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Smoothness of global positive branches of nonlinear elliptic problems over symmetric domains

Hansjörg Kielhöfer

Institut für Mathematik, Universität Augsburg, Universitätsstrasse 8, W-8900 Augsburg, Federal Republic of Germany

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Introduction

In [6] we proved the following theorem: The unbounded global continuum of positive solutions of

$$\Delta u + \lambda f(u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \lambda \in \mathbb{R},$$
$$u = 0 \quad \text{on } \partial\Omega,$$

over some symmetric domain Ω is a smooth curve (in some appropriate function space $\mathbb{R} \times D$).

To be more precise, we require $f(0) = 0, f'(u) > 0$ for $u \geqq 0$, and the global continuum bifurcates from the smallest eigenvalue of the linearization about the trivial solution. The precise conditions on the domain Ω given in [6] are the following: Either Ω is a rectangle or the boundary $\partial\Omega$ is smooth, Ω is symmetric with respect to the x_1- and x_2-axis, Ω is strongly starshaped with respect to the origin, and $\Omega \cap \{x_i > 0\}$, $i = 1, 2$, are both optimal caps in the sense of [4]. Particular examples are convex domains which are symmetric with respect to two orthogonal axes, but we emphasize that convexity of Ω is not necessary.

Triangles, however, do not fulfil these requirements. In this note we present a proof for the smoothness of the global positive continuum when Ω is an equilateral triangle. The same proof holds also if $\partial\Omega$ is smooth, Ω is symmetric with respect to the three symmetry axes of the equilateral triangle, Ω is strongly starshaped with respect to the intersection of these axes, and if each of the three halves of Ω on one respective side of a symmetry axis is an optimal cap in the sense of [4].

It is remarkable that the proof of the smoothness of positive continua requires the condition $f(0) = 0$ (and the oddness of f) only if Ω is a rectangle or an equilateral triangle. In the cases when Ω has a smooth boundary this requirement is not needed. Therefore the proof applies also to continua which do not bifurcate from a trivial solution branch (though we make no comment on the existence of such continua in this note).

The generalization to a "smoothed" regular pentagon or to domains having the symmetry axes of any regular polygon is obvious. In these cases the boundary $\partial\Omega$ has to be smooth since the embedding of Ω into a lattice $\mathscr{L}$ and using appropriate $\mathscr{L}$-periodic functions (see [6, (1.3)] or (1.5) in this note) is not possible. Thus the rectangles and equilateral triangles play a prominent role.

I Smoothness of the global positive continuum

We assume that

$\Omega \subset \mathbb{R}^2$ is the open equilateral triangle

$\{\mathbf{x} = (x_1, x_2), 0 < x_2 < \sqrt{3}x_1, x_2 < \sqrt{3}(1 - x_1)\}$

of sidelength 1 and with the three vertices $(0, 0)$, $(1, 0)$, $(1/2, \sqrt{3}/2)$. (1.1)

In order to study the boundary value problem

$$\Delta u + \lambda f(u) = 0 \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

with some smooth function (C^2 is enough)

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(-u) = -f(u),$$
$$f'(u) > 0, \quad \text{for } u \geqq 0, \tag{1.3}$$

we suggest the following functional analytic setting: We define

$C^{2,\alpha}_{\mathscr{L}}(\mathbb{R}^2) = \{u \in C^{2,\alpha}(\mathbb{R}^2), u(\mathbf{x} + \mathbf{a}) = u(\mathbf{x}) \text{ for all } \mathbf{a} \in \mathscr{L}\}$,

where $\mathscr{L} = \{\mathbf{x} = \alpha_1\mathbf{k}_1 + \alpha_2\mathbf{k}_2, \alpha_1, \alpha_2 \in \mathbb{Z}, \mathbf{k}_1 = (1, 0), \mathbf{k}_2 = (1/2, \sqrt{3}/2)\}$

is the hexagonal lattice,

$C^{0,\alpha}_{\mathscr{L}}(\mathbb{R}^2) = \{u \in C^{0,\alpha}(\mathbb{R}), \text{same as above}\}$. (1.4)

We then use the Banach spaces

$$\begin{aligned} D = \Big\{u \in C^{2,\alpha}_{\mathscr{L}}(\mathbb{R}^2), u(x_1, -x_2) &= u\left(-\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2, \frac{\sqrt{3}}{2}x_1 + \frac{1}{2}x_2\right) \\ &= u\left(-\frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2, -\frac{\sqrt{3}}{2}x_1 + \frac{1}{2}x_2\right) \\ &= -u(x_1, x_2) \quad \text{for all } \mathbf{x} \in \mathbb{R}^2\Big\}, \end{aligned}$$

