\mathcal{F}_c(\zeta,e_k,e_k)]dT + dM_{a_1}(T). \tag{4.26}$$

If $|a_1(0)| \leq \delta_\varepsilon$ for $\delta_\varepsilon \in (0,\varepsilon^{-\frac{1}{3}\kappa})$, then for all $T_0 > 0$ and $p > 0$ there exist a constant $C > 0$ such that

$$\sup_{T\in[0,T_0]}|a_1(T)|^p \leq C\delta_\varepsilon^p, \tag{4.27}$$

and

$$\sup_{T\in[0,T_0]}|\zeta(T)| \leq C(1+\delta_\varepsilon)\sup_{T\in[0,T_0]}|M_{a_1}(T)|. \tag{4.28}$$

Proof. The bound on a_1 follows from Lemma 3.6.3. Note that (cf. (3.31)) in the proof of Lemma 3.6.3, we get

$$|a_1(T)| \leq e^{CT} |a_1(0)| \ \forall\, T \leq T_0. \tag{4.29}$$

To bound ζ we integrate (4.26) from 0 to T in order to obtain

$$\zeta(T) = \int_0^T \mathcal{L}_c \zeta d\tau + 3\int_0^T \mathcal{F}_c(a_1, a_1, \zeta) d\tau + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k} \int_0^T \mathcal{F}_c(\zeta, e_k, e_k) d\tau + M_{a_1}(T).$$

If we define

$$Q_a(\zeta) = \mathcal{L}_c\zeta + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(\zeta, e_k, e_k) \quad \text{and} \quad Q_b(\zeta) = 3\mathcal{F}_c(a_1, a_1, \zeta),$$

then we obtain from Lemma 4.4.1

$$\sup_{T\in[0,T_0]} |\zeta(T)|^2 \leq C(1+\delta_\varepsilon^2) \sup_{T\in[0,T_0]} |M_{a_1}(T)|^2 .$$

Taking the square root on both sides, yields (4.28). □

Remark 4.4.6 *Note that, from now on, we consider $n = 1$ and identify $\mathcal{N}$ with $\mathbb{R}$ using the natural isomorphism $\gamma \cdot e_1 \mapsto \gamma$. Thus, for example $\mathcal{F}_c$ is defined as $\langle \mathcal{F}, e_1\rangle$ and $\mathcal{F}_c^2$ is $\langle \mathcal{F}, e_1\rangle^2$. Moreover, it is easy to check that the quadratic variation of M_{a_1} as a real valued process $\langle M_{a_1}, e_1\rangle$ is given by $\sum_{k=2}^{N} \int_0^T \ominus_k^2(a_1) d\tau$.*

Before we prove the main result let us deduce the approximation g_k of the quadratic variation function $\ominus_k^2$.

Taking the square on both sides of (4.20) and using Lemma 3.5.1, we obtain

$$\int_0^T \ominus_k^2(a_1) d\tau = \int_0^T g_k(a_1) d\tau + \mathcal{O}(\delta_\varepsilon^2 \varepsilon^{1-4\kappa_0}), \tag{4.30}$$

where

$$\begin{aligned} g_k(b_1) \;&=\; \frac{9\sigma^2}{\lambda_k^2} [\mathcal{F}_c(b_1, b_1, e_k)]^2 + \frac{9\sigma^4}{2\lambda_k^3} [\mathcal{F}_c(b_1, e_k, e_k)]^2 \\ &\quad + \sum_{l\neq k}^{N} \frac{18\sigma^4}{\lambda_l(\lambda_k+\lambda_l)^2} [\mathcal{F}_c(b_1, e_k, e_l)]^2 + \theta_1^{(k)} [\mathcal{F}_c(b_1, b_1, e_k)] + \theta_2^{(k)}, \end{aligned} \tag{4.31}$$

with

$$\theta_1^{(k)} = \sum_{l=n+1}^{N} \frac{9\sigma^4 \mathcal{F}_c(e_k, e_l, e_l)}{\lambda_k^2 \lambda_l},$$

and

$$\begin{aligned}
\theta_2^{(k)} \quad = \quad & \frac{11\sigma^6 \mathcal{F}_c^2(e_k)}{4\lambda_k^4} + \sum_{l\neq k}^{N} \frac{9\sigma^6(3\lambda_k^2 + 4\lambda_l\lambda_k + 4\lambda_l^2)\mathcal{F}_c^2(e_k, e_l, e_l)}{4\lambda_k^2\lambda_l(\lambda_k + 2\lambda_l)^2} \\
& + \sum_{l\neq k}^{N} \frac{9\sigma^6 \mathcal{F}_c^2(e_k, e_k, e_l)}{\lambda_k\lambda_l(\lambda_l + 2\lambda_k)^2} + \sum_{l\neq k}^{N} \sum_{j\notin\{l,k\}}^{N} \frac{9\sigma^6 \mathcal{F}_c^2(e_k, e_l, e_j)}{2\lambda_l\lambda_j(\lambda_k + \lambda_l + \lambda_j)^2} \\
& + \sum_{l\neq k}^{N} \frac{\sigma^6(6\lambda_k^2 + 18\lambda_l + 3\lambda_k)\mathcal{F}_c(e_k, e_k, e_l)\mathcal{F}_c(e_k)}{2\lambda_l\lambda_k^3(\lambda_k + 2\lambda_l)} \\
& + \sum_{l\neq k}^{N} \sum_{j\notin\{l,k\}}^{N} \frac{9\sigma^6(4\lambda_l\lambda_j + \lambda_k^2 + \lambda_l\lambda_k)\mathcal{F}_c(e_k, e_l, e_l)\mathcal{F}_c(e_k, e_j, e_j)}{4\lambda_k^2\lambda_l\lambda_j(\lambda_k + 2\lambda_l)(\lambda_k + 2\lambda_j)}.
\end{aligned}$$

Let us state without proof Lemma 6.1 from [9] to bound $M_{a_1}(T) - \tilde{M}_{a_1}(T)$ where the martingale $M_{a_1}(T)$ defined in (4.19) and the martingale $\tilde{M}_{a_1}(T)$ defined in (4.4).

Lemma 4.4.7 *Let $M_{a_1}(T)$ be a continuous martingale with respect to some filtration $(\digamma_T)_{T\geq 0}$. Denote the quadratic variation of M_{a_1} by f and let g be an arbitrary $\digamma_T$-adapted increasing process with $g(0) = 0$. Then, there exists a filtration $\tilde{\digamma}_T$ with $\digamma_T \subset \tilde{\digamma}_T$ and a continuos $\tilde{\digamma}_T$-martingale $\tilde{M}_{a_1}(T)$ with quadratic variation g such that, for every $r_0 < \frac{1}{2}$ there exists a constant C with*

$$\begin{aligned}
\mathbb{E} \sup_{T\in[0,T_0]} \left| M_{a_1}(T) - \tilde{M}_{a_1}(T) \right|^p \quad \leq \quad & C \left(\mathbb{E}\, |g(T_0)|^{2p} \right)^{1/4} \left(\mathbb{E} \sup_{T\in[0,T_0]} |f(T) - g(T)|^p \right)^{r_0} \\
& + \mathbb{E} \sup_{T\in[0,T_0]} |f(T) - g(T)|^{p/2}.
\end{aligned}$$

Remark 4.4.8 *Using the martingale representation theorem, there exists a Brownian motion B with respect to the filtration $\tilde{\digamma}_T$ such that $\tilde{M}_{a_1}(T)$ is given as a stochastic integral as in (4.4).*

Lemma 4.4.9 *Under conditions of Lemma 4.4.7, let $M_{a_1}(T)$ and $\tilde{M}_{a_1}(T)$ are martingales defined in (4.19) and (4.4) with $|a(0)| \leq \delta_\varepsilon$ for some $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$,* *respectively. Let* $f(T) = \int_0^T \sum_{k=2}^N \ominus_k^2(a_1)ds$ *be the quadratic variation of* $M_{b_1}(T)$ *and* $g(T) = \int_0^T \sum_{k=2}^N g_k(a_1)ds$ *be the quadratic variation of the martingale* $\tilde{M}_{a_1}(T)$*, then for* $r_0 = \frac{1}{3}$ *and* $\kappa_0 \le \kappa$ *we obtain*

$$\mathbb{E} \sup_{T\in[0,T_0]} \left| M_{a_1}(T) - \tilde{M}_{a_1}(T) \right|^p \le C\delta_\varepsilon^{\frac{8}{3}p} \varepsilon^{\frac{1}{3}p - \frac{4}{3}p\kappa}. \tag{4.32}$$

Proof. From (4.30), we obtain

$$\begin{aligned} \mathbb{E} \sup_{T\in[0,T_0]} |f(T) - g(T)|^p &= \mathbb{E} \sup_{T\in[0,T_0]} \left| \int_0^T \sum_{k=2}^N [\ominus_k^2(a_1) - g_k(a_1)]ds \right|^p \\ &\le C\delta_\varepsilon^{2p} \varepsilon^{p-4p\kappa_0}, \end{aligned}$$

and as $\theta_i^{(k)}$ are constants

$$|g(T_0)|^{2p} = \left| \int_0^{T_0} \sum_{k=2}^N g_k(s)ds \right|^{2p} \le C \sup_{[0,T_0]} |a_1|^{8p} + C \sup_{[0,T_0]} |a_1|^{4p},$$

using (4.27)in order to obtain

$$\mathbb{E}g(T_0)^{2p} \le C\delta_\varepsilon^{8p}.$$

Applying Lemma 4.4.7, yields (4.32). □

Let us now turn to the proof of the main result.

Definition 4.4.10 *Given* $\delta_\varepsilon \in (0, \varepsilon^{-\frac{1}{3}\kappa})$*. Define the set* $\Omega^{**} \subset \Omega$ *such that all these estimates*

$$\sup_{[0,\tau^{**}]} \|\psi - \mathcal{Q}\|_\alpha < \varepsilon^{2-4\kappa}, \tag{4.33}$$

$$\sup_{[0,\tau^{**}]} \|\psi\|_\alpha < \delta_0 + \varepsilon^{-\frac{1}{2}\kappa}, \tag{4.34}$$

$$\sup_{[0,\tau^{**}]} |R_2| < \varepsilon^{1-6\kappa}, \tag{4.35}$$

$$\sup_{[0,\tau^{**}]} |M_{a_1}| < \varepsilon^{-\frac{1}{2}\kappa}, \tag{4.36}$$

and

$$\sup_{[0,\tau^{**}]} \left| M_{a_1} - \tilde{M}_{a_1} \right| < \delta_\varepsilon^{\frac{8}{3}} \varepsilon^{\frac{1}{3} - \frac{7}{3}\kappa}, \tag{4.37}$$

hold on Ω^{**}.

We will see later that the set Ω^{**} has approximately probability 1 (cf. proof of Theorem 4.2.2) and that $\tau^{**} = T_0$ on Ω^{**}.

The following theorem states that in (4.26), (4.23) we have a good approximation when leaving out the error R_2.

Theorem 4.4.11 *We assume that Assumption 2.2.1, 3.3.1, 3.3.2 and 3.3.3 with all $\alpha_k = \sigma \ \forall\ k$ hold. Let a_1 be a solution of (4.22) and let ζ and a_2 are solution of (4.26) and (4.23), respectively. If the initial condition satisfies $a_2(0) = \zeta(0) = 0$, then for $\kappa < \frac{1}{7}$ there is a constant $C > 0$ such that*

$$\sup_{T\in[0,\tau^{**}]} |a_2(T) - \zeta(T)| \leq C\varepsilon^{1-7\kappa}, \tag{4.38}$$

and

$$\sup_{T\in[0,\tau^{**}]} |a_2(T)| \leq C(1+\delta_\varepsilon)\varepsilon^{-\frac{1}{2}\kappa}, \tag{4.39}$$

on Ω^{**}.

Proof. To prove (4.38) we subtract (4.23) from (4.26) and define $\eta(T) := \zeta(T) - a_2(T)$ to obtain

$$d\eta = [\mathcal{L}_c\eta + 3\mathcal{F}_c(a_1, a_1, \eta) + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k}\mathcal{F}_c(\eta, e_k, e_k)]dT + dR_2.$$

If we take

$$Q_a(\eta) = \mathcal{L}_c\eta + \sum_{k=n+1}^{N} \frac{3\sigma^2}{2\lambda_k}\mathcal{F}_c(\eta, e_k, e_k) \quad \text{and} \quad Q_b(\eta) = 3\mathcal{F}_c(a_1, a_1, \eta),$$

then we obtain from Lemma 4.4.1

$$\sup_{[0,\tau^{**}]} |\eta|^2 \leq C\varepsilon^{-2\kappa} \sup_{[0,\tau^{**}]} |R_2|^2 \quad \text{on } \Omega^{**}. \tag{4.40}$$

From (4.35) we obtain

$$\sup_{[0,\tau^{**}]} |\zeta - a_2| = \sup_{[0,\tau^{**}]} |\eta| \leq C\varepsilon^{1-7\kappa} \quad \text{on } \Omega^{**}.$$

For the second part of the Theorem (cf. (4.39)), consider

$$\sup_{[0,\tau^{**}]} |a_2| \leq \sup_{[0,\tau^{**}]} |\zeta - a_2| + \sup_{[0,\tau^{**}]} |\zeta| \quad \text{on } \Omega^{**}.$$

Using (4.36) together with (4.28) and (4.38), yields (4.39) in case $\kappa < \frac{1}{7}$. □

In the following theorem we approximate the martingale part $\tilde{M}_{a_1}$, that still depends on the fast OU-process. Here we need $n = 1$.

