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Preprint Nr. 04/2012 — 29. März 2012

Institut für Mathematik, Universitätsstraße, D-86135 Augsburg http://www.math.uni-augsburg.de/

Impressum:

Herausgeber:
Institut für Mathematik
Universität Augsburg
86135 Augsburg
http://www.math.uni-augsburg.de/pages/de/forschung/preprints.shtml

ViSdP:
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Full discretization of stochastic Burgers equation with correlated noise

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December 2, 2011

Abstract

The main purpose of this paper is to investigate the spectral Galerkin method for spatial discretization. We combine it with the method introduced by Kloeden, Jentzen & Winkel in [12] for temporal discretization of stochastic partial differential equations and study pathwise convergence. We consider the case of colored noise, instead of the usual space-time white noise that was used before for the spatial discretization. The rate of convergence in uniform topology is estimated for the stochastic Burgers equation. Numerical examples illustrate the estimated convergence rate.

Keywords: stochastic partial differential equations, colored noise, Galerkin approximation, stochastic Burgers equation.

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1 Introduction

In this article the numerical approximation of nonlinear parabolic stochastic partial differential equations (SPDEs) is considered. Following the ideas of Blömker & Jentzen [2] for the case of space-time white noise, a numerical method for simulating nonlinear SPDEs with additive noise for the case of colored noise is proposed and analyzed. The main novelty in this article is to estimate the spatial and temporal discretization error in the L^{∞}-topology in the case of colored noise. This is different from the usual space-time white noise, that was considered before in [2] for spatial discretization.

We consider as forcing term an infinite dimensional stochastic process expanded in the eigenfunctions of the linear operator A present in the SPDE. We focus on the case where the Brownian motions are not independent. This is due to the fact that the spatial covariance operator of the forcing does not commute with A.

In order to illustrate the main result of this article we consider stochastic Burgers equation with Dirichlet boundary conditions on a bounded domain. To be more precise, let $T > 0$, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let the space-time continuous stochastic process $X : [0,T] \times \Omega \to C([0,1], \mathbb{R})$ be the unique solution of the SPDE

$$dX_t = \left[\tfrac{\partial^2}{\partial x^2} X_t - X_t \cdot \tfrac{\partial}{\partial x} X_t\right] dt + dW_t, \qquad X_t(0) = X_t(1) = 0, \quad X_0 = 0, \quad (1)$$

for $t \in [0,T]$ and $x \in (0,1)$. The noise is given by a cylindrical Wiener process W_t, $t \in [0,T]$ defined later.

The existence and uniqueness of solutions of the stochastic Burgers equation was studied by Da Prato & Gatarek in [3] for colored noise. Da Prato & Zabzcyk in [4] and [5] studied (1) for space-time white noise and Gyöngy & Nualart in [6] studied the equation the whole real line.

Alabert and Gyöngy obtained the spatial discretization of this equation in L^2-topology [1]. Recently, Blömker and Jentzen [2] obtained the spatial discretization error in uniform topology by the spectral Galerkin method for the case of space-time white noise.

The spectral Galerkin method has been extensively studied for stochastic partial differential equations with space-time white noise. See for example [10, 13, 14, 15, 16].

Hausenblas investigated the discretization error of semilinear stochastic evolution equations in L^p-spaces, Banach spaces and quasi linear evolution equations driven by nuclear or space time white noise in [8, 9]. Gyöngy and Shardlow in [18, 7] apply finite differences in order to approximate the mild solution of parabolic SPDEs driven by space-time white noise. Yoo investigates the mild solution of parabolic SPDEs by finite differences in [19].

Our aim here is to extend the result of [2]. First we discuss the case of colored noise not diagonal with respect to the eigenfunctions of the Laplacian. Secondly, using the time discretization that was introduced in [12], we obtain an error estimate for the full space-time discretization.

The reminder of this paper is organized as follows. Section 2 gives the setting and the assumptions. In Section 3 we investigate spatial discretization error, and in Section 4 the temporal error is obtained. Finally, in the last section numerical examples are presented.

2 Setting and assumptions

Fix $T > 0$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and both $(V, \|.\|_V)$ and $(W, \|.\|_W)$ be $\mathbb{R}-$Banach spaces. Moreover, let $P_N : V \to V$, $N \in \mathbb{N}$, be a sequence of bounded linear operators.

Throughout this article the following assumptions will be used.

Assumption 1. Let $S : (0,T] \to L(W,V)$ be a continuous mapping satisfyinga

$$\sup_{0<t\leq T} \left(t^\alpha \|S_t\|_{L(W,V)}\right) < \infty, \quad \sup_{N\in\mathbb{N}} \sup_{0\leq t\leq T} \left(t^\alpha N^\gamma \|S_t - P_N S_t\|_{L(W,V)}\right) < \infty, \quad (2)$$

where $\alpha \in [0,1)$ and $\gamma \in (0,\infty)$ are given constants.

Assumption 2. Let $F : V \to W$ be a locally Lischitz continuous mapping, which satisfies

$$\sup_{\|v\|_V, \|w\|_V \leq r} \frac{\|F(v) - F(w)\|_W}{\|v - w\|_V} < \infty \quad (3)$$

for every $r > 0$.
Assumption 3. Let $O : [0,T] \times \Omega \to V$ be a stochastic process with continuous sample paths and

$$\sup_{N \in \mathbb{N}} \sup_{0 \leq t \leq T} N^{\gamma} \| O_t(\omega) - P_N(O_t(\omega)) \|_V < \infty, \tag{4}$$

for every $\omega \in \Omega$, where $\gamma \in (0, \infty)$ is given in Assumption 1.
Assumption 4. Let $X^N : [0,T] \times \Omega \to V, \quad N \in \mathbb{N}$, be a sequence of stochastic processes with continuous sample paths such that

$$\sup_{M \in \mathbb{N}} \sup_{0 \leq s \leq T} \| X_s^M(\omega) \|_V < \infty \tag{5}$$

and

$$X_t^N(\omega) = \int_0^t P_N S_{t-s} F(X_s^N(\omega)) ds + P_N(O_t(\omega)), \tag{6}$$

for every $t \in [0,T], \omega \in \Omega$ and every $N \in \mathbb{N}$.
Blömker and Jentzen [2] obtained the following Theorem.

Theorem 1. *Let Assumptions 1-4 be fulfilled. Then, there exists a unique stochastic process $X : [0,T] \times \Omega \to V$ with continuous sample paths, which fulfills*

$$X_t(\omega) = \int_0^t S_{t-s} F(X_s(\omega)) ds + O_t(\omega), \tag{7}$$

for every $t \in [0,T]$ and every $\omega \in \Omega$. Moreover, there exists a $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $C : [0,\infty) \to \Omega$ such that

$$\sup_{0 \leq t \leq T} \| X_t(\omega) - X_t^N(\omega) \|_V \leq C(\omega) \cdot N^{-\gamma}, \tag{8}$$

holds for every $N \in \mathbb{N}$ and every $\omega \in \Omega$, where $\gamma \in (0,\infty)$ is given in Assumption 1.

3 Spatial discretization for the case of colored noise

Now we will show that Assumptions 1-4 are satisfied for Burgers equation in the case of colored noise. Therefore from Theorem 1 we can conclude convergence of the Galerkin method for this equation. Most of the results are already proven in [2]. We only state the results needed later in the proofs, and the modifications necessary due to the presence of colored noise.

In the reminder of the paper define $V = C^0([0,1])$, $W = H^{-1}(0,1)$. The mapping $\partial : V \to W$ is given by

$$(\partial v)(\varphi) = (v')(\varphi) := - < v, \varphi' >_{L_2} = - \int_0^1 v(x) \varphi'(x) dx$$

for every $v \in V$ and $\varphi \in H^1(0,1)$, is a bounded linear mapping from V to W.
¿From Lemma 4.6 and 4.8 in [2] we have the following Lemmas.