$E = \{u \in C^{0,\alpha}_{\mathscr{L}}(\mathbb{R}^2), \text{same as above}\}$,

endowed with the Hölder norms $\| \ \|_{2,\alpha}$ and $\| \ \|_{0,\alpha}$,

respectively. (1.5)

Then the left side of (1.2) defines as mapping

$$G\colon \mathbb{R}\times D \to E,\ G(\lambda, u) = \Delta u + \lambda f(u)\ ,$$

which is continuously Frechet differentiable,

$$D_{(\lambda,u)}G(\lambda, u)(\mu, h) = \mu f(u) + G_u(\lambda, u)h\ ,$$

$$G_u(\lambda, u)h = \Delta h + \lambda f'(u)h\ . \tag{1.6}$$

The choice of the spaces D and E is motivated by the homogeneous Dirichlet boundary conditions which are automatically fulfilled for all $u \in D$. Geometrically the conditions in (1.5) (together with the $\mathscr{L}$-periodicity) mean inversion-reflections ("odd extensions") across the lines which are given by the three sides of the triangle Ω. It is easy to see that $G(\lambda, \cdot)$ maps D into E. (Here we use that f is odd; see (1.3).)

As shown in [5]

$$G_u(\lambda, u)\colon D \to E \text{ is a Fredholm operator of index zero, whence}$$

$$D_{(\lambda,u)}G(\lambda, u)\colon \mathbb{R}\times D \to E \text{ is a Fredholm operator of index one}$$

$$\text{for all } (\lambda, u) \in \mathbb{R}\times D\ . \tag{1.7}$$

Since $f(0) = 0$, we have the *trivial solution* $(\lambda, u) = (\lambda, 0)$ of $G(\lambda, u) = 0$ for all $\lambda \in \mathbb{R}$. If λ_0 denotes the smallest eigenvalue of

$$G_u(\lambda, 0)h = \Delta h + \lambda f'(0)h = 0, \quad h \in D \tag{1.8}$$

then λ_0 is simple and the corresponding eigenfunction h_0 is positive in Ω:

$$\lambda_0 = \frac{16\pi^2}{3} f'(0)^{-1}$$

$$h_0(x_1, x_2) = \sin\frac{4\pi}{\sqrt{3}}x_2 + \sin 2\pi\left(x_1 - \frac{x_2}{\sqrt{3}}\right) + \sin 2\pi\left(1 - x_1 - \frac{x_2}{\sqrt{3}}\right) \tag{1.9}$$

(see [7]). Standard bifurcation theory then yields a global continuum of solutions of $G(\lambda, u) = 0$ emanating from the trivial solution at $(\lambda_0, 0)$ (see [8], e.g.; the required compactness follows from the compactness of $\Delta^{-1}\colon E \to E$, which, in turn, is due to the compact embedding of D into E). As shown in [5] this global continuum is decomposed into a positive and negative unbounded part. We emphasize that former results on the positivity of this global continuum, as given in [8], e.g., did not admit corners on the boundary of Ω. For later reference we define:

$$\Sigma_0^+ \subset \mathbb{R}\times D \text{ is the global unbounded continuum of positive}$$

$$\text{solutions of } G(\lambda, u) = 0 \text{ emanating at } (\lambda_0, 0)\ . \tag{1.10}$$

We show that this continuum is a smooth curve in $\mathbb{R}\times D$.

Remark 1 The restriction to the space D of $\mathscr{L}$-periodic functions does not rule out any positive solution of (1.2) which is smooth in $\bar{\Omega}$: By the results in [4] positive solutions are symmetric with respect to the three symmetry axes of the triangle. Inversion-reflections across the sides of Ω together with an $\mathscr{L}$-periodic extension then yield a function in D.