Theorem 4.4.12 *Under assumptions of Theorem 4.4.11 . Let a_1 be a solution of (4.22) and let a_2 be a solution of (4.23). Define b_2 in $\mathcal{N}$ as a solution of*

$$db_2 = [\mathcal{L}_c b_2 + 3\mathcal{F}_c(a_1, a_1, b_2) + \sum_{k=2}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(b_2, e_k, e_k)]dT + d\tilde{M}_{a_1}, \qquad (4.41)$$

where $\tilde{M}_{a_1}$ is defined in (4.4). If the initial condition satisfies $\zeta(0) = b_2(0) = 0$, then for every $p > 0$, $\varepsilon \in (0,1)$ and every $\kappa > 0$ there exists a constant C such

$$\sup_{T\in[0,\tau^{**}]} |b_2(T) - \zeta(T)| \leq C\delta_\varepsilon^{\frac{11}{3}} \varepsilon^{\frac{1}{3}-\frac{7}{3}\kappa}. \qquad (4.42)$$

Proof. Subtracting (4.26) from (4.41) and defining $\phi = b_2 - \zeta$ we obtain

$$\begin{aligned}\phi(T) &= \int_0^T \mathcal{L}_c \phi d\tau + 3\int_0^T \mathcal{F}_c(\phi, a_1, a_1) d\tau \\ &\quad + \sum_{k=2}^{N} \frac{3\sigma^2}{2\lambda_k} \int_0^T \mathcal{F}_c(\phi, e_k, e_k)] d\tau + \tilde{M}_{a_1}(T) - M_{a_1}(T).\end{aligned}$$

Let

$$Q_a(\phi) = \mathcal{L}_c \phi + \sum_{k=2}^{N} \frac{3\sigma^2}{2\lambda_k} \mathcal{F}_c(\phi, e_k, e_k) \quad \text{and} \quad Q_b(\phi) = 3\mathcal{F}_c(a_1, a_1, \phi),$$

then all conditions of Lemma 4.4.1 satisfy as follows

$$|Q_a(\phi)| \leq C|\phi| \quad \text{and} \quad |Q_b(\phi)| \leq |a_1|^2 |\phi| \leq C\delta_\varepsilon^2 |\phi| \quad \text{on } \Omega^{**},$$

and from Assumption 3.3.2

$$\langle Q_b(\phi), \phi \rangle \leq 0.$$

Hence, we apply Lemma 4.4.1 to obtain

$$\sup_{[0,\tau^{**}]} |\phi|^2 \leq C(1 + \delta_\varepsilon^2) \sup_{[0,\tau^{**}]} \left| \tilde{M}_{a_1}(T) - M_{a_1}(T) \right|^2.$$

Using (4.37) finishes the proof. □

Finally, we use the results previously obtained to prove the main result of Theorem 4.2.2 for the approximation of the solution of the SPDE (3.1).

Proof of Theorem 4.2.2. We note that provided $\delta_\varepsilon < \varepsilon^{-\frac{1}{3}\kappa}$

$$\Omega \supset \{\tau^{**} = T_0\} \supseteq \{ \sup_{T\in[0,T_0]} \|a_1(T)\|_\alpha < \varepsilon^{-\kappa}, \sup_{T\in[0,T_0]} \|a_2(T)\|_\alpha < \varepsilon^{-\kappa}$$
$$, \sup_{T\in[0,T_0]} \|\psi(T)\|_\alpha < \varepsilon^{-\kappa}\} \supseteq \Omega^{**},$$

where the last inclusion holds due to (4.34) with Lemma 4.4.5 and Theorem 4.4.11. Moreover, $\Omega^* \supset \Omega^{**}$ by the definition if we identify b with a_1. Hence,

$$\mathbb{P}(\Omega^{**}) \geq 1-\mathbb{P}(\sup_{[0,\tau^*]} \|\psi-\mathcal{Q}\|_\alpha \geq \varepsilon^{2-4\kappa})-\mathbb{P}(\sup_{[0,\tau^{**}]} \|\psi\|_\alpha \geq \varepsilon^{-\frac{1}{2}\kappa})-\mathbb{P}(\sup_{[0,\tau^{**}]} \|R_2\|_\alpha \geq \varepsilon^{1-6\kappa})$$
$$-\mathbb{P}(\sup_{[0,\tau^{**}]} \left|M_{a_1} - \tilde{M}_{a_1}\right| \geq \varepsilon^{\frac{1}{3}-\frac{29}{3}\kappa}) - \mathbb{P}(\sup_{[0,\tau^{**}]} |M_{a_1}| \geq \varepsilon^{-\frac{\kappa}{2}}).$$

Using Chebychev inequality and Lemmas 3.4.1, 4.4.3, 4.4.5, 4.4.9 and Corollary 3.4.3, we obtain

$$\mathbb{P}(\Omega^{**}) \geq 1 - C[\varepsilon^{q\kappa} + \varepsilon^{\frac{1}{2}q\kappa - q\kappa_0} + \varepsilon^{\frac{1}{2}q\kappa}] \geq 1 - C\varepsilon^{\frac{1}{2}q\kappa} \geq 1 - C\varepsilon^p, \qquad (4.43)$$

if q is sufficiently large. Now let us turn to the approximation result. Using (3.2) and triangle inequality, yields

$$\begin{aligned}
& \sup_{T\in[0,\tau^{**}]} \left\|u(\varepsilon^{-2}T) - \varepsilon a_1(T) - \varepsilon^2 b_2(T) - \varepsilon\mathcal{Q}(T)\right\|_\alpha \\
= & \sup_{T\in[0,\tau^{**}]} \left\|\varepsilon a(T) - \varepsilon a_1(T) - \varepsilon^2 b_2(T) + \varepsilon\psi(T) - \varepsilon\mathcal{Q}(T)\right\|_\alpha \\
= & \sup_{T\in[0,\tau^{**}]} \left\|\varepsilon^2 a_2(T) - \varepsilon^2 b_2(T) + \varepsilon\psi(T) - \varepsilon\mathcal{Q}(T)\right\|_\alpha \\
\leq & \varepsilon^2 \sup_{[0,\tau^{**}]} \|a_2 - b_2\|_\alpha + \varepsilon \sup_{[0,\tau^{**}]} \left\|\psi - \mathcal{Q}\right\|_\alpha \\
\leq & \varepsilon^2 \sup_{[0,\tau^{**}]} \|a_2 - \zeta\|_\alpha + \varepsilon^2 \sup_{[0,\tau^*]} \|\zeta - b_2\|_\alpha + \varepsilon \sup_{[0,\tau^{**}]} \left\|\psi - \mathcal{Q}\right\|_\alpha .
\end{aligned}$$

From (4.33), (4.38) and (4.42), we obtain

$$\begin{aligned}
& \sup_{t\in[0,\varepsilon^{-2}T_0]} \left\|u(t) - \varepsilon a_1(\varepsilon^2 t) - \varepsilon^2 b_2(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\right\|_\alpha \\
= & \sup_{t\in[0,\varepsilon^{-2}\tau^{**}]} \left\|u(t) - \varepsilon a_1(\varepsilon^2 t) - \varepsilon^2 b_2(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\right\|_\alpha \\
\leq & C\varepsilon^{\frac{7}{3}-7\kappa} \text{ on } \Omega^{**}.
\end{aligned}$$

Thus,

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\Big\|u(t)-\varepsilon a_1(\varepsilon^2 t)-\varepsilon^2 b_2(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha > C\varepsilon^{\frac{7}{3}-7\kappa}\Big) \le 1-\mathbb{P}(\Omega^{**}).$$

Using (4.43), yields (4.8). □

Proof of Corollary 4.2.3. We follow exactly the same proof as in Corollary 3.3.7. □

4.5 Applications

To apply our main theorem, we will take the Swift-Hohenberg equation (4.5) with respect to Neumann boundary conditions on the interval $[0,\pi]$ and Ginzburg-Landau / Allen-Cahn equation (3.42) as examples and we will discuss several cases depending on the form of the noise.

4.5.1 Swift-Hohenberg Equation

Define

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}} & \text{if } k=0, \\ \sqrt{\frac{2}{\pi}}\cos(kx) & \text{if } k>0, \end{cases}$$

and

$$\mathcal{H} = L^2([0,\pi]) \text{ and } \mathcal{N} = \text{span}\{\cos\}.$$

In this case our main theorem states that the solution of (4.5) is

$$u(t,x) \simeq \varepsilon\gamma_1(\varepsilon^2 t)\cos(x) + \varepsilon^2\gamma_2(\varepsilon^2 t)\cos(x) + \varepsilon\mathcal{Z}_k(\varepsilon^2 t)\cos(kx) + \mathcal{O}(\varepsilon^{3-}),$$

where γ_1 and γ_2 are the solution of the following amplitude equations, we will discuss three cases depending on the noise as follow,

First case. If the noise is a constant in the space, i.e.

$$W(t) = \sigma\beta_0(t)\ ,$$

then

$$\gamma_1^{\backprime} = \Big(\nu - \frac{3\sigma^2}{2}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = \Big[\Big(\nu - \frac{3\sigma^2}{2}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2\Big]\, dT + \frac{3\sigma^2}{\sqrt{2}}\gamma_1 dB.$$

Second case: If the noise acts on $\cos(kx)$ for one $k \in \{2, 4, 5, 6, ...\}$, then

$$\gamma_1^{\backprime} = \Big(\nu - \frac{3\sigma^2}{2(k^2-1)^2}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = \Big[\Big(\nu - \frac{3\sigma^2}{2(k^2-1)^2}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2\Big]\, dT + \frac{3\sigma^2}{2\sqrt{2}(k^2-1)^3}\gamma_1 dB.$$

Third case: If the noise takes the form

$$W(t) = \sigma\beta_3(t)\cos(3x)\ ,$$

then

$$\gamma_1^{\backprime} = \Big(\nu - \frac{3\sigma^2}{128}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = \Big[\Big(\nu - \frac{3\sigma^2}{128}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2\Big]\, dT + \frac{3\sigma}{256}\gamma_1\sqrt{(\gamma_1^2 + \frac{\sigma^2}{32})}dB.$$

4.5.2 Ginzburg-Landau / Allen-Cahn Equation

First case $r = 0$ (i.e., the Ginzburg-Landau Equation (3.42) subject to Neumann boundary conditions on the interval $[0, \pi]$). In this case, our main theorem states that the solution of (3.42) takes the form

$$u(t,x) \simeq \varepsilon\gamma_1(\varepsilon^2 t) + \varepsilon^2\gamma_2(\varepsilon^2 t) + \varepsilon\sum_{k=1}^{N}\mathcal{Z}_k(\varepsilon^2 t)\cos(kx) + \mathcal{O}(\varepsilon^{3-}),$$

where γ_1 and γ_2 , we will discuss two cases depending on the noise, are the solution of the amplitude equations

First case. The noise acts on $\cos(kx)$ for one $k \in \{1, 2, ...\}$, in this case

$$\gamma_1^{\backprime} = \Big(\nu - \frac{3\alpha_k^2}{2\pi k^2}\Big)\gamma_1 - \gamma_1^3,$$

and

$$d\gamma_2 = \Big[\Big(\nu - \frac{3\sigma^2}{4\pi k^2}\Big)\gamma_2 - 3\gamma_1^2\gamma_2\Big]\, dT + \frac{3\sigma^2}{\sqrt{2}\pi k^3}\gamma_1 dB.$$

Second case: The noise takes the form

$$W(t) = \sum_{k=1}^{N} \sigma\beta_k(t)e_k \ .$$

In this case

$$\gamma_1^{\backprime} = \Big(\nu - \sum_{k=1}^{N}\frac{3\sigma^2}{2\pi k^2}\Big)\gamma_1 - \gamma_1^3,$$

and

$$d\gamma_2 = \Bigg[\Big(\nu - \sum_{k=1}^{N}\frac{3\sigma^2}{2\pi k^2}\Big)\gamma_2 - 3\gamma_1^2\gamma_2\Bigg]\,dT + \frac{3\sigma^2}{\sqrt{2\pi}}\Big(\sum_{k=1}^{N}\frac{1}{k^6}\Big)^{\frac{1}{2}}\gamma_1 dB.$$

Remark 4.5.1 *Either γ_1 tends to be 0 in case $\nu < \sum_{k=1}^{N}\frac{3\sigma^2}{2\pi k^2}$ and γ_2 tends thus to 0, too. Or $\gamma_1^2 \approx (\nu - \sum_{k=1}^{N}\frac{3\sigma^2}{2\pi k^2}) > 0$ for large T, and thus γ_2 behaves like an Ornstein-Uhlenbeck process.*

Second case $r = 1$ (i.e., the Ginzburg-Landau Equation (3.42) subject to Dirichlet boundary conditions on the interval $[0, \pi]$). In this case, our main theorem states that the solution of (3.42) takes the form

$$u(t,x) \simeq \varepsilon\gamma_1(\varepsilon^2 t)\sin(x) + \varepsilon^2\gamma_2(\varepsilon^2 t)\sin(x) + \varepsilon\mathcal{Z}_k(\varepsilon^2 t)\sin(kx) + \mathcal{O}(\varepsilon^{3-}),$$

where γ_1 and γ_2 , we will discuss three cases depending on the noise, are the solution of