Lemma 2. *The mapping* $S : (0,T] \to L(H^{-1}(0,1), C^0([0,1]))$ *given by*

$$(S_t(w))(x) = \sum_{n=1}^{N} \Big(2 \cdot e^{-n^2\pi^2 t} \cdot w(sin(n\pi(.))) \cdot sin(n\pi x) \Big)$$

for every $x \in [0,1]$, $w \in H^{-1}(0,1)$ *and every* $t \in (0,T]$, *is well defined and satisfies Assumption 1.*

¿From Assumption 1 we derive

$$\sup_{0<t\le T} \big(t^{\alpha} \|S_t \partial\|_{L(V,V)} \big) < \infty\,, \tag{9}$$

where α was introduced in Assumption 1.
Remark 1 As we can see from Lemma 4.6 in [2], Assumption 1 is satisfied for $\alpha = \frac{3}{4}$, and $\gamma \in [0, \frac{1}{2})$.

Lemma 3. *The mapping* $F : C^0([0,1]) \to H^{-1}(0,1)$, $F(v) = \partial_x(v^2)$ *for every* $v \in C^0([0,1])$ *satisfies Assumption 2.*

In the following we present details on the Q-Wiener process W for the colored noise, in order to prove Assumption 3 later. Here we focus on a d-dimensional setting, while the result needed later is for $d = 1$. Let $\beta^i : [0,T] \times \Omega \to \mathbb{R}, i \in \mathbb{N}^d$, be a family of Brownian motions that are not necessarily independent. They are correlated as given by

$$\mathbb{E}(\beta^k(t)\beta^l(t)) = < Qe_k, e_l > t, \quad k = (k_1, ..., k_d) \in \mathbb{N}^d, \quad t > 0, l = (l_1, ..., l_d) \in \mathbb{N}^d,$$

where for every $k \in \mathbb{N}^d$

$$e_k : [0,1]^d \to \mathbb{R}, \quad e_k(x) = 2^{\frac{d}{2}} \sin(k_1 \pi x_1) ... \sin(k_d \pi x_d), \quad x \in [0,1]^d,$$

are smooth functions. Furthermore, Q is a symmetric non-negative operator, such that

$$< Qe_k, e_l > = \int_0^1 \int_0^1 e_k(x) e_l(y) q(x-y) dy dx, \tag{10}$$

for $k, l \in \mathbb{N}^d$ and some positive definite function q.

Moreover, for every $k \in \mathbb{N}^d$ define the real numbers $\lambda_k = \pi^2(k_1^2 + ... + k_d^2) \in \mathbb{R}$.

Lemma 4. *Assume for one* $\rho > 0$ *and dimension* $d \in \{1,2,3\}$ *that*

$$\sum_{i \in N^d} \sum_{j \in N^d} \|i\|_2^{\rho-1} \|j\|_2^{\rho-1} |\langle Qe_i, e_j \rangle| < \infty.$$

Then there exists a stochastic process $O : [0,T] \times \Omega \to V$, *which satisfies*

$$\begin{aligned} \sup_{0 \le t_1 \le t_2 \le T} \frac{\big\| O_{t_2}(\omega) - O_{t_1}(\omega) \big\|_V}{(t_2 - t_1)^{\theta}} &< \infty, \\ \sup_{N \in \mathbb{N}} \sup_{0 \le t \le T} N^{\gamma} \|O_t(\omega) - P_N(O_t(\omega))\|_V &< \infty, \end{aligned} \tag{11}$$

for every $\omega \in \Omega$, *every* $\theta \in (0, \min\{\frac{1}{2}, \frac{\rho}{2}\})$, *every* $\gamma \in (0, \rho)$. *Furthermore,* O *satisfies*

$$\mathbb{P}\Big[\lim_{N\to\infty} \sup_{0 \le t \le T} \Big\| O_t - \sum_{i \in \{1,...,N\}^d} \Big(-\lambda_i \int_0^t e^{-\lambda_i(t-s)} \beta_s^i ds + \beta_t^i \Big) e_i \Big\|_V = 0 \Big] = 1,$$

$$\sup_{N\in\mathbb{N}}\Big[\Big(\mathbb{E}\Big[\sup_{0\le t\le T}\|O_t-P_NO_t\|_V^p\Big]\Big)^{\frac{1}{p}}N^\gamma\Big]+\sup_{0\le t_1\le t_2\le T}\frac{\Big(\mathbb{E}\Big[\|O_{t_2}-O_{t_1}\|_V^p\Big]\Big)^{\frac{1}{p}}}{(t_2-t_1)^\theta}<\infty$$

for every $p\in[1,\infty)$, and every $\gamma\in(0,\rho)$.

We need some technical Lemmas first, in order to prove this Lemma.

Lemma 5. *For every $t_1,t_2\in[0,T]$ with $t_1\le t_2$, and every $r\in(0,1)$ we have*

$$\begin{aligned}\mathbb{E}\Big(&\int_0^{t_2}e^{-\lambda_i(t_2-s)}d\beta_s^i-\int_0^{t_1}e^{-\lambda_i(t_1-s)}d\beta_s^i\Big)\Big(\int_0^{t_2}e^{-\lambda_j(t_2-s)}d\beta_s^j-\int_0^{t_1}e^{-\lambda_j(t_1-s)}d\beta_s^j\Big)\\&\le 2(\lambda_i+\lambda_j)^{r-1}(t_2-t_1)^r\langle Qe_i,e_j\rangle\end{aligned}\tag{12}$$

for all $i,j\in\mathbb{N}^d$.

Proof. Fix $t_1,t_2\in[0,T]$ with $t_1\le t_2$, and $i,j\in\mathbb{N}^d$. Define $\Delta t=t_2-t_1$ and $\Lambda_{ij}=\lambda_i+\lambda_j$. We obtain

$$\begin{aligned}&\mathbb{E}\Big(\int_0^{t_2}e^{-\lambda_i(t_2-s)}d\beta_s^i-\int_0^{t_1}e^{-\lambda_i(t_1-s)}d\beta_s^i\Big)\cdot\Big(\int_0^{t_2}e^{-\lambda_j(t_2-s)}d\beta_s^j-\int_0^{t_1}e^{-\lambda_j(t_1-s)}d\beta_s^j\Big)\\&=\mathbb{E}\Big(\int_{t_1}^{t_2}e^{-\lambda_i(t_2-s)}d\beta_s^i+(e^{-\lambda_i\Delta t}-1)\int_0^{t_1}e^{-\lambda_i(t_1-s)}d\beta_s^i\Big)\\&\qquad\qquad\times\Big(\int_{t_1}^{t_2}e^{-\lambda_j(t_2-s)}d\beta_s^j+(e^{-\lambda_j\Delta t}-1)\int_0^{t_1}e^{-\lambda_j(t_1-s)}d\beta_s^j\Big)\\&=\int_0^{\Delta t}e^{-\Lambda_{ij}s}\langle Qe_i,e_j\rangle ds\\&\qquad+\Big(e^{-\Lambda_{ij}\Delta t}-e^{-\lambda_i\Delta t}-e^{-\lambda_j\Delta t}+1\Big)\cdot\langle Qe_i,e_j\rangle\cdot\frac{1-e^{-\Lambda_{ij}t_1}}{\Lambda_{ij}}\\&=\Big(1-e^{-\Lambda_{ij}\Delta t}+(e^{-\Lambda_{ij}\Delta t}-e^{-\lambda_i\Delta t}-e^{-\lambda_j\Delta t}+1)(1-e^{-\Lambda_{ij}t_1})\Big)\cdot\frac{\langle Qe_i,e_j\rangle}{\Lambda_{ij}}\\&\le\big(1-e^{-\Lambda_{ij}\Delta t}+(1-e^{-\Lambda_{ij}\Delta t})(1-e^{-\Lambda_{ij}t_1})\big)\cdot\frac{\langle Qe_i,e_j\rangle}{\Lambda_{ij}}\\&\le 2\cdot\frac{1-e^{-\Lambda_{ij}\Delta t}}{\Lambda_{ij}}\cdot\langle Qe_i,e_j\rangle.\end{aligned}$$