Lemma 1 *If $v \in N(G_u(\lambda, u))$ for some $(\lambda, u) \in \Sigma_0^+$, then v is symmetric with respect to the three symmetry axes of the triangle Ω.*

Proof. Choose one such symmetry axis and decompose any function over $\mathbb{R}^2$ into its components in $S^{(+)}$ and $S^{(-)}$ which denote the symmetric and anti-symmetric functions with respect to that axis. By [4]:

$$\text{If } (\lambda, u) \in \Sigma_0^+ \text{ then } u \in S^{(+)} \cap D . \tag{1.11}$$

For the sake of convenience we move the triangle Ω in the plane such that the symmetry axis under consideration is the x_2-axis and that its base is on the x_1-axis. Then, in addition

$$\begin{aligned} &G_u(\lambda, u)u_{x_1} = \Delta u_{x_1} + \lambda f'(u)u_{x_1} = 0 \\ &u_{x_1} \in S^{(-)}, \quad u_{x_1} < 0 \text{ in } \bar{\Omega} \cap \{x_1 > 0, x_2 > 0\} , \\ &u_{x_1} = 0 \text{ on } \{x_1 = 0\} \text{ and on } \{x_2 = 0\} , \end{aligned} \tag{1.12}$$

(see [4, Theorem 3.2]). Assume now that $v \in N(G_u(\lambda, u))$ has a nonzero component in $S^{(-)}$. By (1.11)

$$G_u(\lambda, u) \colon S^{(\pm)} \cap D \to S^{(\pm)} \cap E, \text{ respectively} , \tag{1.13}$$

and therefore (denoting this component again by v)

$$\Delta v + \lambda f'(u)v = 0, \quad v \in S^{(-)} \cap D . \tag{1.14}$$

We define the following open component in $\Omega^{(-)} = \Omega \cap \{x_1 > 0\}$:

$$\begin{aligned} &\Omega_+ = \operatorname{comp}\{\mathbf{x} \in \Omega^{(-)}, v(\mathbf{x}) > 0\} \\ &\text{which is a nodal domain of } v . \end{aligned} \tag{1.15}$$

Replacing, if necessary, v by $-v$, the domain Ω_+ is nonempty and $\partial\Omega_+$ is piecewise smooth allowing the application of Green's formula (see [3], e.g.):

$$\begin{aligned} &0 = \int_{\partial\Omega_+} \left(\frac{\partial u_{x_1}}{\partial n} v - u_{x_1} \frac{\partial v}{\partial n} \right) = - \int_{\Gamma_+} u_{x_1} \frac{\partial v}{\partial n}, \text{ where} \\ &\Gamma_+ = \partial\Omega_+ \setminus [\{x_1 = 0\} \cup \{x_2 = 0\}] . \end{aligned} \tag{1.16}$$

Observe simply that $v = 0$ on $\partial\Omega_+$ ($v = 0$ for $x_1 = 0$) and that $u_{x_1} = 0$ on both axes. But the Hopf boundary Lemma (see [4], e.g.) yields

$$\frac{\partial v}{\partial n} < 0 \text{ on } \Gamma_+ \setminus \{\text{corners}\} \tag{1.17}$$

and $u_{x_1} < 0$ on Γ_+ (see (1.12)) contradicts (1.16) since $\Gamma_+ \setminus \{\text{corners}\}$ has a positive boundary measure.

Theorem 1 *Under the assumptions* (1.3) *the global unbounded continuum Σ_0^+ of positive solutions of $G(\lambda, u) = 0$ emanating at $(\lambda_0, 0)$ is a smooth curve of class of C^k if f is of class $C^{k+1} (k \geqq 1)$.*

Proof. Let $(\lambda, u)\in\Sigma_0^+$ and set $u_\alpha(\mathbf{x}) = u(\alpha\mathbf{x})$. Then $\Delta u_\alpha + \alpha^2\lambda f(u_\alpha) = 0$ for $\alpha\in\mathbb{R}$. Differentiation of that equation with respect to α yields:

$$w(\mathbf{x}) = \frac{d}{d\alpha} u_\alpha(\mathbf{x})|_{\alpha=1} = \mathbf{x}\cdot\nabla u(\mathbf{x}) \quad \text{solves}$$

$$\Delta w + \lambda f'(u)w = -2\lambda f(u)\,. \tag{1.18}$$

We assume again that Ω is moved such that the x_2-axis is a symmetry axis and that the base is on the x_1-axis. By (1.2) and the Hopf Boundary Lemma

$$w\in S^{(+)} \text{ and } w<0 \text{ on } \partial\Omega\cap\{x_1>0, x_2>0\}\,,$$
$$w=0 \text{ on } \partial\Omega\cap\{x_2=0\}\,. \tag{1.19}$$

(It is here where we use the fact that more general domains described in Corollary 1 below are strongly starshaped. If they are centered at $\mathbf{0}$ then this means, by definition, that $\mathbf{x}\cdot\mathbf{n}>0$ where $\mathbf{x}\in\partial\Omega$ and $\mathbf{n}$ denotes the outer normal unit vector at $\mathbf{x}$.)