First case. The noise acts on $\sin(kx)$ for one $k \in \{2, 4, 5, 6, ...\}$, in this case

$$\gamma_1^{\backprime} = \Big(\nu - \frac{3\sigma^2}{4(k^2-1)}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = \Bigg[\Big(\nu - \frac{3\sigma^2}{4(k^2-1)}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2\Bigg]\,dT + \frac{3\sigma^2}{2\sqrt{2(k^2-1)^3}}\gamma_1 dB.$$

Second case: The noise acts on $\sin(3x)$, in this case

$$\gamma_1^{\backprime} = \Big(\nu - \frac{3\sigma^2}{32}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = \Bigg[\Big(\nu - \frac{3\sigma^2}{32}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2\Bigg]\,dT + \frac{3\sigma}{32}\gamma_1\sqrt{(\gamma_1^2 + \frac{\sigma^2}{16}})dB.$$

Third case: The noise takes the form

$$W(t) = \sum_{k=2}^{3} \sigma\beta_k(t)e_k \ ,$$

in this case

$$\gamma_1^{\backprime} = \Big(\nu - \sum_{k=2}^{3} \frac{3\sigma^2}{4(k^2-1)}\Big)\gamma_1 - \frac{3}{4}\gamma_1^3,$$

and

$$d\gamma_2 = \left[\Big(\nu - \sum_{k=2}^{3} \frac{3\sigma^2}{4(k^2-1)}\Big)\gamma_2 - \frac{3}{4}\gamma_1^2\gamma_2\right] dT + d\tilde{M},$$

where

$$d\tilde{M} = \frac{\sigma}{32}\left(9\gamma_1^4 + \frac{3713\sigma^2}{84}\gamma_1^2 + \frac{16819\sigma^4}{19494}\right)^{1/2} dB.$$

Chapter 5

Modulation Equation for the Stochastic Swift-Hohenberg Equation

5.1 Introduction

We consider the stochastic Swift-Hohenberg equation on an unbounded domain near its change of stability. This equation has been used as a toy model for the convective instability in Rayleigh-Bénard problem (see [16] or [22]). Now it is one of the celebrated models in the theory of pattern formation. For a scalar field $U(t,x)$ it takes the form

$$\partial_t U = \mathcal{L}U + \varepsilon^2 \nu U - U^3 + \varepsilon\sigma\partial_t\beta, \tag{SH}$$

where the linear differential operator is $\mathcal{L} = -(1+\partial_x^2)^2$ and its eigenvalues are $-\lambda_k = -(1-k^2)^2$ for $k \in \mathbb{R}$ corresponding to eigenfunctions e^{ikx}. The noise is the derivative of a standard Brownian motion $\{\beta(t)\}_{t\geq 0}$ in $\mathbb{R}$. In this article we restrict ourselves to the case of noise constant in space, because on one hand, this is the case where we are able to study the stabilization effect. On the other hand noise in space and time may lead to spatially unbounded solutions of (SH). So, this result is only the starting point for modulation equations on unbounded domains. The stochastic Swift-Hohenberg model was first studied in the context of amplitude equations with non-degenerate noise in [10] and later in [6].

For (SH) on the whole real line with degenerate additive noise, Axel Hutt and collaborators [23], [24] used a formal argument based on center manifold theory. They showed that noise constant in space leads to a deterministic amplitude equation, which is stabilized by the impact of additive noise. The aim of this chapter is to make their results rigorous.

Blömker, Hairer, and Pavliotis [8] considered the stochastic Swift-Hohenberg Equation (SH) near its change of stability on a large domain $[-L/\varepsilon, L/\varepsilon]$ with additive noise, where the noise is assumed to be real-valued homogeneous space-time noise. They showed that, under appropriate scaling, its solutions can be approximated by the solution A of the stochastic Ginzburg-Landau equation.

$$U(t,x) \approx \varepsilon A(\varepsilon^2 t, \varepsilon x)e^{ix} + c.c.$$

One severe problem is, that solutions of stochastic PDEs are not very regular in space and time. They are at most Hölder continuous and only for (SH) we have one spacial derivative. In [8] the amplitude $A(T)$ was shown to split into a more regular H^1-part and a Gaussian.

For the deterministic Swift-Hohenberg equation on an unbounded domain (i.e., $\sigma = 0$). Kirrmann, Mielke, and Schneider [26] approximated solutions of the Swift-Hohenberg equation via the Ginzburg-Landau equation

$$\partial_T A = 4\partial_X^2 A + \nu A - 3|A|^2 A,$$

but this method of approximation depends on high regularity of the modulation equation, as they needed $A \in C_b^{1,4}([0,T] \times \mathbb{R})$, which means one bounded derivative in time and four bounded spatial derivatives. For more results on the deterministic Swift-Hohenberg equation, see for instance [15], [33], [34] and [43].

Our method of approximation relies on very low regularity of the modulation equation, which is necessary when turning to spatial noise. Unfortunately, we still need too much regulatity for A, as we need $A \in C^0([0,T], \mathcal{H}^{1/2+})$. But as a solution of the stochastic Ginzburg-Landau, A is at most Hölder continuous with exponent less than $1/2$.

The main aim of this chapter is to show that the solution $U(t,x)$ of (SH) is well approximated by

$$U(t,x) \simeq \varepsilon A(\varepsilon^2 t, \varepsilon x)e^{ix} + \varepsilon \bar{A}(\varepsilon^2 t, \varepsilon x)e^{-ix} + \varepsilon \mathcal{Z}_\varepsilon(\varepsilon^2 t) \ ,$$

where the complex amplitude $A(T, X)$ is the solution of the Ginzburg-Landau equation

$$\partial_T A = 4\partial_X^2 A + (\nu - \tfrac{3}{2}\sigma^2)A - 3|A|^2 A \;, \tag{GL}$$

and

$$\mathcal{Z}_\varepsilon(T) = \varepsilon^{-1}\sigma \int_0^T e^{-\varepsilon^{-2}(T-\tau)} d\tilde{\beta}(\tau) \;, \tag{5.1}$$

is a fast Ornstein-Uhlenbeck process (OU, for short) with $\tilde{\beta}(T) := \varepsilon\beta(\varepsilon^{-2}T)$ being a rescaled version of the Brownian motion.

The remainder of this chapter is organised as follows. In the next section we define the standard fractional Sobolev space $\mathcal{H}^\alpha$. We also state and prove the relation between the norm in $\mathcal{H}^\alpha$ and the norm in $C^0(\mathbb{R})$. In Section 5.3 we give a formal derivation of the modulation equation and state the main result. In Section 5.4 we recall the Green's functions $\mathcal{G}_t(x)$ of the Swift-Hohenberg operator, and give estimates on it. In Section 5.5 we bound the Ornstein-Uhlenbeck process $\mathcal{Z}_\varepsilon(T)$. Finally, in Section 5.6 we give the proof of the main result.

5.2 The $\mathcal{H}^\alpha$-Space

In this section we define the well known Sobolev space $\mathcal{H}^\alpha$, where we rely on weighted L^2-norms of Fourier transforms. We also recall the relation between the norm in $\mathcal{H}^\alpha$ and the norm in $C^0(\mathbb{R})$ by stating the Sobolev embedding theorem.

Definition 5.2.1 *For $\alpha \in \mathbb{R}$, we define the space $\mathcal{H}^\alpha$ by*

$$\mathcal{H}^\alpha = \left\{ u : \mathbb{R} \to \mathbb{R} : \int_{-\infty}^{\infty} \left(1 + y^2\right)^\alpha |\mathcal{F}(u)(y)|^2 \, dy < \infty \right\},$$

with norm

$$\|u\|_\alpha^2 = \int_{-\infty}^{\infty} \left(1 + y^2\right)^\alpha |\mathcal{F}(u)(y)|^2 \, dy,$$

where $\mathcal{F}(u)$ is the Fourier transform of u, which takes the form

$$\mathcal{F}(u)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(k) e^{-iky} dk.$$

Note that in the space $\mathcal{H}^\alpha$ functions still decay to 0 at ∞. Thus if $A \in \mathcal{H}^\alpha$ we are still in a setting, where the solutions of (SH) and the amplitude A decay to 0 for $|x| \to \infty$.

Let us now consider semigroups in the space $\mathcal{H}^\alpha$.

Lemma 5.2.2 *Let $\mathcal{A}$ be a non-positive operator with eigenvalues $P(k)$ such that $P(k) \leq 0$ defined by $\mathcal{F}(\mathcal{A}u) = P(\cdot)\mathcal{F}(u)$. Then for $t \geq 0$ and $u \in \mathcal{H}^\alpha$*

$$\left\|e^{t\mathcal{A}}u\right\|_\alpha \leq \|u\|_\alpha \,. \tag{5.2}$$

It is well known that $e^{t\mathcal{A}}$ defined by $\mathcal{F}(e^{t\mathcal{A}}u) = e^{tP}\mathcal{F}(u)$ generates a contraction semigroup.

Proof. We note from Definition 5.2.1 that ($e^{-2tP(k)} \leq 1$)

$$\begin{aligned}\left\|e^{t\mathcal{A}}u\right\|_\alpha^2 &= \int_{-\infty}^{\infty} \left(1+y^2\right)^\alpha \left|\mathcal{F}(e^{t\mathcal{A}}u)(y)\right|^2 dy \\ &= \int_{-\infty}^{\infty} \left(1+y^2\right)^\alpha \left|e^{-tP(k)}\mathcal{F}(u)(y)\right|^2 dy \leq \|u\|_\alpha^2 \,.\end{aligned}$$

□

The next Lemma states the relation between the norm $\|\cdot\|_\alpha$ and the supremum-norm in $C^0(\mathbb{R})$.

Lemma 5.2.3 *For $\alpha > \frac{1}{2}$ there is a constant $C > 0$ such that*

$$\|u\|_\infty \leq C \|u\|_\alpha \quad \textit{for all } u \in \mathcal{H}^\alpha. \tag{5.3}$$

Proof. Using Sobolev Embedding Theorems (See Theorem 5.4 in [2]), yields (5.3). □

The following lemma is necessary in order to estimate the nonlinearity. It states that $\mathcal{H}^\alpha$ is up to the constant a Banach algebra for $\alpha > \frac{1}{2}$.

Lemma 5.2.4 *For $\alpha > \frac{1}{2}$ and $m \in \mathbb{N}$ there exist a constant $C > 0$ such that*

$$\|u^m\|_\alpha \leq C \|u\|_\alpha^m \,, \textit{ for } u \in \mathcal{H}^\alpha. \tag{5.4}$$

For simplicity of presentation, here let us give an elementary proof of (5.4) in case of $\frac{1}{2} < \alpha \leq 1$. For the complete proof, see proof of Theorem 4 in [42].

Proof. To prove (5.4) in case $\frac{1}{2} < \alpha \leq 1$, we study two cases depending on α.

First case, $\frac{1}{2} < \alpha < 1$. In this case, we use that the norm in $\mathcal{H}^\alpha$ is equivalent to

$$\|D^\alpha u\|_{L^2} = \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(x+y) - u(y)|^2}{|y|^{2\alpha+1}} dy dx\right)^{1/2},$$

where D^α is a fractional derivative. We note that, if $u = f \cdot g$, then

$$\begin{aligned} u(x+y) - u(y) &= f(x+y)g(x+y) - f(y)g(y) \\ &= f(x+y)\left[g(x+y) - g(y)\right] + g(y)\left[f(x+y) - f(y)\right]. \end{aligned}$$

Hence,

$$\|D^\alpha fg\|_{L^2} \le \|f\|_\infty \|D^\alpha g\|_{L^2} + \|g\|_\infty \|D^\alpha f\|_{L^2}.$$

From Lemma 3.5.1 in [1], there exist a constant $C > 0$ such that

$$C^{-1}\|u\|_\alpha \le \|u\|_{L^2} + \|D^\alpha u\|_{L^2} \le C\|u\|_\alpha.$$

Thus

$$\begin{aligned} \|u^m\|_\alpha &= \|u^m\|_{L^2} + \|D^\alpha u^m\|_{L^2} \\ &\le \left\|u^{m-1}\right\|_\infty \|u\|_{L^2} + \left\|u^{m-1}\right\|_\infty \|D^\alpha u\|_{L^2} + \|u\|_\infty \left\|D^\alpha u^{m-1}\right\|_{L^2} \\ &\le \|u\|_\infty^{m-1} \|u\|_{L^2} + m\|u\|_\infty^{m-1} \|D^\alpha u\|_{L^2} \\ &\le m\|u\|_\infty^{m-1} \left(\|u\|_{L^2} + \|D^\alpha u\|_{L^2}\right) \\ &= m\|u\|_\infty^{m-1} \|u\|_\alpha. \end{aligned}$$

Using Lemma 5.2.3, yields (5.4).