Therefore, for every $r\in(0,1)$ we derive

$$\begin{aligned}&\mathbb{E}\left(\int_0^{t_2}e^{-\lambda_i(t_2-s)}d\beta_s^i-\int_0^{t_1}e^{-\lambda_i(t_1-s)}d\beta_s^i\right)\cdot\left(\int_0^{t_2}e^{-\lambda_j(t_2-s)}d\beta_s^j-\int_0^{t_1}e^{-\lambda_j(t_1-s)}d\beta_s^j\right)\\&\le 2\cdot\left(\sup_{x>0}\frac{1}{x}(1-e^{-x})\right)^r\cdot\Lambda_{ij}^{r-1}(\Delta t)^r\cdot|\langle Qe_i,e_j\rangle|\\&=2\cdot\Lambda_{ij}^{r-1}(\Delta t)^r\cdot|\langle Qe_i,e_j\rangle|\,.\end{aligned}$$

□

Lemma 6. *For every $t_1, t_2 \in [0,T]$, with $t_1 \le t_2$, $N \in \mathbb{N}$, $p \in [1,\infty)$ and every $\alpha, \theta \in (0, \frac{1}{2}]$ we have*

$$\Big(\mathbb{E}\Big[\sup_{x\in[0,1]^d} |O_{t_2}^N(x) - O_{t_1}^N(x)|^p\Big]\Big)^{\frac{1}{p}} \le C \sum_{i,j\in I_N} \|i\|_2^{2\theta+2\alpha-1}\|j\|_2^{2\theta+2\alpha-1}|\langle Qe_i, e_j\rangle|(t_2-t_1)^\theta$$

where $C = C(d,p,\alpha,\theta)$ is a constant depending only on d, p, α and θ. The stochastic process $O^N : [0,T]\times\Omega \to C([0,1]^d)$ is given by

$$O_t^N = \sum_{i\in I_N} \int_0^t e^{-\lambda_i(t-s)} d\beta_s^i \cdot e_i \tag{13}$$

for every $t \in [0,T]$ and every $N \in \mathbb{N}$, where $I_N = \{1, ..., N\}^d$.

Proof. Consider first

$$\begin{aligned}&(O_{t_2}^N(x) - O_{t_1}^N(x)) - (O_{t_2}^N(y) - O_{t_1}^N(y))\\ &= \sum_{i\in I_N}\left(\int_0^{t_2} e^{-\lambda_i(t_2-s)}d\beta_s^i - \int_0^{t_1} e^{-\lambda_i(t_1-s)}d\beta_s^i\right).(e_i(x)-e_i(y)),\end{aligned}$$

$\mathbb{P}-$a.s. for every $x, y \in [0,1]^d$. Hence, expanding the square of the series as a double sum and using Lemma 5 we obtain (again with $\Delta t = t_2 - t_1$ and $\Lambda_{ij} = \lambda_i + \lambda_j$)

$$\begin{aligned}&\mathbb{E}|(O_{t_2}^N(x) - O_{t_1}^N(x)) - (O_{t_2}^N(y) - O_{t_1}^N(y))|^2\\ &\le \sum_{i,j\in I_N} \Lambda_{ij}^{2\theta-1}(\Delta t)^{2\theta}|\langle Qe_i,e_j\rangle|\cdot|(e_i(x)-e_i(y))(e_j(x)-e_j(y))|\\ &\le C\sum_{i,j\in I_N} \Lambda_{ij}^{2\theta-1}(\Delta t)^{2\theta}|\langle Qe_i,e_j\rangle|\cdot(\|i\|_2^2\|x-y\|_2^2)^\alpha(\|j\|_2^2\|x-y\|_2^2)^\alpha\\ &\le C(\Delta t)^{2\theta}\|x-y\|_2^{4\alpha}\sum_{i,j\in I_N}(\|i\|_2^2+\|j\|_2^2)^{2\theta-1}\|i\|_2^{2\alpha}\|j\|_2^{2\alpha}|\langle Qe_i,e_j\rangle|\ ,\end{aligned}$$

where we used that e_k is bounded and Lipschitz. Therefore,

$$\begin{aligned}&\mathbb{E}|(O_{t_2}^N(x) - O_{t_1}^N(x)) - (O_{t_2}^N(y) - O_{t_1}^N(y))|^2\\ &\le C(\Delta t)^{2\theta}\|x-y\|_2^{4\alpha}\sum_{i,j\in I_N}\|i\|_2^{2\theta+2\alpha-1}\|j\|_2^{2\theta+2\alpha-1}|\langle Qe_i,e_j\rangle|.\end{aligned} \tag{14}$$

Again from Lemma 5 we derive in a similar way for every $x \in [0,1]^d$

$$\begin{aligned}\mathbb{E}\Big[|O_{t_2}^N(x) - O_{t_1}^N(x)|^2\Big] &\le C\sum_{i,j\in I_N}\Lambda_{ij}^{2\theta-1}(\Delta t)^{2\theta}|\langle Qe_i,e_j\rangle|\\ &\le C\sum_{i,j\in I_N}(\|i\|_2^2+\|j\|_2^2)^{2\theta-1}(\Delta t)^{2\theta}|\langle Qe_i,e_j\rangle|.\end{aligned} \tag{15}$$

The Sobolev embedding of the fractional space $W^{\alpha,p}$ into $C^0([0,1]^d)$ given in Theorem 1 in Section 2.2.4 in [17] yields

$$
\begin{aligned}
&\mathbb{E}\Big[\|O^N_{t_2}-O^N_{t_1}\|^p_{C^0([0,1]^d)}\Big]\\
&\le C\int_{(0,1)^d}\int_{(0,1)^d}\frac{\Big(\mathbb{E}\Big[\big|(O^N_{t_2}(x)-O^N_{t_1}(x))-(O^N_{t_2}(y)-O^N_{t_1}(y))\big|^2\Big]\Big)^{\frac{p}{2}}}{\|x-y\|_2^{d+p\alpha}}dxdy\\
&\qquad\qquad+C\int_{(0,1)^d}\Big(\mathbb{E}\Big[\big|O^N_{t_2}(x)-O^N_{t_1}(x)\big|^2\Big]\Big)^{\frac{p}{2}}dx,
\end{aligned}
$$

where we have used Gaussianity for the p-th moment. In the following, for shorthand notation, all spatial integrals are over $(0,1)^d$.

Therefore, by (14) and (15)

$$
\begin{aligned}
&\mathbb{E}\Big[\|O^N_{t_2}-O^N_{t_1}\|^p_{C^0([0,1]^d)}\Big]\\
\le& C\int\int\frac{\big((\Delta t)^{2\theta}\|x-y\|_2^{4\alpha}\big)^{\frac{p}{2}}}{\|x-y\|_2^{d+p\alpha}}dxdy\Big(\sum_{i,j\in I_N}(\|i\|_2\|j\|_2)^{2\theta+2\alpha-1}|\langle Qe_i,e_j\rangle|\Big)^{\frac{p}{2}}\\
&\quad+C\int(\Delta t)^{p\theta}\Big(\sum_{i,j\in I_N}\|i\|_2^{2\theta-1}\|j\|_2^{2\theta-1}|\langle Qe_i,e_j\rangle|\Big)^{\frac{p}{2}}dx\\
\le& C\Big(1+\int\int\|x-y\|_2^{\alpha p-d}dxdy\Big)\cdot(\Delta t)^{p\theta}\cdot\Big(\sum_{i,j\in I_N}(\|i\|_2\|j\|_2)^{2\theta+2\alpha-1}|\langle Qe_i,e_j\rangle|\Big)^{\frac{p}{2}}.
\end{aligned}
$$