Due to the Fredholm property (1.7) Theorem 1 is proved if we have shown that

$$D_{(\lambda,u)}G(\lambda,u)\colon \mathbb{R}\times D\to E \text{ is surjective}$$
$$\text{for all } (\lambda,u)\in\Sigma_0^+\,. \tag{1.20}$$

(see [9, Chap. 4], e.g.). Again by the Fredholm property of $G_u(\lambda, u)$, (1.20) is true provided

(a) $\dim N(G_u(\lambda,u)) \leqq 1$ and

(b) $\dim N(G_u(\lambda,u)) = 1$ implies $f(u)\notin R(G_u(\lambda,u))$. (1.21)

We show (1.21.b) by contradiction. Assume that there is some $v\in S^{(+)}\cap D$ such that

$$\Delta v + \lambda f'(u)v = 2\lambda f(u)\,. \tag{1.22}$$

(Observe that $f(u)\in S^{(+)}\cap E$ and apply (1.13).) Then, by (1.18, 19),

$$w_0 = w + v\in S^{(+)} \quad \text{solves}$$
$$\Delta w_0 + \lambda f'(u)w_0 = 0 \quad \text{in } \bar\Omega$$
$$w_0 < 0 \quad \text{on } \partial\Omega\cap\{x_1>0, x_2>0\}\,,$$
$$w_0 = 0 \quad \text{on } \partial\Omega\cap\{x_2=0)\,. \tag{1.23}$$

Let $0 \neq v_0\in N(G_u(\lambda,u))$, i.e. by Lemma 1

$$\Delta v_0 + \lambda f'(u)v_0 = 0, \quad v_0\in S^{(+)}\cap D\,. \tag{1.24}$$

Then Green's formula over $\Omega^{(-)}$ yields

$$\int_\Gamma w_0 \frac{\partial v_0}{\partial n} = 0, \quad \text{where } \Gamma = \partial\Omega\cap\{x_1>0, x_2>0\}\,. \tag{1.25}$$

Observe that the symmetry $S^{(+)}$ together with the boundary conditions of v_0 and of w_0 make vanish all other terms. Since $w_0 < 0$ on Γ, (1.25) implies that

$$\frac{\partial v_0}{\partial n} \text{ changes sign on } \Gamma \text{ or a nodal line of } v_0 \text{ meets } \Gamma . \tag{1.26}$$

The symmetry of v_0 insured by Lemma 1 implies, in turn,

$$\text{that a nodal domain of } v_0 \text{ is completely contained in one (symmetric) half of } \Omega . \tag{1.27}$$

Let $S^{(-)}$ be the symmetry class of functions over $\mathbb{R}^2$ defined by that axis which bisects Ω according (1.27). The eigenvalue problem

$$\begin{aligned} \Delta h + \mu f'(u)h = 0 \quad & \text{in } \Omega , \\ h = 0 \quad & \text{on } \partial\Omega , \\ h \in S^{(-)} , & \end{aligned} \tag{1.28}$$

has a smallest simple eigenvalue $\mu_0^{(-)}(u)$ with an eigenfunction $h_0^{(-)}(u)$ whose nodal domains in Ω are precisely the two symmetric halves and the only nodal line in Ω is that symmetry axis between them. This is shown by Courant's minimax principle (see [2, Chap. VI]): To the smallest simple eigenvalue of $\Delta h + \mu f'(u)h = 0$ over the half triangle with homogeneous Dirichlet boundary conditions belongs a positive weak eigenfunction. By repeated inversion-reflections across the symmetry axis and across all sides of each triangle we get an $\tilde{\mathscr{L}}$-periodic function over $\mathbb{R}^2$ where $\tilde{\mathscr{L}} = \{\alpha_1 \tilde{\mathbf{k}}_1 + \alpha_2 \tilde{\mathbf{k}}_2, \alpha_i \in \mathbb{Z}\}$ with $\tilde{\mathbf{k}}_1 = 3\mathbf{k}_1 = (3, 0)$, $\tilde{\mathbf{k}}_2 = 3\mathbf{k}_2 = (3/2, 3\sqrt{3}/2)$. By the symmetry of u (see (1.11)) and by $f'(-u) = f'(u)$ this extended function still solves $\Delta h + \mu_0^{(-)} f'(u)h = 0$ in the weak sense. By *interior* regularity theory this weak eigenfunction $h_0^{(-)}(u)$ is everywhere smooth and solves (1.28).