Second case, $\alpha = 1$. In this case

$$\begin{aligned} \|u^m\|_{\mathcal{H}^1} &= \|u^m\|_{L^2} + \|Du^m\|_{L^2} \\ &\le \|u\|_\infty^{m-1} \|u\|_{L^2} + m\|u\|_\infty^{m-1} \|Du\|_{L^2} \\ &\le m\|u\|_\infty^{m-1} \|u\|_{\mathcal{H}^1}. \end{aligned}$$

Using Lemma 5.2.3, yields (5.4). □

5.3 Formal Derivation and the Main Result

In this section let us discuss a formal derivation of the amplitude equation or modulation equation corresponding to Equation (SH). This is based on the approach in [26] and uses high regularity of the amplitude A. Let us first rescale (SH). If we assume that

$$U(t,x) = \varepsilon u(\varepsilon^2 t, \varepsilon x),$$

then Equation (SH) takes the form

$$\partial_T u = \mathcal{L}_\varepsilon u + \nu u - u^3 + \varepsilon^{-1}\sigma\partial_T\tilde{\beta}(T), \qquad (SH_\varepsilon)$$

with differential operator $\mathcal{L}_\varepsilon = -\varepsilon^{-2}(1+\varepsilon^2\partial_X^2)^2$ on the slow time $T = \varepsilon^2 t$ and the "slow" space $X = \varepsilon x$. Now define w via

$$u(T,X) = w(T,X) + \mathcal{Z}_\varepsilon(T), \qquad (5.5)$$

where $\mathcal{Z}_\varepsilon$ was defined in (5.1). Plugging (5.5) into (SH_ε), we obtain

$$\partial_T w = \mathcal{L}_\varepsilon w + \nu w - w^3 - 3w^2\mathcal{Z}_\varepsilon - 3w\mathcal{Z}_\varepsilon^2 + \nu\mathcal{Z}_\varepsilon - \mathcal{Z}_\varepsilon^3. \qquad (5.6)$$

Leaving out the error term for simplicity of presentation, we make the following ansatz:

$$w_A(T,X) = A(T,X)e^{ix} + \varepsilon^2 B(T,X)e^{2ix} + \varepsilon^2 H(T,X)e^{3ix} + \varepsilon^2 J(T,X) + c.c., \qquad (5.7)$$

where $c.c.$ denotes the complex conjugate. The higher order terms of order $\mathcal{O}(\varepsilon^2)$ are used to cancel various terms that appear due to the nonlinearity. We assume that all functions are sufficiently smooth.

Plugging (5.7) into (5.6) and using the relation

$$\begin{aligned}\mathcal{L}_\varepsilon\big(f(X)e^{i\frac{n}{\varepsilon}X}\big) = -[&\varepsilon^{-2}(1-n^2)^2 f + 4i\varepsilon^{-1}n(1-n^2)f' \\ &+ (2-6n^2)f'' + 4i\varepsilon n f''' + \varepsilon^2 f''''] \cdot e^{i\frac{n}{\varepsilon}X} ,\end{aligned} \qquad (5.8)$$

in order to obtain

$$\begin{aligned}&\partial_T A e^{ix} + \varepsilon^2\partial_T B e^{2ix} + \varepsilon^2\partial_T H e^{3ix} + \varepsilon^2\partial_T J + c.c. \\ &= [4A'' - 4i\varepsilon A''' - \varepsilon^2 A'''']e^{ix} - [9B - 24i\varepsilon B' - 22\varepsilon^2 B'' \\ &+ 8i\varepsilon^3 B''' + \varepsilon^4 B'''']e^{2ix} - [64H - 96i\varepsilon H' - 52\varepsilon^2 H'' \\ &+ 12i\varepsilon^3 H''' + \varepsilon^4 H'''']e^{3ix} - [J + 2\varepsilon^2 J'' + \varepsilon^4 J''''] \\ &+ \nu[Ae^{ix} + \varepsilon^2(Be^{2ix} + He^{3ix} + J + c.c.)] \\ &- [(Ae^{ix} + \bar{A}e^{-ix}) + \varepsilon^2(Be^{2ix} + He^{3ix} + J + c.c.)]^3 \\ &- 3\mathcal{Z}_\varepsilon[(Ae^{ix} + \bar{A}e^{-ix}) + \varepsilon^2(Be^{2ix} + He^{3ix} + J + c.c.)]^2 \\ &- 3\mathcal{Z}_\varepsilon^2[(Ae^{ix} + \bar{A}e^{-ix}) + \varepsilon^2(Be^{2ix} + He^{3ix} + J + c.c.)] \\ &+ c.c. + \nu\mathcal{Z}_\varepsilon - \mathcal{Z}_\varepsilon^3 .\end{aligned}$$

Hence,

$$\begin{aligned}\partial_T A e^{ix} + c.c. = &[4A'' - 4i\varepsilon A''']e^{ix} - [9B - 24i\varepsilon B']e^{2ix} \\ &- [64H - 96i\varepsilon H']e^{3ix} + \nu A e^{ix} - A^3 e^{3ix} \\ &- 3|A|^2 A e^{ix} - 3\mathcal{Z}_\varepsilon A^2 e^{2ix} - 3\mathcal{Z}_\varepsilon^2 A e^{ix} + c.c. \\ &- J + \nu \mathcal{Z}_\varepsilon - \mathcal{Z}_\varepsilon^3 - 6\mathcal{Z}_\varepsilon |A|^2 + \mathcal{O}(\varepsilon^2)\,.\end{aligned}$$

Removing all unwanted $\mathcal{O}(1)$-terms by defining

$$B = \frac{-1}{3}\mathcal{Z}_\varepsilon A^2\,,\; H = \frac{-1}{64}A^3 \;\text{ and }\; J = \nu\mathcal{Z}_\varepsilon - \mathcal{Z}_\varepsilon^3 - 6\mathcal{Z}_\varepsilon |A|^2\,, \tag{5.9}$$

we obtain

$$\begin{aligned}\partial_T A e^{ix} + c.c. = &[4A'' - 4i\varepsilon A''' + \nu A - 3|A|^2 A - 3\mathcal{Z}_\varepsilon^2 A]e^{ix} \\ &+ 24i\varepsilon B' e^{2ix} + 96i\varepsilon H' e^{3ix} + c.c. + \mathcal{O}(\varepsilon^2)\,.\end{aligned} \tag{5.10}$$

Before we proceed this formal derivation, let us state the following two lemmas on the approximation of $\mathcal{Z}_\varepsilon$. In the following we will rely on the important fact that due to averaging we can replace $\mathcal{Z}_\varepsilon^2$ approximately by the constant $\sigma^2/2$. Here we state the result in a way, which is useful for the mild formulation later.

Lemma 5.3.1 *For every $\kappa_0 > 0$ and $p > 1$ there is a constant $C > 0$, depending only on p, σ, κ_0, and T_0, such that*

$$\mathbb{E} \sup_{T\in[0,T_0]} |\mathcal{Z}_\varepsilon(T)|^p \leq C\varepsilon^{-\kappa_0}\,,$$

where the fast OU $\mathcal{Z}_\varepsilon(T)$ is defined in (5.1).

Lemma 5.3.2 *Let y be a complex function with $y = \mathcal{O}(\varepsilon^{-r})$ in $\mathcal{H}^\alpha$ and initial condition $\|y(0)\|_\infty = \mathcal{O}(\varepsilon^{-r})$ for some $r \geq 0$.*

If $Y(T,s) = e^{4(T-s)\partial_X^2} y(s)$ and $dY(T,s) = e^{4(T-s)\partial_X^2} G(s)ds$ with $G = \mathcal{O}(\varepsilon^{-r})$ in $\mathcal{H}^\alpha$, then for any small $\kappa_0 \in (0,1)$

$$\int_0^T Y(T,s)\{\mathcal{Z}_\varepsilon^2 - \tfrac{\sigma^2}{2}\}d\tau = \mathcal{O}(\varepsilon^{1-r-2\kappa_0}). \tag{5.11}$$

These two lemmas will be proved in Section 5.5.

Now let us complete our formal derivation. Collecting all coefficients in front of e^{ix} in (5.10), yields

$$\partial_T A = 4A'' + \nu A - 3|A|^2 A - 3\mathcal{Z}_\varepsilon^2 A + \mathcal{O}(\varepsilon).$$

Using the averaging result of Lemma 5.3.2, we obtain

$$\partial_T A = 4\partial_X^2 A + \left(\nu - \frac{3\sigma^2}{2}\right) A - 3|A|^2 A + \mathcal{O}(\varepsilon^{1-}).$$

Neglecting all small terms in ε, yields (GL).

The main result of the chapter is the following approximation result for the stochastic Swift-Hohenberg Equation (SH) through the Ginzburg-Landau Equation (GL).

Theorem 5.3.3 *(Approximation) Let $U(t,x)$ be a solution of (SH), $w_A(T,X)$ the formal approximation defined as*

$$w_A(T,X) = A(T,X)e^{iX\frac{1}{\varepsilon}} + c.c, \tag{5.12}$$

where $A(T,X)$ is a solution of (GL) such that $A \in C^0([0,T_0],\mathcal{H}^\alpha)$ for $\alpha > \frac{1}{2}$. Suppose for the initial condition $\|U(0) - \varepsilon A(0)e^{ix} - \varepsilon\bar{A}(0)e^{-ix}\|_\infty \le d\varepsilon^{1-3\kappa_0}\phi_\varepsilon$ for some fixed $d > 0$ and for $\kappa_0 \in (0, \frac{1}{8})$ such that $\varepsilon^{-8\kappa_0}\phi_\varepsilon^2 \to 0$ for $\varepsilon \to 0$.

Then for each $T_0 > 0$ and $p > 1$ there exist $C > 0$, depending on $\sup_{[0,T_0]}\|A\|_\alpha$, such that

$$\mathbb{P}\Big\{\sup_{t\in[0,\varepsilon^{-2}T_0]} \big\|U(t,x) - \varepsilon w_A(\varepsilon^2 t, \varepsilon x) - \varepsilon\mathcal{Z}_\varepsilon(\varepsilon^2 t)\big\|_\infty > C\varepsilon^{1-4\kappa_0}\phi_\varepsilon\Big\} \le C\varepsilon^p, \tag{5.13}$$

where $\mathcal{Z}_\varepsilon(T)$ is the fast OU defined in (5.1) and

$$\phi_\varepsilon^2 = \begin{cases} \varepsilon^2 & \text{if } \alpha > 3/2, \\ \varepsilon^2 \ln(1/\varepsilon) & \text{if } \alpha = 3/2, \\ \varepsilon^{2\alpha-1} & \text{if } \alpha < 3/2. \end{cases} \tag{5.14}$$

5.4 Green's Function and Semigroup Estimation

For the first part of this section we follow the ideas of Collet and Eckmann [15] which they apply to a slightly different operator. We define the Green's functions $\mathcal{G}_t(x)$ of the Swift-Hohenberg operator, and we give estimates on it.

Definition 5.4.1 *Define the Green's function $\mathcal{G}_t(x)$ of the operator $\mathcal{L}$ for $t > 0$ and $x \in \mathbb{R}$ as*

$$\mathcal{G}_t(x) = \int_{-\infty}^{\infty} e^{ikx} e^{-t(1-2k^2+k^4)} dk. \tag{5.15}$$

The next lemma states that the Green's function $\mathcal{G}_t(x)$ is bounded with respect to the norm $\|\cdot\|_{L^1}$.

Lemma 5.4.2 *There exists a constant $C > 0$ such that for all $t > 0$*

$$\|\mathcal{G}_t\|_{L^1} \leq C\,. \tag{5.16}$$

In order to prove this lemma, let us state and prove the following two lemmas:

Lemma 5.4.3 *Define the function $g_\tau(y)$ as*

$$g_\tau(y) = \int_{-\infty}^{\infty} e^{imy} e^{-Q_1(m,\tau)} dm,$$

where $Q_1(m,\tau) = \tau^{-2} - 2m^2 + \tau^2 m^4$. Then there exists a constant $C > 0$ such that for $0 < \tau \leq 1$

$$\sup_{y\in\mathbb{R}} \left|(4+y^2) g_\tau(y)\right| \leq C\,.$$

Proof. Using integration by parts, we obtain

$$\begin{aligned}(4+y^2)g_\tau(y) &= \int_{-\infty}^{\infty} P_1(m,\tau) e^{imy} e^{-Q_1(m,\tau)} dm \\ &= \int_{0}^{\infty} P_1(m,\tau) e^{imy} e^{-Q_1(m,\tau)} dm + \int_{-\infty}^{0} P_1(m,\tau) e^{imy} e^{-Q_1(m,\tau)} dm \\ &:= I_1 + I_2,\end{aligned}$$

where

$$P_1(m,\tau) = 12m^2\tau^2 - 16m^6\tau^4 + 32m^4\tau^2 - 16m^2.$$

For $m \geq 0$ and $0 < \tau \leq 1$ we not that

$$Q_1(m,\tau) = (m\tau - 1)^2 \underbrace{(m+\tau^{-1})^2}_{\geq \tau^{-2}} \geq (m - \tau^{-1})^2\,,$$

and

$$P_1(m,\tau) = \tau^2 m^2 [12 - 16(m-\tau^{-1})^2(1+\tau m)^2]\,.$$

Hence,

$$|P_1(m,\tau)| \le C[1+(\tau m)^4][1+(m-\tau^{-1})^2] \; .$$

Thus,

$$\left|P_1(m+\tau^{-1},\tau)\right| \le C[1+(\tau m+1)^4][1+m^2] \le C(1+m^6) \; .$$