By the fact that

$$\int\int(\|x-y\|_2)^{-\alpha}dxdy\le\frac{(3d)^d}{d-\alpha}$$

for every $\alpha\in(0,d)$, with arbitrary $d\in\mathbb{N}$, we derive

$$\Big(\mathbb{E}\|O^N_{t_2}-O^N_{t_1}\|^p_{C^0([0,1]^d)}\Big)^{\frac{1}{p}}\le C\Big(\sum_{i,j\in I_N}(\|i\|_2\|j\|_2)^{2\theta+2\alpha-1}|\langle Qe_i,e_j\rangle|\Big)^{\frac{1}{2}}(\Delta t)^{\theta}.$$

□

Lemma 7. *For every $N,M\in\mathbb{N}$, $N\ge M$, $p\in[1,\infty)$ and every $\alpha\in(0,\frac{1}{2}]$ we have*

$$\Big(\mathbb{E}\sup_{0\le t\le T}\|O^N_t-O^M_t\|^p_{C^0([0,1]^d)}\Big)^{\frac{1}{p}}\le C\Big(\sum_{i,j\in I_N\setminus I_M}\|i\|_2^{4\alpha-1}\|j\|_2^{4\alpha-1}|\langle Qe_i,e_j\rangle|\Big)^{\frac{1}{2}},$$

where $I_N=\{1,...,N\}^d$, $I_M=\{1,...,M\}^d$ and $C=C(d,p,\alpha,\theta)$ is a constant only depending on d,p,α, and θ.

Proof. Throughout this proof we assume $\alpha\in(0,\frac{1}{2})$ and $p>\frac{1}{\alpha}$. Moreover, $N>M$ is fixed. Define for every $t\in[0,T]$

$$Y^{N,M}_t=\sum_{i\in I_N\setminus I_M}\int_0^t(t-s)^{-\alpha}e^{-\lambda_i(t-s)}d\beta^i_se_i.$$

The celebrated factorization method [4] yields

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\|O_t^N-O_t^M\|^p_{C^0([0,1]^d)}\\
&=\mathbb{E}\sup_{0\le t\le T}\Big\|\frac{\sin(\pi\alpha)}{\pi}\int_0^t(t-s)^{\alpha-1}S_{t-s}Y_s^{N,M}ds\Big\|^p_{C^0([0,1]^d)}\\
&\le \mathbb{E}\sup_{0\le t\le T}\Big\|\int_0^t(t-s)^{\alpha-1}S_{t-s}Y_s^{N,M}ds\Big\|^p_{C^0([0,1]^d)}.
\end{aligned}
$$

Therefore, using Hölder inequality and boundedness of $\|S_t\|_{L(C^0([0,1]^d))}$ yields

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le t\le T}\|O_t^N-O_t^M\|^p_{C^0([0,1]^d)}\\
&\le \sup_{0\le t\le T}\Big(\int_0^t(t-s)^{\frac{p(\alpha-1)}{p-1}}ds\Big)^{p-1}\cdot\mathbb{E}\int_0^T\|Y_s^{N,M}\|^p_{C^0([0,1]^d)}ds\\
&\le C\int_0^T\mathbb{E}\|Y_s^{N,M}\|^p_{C^0([0,1]^d)}ds.
\end{aligned}
$$

Hence,

$$
\Big(\mathbb{E}\sup_{0\le t\le T}\|O_t^N-O_t^M\|^p_{C^0([0,1]^d)}\Big)^{\frac{1}{p}}\le C\sup_{0\le t\le T}(\mathbb{E}\|Y_t^{N,M}\|^p_{C^0([0,1]^d)})^{\frac{1}{p}}. \tag{16}
$$

Again using the embedding of $W^{\alpha,p}$ into C^0

$$
\begin{aligned}
\mathbb{E}\|Y_t^{N,M}\|^p_{C^0([0,1]^d)}\le{}& C\int_{(0,1)^d}\int_{(0,1)^d}\frac{\Big(\mathbb{E}|Y_t^{N,M}(x)-Y_t^{N,M}(y)|^2\Big)^{\frac{p}{2}}}{\|x-y\|_2^{d+p\alpha}}dxdy\\
&+C\int_{(0,1)^d}\Big(\mathbb{E}|Y_t^{N,M}(x)|^2\Big)^{\frac{p}{2}}dx.
\end{aligned}\tag{17}
$$

For the first term on the right side of (17) we proceed completely analogous to Lemma 6, in order to obtain

$$
\begin{aligned}
&\mathbb{E}|Y_t^{N,M}(x)-Y_t^{N,M}(y)|^2\\
&\le C\sum_{i,j\in I_N\setminus I_M}\int_0^\infty s^{-2\alpha}e^{-s}ds\cdot(\lambda_i+\lambda_j)^{2\alpha-1}\cdot|\langle Qe_i,e_j\rangle|\cdot\|i\|_2^{2\alpha}\|j\|_2^{2\alpha}\|x-y\|_2^{4\alpha}.
\end{aligned}
$$

Therefore,

$$
\mathbb{E}|Y_t^{N,M}(x)-Y_t^{N,M}(y)|^2\le C\sum_{i,j\in I_N\setminus I_M}\frac{|\langle Qe_i,e_j\rangle|}{\|i\|_2^{1-4\alpha}\|j\|_2^{1-4\alpha}}\|x-y\|_2^{4\alpha}. \tag{18}
$$

For the second term on the right hand side of (17) we establish

$$
\begin{aligned}
\mathbb{E}|Y_t^{N,M}(x)|^2&\le\sum_{i,j\in I_N\setminus I_M}\int_0^t(t-s)^{-2\alpha}e^{-(\lambda_i+\lambda_j)(t-s)}ds|\langle Qe_i,e_j\rangle||e_i(x)||e_j(x)|\\
&\le C\sum_{i,j\in I_N\setminus I_M}\|i\|_2^{2\alpha-1}\|j\|_2^{2\alpha-1}|\langle Qe_i,e_j\rangle|.
\end{aligned}\tag{19}
$$

Hence using (18) and (19) we obtain from (17)

$$\sup_{0\le t\le T}(\mathbb{E}\|Y_t^{N,M}\|^p_{C^0([0,1]^d)})^{\frac{1}{p}} \le C\Big(\sum_{i,j\in I_N\setminus I_M}\|i\|_2^{4\alpha-1}\|j\|_2^{4\alpha-1}|\langle Qe_i,e_j\rangle|\Big)^{\frac{1}{2}}. \quad (20)$$

Finally, (16) and (20) yield

$$\Big(\mathbb{E}\sup_{0\le t\le T}\|O_t^N(x)-O_t^M(x)\|^p_{C^0([0,1]^d)}\Big)^{\frac{1}{p}} \le C\Big(\sum_{i,j\in I_N\setminus I_M}\|i\|_2^{4\alpha-1}\|j\|_2^{4\alpha-1}\langle Qe_i,e_j\rangle\Big)^{\frac{1}{2}}.$$

□

Now we are ready to present the remaining parts of the proof of Lemma 4.