Using Courant's comparison principle (see [2, Chap. VI]) we conclude that

$$\mu_0^{(-)}(u) \text{ is smaller than } \lambda, \text{ (see (1.24, 27, 28))} . \tag{1.29}$$

Since $\mu_0^{(-)}(u)$ depends continuously on u (as a function from D into $\mathbb{R}$, see [2, Chap. VI]), since λ_0 is smaller than $\mu_0^{(-)}(0)$ (which is the second eigenvalue of (1.8)), and since Σ_0^+ is connected, the Intermediate Value Theorem guarantees the existence of some

$$(\bar{\lambda}, \bar{u}) \in \Sigma_0^+ \text{ such that } \bar{\lambda} = \mu_0^{(-)}(\bar{u}) . \tag{1.30}$$

The corresponding eigenfunction $v_0^{(-)}(u)$ is in $S^{(-)} \cap N(G_u(\bar{\lambda}, \bar{u}))$, contradicting Lemma 1.

We show (1.21.a) also by contradiction. Assume

$$v_1, v_2 \in N(G_u(\lambda, u)) \text{ which are linearly independent} . \tag{1.31}$$

Using again the function w of (1.18, 19), Green's formula over $\Omega^{(-)}$ yields this time

$$\int_{\Gamma} w \frac{\partial v_i}{\partial n} = 2\lambda \int_{\Omega^{(-)}} f(u) v_i, \quad i = 1, 2, \tag{1.32}$$

(for Γ see (1.25), and observe that $v_i \in S^{(+)}$ by Lemma 1). But (1.32) implies the existence of some $v_0 = \alpha_1 v_1 + \alpha_2 v_2 \neq 0$ in $S^{(+)} \cap N(G_u(\lambda, u))$ such that

$$\int_{\Gamma} w \frac{\partial v_0}{\partial n} = 0. \tag{1.33}$$

This leads to a contradiction in the same way as (1.25) did, and Theorem 1 is proved.

The same ideas of the proof yield

Corollary 1 *Let $\Omega \subset \mathbb{R}^2$ be a domain having the following properties*: (i) *$\partial\Omega$ is smooth ($C^{2,\alpha}$ is sufficient)*, (ii) *Ω is symmetric with respect to the three axis of the equilateral triangle*, (iii) *Ω is strongly starshaped with respect to its center (its definition is given after* (1.19)), (iv) *each half of Ω on one respective side of a symmetry axis is an optimal cap in the sense of* [4]. (*The latter means that the reflections of all caps cut off from Ω by parallel lines to the symmetry axis are in Ω.*)

Then any continuum Σ_0^+ of positive solutions of

$$\begin{aligned} \Delta u + \lambda f(u) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.34}$$

with $f'(u) > 0$ for $u \geqq 0$ is a smooth curve of class C^k if f is of class C^{k+1}.

Notice that we do not require that $f(0) = 0$ and that f is odd. The functional analytic setting is the usual one: $D = C^{2,\alpha}(\bar{\Omega}) \cap \{u = 0 \text{ on } \partial\Omega\}$, $E = C^{0,\alpha}(\bar{\Omega})$, and the smoothness of the boundary $\partial\Omega$ guarantees via interior and boundary estimates the Fredholm property (1.7). If in addition $f(0) = 0$, there actually exists a global positive continuum emanating at the smallest eigenvalue of (1.8): the corresponding eigenfunction is positive (see Courant's minimax principle [2] or Krein–Rutman's Theorem in [1], e.g.), and the positivity of the continuum is already shown in [8]. The results of [4] imply that positive solutions are symmetric with respect to all three symmetry axes. The argument for (1.28) is the following: the positive weak eigenfunction over the half domain is in fact smooth since an inversion-reflection across the symmetry axis yields a weak eigenfunction over Ω with smooth boundary. The remaining arguments of the proofs of Lemma 1 and Theorem 2 can be taken without any change.

Finally we remark that Corollary 1 holds as well if Ω has the reflection symmetries of any regular polygon together with the properties (i, iii, iv).

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