Now we bound I_1 and I_2 separately. For the first integral I_1 we obtain

$$\begin{aligned} I_1 &= \int_{-\tau^{-1}}^{\infty} P_1(r+\tau^{-1},\tau)e^{i(r+\tau^{-1})y}e^{-Q_1(r+\tau^{-1},\tau)}dr \\ &\le \int_{-\tau^{-1}}^{\infty} P_1(r+\tau^{-1},\tau)e^{i(r+\tau^{-1})y}e^{-r^2}dr \; , \end{aligned}$$

where we substituted $r = m - \tau^{-1}$. Thus

$$|I_1| \le \int_{-\tau^{-1}}^{\infty} (c+cr^6)e^{-r^2}dr \le \int_{-\infty}^{\infty} (c+cr^6)e^{-r^2}dr = C.$$

For the second integral I_2, we put $-m$ instead of m to obtain

$$I_2 = \int_0^{\infty} P_1(m,\tau)e^{-imy}e^{-Q_1(m,\tau)}dm,$$

where P_1 and Q_1 are even polynomials in m. Analogously to the first integral, we derive

$$|I_2| \le C.$$

Hence, from the bounds on I_1 and I_2 we obtain

$$\sup_{y\in\mathbb{R}} \left|(4+y^2)g_\tau(y)\right| \le C \quad \text{for} \quad 0<\tau\le 1.$$

□

Lemma 5.4.4 *Define the function $h_\eta(y)$ as*

$$h_\eta(y) = \int_{-\infty}^{\infty} e^{iky}e^{-Q_2(k,\eta)}dk,$$

where $Q_2(k,\eta) = \eta^4 - 2\eta^2k^2 + k^4$. Then there exists a constant $C > 0$ such that for $0 < \eta < 1$

$$\sup_{y\in\mathbb{R}} \left|(1+y^2)h_\eta(y)\right| \le C \; .$$

Proof. Using integration by parts, we obtain

$$\begin{aligned}(1+y^2)h_\eta(y) &= \int_{-\infty}^{\infty} P_2(k,\eta)e^{iky}e^{-Q_2(k,\eta)}dk \\ &= \int_{1}^{\infty} P_2(k,\eta)e^{iky}e^{-Q_2(k,\eta)}dk + \int_{-\infty}^{-1} P_2(k,\eta)e^{iky}e^{-Q_2(k,\eta)}dk \\ &+ \int_{-1}^{1} P_2(k,\eta)e^{iky}e^{-Q_2(k,\eta)}dk \\ &:= II_1 + II_2 + II_3,\end{aligned}$$

where

$$P_2(k,\eta) = 1 + 12k^2 - 4\eta^2 - 16k^6 + 32k^4\eta^2 - 16k^2\eta^4.$$

We note that for $k \geq 1$ and $0 < \eta < 1$

$$Q_2(k,\eta) = (k-\eta)^2\underbrace{(k+\eta)^2}_{\geq 1} \geq (k-\eta)^2,$$

and

$$|P_2(k,\tau)| \leq c(1+k^6).$$

We now bound all three terms separately. To bound II_1 and II_2, we follow the same steps as in the case of Lemma 5.4.3. For the third term

$$\begin{aligned}|II_3| &\leq \int_{-1}^{1} |P_2(k,\eta)| \left|e^{-Q_2(k,\eta)}\right| dk \leq \int_{-1}^{1} |P_2(k,\eta)|\, dk \\ &\leq c\int_{-1}^{1}(1+k^6)dk = C.\end{aligned}$$

Hence, combining all three estimates on II_1, II_2 and II_3 we obtain for $0 < \eta < 1$ that

$$\sup_{y\in\mathbb{R}} \left|(1+y^2)h_\eta(y)\right| \leq C.$$

□

Proof of Lemma 5.4.2. In order to prove (5.16), we consider two cases:
First case $t \geq 1$. In this case we note that

$$\mathcal{G}_t(x) = \tau g_\tau(\tau x),$$

with $\tau = t^{-\frac{1}{2}}$ and

$$\begin{aligned}
\|\mathcal{G}_t\|_{L^1} &= \int_{-\infty}^{\infty} |\mathcal{G}_t(x)|\,dx = \int_{-\infty}^{\infty} |\tau g_\tau(\tau x)|\,dx \\
&= \int_{-\infty}^{\infty} |g_\tau(y)|\,dy = \int_{-\infty}^{\infty} \frac{1}{4+y^2}\left|(4+y^2)g_\tau(y)\right|dy \\
&\leq \sup_{y\in\mathbb{R}} \left|(4+y^2)g_\tau(y)\right| \int_{-\infty}^{\infty} \frac{1}{4+y^2}dy \\
&\leq C \sup_{y\in\mathbb{R}} \left|(4+y^2)g_\tau(y)\right| \ ,
\end{aligned}$$

where $y = \tau x$. Using Lemma 5.4.3, we obtain for $t \geq 1$

$$\|\mathcal{G}_t\|_{L^1} \leq C \ . \tag{5.17}$$

Second case $t \in (0,1)$. In this case we note that

$$\mathcal{G}_t(x) = \eta^{-1} h_\eta(\eta^{-1}x),$$

with $\eta = t^{\frac{1}{4}}$ and

$$\begin{aligned}
\|\mathcal{G}_t\|_{L^1} &= \int_{-\infty}^{\infty} \frac{1}{1+y^2}\left|(1+y^2)h_\eta(y)\right|dy \\
&\leq \sup_{y\in\mathbb{R}} \left|(1+y^2)h_\eta(y)\right| \int_{-\infty}^{\infty} \frac{1}{1+y^2}dy \\
&\leq C \sup_{y\in\mathbb{R}} \left|(1+y^2)h_\eta(y)\right| \ ,
\end{aligned}$$

where $y = \eta^{-1}x$. Using Lemma 5.4.4, we obtain for $t \in (0,1)$

$$\|\mathcal{G}_t\|_{L^1} \leq C \ . \tag{5.18}$$

Combining (5.17) and (5.18), yields (5.16) for all $t > 0$. □

Lemma 5.4.5 *There exists a constant $C > 0$ such that*

$$\|e^{t\mathcal{L}}u\|_\infty \leq C\|u\|_\infty \quad \textit{for all } t \geq 0 \textit{ and } u \in C^0(\mathbb{R}). \tag{5.19}$$

Proof. Let $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Then

$$\begin{aligned}
e^{t\mathcal{L}}u(x) &= \mathcal{F}^{-1}\mathcal{F}e^{t\mathcal{L}}u(x) = \mathcal{F}^{-1}(e^{-t\lambda_k}\mathcal{F}u(x)) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(x-y)}e^{-t\lambda_k}u(y)dydk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} u(y)\mathcal{G}_t(x-y)dy \ .
\end{aligned} \tag{5.20}$$

We obtain

$$\left\|e^{t\mathcal{L}}u\right\|_\infty \leq C\left\|u\right\|_\infty \left\|\mathcal{G}_t\right\|_{L^1} \ .$$

Using Lemma 5.4.2, yields (5.19). □

Corollary 5.4.6 *For $T \geq 0$, there exists a constant $C > 0$ such that*

$$\left\|e^{T\mathcal{L}_\varepsilon}u\right\|_\infty \leq C\|u\|_\infty \quad \textit{for all } T \geq 0 \textit{ and } u \in C^0(\mathbb{R}).$$

Proof.

$$\begin{aligned} e^{T\mathcal{L}_\varepsilon}u(X) &= e^{\varepsilon^{-2}T(1+(\varepsilon\partial_X)^2)^2}u(X) = e^{\varepsilon^{-2}T(1+\partial_X^2)^2}u(\varepsilon X) \\ &= e^{\varepsilon^{-2}T\mathcal{L}}u(\varepsilon X) = e^{t\mathcal{L}}u_\varepsilon(X)\ , \end{aligned}$$

where $u_\varepsilon(X) = u(\varepsilon X)$. Using Lemma 5.4.5, we obtain

$$\left\|e^{T\mathcal{L}_\varepsilon}u\right\|_\infty = \left\|e^{t\mathcal{L}}u_\varepsilon\right\|_\infty \leq C\left\|u_\varepsilon\right\|_\infty = C\left\|u\right\|_\infty \ .$$

□

The following lemma provides a result on how to change from semigroup $e^{T\mathcal{L}_\varepsilon}$ to $e^{4T\partial_X^2}$ when they are applied to $Ae^{iX\varepsilon^{-1}}$.

Lemma 5.4.7 *There is a constant $C > 0$ such that for all $T > 0$ and all $A \in \mathcal{H}^\alpha$ with $\alpha > \frac{1}{2}$*

$$\sup_{X\in\mathbb{R}}\left|e^{T\mathcal{L}_\varepsilon}A(X)e^{iX\varepsilon^{-1}} - (e^{4T\partial_X^2}A)(X)e^{iX\varepsilon^{-1}}\right| \leq C\|A\|_\alpha\phi_\varepsilon,$$

where ϕ_ε is defined in (5.14).

Proof. We write $e^{t\mathcal{L}}A(\varepsilon x)e^{ix}$ as a convolution with the Green's function of $\mathcal{L}$, as in (5.20),

$$\begin{aligned} e^{t\mathcal{L}}A(\varepsilon x)e^{ix} &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(x-y)}e^{-t\lambda_k}A(\varepsilon y)e^{iy}dy\,dk \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k-1)(x-y)}e^{-t\lambda_k}A(\varepsilon y)dy\,dk\cdot e^{ix} \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(\varepsilon x-y)}e^{-t\lambda_{\varepsilon k+1}}A(y)dy\,dk\cdot e^{ix}, \end{aligned}$$

where we used the substitution $y = \varepsilon y$ and $k = \varepsilon^{-1}(k-1)$. Hence,

$$e^{T\mathcal{L}_\varepsilon} A(X) e^{iX\varepsilon^{-1}} = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik(\varepsilon x - y)} e^{-T(\varepsilon k^2 + 2k)^2} A(y) dy\, dk \cdot e^{ix}. \quad (5.21)$$

Analogously, we can write $(e^{4T\partial_X^2} A)(X) \cdot e^{iX\varepsilon^{-1}}$ as

$$(e^{4T\partial_X^2} A)(X) e^{iX\varepsilon^{-1}} = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik(\varepsilon x - y)} e^{-4Tk^2} A(y) dy\, dk \cdot e^{ix}. \quad (5.22)$$

Let

$$\Theta = e^{T\mathcal{L}_\varepsilon} A(X) e^{iX\varepsilon^{-1}} - (e^{4T\partial_X^2} A)(X) \cdot e^{iX\varepsilon^{-1}}.$$

Hence,

$$\begin{aligned} \Theta &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(y) e^{ik(\varepsilon x - y)} \left[e^{-T(\varepsilon k^2 + 2k)^2} - e^{-4Tk^2} \right] dy\, dk \cdot e^{ix} \\ &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(A)(k) \left[e^{-T(\varepsilon k^2 + 2k)^2} - e^{-4Tk^2} \right] e^{i\varepsilon k x} dk \cdot e^{ix}. \end{aligned}$$

Using Cauchy-Schwarz inequality, yields

$$|\Theta|^2 \leq C \|A\|_\alpha^2 \int_{-\infty}^{\infty} \Psi(k) dk,$$

where

$$\Psi(k) = \tfrac{1}{(1+k^2)^\alpha} e^{-8Tk^2} \left[e^{-T(\varepsilon^2 k^4 + 4\varepsilon k^3)} - 1 \right]^2.$$

In order to bound Θ it is sufficient to bound

$$\begin{aligned} \int_{-\infty}^{\infty} \Psi(k) dk &= \int_0^{\frac{1}{2}\varepsilon^{-1}} \Psi(k) dk + \int_{-\frac{1}{2}\varepsilon^{-1}}^{0} \Psi(k) dk \\ &\quad + \int_{\frac{1}{2}\varepsilon^{-1}}^{\infty} \Psi(k) dk + \int_{-\infty}^{-\frac{1}{2}\varepsilon^{-1}} \Psi(k) dk \\ &:= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where we consider all terms separately. For I_1, we note that $\varepsilon k^3(\varepsilon k + 4)$ is non-negative for all $k \in [0, \frac{1}{2}\varepsilon^{-1}]$. Thus, we can use the following inequality, which follows directly from the intermediate value theorem:

$$|e^x - 1| \leq |x| \max\{1, e^x\}. \quad (5.23)$$

Hence,

$$\begin{aligned} I_1 &\leq \int_0^{\frac{1}{2}\varepsilon^{-1}} \tfrac{1}{(1+k^2)^\alpha} e^{-8Tk^2} \left[\varepsilon T k^3(\varepsilon k+4)\right]^2 dk \\ &\leq C\varepsilon^2 \int_0^{\frac{1}{2}\varepsilon^{-1}} \tfrac{k^2}{(1+k^2)^\alpha} \left(Tk^2\right)^2 e^{-8Tk^2} dk, \end{aligned}$$

where we used $(\varepsilon k + 4) < 5$ for all $k \in [0, \frac{1}{2}\varepsilon^{-1}]$. Now, using the fact

$$\sup_{z>0} \left\{ z^m e^{-z} \right\} < \infty \quad \text{for all } \; m \geq 0, \tag{5.24}$$

we get

$$I_1 \leq C\varepsilon^2 \int_0^{\frac{1}{2}\varepsilon^{-1}} \frac{k^2}{(1+k^2)^\alpha} dk \leq C\varepsilon^2 + C\varepsilon^2 \int_1^{\frac{1}{2}\varepsilon^{-1}} k^{2-2\alpha} dk \leq C\phi_\varepsilon^2.$$