Proof. **(Proof of Lemma 4)**. ¿From Lemma 7 we obtain

$$\begin{aligned}\Big(\mathbb{E}\Big[\sup_{0\le t\le T}\|O_t^N-O_t^M\|^p_{C^0([0,1]^d)}\Big]\Big)^{\frac{1}{p}} &\le C\Big(\sum_{i,j\in\mathbb{N}^d\setminus I_M}\|i\|_2^{4\alpha-1}\|j\|_2^{4\alpha-1}|\langle Qe_i,e_j\rangle|\Big)^{\frac{1}{2}}\\ &\le CM^{4\alpha-\rho}\Big(\sum_{i,j\in\mathbb{N}^d}\|i\|_2^{\rho-1}\|j\|_2^{\rho-1}|\langle Qe_i,e_j\rangle|\Big)^{\frac{1}{2}}\end{aligned}$$

for every $N,M\in\mathbb{N}$ with $N\ge M$, $p\in[1,\infty)$, and $\alpha\in(0,\min\{\frac{1}{2},\frac{\rho}{4}\})$. The processes O^N form a Cauchy sequences in

$$V_p := L^p((\Omega,\mathcal{F},\mathbb{P}),(C^0([0,T]\times[0,1]^d))).$$

Hence, there exists a stochastic process $\tilde{O}:[0,T]\times\Omega\to C^0([0,1]^d)$ with $\tilde{O}\in V_p$ and

$$\Big(\mathbb{E}\sup_{0\le t\le T}\|\tilde{O}_t-O_t^N\|^p_{C^0([0,1]^d)}\Big)^{\frac{1}{p}} \le CN^{4\alpha-\rho}\Big(\sum_{i,j\in\mathbb{N}^d}\|i\|_2^{\rho-1}\|j\|_2^{\rho-1}|\langle Qe_i,e_j\rangle|\Big)^{\frac{1}{2}}$$

for every $N\in\mathbb{N}$, $p\in[1,\infty)$, and $\alpha\in(0,\min\{\frac{1}{2},\frac{\rho}{4}\})$.
Therefore,

$$\sup_{N\in\mathbb{N}}\Big\{N^\gamma\Big(\mathbb{E}\sup_{0\le t\le T}\|\tilde{O}_t-O_t^N\|^p_{C^0([0,1]^d)}\Big)^{\frac{1}{p}}\Big\}<\infty$$

for every $\gamma\in(0,\rho)$ and every $p\in[1,\infty)$. This yields (Lemma 1 in [11])

$$\mathbb{P}\Big[\sup_{N\in\mathbb{N}}\sup_{0\le t\le T}\Big\{N^\gamma\|\tilde{O}_t-O_t^N\|_{C^0([0,1]^d)}\Big\}<\infty\Big]=1.$$

In particular,

$$\mathbb{P}\left[\lim_{N\to\infty}\sup_{0\le t\le T}\|\tilde{O}_t-O_t^N\|_{C^0([0,1]^d)}=0\right]=1$$

and

$$\mathbb{P}\left[\sup_{N\in\mathbb{N}}\sup_{0\le t\le T}\Big\{N^\gamma\|\tilde{O}_t-P_N\tilde{O}_t\|_{C^0([0,1]^d)}\Big\}<\infty\right]=1.$$

¿From Lemma 6 we derive

$$\Big(\mathbb{E}\big\|O^N_{t_2}-O^N_{t_1}\big\|^p_{C^0([0,1]^d)}\Big)^{\frac{1}{p}}$$
$$\leq C\Big(\sum_{i,j\in I_N}(\|i\|_2\|j\|_2)^{2\theta+2(\frac{\rho}{2}-\theta)-1}\,|\langle Qe_i,e_j\rangle|\Big)^{\frac{1}{2}}|t_2-t_1|^\theta$$
$$\leq C\Big(\sum_{i,j\in I_N}\|i\|_2^{\rho-1}\|j\|_2^{\rho-1}|\langle Qe_i,e_j\rangle|\Big)^{\frac{1}{2}}|t_2-t_1|^\theta$$

for every $t_1,t_2\in[0,T]$, $N\in\mathbb{N}$, and $\theta\in(0,\frac{\rho}{2})$. Provided $\theta\leq\frac{1}{2}$ this furnishes

$$\Big(\mathbb{E}\big\|\tilde{O}_{t_2}-\tilde{O}_{t_1}\big\|^p_{C^0([0,1]^d)}\Big)^{\frac{1}{p}}\leq C\Big(\sum_{i,j\in I_N}\|i\|_2^{\rho-1}\|j\|_2^{\rho-1}|\langle Qe_i,e_j\rangle|\Big)^{\frac{1}{2}}|t_2-t_1|^\theta.$$

Hence, for every $\theta\in(0,\min\{\frac{1}{2},\frac{\rho}{2}\})$

$$\mathbb{P}\Big[\sup_{0\leq t_1,t_2\leq T}\frac{\big\|\tilde{O}_{t_2}-\tilde{O}_{t_1}\big\|_{C^0([0,1]^d)}}{|t_2-t_1|^\theta}<\infty\Big]=1.$$

Therefore,

$$\mathbb{P}\Big[\forall\theta\in(0,\min\{\tfrac{1}{2},\tfrac{\rho}{2}\}):\ \sup_{0\leq t_1,t_2\leq T}\frac{\big\|\tilde{O}_{t_2}-\tilde{O}_{t_1}\big\|_{C^0([0,1]^d)}}{|t_2-t_1|^\theta}<\infty\Big]=1.$$

In conclusion, this shows the existence of a process $O:[0,T]\times\Omega\to C^0([0,1]^d)$, which satisfies

$$\sup_{0\leq t_1,t_2\leq T}\frac{\big\|O_{t_2}(\omega)-O_{t_1}(\omega)\big\|_{C^0([0,1]^d)}}{|t_2-t_1|^\theta}<\infty,$$

and

$$\sup_{N\in\mathbb{N}}\sup_{0\leq t\leq T}\Big(N^\gamma\big\|O_t(\omega)-P_NO_t(\omega)\big\|_{C^0([0,1]^d)}\Big)<\infty$$

for every $\omega\in\Omega$, $\theta\in(0,\min\{\frac{1}{2},\frac{\rho}{2}\})$, and $\gamma\in(0,\rho)$. Moreover, O is indistinguishable from $\tilde{O}$, i.e.,

$$\mathbb{P}\Big[\forall t\in[0,T]:\ O_t=\tilde{O}_t\Big]=1.$$

□

Summarizing our results, we can state the following Lemma:

Lemma 8. *Assume* $\rho>0, d\in\{1,2,3\}$ *and*

$$\sum_{i,j\in N^d}\|i\|_2^{\rho-1}\|j\|_2^{\rho-1}|\langle Qe_i,e_j\rangle|<\infty.$$

Furthermore, suppose that $\xi:\Omega\to V$ *is* $\mathcal{F}/V$*-measurable with*

$$\sup_{N\in\mathbb{N}}\big(N^\rho\big\|\xi(\omega)-P_N(\xi(\omega))\big\|_V\big)<\infty$$

for every $\omega \in \Omega$. Then there exists a stochastic process $O : [0,T] \times \Omega \to V$ with continuous sample paths, satisfying

$$\mathbb{P}\left[\lim_{N\to\infty} \sup_{0<t<T} \left\| O_t - S_t\xi - \sum_{i\in I_N} \Big(-\lambda_i \int_0^t e^{-\lambda_i(t-s)} \beta_s^i ds + \beta_t^i \Big) e_i \right\|_V = 0 \right] = 1$$

and

$$\sup_{N\in\mathbb{N}} \sup_{0\leq t\leq T} \left\{ N^\gamma \left\| O_t(\omega) - P_N(O_t(\omega)) \right\|_V \right\} < \infty$$

for every $\omega \in \Omega$ and $\gamma \in (0,\rho)$.

In particular O satisfies Assumptions 3 for every $\gamma \in (0,\rho)$.

Note that the process O in the previous Lemma 8 is the solution of the following linear SPDE

$$dO_t = \Delta O_t dt + dW_t, \quad O_t|_{\partial(0,1)^d} = 0, \quad O_0 = \xi,$$

for $t \in [0,T]$, where W is a Q-Wiener process.

Lemma 9. *Let $V = C^0([0,1])$, $W = H^{-1}((0,1))$ and $S : (0,T] \to L(W,V)$, and $F : V \to W$ be given by Lemmas 2, 3. Let $O : [0,T]\times\Omega \to V$ be a stochastic process with continuous sample paths with*

$$\sup_{N\in\mathbb{N}} \sup_{0\leq t\leq T} \|P_N(O_t(\omega))\|_V < \infty$$

for every $\omega \in \Omega$. Then Assumption 4 is fulfilled.

Proof. The proof is exactly the same as the one of Lemma 4.9 in [2]. □

4 Time discretization

For time discretization of the finite dimensional SDEs (6) we consider the method introduced by Jentzen, Kloeden and Winkel in [12]. Consider the discretization scheme for the Burgers equation, i.e., $F(u) = \partial_x u^2$ in one dimension. This is for simplicity of presentation only, as we need to bound various terms depending on X^N and $F(X^N)$.

Through this section assume $\rho > 0$, is such that

$$\sum_{i,j\in N^d} \|i\|_2^{\rho-1} \|j\|_2^{\rho-1} |\langle Qe_i, e_j\rangle| < \infty. \tag{21}$$

Moreover assume $\theta \in (0, \min\{\frac{1}{2}, \frac{\rho}{2}\})$. For the time discretization we define the mapping $Y_m^{N,M} : \Omega \to V$ for $m \in \{1, ..., M\}$ by

$$Y_{m+1}^{N,M}(\omega) = S_{\Delta t}\Big(Y_m^{N,M}(\omega) + \Delta t (P_N F)(Y_m^{N,M}(\omega))\Big) + P_N\Big(O_{(m+1)\Delta t}(\omega) - S_{\Delta t} O_{m\Delta t}(\omega)\Big). \tag{22}$$

The purpose of this section is to consider the discretization error in time

$$\|X_{m\Delta t}^N(\omega) - Y_m^{N,M}(\omega)\|_V,$$

where

$$X^N_{m\Delta t}(\omega) = \int_0^{m\Delta t} P_N S_{m\Delta t - s} F(X^N_s(\omega))ds + O^N_{m\Delta t}(\omega),$$

is the solution of the spatial discretization, which is evaluated at the grid points.

Recall that as we proved in the last section Assumptions 1-4 are satisfied for the stochastic Burgers equation in one dimension.

Lemma 10. *Let $X^N : [0,T] \times \Omega \to V$ be the unique adapted stochastic process with continuous sample paths, defined in Assumption 4. Assume that $O^N : [0,T] \times \Omega \to C^0([0,1]^d)$ is the stochastic process defined in (13). Then we obtain*

$$\left\| (X^N_{t_2}(\omega) - O^N_{t_2}(\omega)) - (X^N_{t_1}(\omega) - O^N_{t_1}(\omega)) \right\|_V \leq C(\omega)(t_2 - t_1)^{\frac{1}{4}}$$

for every $\omega \in \Omega$ and all $t_1, t_2 \in [0,T]$, with $t_1 < t_2$ where C is a finite random variable $C : \Omega \to [0,\infty)$.

Proof. For every $0 \leq t_1 \leq t_2 \leq T$ we have

$$\begin{aligned}
&\left\| X^N_{t_2}(\omega) - O^N_{t_2}(\omega) - (X^N_{t_1}(\omega) - O^N_{t_1}(\omega)) \right\|_V \\
&= \left\| \int_0^{t_2} P_N S_{t_2 - s} F(X^N_s(\omega))ds - \int_0^{t_1} P_N S_{t_1 - s} F(X^N_s(\omega))ds \right\|_V \\
&= \left\| \int_{t_1}^{t_2} P_N S_{t_2 - s} F(X^N_s(\omega))ds + \int_0^{t_1} (S_{t_2 - s} - S_{t_1 - s}) P_N F(X^N_s(\omega))ds \right\|_V \\
&\leq \int_{t_1}^{t_2} \|P_N S_{t_2 - s} \partial\|_{L(V,V)} \|(X^N_s(\omega))^2\|_V ds + \left\| \int_0^{t_1} S_{t_1 - s}(S_{t_2 - t_1} - I) P_N F(X^N_s(\omega))ds \right\|_V .
\end{aligned}$$

¿From (9) and using the fact that S_t is the semigroup generated by Laplacian operator, Δ, we conclude

$$\begin{aligned}
&\|X^N_{t_2}(\omega) - O^N_{t_2}(\omega) - (X^N_{t_1}(\omega) - O^N_{t_1}(\omega))\|_V \\
&\leq C_1(\omega) \int_{t_1}^{t_2} (t_2 - s)^{-\frac{3}{4}} ds + \int_0^{t_1} \|P_N S_{t_1 - s} \Delta^{\frac{1}{4}}\|_{L(W,V)} \|(S_{t_2 - t_1} - I)\Delta^{-\frac{1}{4}}\|_{L(W,V)} \|F(X^N_s(\omega))\|_W ds \\
&\leq 4C_1(\omega)(t_2 - t_1)^{\frac{1}{4}} + \int_0^{t_1} (t_1 - s)^{-\frac{1}{4}} ds (t_2 - t_1)^{\frac{1}{4}} \|F(X^N_s(\omega))\|_W \\
&\leq 4C_1(\omega)(t_2 - t_1)^{\frac{1}{4}} + C_2(\omega)(t_2 - t_1)^{\frac{1}{4}} T^{\frac{3}{4}} \\
&\leq C(\omega)(t_2 - t_1)^{\frac{1}{4}} .
\end{aligned}$$

where $C_1(\omega) = \sup_{M \in \mathbb{N}} \sup_{0 \leq s \leq T} \|X^M_s(\omega)\|^2_V$, $C_2(\omega) = \sup_{M \in \mathbb{N}} \sup_{0 \leq s \leq T} \|F(X^M_s(\omega))\|_W$ are finite due to Assumptions 4 and 2, and therefore C is an almost surely finite random variable $C : \Omega \to [0,\infty)$. □

Before we begin with the first part of the error, we define

$$\begin{aligned}
R(\omega) := &\sup_{N \in \mathbb{N}} \sup_{0 \leq s \leq T} \|F(X^N_s(\omega))\|_W + \sup_{N \in \mathbb{N}} \sup_{0 \leq s \leq T} \|X^N_s(\omega)\|_V \\
&+ \sup_{0 \leq t_1, t_2 \leq T} \|O_{t_2}(\omega) - O_{t_1}(\omega)\|_V |t_2 - t_1|^{-\theta} \\
&+ \sup_{N \in \mathbb{N}} \sup_{0 \leq t_1, t_2 \leq T} \|X^N_{t_2}(\omega) - O^N_{t_2}(\omega) - (X^N_{t_1}(\omega) - O^N_{t_1}(\omega))\|_V |t_2 - t_1|^{-\frac{1}{4}},
\end{aligned}$$

where from Assumption 4, Lemma 4 and Lemma 10, $R : \Omega \to \mathbb{R}$ is a finite random variable.

The main result of this section is stated below.

Theorem 11. *For $m \in \{0, 1, ..., M\}$ and every $M, N \in \mathbb{N}$, there exists a finite random variable $C : \Omega \to [0, \infty)$ such that*

$$\|X^N_{m\Delta t}(\omega) - Y^{N,M}_m(\omega)\|_V \leq C(\omega)(\Delta t)^{\min(\frac{1}{4},\theta)},$$

where $X^N : [0, T] \times \Omega \to V$ is the unique adapted stochastic process with continuous sample paths, defined in Assumption 4, and $Y^{N,M}_m : \Omega \to V$, for $m \in \{0, 1, ..., M\}$, and $N, M \in \mathbb{N}$, is given in (22).

Proof. For the proof it is sufficient to prove the result for sufficiently small $|t_2 - t_1|$. Due to (6) we have

$$\begin{aligned} X^N_{m\Delta t}(\omega) &= \int_0^{m\Delta t} P_N S_{m\Delta t - s} F(X^N_s(\omega)) ds + O^N_{m\Delta t}(\omega) \\ &= \sum_{k=0}^{m-1} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - s} F(X^N_s(\omega)) ds + O^N_{m\Delta t}(\omega), \end{aligned} \tag{23}$$

for every $m \in \{0, 1, ..., M\}$, and every $M \in \mathbb{N}$.