Let us now turn to I_2. Substituting $k = -k$, yields

$$I_2 = \int_0^{\frac{1}{2}\varepsilon^{-1}} \frac{1}{(1+k^2)^\alpha} e^{-8Tk^2} \left[e^{\varepsilon T k^3(4-\varepsilon k)} - 1 \right]^2 dk.$$

We note that $\varepsilon k^3(4-\varepsilon k)$ is non-negative for all $k \in [0, \frac{1}{2}\varepsilon^{-1}]$. Using (5.23), yields

$$\begin{aligned} I_2 &\leq \int_0^{\frac{1}{2}\varepsilon^{-1}} \tfrac{1}{(1+k^2)^\alpha} e^{-8Tk^2} \left[4\varepsilon T k^3 e^{4\varepsilon T k^3} \right]^2 dk \\ &\leq \varepsilon^2 \int_0^{\frac{1}{2}\varepsilon^{-1}} \frac{k^2}{(1+k^2)^\alpha} \left(4Tk^2\right)^2 e^{-4Tk^2} dk, \end{aligned}$$

where we used $\varepsilon k \leq \frac{1}{2}$ for all $k \in [0, \frac{1}{2}\varepsilon^{-1}]$. Now (5.24) implies

$$I_2 \leq C\varepsilon^2 \int_0^{\frac{1}{2}\varepsilon^{-1}} \frac{k^2}{(1+k^2)^\alpha} dk \leq C\phi_\varepsilon^2.$$

To bound I_3:

$$\begin{aligned} I_3 &\leq C \int_{\frac{1}{2}\varepsilon^{-1}}^\infty \frac{1}{(1+k^2)^\alpha} \left[e^{-T(\varepsilon k^2+2)^2} + e^{-8Tk^2} \right]^2 dk \\ &\leq C \int_{\frac{1}{2}\varepsilon^{-1}}^\infty \frac{1}{(1+k^2)^\alpha} dk \leq C\varepsilon^{2\alpha-1} \;\text{ for } \alpha > \tfrac{1}{2}. \end{aligned}$$

Analogously for I_4 :

$$I_4 \leq C\varepsilon^{2\alpha-1} \ \text{ for } \alpha > \tfrac{1}{2}.$$

Collecting all four results together, we obtain$\|\Theta\|_\infty^2 \leq C\|A\|_\alpha^2 \phi_\varepsilon^2$. □

Let us now state a bound for the semigroup $e^{T\mathcal{L}_\varepsilon}$, when applied to $B(X)e^{in\varepsilon^{-1}X}$. The case $n = \pm 1$ was treated in Lemma 5.4.7 before.

Lemma 5.4.8 *Let $n \in \mathbb{Z} \setminus \{\pm 1\}$ and $\alpha > \frac{1}{2}$. There are two constants $C > 0$ and $c_n > 0$, depending on n, such that, for $T > 0$ and $B \in \mathcal{H}^\alpha$,*

$$\sup_{X\in\mathbb{R}} \left| e^{T\mathcal{L}_\varepsilon} B(X) e^{in\varepsilon^{-1}X} \right|^2 \leq C \|B\|_\alpha^2 \{e^{-c_n\varepsilon^{-2}T} + \varepsilon^{2\alpha-1}\}. \tag{5.25}$$

Proof. Writing $e^{t\mathcal{L}}B(\varepsilon x)e^{inx}$ as a convolution with the Green's function of $\mathcal{L}$ as in Lemma 5.4.7

$$\begin{aligned} e^{t\mathcal{L}}B(\varepsilon x)e^{inx} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(x-y)}e^{-t\lambda_k}B(\varepsilon y)e^{iny}dydk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k-n)(x-y)}e^{-t\lambda_k}B(\varepsilon y)dydk \cdot e^{inx} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(\varepsilon x-y)}e^{-t\lambda_{\varepsilon k+n}}B(y)dydk \cdot e^{inx}, \end{aligned}$$

where we used the substitution $y = \varepsilon y$ and $k = \varepsilon^{-1}(k-n)$. Hence, using the definition of λ_k and $X = \varepsilon x$, we obtain

$$e^{T\mathcal{L}_\varepsilon}B(X)e^{in\varepsilon^{-1}X} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(X-y)}e^{-t[1-(\varepsilon k+n)^2]^2}B(y)dydk \cdot e^{in\varepsilon^{-1}X}.$$

Taking the absolute value $|\cdot|$ on both sides and using Cauchy-Schwarz inequality, yields

$$\left| e^{T\mathcal{L}_\varepsilon}B(X)e^{in\varepsilon^{-1}X} \right|^2 \leq C\|B\|_\alpha^2 \int_{-\infty}^{\infty} \frac{1}{(1+k^2)^\alpha} e^{-2t[1-(\varepsilon k+n)^2]^2} dk. \tag{5.26}$$

Now, we want to bound the integral in (5.26)

$$\int_{-\infty}^{\infty} \Phi(k)dk \leq \int_0^{\frac{1}{2\varepsilon}} \Phi(k)dk + \int_{\frac{-1}{2\varepsilon}}^0 \Phi(k)dk + 2\int_{\frac{1}{2\varepsilon}}^{\infty} \frac{1}{(1+k^2)^\alpha}dk,$$

with

$$\Phi(k) = \frac{1}{(1+k^2)^\alpha}e^{-2tq(k)} \quad \text{and} \quad q(k) = [1-(\varepsilon k+n)^2]^2.$$

Now, let us bound $q(k)$ on $[0, \pm\frac{1}{2\varepsilon}]$. We consider several cases depending on n and k.

First case $n = 0$ and $k \in [-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}]$. In this case as $|k| \leq \frac{1}{2\varepsilon}$

$$q(k) = [1 - \varepsilon^2 k^2]^2 \geq \frac{9}{16}.$$

Second case $n \geq 2$ and $k \in [0, \frac{1}{2\varepsilon}]$, $\big($or $n \leq -2$ and $k \in [\frac{-1}{2\varepsilon}, 0]\big)$. In this case as $\varepsilon k \geq 0$

$$q(k) = [n + 1 + \varepsilon k]^2 [n - 1 + \varepsilon k]^2 \geq [n + 1]^2 [n - 1]^2 = \left(n^2 - 1\right)^2 .$$

Third case $n \geq 2$ and $k \in [\frac{-1}{2\varepsilon}, 0]$, $\big($or $n \leq -2$ and $k \in [0, \frac{1}{2\varepsilon}]\big)$. In this case as $k \leq \frac{1}{2\varepsilon}$

$$q(k) = [n - 1 - \varepsilon k]^2 [n + 1 - \varepsilon k]^2 \geq \frac{1}{16}[n + \frac{1}{2}]^2.$$

We deduce from the previous three cases that

$$q(k) \geq \frac{1}{2} c_n > 0 \, .$$

Thus,

$$\begin{aligned}
\int_{-\infty}^{\infty} \Phi(k) dk &\leq 2 \int_0^{\frac{1}{2\varepsilon}} \frac{1}{(1 + k^2)^\alpha} e^{-c_n t} dk + 2 \int_{\frac{1}{2\varepsilon}}^{\infty} \frac{1}{(1 + k^2)^\alpha} dk \\
&\leq 2 e^{-c_n t} \int_0^{\infty} \frac{1}{(1 + k^2)^\alpha} dk + 2 \int_{\frac{1}{2\varepsilon}}^{\infty} k^{-2\alpha} dk \\
&\leq C e^{-c_n t} + C \varepsilon^{2\alpha - 1}. \qquad (5.27)
\end{aligned}$$

Plugging (5.27) into (5.26), yields (5.25). □

5.5 General Bounds on $\mathcal{Z}_\varepsilon$

In this section, we prove Lemmas 5.3.1 and 5.3.2.

Proof of Lemma 5.3.1. See the first part of the proof of Lemma 3.4.2 with $\lambda_k = 1$. □

Proof of Lemma 5.3.2. First, we note from Lemma 5.2.2 that

$$\mathbb{E} \sup_{s \in [0, T_0]} \|Y(T, s)\|_\alpha^p = \mathbb{E} \sup_{s \in [0, T_0]} \left\| e^{(T-s)\mathcal{A}} y(s) \right\|_\alpha^p \leq C \mathbb{E} \sup_{[0, T_0]} \|y\|_\alpha^p \leq C \varepsilon^{-pr}.$$

Applying Itô formula to $Y\mathcal{Z}_\varepsilon^2$, yields

$$\begin{aligned} d\left(Y\mathcal{Z}_\varepsilon^2\right) &= \mathcal{Z}_\varepsilon^2 dY + 2Y\mathcal{Z}_\varepsilon d\mathcal{Z}_\varepsilon + Y\left(d\mathcal{Z}_\varepsilon\right)^2 \\ &= G\mathcal{Z}_\varepsilon^2 ds - 2\varepsilon^{-2}Y\mathcal{Z}_\varepsilon^2 ds + 2\varepsilon^{-1}\sigma\mathcal{Z}_\varepsilon Y d\tilde{\beta} + \varepsilon^{-2}\sigma^2 Y ds. \end{aligned}$$

$$d\left(Y\mathcal{Z}_\varepsilon^2\right) = e^{4(T-s)\partial_X^2}G(s)\mathcal{Z}_\varepsilon^2 ds - 2\varepsilon^{-2}Y\mathcal{Z}_\varepsilon^2 ds + 2\varepsilon^{-1}\sigma\mathcal{Z}_\varepsilon d\tilde{\beta} + \varepsilon^{-2}\sigma^2 Y ds.$$

Integrating from 0 to T, taking $\|\cdot\|_\infty^p$ norms, and using triangle inequality, yields

$$\begin{aligned} \Big\|\int_0^T Y\{\mathcal{Z}_\varepsilon^2 - \tfrac{\sigma^2}{2}\}ds\Big\|_\infty^p &\le c\varepsilon^{2p}\left\|Y\mathcal{Z}_\varepsilon^2\right\|_\infty^p + c\varepsilon^{2p}\Big\|\int_0^T e^{4(T-s)\partial_X^2}G(s)\mathcal{Z}_\varepsilon^2 ds\Big\|_\infty^p \\ &\quad + c\varepsilon^p\Big\|\int_0^T Y\mathcal{Z}_\varepsilon d\tilde{\beta}(s)\Big\|_\infty^p \\ &\le C\varepsilon^{2p-pr}\sup_{[0,T_0]}|\mathcal{Z}_\varepsilon|^{2p} + c\varepsilon^p\Big\|\int_0^T Y(T,s)\mathcal{Z}_\varepsilon d\tilde{\beta}(s)\Big\|_\infty^p. \end{aligned}$$

Taking expectation after supremum on both sides, we obtain

$$\mathbb{E}\sup_{[0,T_0]}\Big\|\int_0^T Y\{\mathcal{Z}_\varepsilon^2 - \tfrac{\sigma^2}{2}\}ds\Big\|_\infty^p \le C\varepsilon^{2p-pr-2\kappa_0} + C\varepsilon^p\mathbb{E}\sup_{[0,T_0]}\Big\|\int_0^T Y(T,s)\mathcal{Z}_\varepsilon d\tilde{\beta}(s)\Big\|_\infty^p. \tag{5.28}$$

In order to obtain (5.11), let us bound the last term on the right hand side of (5.28). Using Sobolev embedding from Lemma 5.2.3, yields

$$\mathbb{E}\sup_{[0,T_0]}\Big\|\int_0^T Y(T,s)\mathcal{Z}_\varepsilon(s)d\tilde{\beta}(s)\Big\|_\infty^p \le \mathbb{E}\sup_{[0,T_0]}\Big\|\int_0^T Y(T,s)\mathcal{Z}_\varepsilon(s)d\tilde{\beta}(s)\Big\|_\alpha^p.$$

By a variant Burkholder-Davis-Gundy theorem (see, Theorem 1.2.5 in [32] or the paper of Hausenblas and Seidler [21]), we obtain for $p \ge 2$

$$\begin{aligned} \mathbb{E}\sup_{[0,T_0]}\Big\|\int_0^T e^{4(T-s)\partial_X^2}y(s)\mathcal{Z}_\varepsilon(s)d\tilde{\beta}(s)\Big\|_\infty^p &\le C\mathbb{E}\Big(\int_0^{T_0}\|y(s)\mathcal{Z}_\varepsilon(s)\|_\alpha^2 ds\Big)^{\frac{p}{2}} \\ &\le C\mathbb{E}\Big(\int_0^{T_0}|\mathcal{Z}_\varepsilon(s)|^2\|y(s)\|_\alpha^2 ds\Big)^{\frac{p}{2}} \\ &\le C\varepsilon^{-pr-\kappa_0}. \end{aligned}$$

□

As a final result in this section, we prove an averaging result for a mild formulation of (GL).

Lemma 5.5.1 *If A is a solution of (GL) with $\sup_{[0,T_0]} \|A\|_\alpha \le C$, then*

$$\int_0^T e^{4(T-s)\partial_X^2} A(s)\{\mathcal{Z}_\varepsilon^2(s) - \tfrac{\sigma^2}{2}\} ds = \mathcal{O}(\varepsilon^{1-2\kappa_0}), \tag{5.29}$$

for any $\kappa_0 > 0$.