The mapping $Y^N_m : \Omega \to V,\ m = 1, 2, ..., M$ is defined by

$$Y^N_m(\omega) = \sum_{k=0}^{m-1} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - k\Delta t} F(X^N_{k\Delta t}(\omega)) ds + O^N_{m\Delta t}(\omega). \tag{24}$$

Our aim is to bound $\|X^N_{m\Delta t}(\omega) - Y^{N,M}_m(\omega)\|_V$. Therefore, we first estimate the difference of the true solution to Y^N_m

$$\|X^N_{m\Delta t}(\omega) - Y^N_m(\omega)\|_V \tag{25}$$

for every $m \in \{0, 1, ..., M\}$ and then the difference between Y^N_m and the full discretization in time

$$\|Y^N_m(\omega) - Y^{N,M}_m(\omega)\|_V. \tag{26}$$

For the first error in (25) we have

$$\begin{aligned} & X^N_{m\Delta t}(\omega) - Y^N_m(\omega) \\ &= \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - s} F(X^N_s(\omega)) ds \\ &\quad - \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - k\Delta t} F(X^N_{k\Delta t}(\omega)) ds \\ &\quad + \int_{(m-1)\Delta t}^{m\Delta t} P_N S_{m\Delta t - s} F(X^N_s(\omega)) ds - \int_{(m-1)\Delta t}^{m\Delta t} P_N S_{\Delta t} F(X^N_{k\Delta t}(\omega)) ds. \end{aligned} \tag{27}$$

Let us now bound the last two integrals in (27). For the first one, we derive

$$\begin{aligned}
&\Big\| \int_{(m-1)\Delta t}^{m\Delta t} P_N S_{m\Delta t-s} F(X_s^N(\omega)) ds \Big\|_V \\
&\qquad = \Big\| \int_{(m-1)\Delta t}^{m\Delta t} P_N S_{m\Delta t-s} \partial (X_s^N(\omega))^2 ds \Big\|_V \\
&\qquad \leq \int_{(m-1)\Delta t}^{m\Delta t} \| P_N S_{m\Delta t-s} \partial \|_{L(V,V)} \cdot \| X_s^N(\omega)\|_V^2 ds \\
&\qquad \leq \sup_{0\leq s\leq t} \| X_s^N(\omega)\|_V^2 \int_{(m-1)\Delta t}^{m\Delta t} (m\Delta t - s)^{-\frac{3}{4}} ds \\
&\qquad \leq R^2(\omega)(\Delta t)^{\frac{1}{4}} \ .
\end{aligned}$$

For the second one we get

$$\begin{aligned}
\Big\| \int_{(m-1)\Delta t}^{m\Delta t} P_N S_{\Delta t} F(X_{k\Delta t}^N(\omega)) ds \Big\|_V &= \Big\| \int_{(m-1)\Delta t}^{m\Delta t} P_N S_{\Delta t} \partial (X_{k\Delta t}^N(\omega))^2 ds \Big\|_V \\
&\leq \int_{(m-1)\Delta t}^{m\Delta t} \| P_N S_{\Delta t} \partial \|_{L(V,V)} \cdot \| X_{k\Delta t}^N(\omega)\|_V^2 ds \\
&\leq \sup_{0\leq s\leq t} \| X_s^N(\omega)\|_V^2 \int_{(m-1)\Delta t}^{m\Delta t} (\Delta t)^{-\frac{3}{4}} ds \\
&\leq R^2(\omega)(\Delta t)^{\frac{1}{4}}.
\end{aligned}$$

Therefore, we can conclude

$$\begin{aligned}
&\| X_{m\Delta t}^N(\omega) - Y_m^N(\omega)\|_V \\
&\leq \Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t-s} (F(X_s^N(\omega)) - F(X_{k\Delta t}^N(\omega))) ds \Big\|_V \\
&+ \Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} (P_N S_{m\Delta t-s} - P_N S_{(m\Delta t - k\Delta t)}) F(X_{k\Delta t}^N(\omega)) ds \Big\|_V \\
&+ R^2(\omega)(\Delta t)^{\frac{1}{4}}.
\end{aligned}$$

Thus inserting the nonlinearity with the Ornstein-Uhlenbeck process in the first term yields for every $m \in \{0, 1, ..., M\}$,

$$\begin{aligned}
&\Big\| X_{m\Delta t}^N(\omega) - Y_m^N(\omega) \Big\|_V \\
&\leq \Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t-s} \Big(F(X_s^N(\omega)) - F\big(X_{k\Delta t}^N(\omega) + O_s^N(\omega) - O_{k\Delta t}^N(\omega)\big)\Big) ds \Big\|_V \\
&+ \Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t-s} \Big(F(X_{k\Delta t}^N(\omega) + O_s^N(\omega) - O_{k\Delta t}^N(\omega)) - F(X_{k\Delta t}^N(\omega))\Big) ds \Big\|_V \\
&+ \Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} \Big(P_N S_{m\Delta t-s} - P_N S_{m\Delta t - k\Delta t}\Big) F(X_{k\Delta t}^N(\omega)) ds \Big\|_V \\
&+ R^2(\omega)(\Delta t)^{\frac{1}{4}}.
\end{aligned} \tag{28}$$

For the first term in (28) by using Lemma 10 together with $\|P_N S_{t-s}\partial u\|_V \leq C(t-s)^{-\frac{3}{4}}\|u\|_V$ we conclude

$$\Big\|\sum_{k=0}^{m-2}\int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t-s}\Big(F(X_s^N(\omega)) - F\big(X_{k\Delta t}^N(\omega) + O_s^N(\omega) - O_{k\Delta t}^N(\omega)\big)\Big)ds\Big\|_V$$
$$\leq \sum_{k=0}^{m-2}\int_{k\Delta t}^{(k+1)\Delta t}(m\Delta t - s)^{-\frac{3}{4}}\Big\|X_s^N(\omega) - (X_{k\Delta t}^N(\omega) + O_s^N(\omega) - O_{k\Delta t}^N(\omega))\Big\|_V$$
$$\cdot\Big\|X_s^N(\omega) + (X_{k\Delta t}^N(\omega) + O_s^N(\omega) - O_{k\Delta t}^N(\omega))\Big\|_V ds$$
$$\leq R(\omega)\sum_{k=0}^{m-2}\int_{k\Delta t}^{(k+1)\Delta t}(m\Delta t - s)^{-\frac{3}{4}}(s-k\Delta t)^{\frac{1}{4}}(2R(\omega) + R(\omega)(s-k\Delta t)^{\theta})ds$$
$$\leq 2C(R(\omega),T)(\Delta t)^{\frac{1}{4}}, \tag{29}$$

where the constant depends on R and T.

For the second term in (28) we derive

$$\Big\|\sum_{k=0}^{m-2}\int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t-s}\Big(F\big(X_{k\Delta t}^N(\omega) + O_s^N(\omega) - O_{k\Delta t}^N(\omega)\big) - F(X_{k\Delta t}^N(\omega))\Big)ds\Big\|_V$$
$$\leq 2\sum_{k=0}^{m-2}\int_{k\Delta t}^{(k+1)\Delta t}\Big\|P_N S_{m\Delta t-s}\partial\Big(X_{k\Delta t}^N(\omega)\cdot(O_s^N(\omega) - O_{k\Delta t}^N(\omega))\Big)\Big\|_V ds$$
$$+ \sum_{k=0}^{m-2}\int_{k\Delta t}^{(k+1)\Delta t}\Big\|P_N S_{m\Delta t-s}\partial\Big((O_s^N(\omega) - O_{k\Delta t}^N(\omega))^2\Big)\Big\|_V ds$$
$$\leq 2\sum_{k=0}^{m-2}\int_{k\Delta t}^{(k+1)\Delta t}\|P_N S_{m\Delta t-s}\partial\|_{L(V,V)}\|X_{k\Delta t}^N(\omega)\|_V\|(O_s^N(\omega) - O_{k\Delta t}^N(\omega))\|_V ds$$
$$+ \sum_{k=0}^{m-2}\int_{k\Delta t}^{(k+1)\Delta t}\|P_N S_{m\Delta t-s}\partial\|_{L(V,V)}\cdot\|(O_s^N(\omega) - O_{k\Delta t}^N(\omega))^2\|_V ds$$
$$\leq 2R^2(\omega)\sum_{k=0}^{m-2}\int_{k\Delta t}^{(k+1)\Delta t}(m\Delta t - (k+1)\Delta t)^{-\frac{3}{4}}.(s-k\Delta t)^{\theta}ds$$
$$+ R^2(\omega)\sum_{k=0}^{m-2}\int_{k\Delta t}^{(k+1)\Delta t}(m\Delta t - (k+1)\Delta t)^{-\frac{3}{4}}(s-k\Delta t)^{2\theta}ds$$
$$\leq C(R(\omega),\theta)(\Delta t)^{\theta},$$

where the constant depends on R and θ.