Proof. Define for $s \in [0, T]$

$$Y(T, s) = e^{4(T-s)\partial_X^2} A(s),$$

with

$$dY = (-4\partial_X^2) e^{4(T-s)\partial_X^2} A(s) ds + e^{4(T-s)\partial_X^2} dA.$$

Using (GL), we obtain

$$dY = e^{4(T-s)\partial_X^2} \left[(\nu - \tfrac{3}{2}\sigma^2)A - 3|A|^2 A\right] ds = e^{4(T-s)\partial_X^2} G(s) ds.$$

Using Lemmas 5.2.3, 5.2.2 and 5.2.4, we derive

$$\|G\|_\infty \le C \|G\|_\alpha \le C\|A\|_\alpha + C\|A\|_\alpha^3.$$

Thus

$$\sup_{[0,T_0]} \|G\|_\infty \le C.$$

Now applying Lemma 5.3.2, yields (5.29). □

5.6 Main Results

In this section, we give the proof of the main result.

Definition 5.6.1 *Define the residual $\rho(T)$ as*

$$\rho(T) = w_A(T) - e^{T\mathcal{L}_\varepsilon} w_A(0) - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon} \left[\nu(w_A + \mathcal{Z}_\varepsilon) - (w_A + \mathcal{Z}_\varepsilon)^3\right] ds, \tag{5.30}$$

where w_A is defined in (5.12).

Lemma 5.6.2 *If* $\sup_{[0,T_0]} \|A\|_\alpha \leq C$ *for* $\alpha > \frac{1}{2}$, *then for all* $p > 1$ *there is a constant* $C > 0$ *such that*

$$\mathbb{E} \sup_{T\in[0,T_0]} \|\rho(T)\|_\infty^p \leq C\varepsilon^{-3p\kappa_0}\phi_\varepsilon^p, \tag{5.31}$$

where ϕ_ε *is defined in (5.14).*

Proof. From (5.12), we obtain

$$\begin{aligned}\rho(T) &= A(T)e^{ix} + c.c. - e^{T\mathcal{L}_\varepsilon}A(0)e^{ix} + c.c. - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}\nu(Ae^{ix} + c.c.)ds \\ &\quad + \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}(Ae^{ix} + c.c. + \mathcal{Z}_\varepsilon)^3 ds - \nu\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}\mathcal{Z}_\varepsilon ds.\end{aligned}$$

Hence,

$$\begin{aligned}\rho(T) &= A(T)e^{ix} - e^{T\mathcal{L}_\varepsilon}A(0)e^{ix} - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}(\nu A - 3A\mathcal{Z}_\varepsilon^2 - 3|A|^2 A)e^{ix}ds \\ &\quad + \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}A^3e^{3ix}ds + 3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}A^2\mathcal{Z}_\varepsilon e^{2ix}ds \\ &\quad +3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}|A|^2\mathcal{Z}_\varepsilon e^{2ix}ds + c.c. \\ &\quad -\nu\int_0^T e^{-(T-s)\mathcal{L}_\varepsilon}\mathcal{Z}_\varepsilon ds + \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}\mathcal{Z}_\varepsilon^3 ds.\end{aligned}$$

Using Lemma 5.4.7, we obtain

$$\begin{aligned}\rho(T) &= \left[A(T) - e^{4T\partial_X^2}A(0) - \int_0^T e^{4(T-s)\partial_X^2}(\nu A - 3A\mathcal{Z}_\varepsilon^2 - 3|A|^2 A)ds\right]e^{ix} \\ &\quad + \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}A^3e^{3ix}ds + 3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}A^2\mathcal{Z}_\varepsilon e^{2ix}ds \\ &\quad +3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}|A|^2\mathcal{Z}_\varepsilon e^{2ix}ds + c.c. - \nu\int_0^T e^{-\varepsilon^{-2}(T-s)}\mathcal{Z}_\varepsilon ds \\ &\quad + \int_0^T e^{-\varepsilon^{-2}(T-s)}\mathcal{Z}_\varepsilon^3 ds + \mathcal{O}(\varepsilon^{-3\kappa_0}\phi_\varepsilon).\end{aligned}$$

From (GL) we have

$$\begin{aligned}\rho(T) &= \left[3\int_0^T e^{4(T-s)\partial_X^2}A(\mathcal{Z}_\varepsilon^2 - \frac{1}{2}\sigma^2)ds\right]e^{ix} + \int_0^T e^{(T-s)\mathcal{L}_\varepsilon}A^3e^{3ix}ds \\ &\quad +3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}A^2\mathcal{Z}_\varepsilon e^{2ix}ds + 3\int_0^T e^{(T-s)\mathcal{L}_\varepsilon}|A|^2\mathcal{Z}_\varepsilon e^{2ix}ds \\ &\quad +c.c. + \mathcal{O}(\varepsilon^{-3\kappa_0}\phi_\varepsilon).\end{aligned}$$

Taking the norm $\|\cdot\|_\infty^p$ on both sides and using Lemma 5.4.8 in order to obtain

$$\begin{aligned}\|\rho\|_\infty^p \;\leq\; & C\left\|\int_0^T e^{4(T-s)\partial_X^2} A(\mathcal{Z}_\varepsilon^2-\frac{1}{2}\sigma^2)ds\right\|_\infty^p \\ & +C\left(\varepsilon^{2p}+\varepsilon^{p\alpha-\frac{p}{2}}\right)\left[\left\|A^3\right\|_\alpha^p+|\mathcal{Z}_\varepsilon|^p\left\|A^2\right\|_\alpha^p+|\mathcal{Z}_\varepsilon|^p\left\||A|^2\right\|_\alpha^p\right] \\ & +C\varepsilon^{-3p\kappa_0}\phi_\varepsilon^p.\end{aligned}$$

Taking expectation after supremum and using the bound on $\mathcal{Z}_\varepsilon$ from Lemma 5.3.1, the fact $\mathcal{H}^\alpha$ is a Banach Algebra from Lemma 5.2.4 and averaging result for a mild formulation from Lemma 5.5.1, yields (5.31). □

Definition 5.6.3 *Define the set $\Omega_0 \subset \Omega$ such that all these estimates*

$$\sup_{T\in[0,T_0]} |\mathcal{Z}_\varepsilon(T)| < \varepsilon^{-\kappa_0}\ , \tag{5.32}$$

$$\left|\int_0^{T_0}\{|\mathcal{Z}_\varepsilon|^2-\frac{\sigma^2}{2}\}d\tau\right| < \varepsilon^{1-3\kappa_0}\ , \tag{5.33}$$

and

$$\sup_{T\in[0,T_0]} \|\rho(T)\|_\infty < \varepsilon^{-4\kappa_0}\phi_\varepsilon,$$

hold on Ω_0.

Corollary 5.6.4 *For all $p>0$ there exist a constant C_p such that on Ω_0*

$$\mathbb{P}(\Omega_0) \geq 1-C_p\varepsilon^p \quad \textit{for all} \quad \varepsilon\in(0,1). \tag{5.34}$$

Proof. We note that

$$\mathbb{P}(\Omega_0) \geq 1-\mathbb{P}(\sup_{[0,T_0]}|\mathcal{Z}_\varepsilon(T)|\geq\varepsilon^{-\kappa_0})-\mathbb{P}(\int_0^{T_0}\{|\mathcal{Z}_\varepsilon|^2-\frac{\sigma^2}{2}\}d\tau\geq\varepsilon^{1-3\kappa_0})$$
$$-\mathbb{P}(\sup_{[0,T_0]}|\rho(T)|\geq\varepsilon^{-4\kappa_0}\phi_\varepsilon).$$

Using Chebychev's inequality

$$\begin{aligned}\mathbb{P}(\Omega_0) \geq 1 & -\varepsilon^{q\kappa_0}\mathbb{E}\sup_{[0,T_0]}|\mathcal{Z}_\varepsilon|^q-\varepsilon^{4q\kappa_0}\phi_\varepsilon^{-q}\mathbb{E}\sup_{[0,T_0]}\|\rho\|_\infty^q \\ & -\varepsilon^{-q+3q\kappa_0}\mathbb{E}(\int_0^{T_0}\{|\mathcal{Z}_\varepsilon|^2-\frac{\sigma^2}{2}\}d\tau)^q.\end{aligned}$$

From Lemmas 5.3.1,5.3.2 and 5.6.2, we obtain

$$\mathbb{P}(\Omega_0) \geq 1 - C_q \varepsilon^{q\kappa_0 - \kappa_0} - C_q \varepsilon^{q\kappa_0}.$$

For sufficiently large q, we obtain

$$\mathbb{P}(\Omega_0) \geq 1 - C_p \varepsilon^p \quad \text{for all} \quad p > 0.$$

□

Finally, we use the results previously obtained to prove the main result of Theorem 5.3.3 for the approximation of the solution of the SPDE (SH_ε).

Proof of Theorem 5.3.3. Define

$$R(T) = u(T) - w_A(T) - \mathcal{Z}_\varepsilon(T). \tag{5.35}$$

Integrating (SH_ε) from 0 to T, we obtain

$$u(T) = e^{T\mathcal{L}_\varepsilon} u(0) + \nu \int_0^T e^{(T-s)\mathcal{L}_\varepsilon} u(s) ds - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon} u(s)^3 ds + \mathcal{Z}_\varepsilon(T). \tag{5.36}$$

Substituting from (5.35) into (5.36), we obtain

$$\begin{aligned} R(T) = e^{T\mathcal{L}_\varepsilon} R(0) &+ \nu \int_0^T e^{(T-s)\mathcal{L}_\varepsilon} R ds - 3 \int_0^T e^{(T-s)\mathcal{L}_\varepsilon} \mathcal{Z}_\varepsilon R^2 ds \\ &- 3 \int_0^T e^{(T-s)\mathcal{L}_\varepsilon} \mathcal{Z}_\varepsilon^2 R ds - \int_0^T e^{(T-s)\mathcal{L}_\varepsilon} R^3 ds - 3 \int_0^T e^{(T-s)\mathcal{L}_\varepsilon} w_A^2 R ds \\ &- 6 \int_0^T e^{(T-s)\mathcal{L}_\varepsilon} w_A \mathcal{Z}_\varepsilon R ds - 3 \int_0^T e^{(T-s)\mathcal{L}_\varepsilon} w_A R^2 ds + \rho(T), \end{aligned}$$

where the residual $\rho(T)$ is defined in (5.30). Taking the norm $\|\cdot\|_\infty$ on both sides and using Corollary 5.4.6, yields on Ω_0

$$\begin{aligned} \|R(T)\|_\infty \leq C \|R(0)\|_\infty &+ C \int_0^T \|R\|_\infty ds + C \int_0^T |\mathcal{Z}_\varepsilon| \|R\|_\infty^2 ds \\ &+ C \int_0^T \left|\mathcal{Z}_\varepsilon^2\right| \|R\|_\infty ds + C \int_0^T \|R\|_\infty^3 ds + C \int_0^T |\mathcal{Z}_\varepsilon| \|R\|_\infty ds \\ &+ C \int_0^T \|R\|_\infty^2 ds + C\varepsilon^{-4\kappa_0} \phi_\varepsilon . \end{aligned}$$

where we used $\|w_A\|_\infty \le C$. As long as $\|R(T)\|_\infty \le D\varepsilon^{-4\kappa_0}\phi_\varepsilon$, we obtain

$$\begin{aligned}
\|R(T)\|_\infty &\le (C\varepsilon^{\kappa_0}d + C)\varepsilon^{-4\kappa_0}\phi_\varepsilon \\
&\quad + C[1 + D\varepsilon^{-4\kappa_0}\phi_\varepsilon + \left|\mathcal{Z}_\varepsilon^2\right| + D^2\varepsilon^{-8\kappa_0}\phi_\varepsilon^2 + |\mathcal{Z}_\varepsilon|]\int_0^T \|R\|_\infty\, ds \\
&\le C_1\varepsilon^{-4\kappa_0}\phi_\varepsilon + C[\frac{3}{2} + D\varepsilon^{-4\kappa_0}\phi_\varepsilon + \frac{1}{2}|\mathcal{Z}_\varepsilon|^2 + D^2\varepsilon^{-8\kappa_0}\phi_\varepsilon^2]\int_0^T \|R\|_\infty\, ds \\
&\le C_1\varepsilon^{-4\kappa_0}\phi_\varepsilon + \int_0^T [C_2 + \frac{1}{2}C|\mathcal{Z}_\varepsilon|^2]\,\|R\|_\infty\, ds\;,
\end{aligned}$$

where $C_1 = C\varepsilon^{\kappa_0}d + C$ and

$$C[\frac{3}{2} + D\varepsilon^{-4\kappa_0}\phi_\varepsilon + D^2\varepsilon^{-8\kappa_0}\phi_\varepsilon^2] \le C[2 + \frac{3}{2}D^2\varepsilon^{-8\kappa_0}\phi_\varepsilon^2] = C_2.$$

Note that by Assumption on κ_0, we can choose C_2 independent of D, provided $\varepsilon > 0$ is sufficiently small. Using Gronwall's inequality, we obtain

$$\begin{aligned}
\|R(T)\|_\infty &\le C_1\varepsilon^{-4\kappa_0}\phi_\varepsilon[1 + \int_0^T [C_2 + \tfrac{1}{2}C|\mathcal{Z}_\varepsilon|^2]\exp\{\int_s^T [C_2 + \tfrac{1}{2}C|\mathcal{Z}_\varepsilon|^2]dr\}ds] \\
&\le C_1\varepsilon^{-4\kappa_0}\phi_\varepsilon[1 + \int_0^{T_0} [C_2 + \tfrac{1}{2}C|\mathcal{Z}_\varepsilon|^2]\exp\{C_2T + \tfrac{1}{2}C\int_0^{T_0} |\mathcal{Z}_\varepsilon|^2\, dr\}ds].
\end{aligned}$$

Taking the supremum over $[0, T_\star]$ yields

$$\sup_{T\in[0,T_\star]} \|R(T)\|_\infty \le C_1\varepsilon^{-4\kappa_0}\phi_\varepsilon[1 + \tilde{C}_2] \quad \text{on } \Omega_0\;, \tag{5.37}$$

where we used (see (5.33))

$$\int_0^{T_0} |\mathcal{Z}_\varepsilon|^2 d\tau \le \varepsilon^{1-3\kappa_0} + \tfrac{\sigma^2}{2}T_0 \le \tilde{C} \quad \text{on } \Omega_0 \tag{5.38}$$

and defined

$$\tilde{C}_2 = (C_2T_0 + \tfrac{1}{2}C\tilde{C})e^{(C_2T_0 + \frac{1}{2}C\tilde{C})}\,.$$

Now fix $D > C_1[1 + \tilde{C}_2]$. Hence, (5.37) shows that

$$\sup_{T\in[0,T_\star]} \|R(T)\|_\infty < D\varepsilon^{-4\kappa_0}\phi_\varepsilon\,.$$

Hence, $T_\star = T_0$ and finally

$$\begin{aligned}
\sup_{t\in[0,\varepsilon^{-2}T_0]} \left\|U(t,x) - \varepsilon w_A(\varepsilon^2 t, \varepsilon x) - \varepsilon\mathcal{Z}_\varepsilon(\varepsilon^2 t)\right\|_\infty &\le \varepsilon \sup_{T\in[0,T_0]} \|R(T)\|_\infty \\
&\le C\varepsilon^{1-4\kappa_0}\phi_\varepsilon.
\end{aligned}$$

Thus,

$$\mathbb{P}\Big\{\sup_{t\in[0,\varepsilon^{-2}T_0]}\big\|U(t,x)-\varepsilon w_A(\varepsilon^2 t,\varepsilon x)-\varepsilon\mathcal{Z}_\varepsilon(\varepsilon^2 t)\big\|_\infty > C\varepsilon^{1-4\kappa_0}\phi_\varepsilon\Big\}\leq 1-\mathbb{P}(\Omega_0).$$

Using (5.34), yields (5.13). □

Bibliography

[1] D. R. Adams and L. I. Hedberg. *Function Spaces and Potential Theory*. Springer, 1996.

[2] R. A. Adams. *Sobolev Spaces*. Academic Press, New York, 1975.

[3] D. Blömker. Amplitude equations for locally cubic non-autonomous nonlinearities. *SIAM Journal on Applied Dynamical Systems*, 2:464–486, 2003.

[4] D. Blömker. Approximation of the stochastic Rayleigh-Bénard problem near the onset of convection and related problems. *Stochastic and Dynamics*, 5:441–474, 2005.

[5] D. Blömker. *Amplitude equations for stochastic partial differential equations*. World Scientific, Interdisciplinary mathematical sciences-Vol.3, 2007.

[6] D. Blömker and M. Hairer. Multiscale expansion of invariant measures for SPDEs. *Commun. Math. Phys.*, 251(3):515–555, 2004.

[7] D. Blömker and M. Hairer. Amplitude equations for SPDEs: approximate centre manifolds and invariant measures. *In Duan J. and Waymire, E. C. (eds.)*, The IMA Volumes in Mathematics and its Applications 140:41–59, 2005.

[8] D. Blömker, M. Hairer, and G. A. Pavliotis. Modulation equations: Stochastic bifurcation in large domains. *Commun. Math. Phys.*, 258:479–512, 2005.

[9] D. Blömker, M. Hairer, and G. A. Pavliotis. Multiscale analysis for stochastic partial differential equations with quadratic nonlinearities. *Nonlinearity*, 20:1–25, 2007.

[10] D. Blömker, S. Maier-Paape, and G. Schneider. The stochastic Landau equation as an amplitude equation. *Discrete Contin. Dyn. Syst. Ser B*, 1:527–541, 2001.

[11] D. Blömker and W. W. Mohammed. Amplitude equation for SPDEs with quadratic nonlinearities. *Electronic Journal of Probability*, 14(88):2527–2550, 2009.

[12] D. Blömker and W. W. Mohammed. Modulation equation for the stochastic Swift-Hohenberg equation. *Preprint*, 2011.

[13] D. Blömker and W. W. Mohammed. Amplitude equations for SPDEs with cubic nonlinearities. *Stochastics: An International Journal of Probability and Stochastic Processes*, will appear.

[14] A. Chekhlov and V. Yakhot. Kolmogorov turbulence in a random-force-driven burgers equation: Anomalous scaling and probability density functions. *Phys. Rev. E*, 52(5):5681–5684, 1995.

[15] P. Collet and J.-P. Eckmann. The time dependent amplitude equation for the Swift-Hohenberg problem. *Comm. Math. Physics*, 132:139–153, 1990.

[16] M. C. Cross and P. C. Hohenberg. Pattern formation outside of equilibrium. *Reviews of Modern Physics*, 65(3):851–1112, 1993.

[17] Z. R. Cuerno and A.-L. Barabási. Dynamic scaling of ion-sputtered surfaces. *Phys. Rev. Lett.*, 74:4746–4749, 1995.

[18] A. de Bouard and A. Debussche. Random modulation of solitons for the stochastic korteweg-de vries equation. *Ann. I. H. Poincare-An*, 24:251–278, 2007.

[19] A. Doelman, B. Sandstede, A. Scheel, and G. Schneider. *The dynamics of modulated wave trains*, volume 199. Memoirs of the Amerian Mathematical Society, New York, 2009.

[20] H. Haken. *Synergetics: An introduction. Nonequilibrium phase transition and self-organization in Physics, Chemistry and Biology*. Springer Ser. Synergetics, Springer-Verlag, Berlin, 1983.

[21] E. Hausenblas and J. Seidler. A note on maximal inequality for stochastic convolutions. *Czechoslovak Mathematical Journal*, 51(4):785–790, 2001.

[22] P. C. Hohenberg and J. B. Swift. Effects of additive noise at the onset of Rayleigh-Bénard convection. *Physical Review A.*, 46:4773–4785, 1992.

[23] A. Hutt. Additive noise may change the stability of nonlinear systems. *Europhys. Lett. 84*, 34003:1–4, 2008.

[24] A. Hutt, A. Longtin, and L. Schimansky-Geier. Additive global noise delays Turing bifurcations. *Physical Review Letters 98*, 230601:1–4, 2007.

[25] A. Hutt, A. Longtin, and L. Schimansky-Geier. Additive noise-induced turing transitions in spatial systems with application to neural fields and the swift-hohenberg equation. *Physica D*, 237:755–773, 2008.

[26] P. Kirrmann, G. Schneider, and A. Mielke. The validity of modulation equations for extended systems with cubic nonlinearities. *Proceedings of the Royal Society of Edinburgh*, 122A:85–91, 1992.

[27] M. Kovács and S. Larsson. Introduction to stochastic partial differential equations. *Lecture notes. Department of Mathematical Science, Chalmers University of Technology and University of Gothenburg*, 2008. http://www.maths.otago.ac.nz/ mkovacs/spdenotes.pdf.

[28] B. Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. Universitext. Springer-Verlag, Berlin,, 1998.

[29] Z.-W. Lai and S. Das Sarma. Kinetic growth with surface relaxation: Continuum versus atomistic models. *Phys. Rev. Lett.*, 66:2348–2351, 1991.

[30] K. B. Lauritsen, R. Cuerno, and H.A. Makse. Noisy Kuramoto-Sivashinsky equation for an erosion model. *Phys. Rev. E*, 54:3577–3580, 1996.

[31] C. Liu. A fourth-order parabolic equation in two space dimensions. *Mathematical Methods in the Applied Sciences*, 30:1913–1930, 2007.

[32] K. Liu. *Stability of infinite dimensional stochastic differential equations with applications*, volume 135. Chapman and Hall /CRC Monographs, 2006.

[33] A. Mielke and G. Schneider. Attractors for modulation equations on unbounded domains - existence and comparison. *Nonlinearity*, 8:743–768, 1995.

[34] A. Mielke, G. Schneider, and A. Ziegra. Comparison of interial manifolds and application to modulated systems. *Math. Nachr.*, 214:53–69, 2000.

[35] G. Da Prato, S. Kwapien, and J. Zabczyk. Regularity of solutions of linear stochastic equations in Hilbert spaces. *Stochastics*, 23:1–23, 1987.

[36] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1992.

[37] C. Prévôt and M. Röckner. *A concise course on stochastic partial differential equations, Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[38] M. Raible, S. G. Mayr, S. J. Linz, M. Moske, P. Hänggi, and K. Samwer. Amorphous thin film growth: Theory compared with experiment. *Europhysics Letters*, 50:61–67, 2000.

[39] M. Reed and B. Simon. *Methods of modern mathematical physics*. Academic Press, 1972.

[40] A. J. Roberts. A step towards holistic discretisation of stochastic partial differential equations. *ANZIAM J.*, 45:C1–C15, 2003.

[41] A. J. Roberts and Wei Wang. Macroscopic reduction for stochastic reaction-diffusion equations. *Preprint. arXiv:0812.1837v1*, 2008.

[42] T. Runst and W. Sickel. *Sobolev Spaces of fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*. Walter de Gruyter. Berlin. New York, 1996.

[43] G. Schneider. The validity of generalized Ginzburg-Landau equations. *Math. Methods Appl. Sci.*, 19(9):717–736, 1996.

[44] G. Schneider and H. Uecker. The amplitude equation for the first instability of electro-convection in nematic liquid crystals in case of two unbounded space directions. *Nonlinearity*, 20:1361–1386, 2007.

[45] D. Siegert and M. Plischke. Solid-on-solid models of moleular-beam epitaxy. *Physics Rev.*, E50:917–931, 1994.

[46] D. Walgeraef. *Spatio-temporal pattern formation : With examples from physics, chemistry, and materials science (partially ordered systems)*. Springer-Verlag, New York, 1997.

[47] J. Weidmann. *Linear operators in Hilbert spaces*. Springer-Verlag, New York, 1980.

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Curriculum Vitae

Personal Details

Name	Wael Wagih Elbayoumi Mohammed
Date of birth	20/3/1978
Current job	Assistant Lecture, Mathematics Department,
	Faculty of Science, Mansoura University
Marital status	Married and have 3 children

Scientific Qualifications

1997-2000	B.Sc., Faculty of Science,
	Mansoura University, Egypt
2004-2007	Master, Faculty of Science,
	Mansoura University, Egypt
2007-till now	PhD student in Augsburg University, Germany

HOW TO CONTACT

E-mail address wael.mohammed@math.uni-augsburg.de
wwelhadad@mans.edu.eg
wwelhadad@yahoo.com

Computer Experience

International computer driving licence (ICDL)

References

Prof. Dr. Dirk Blömker	Institut für Mathematik, Universität Augsburg, Germany
Prof. Dr. E. M. Elabbasy	Department of Mathematics, Faculty of Science, Mansoura University, Egypt

Conference / Workshop

- Random Dynamical Systems. 17 -19 November 2008, Bielefeld, Germany
- Stochastic Partial Differential Equations (SPDEs) and their Applications. 29 March - 1 April 2010, Cambridge, UK
- Stochastic Partial Differential Equations: Approximation, Asymptotics and Computation. 28 Jun - 2 Jul 2010, Cambridge, UK
- Coherent Structures in Evolutionary Equations. 12-16 July 2010, Leiden, Netherlands
- 6th PhD Student Conference in Stochastics. 30 Sept-2 Oct 2010, Zurich, Switzerland
- The Ergodic Theory of Markov Processes, 24-30 Oct 2010, Oberwalfach, Germany
- SIAM Conference on Applications of Dynamical Systems, 22-26 May 2011, Snowbird, Utah, USA

List of Publication

- Wael Wagih Elbayoumi Mohammed
  Oscillation of Certain Types of Second Order Differential Equations, Master Thesis, Mansoura University (2007)

- E. M. Elabbasy and W. W. Elhaddad
  Oscillation of Second-Order Nonlinear Differential Equations With Damping Term, E. J. of Qualitative Theory of Diff. Equ., Number 25 (2007)

- E. M. Elabbasy and W. W. Elhaddad
  Oscillation Criteria for Nonlinear Differential Equations of Second Order With Damping Term, Serdica Mathematical Journal, Volume 34, Number 2 (2008)

- Dirk Blömker and Wael W. Mohammed
  Amplitude equation for SPDEs with quadratic nonlinearities, Electronic Journal of Probability, Volume 14, Number 88 (2009).

- Dirk Blömker and Wael W. Mohammed
  Amplitude equation for SPDEs with Cubic nonlinearities, Stochastics: An International Journal of Probability and Stochastic Processes, (will appear).

- Dirk Blömker and Wael W. Mohammed
  Modulation Equation for Stochastic Swift-Hohenberg Equation, Preprint, (2011).

- Dirk Blömker, Wael W. Mohammed, Christian Nolde and Franz Wöhrl
  Numerical Study of Amplitude Equations for SPDEs with Degenerate Forcing, Preprint, (2011).