Finally, for the third term in (28) again by using this fact that S_t is the semigroup generated by Laplacian, we have

$$\begin{aligned}
&\Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} (P_N S_{m\Delta t - s} - P_N S_{m\Delta t - k\Delta t}) F(X^N_{k\Delta t}(\omega)) ds \Big\|_V \\
&\le \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} \Big\| P_N S_{m\Delta t - k\Delta t} (S_{k\Delta t - s} - I) F(X^N_{k\Delta t}(\omega)) \Big\|_V ds \\
&\le \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} (m\Delta t - k\Delta t)^{-1} (k\Delta t - s) \| F(X^N_{k\Delta t}(\omega)) \|_W ds \\
&\le C(R(\omega), T)\Delta t,
\end{aligned} \tag{30}$$

where we used $\|P_N \Delta S_t\|_{L(W,V)} \le Ct^{-1}$, together with $\|\Delta^{-1}(S_t - I)\|_{L(W,V)} \le t$. Hence from (29) and (30) we derive

$$\|X^N_{m\Delta t}(\omega) - Y^N_m(\omega)\|_V \le C(R(\omega), R^2(\omega), \theta, T)(\Delta t)^{\min\{\frac{1}{4}, \theta\}}. \tag{31}$$

Let us now turn to the second error term in (26). Note that $Y^{N,M}_m : \Omega \to V$ satisfies

$$Y^{N,M}_m(\omega) = \sum_{k=0}^{m-1} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - k\Delta t} F(Y^{N,M}_k(\omega)) ds + P_N O_{m\Delta t}(\omega). \tag{32}$$

Thus by using $\|P_N S_t \partial\|_{L(V,V)} \le Ct^{-\frac{3}{4}}$, we can estimate

$$\begin{aligned}
&\|Y^N_m - Y^{N,M}_m\|_V = \Big\| \sum_{k=0}^{m-1} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - k\Delta t} (F(X^N_{k\Delta t}) - F(Y^{N,M}_k)) \Big\|_V \\
&\le \sum_{k=0}^{m-1} \int_{k\Delta t}^{(k+1)\Delta t} (m\Delta t - k\Delta t)^{-\frac{3}{4}} \|(X^N_{k\Delta t} - Y^{N,M}_k)^2 + 2X^N_{k\Delta t}(X^N_{k\Delta t} - Y^{N,M}_k)\|_V ds \\
&\le \sum_{k=0}^{m-1} \Delta t (m\Delta t - k\Delta t)^{-\frac{3}{4}} \Big(\|X^N_{k\Delta t} - Y^{N,M}_k\|_V^2 + 2R(\omega)\|X^N_{k\Delta t} - Y^{N,M}_k\|_V \Big).
\end{aligned} \tag{33}$$

Combining (31) with (33), we have

$$\begin{aligned}
\|X^N_{m\Delta t}(\omega) - Y^{N,M}_m(\omega)\|_V &\le C\Big(R(\omega), \theta, T\Big)(\Delta t)^{\min\{\frac{1}{4}, \theta\}} \\
&+ \sum_{k=0}^{m-1} \|X^N_{k\Delta t}(\omega) - Y^{N,M}_k(\omega)\|_V^2 + 2R(\omega) \sum_{k=0}^{m-1} \|X^N_{k\Delta t}(\omega) - Y^{N,M}_k(\omega)\|_V.
\end{aligned} \tag{34}$$

If we assume that for some $\delta > 0$ fixed later

$$\sup_{0 \le k \le M} \|X^N_{k\Delta t}(\omega) - Y^{N,M}_k(\omega)\|_V \le \delta, \tag{35}$$

then

$$\begin{aligned}
\|X^N_{m\Delta t}(\omega) - Y^{N,M}_m(\omega)\|_V &\le C\Big(R(\omega), \theta, T\Big)(\Delta t)^{\min\{\frac{1}{4}, \theta\}} \\
&+ (\delta + 2R(\omega)) \sum_{k=0}^{m-1} \|X^N_{k\Delta t}(\omega) - Y^{N,M}_k(\omega)\|_V.
\end{aligned} \tag{36}$$

Then by the discrete Gronwalls Lemma we can conclude

$$\|X^N_{m\Delta t}(\omega) - Y^{N,M}_m(\omega)\|_V \leq e^{(m-1)(\delta+2R(\omega))} C\Big(R(\omega), \theta, T\Big)(\Delta t)^{\min\{\frac{1}{4},\theta\}}.$$

In order to verify (35) we need

$$e^{(m-1)(\delta+2R)} C\Big(R(\omega), \theta, T\Big)(\Delta t)^{\min\{\frac{1}{4},\theta\}} \leq \delta,$$

which is true for any $\delta > 0$ provided Δt is sufficiently small. This finishes the proof of the time discretization. □

¿From Theorem 1 for the spatial discretization error we verified in Section 3

$$\|X_{m\Delta t}(\omega) - X^N_{m\Delta t}(\omega)\|_V \leq C(\omega) \cdot N^{-\gamma}, \tag{37}$$

and from Theorem 11 for the temporal discretization error we just established

$$\|X^N_{m\Delta t}(\omega) - Y^{N,M}_m(\omega)\|_V \leq C\Big(R(\omega), \theta, T\Big)(\Delta t)^{\min\{\frac{1}{4},\theta\}}.$$

Therefore we have proved the following Theorem for the stochastic Burgers equation.

Theorem 12. *Assume $\rho > 0$ such that*

$$\sum_{i,j \in N} \|i\|_2^{\rho-1} \|j\|_2^{\rho-1} |\langle Qe_i, e_j\rangle| < \infty.$$

Let $X : [0,T] \times \Omega \to V$ be the solution of SPDE (7) and $Y^{N,M}_m : \Omega \to V$, $m \in \{0, 1, ..., M\}, M, N \in \mathbb{N}$ the numerical solution given by (22). Fix $\theta \in (0, \min\{\frac{1}{2}, \frac{\rho}{2}\})$ and $\gamma \in [0, \frac{1}{2})$.

Then there exists a finite random variable $C : \Omega \to [0, \infty)$ such that

$$\|X_{m\Delta t}(\omega) - Y^{N,M}_m(\omega)\|_V \leq C(\omega)\left(N^{-\gamma} + (\Delta t)^{\min\{\frac{1}{4},\theta\}}\right) \tag{38}$$

for all $m \in \{0, 1, ..., M\}$ and every $M, N \in \mathbb{N}$.

5 Numerical results

In this section we consider the numerical solution of stochastic Burgers equation by the method given in (22).

Example 1. Consider the stochastic evolution equation (7) with $S : (0,T] \to L(W,V)$, $F : V \to W$ given by Lemma 2, Lemma 3 for $T = 1$, $d = 1$, and $(\xi(\omega))(x) = \frac{6}{5}\sin(x)$, for all $x \in [0, \pi]$. We assume that $O : [0,T] \times \Omega \to V$ is given by Lemma 4 where the Brownian motion $\beta^i : [0,T] \times \Omega \to \mathbb{R}, i \in \mathbb{N}^d$, are dependent by the relation

$$\mathbb{E}(\beta^k \beta^l) = < Qe_k, e_l >, \quad k, l \in \mathbb{N}, \tag{39}$$

where the covariance operator Q is explicitly given as a convolution operator

$$< Qe_k, e_l > = \int_0^\pi \int_0^\pi e_k(x) e_l(y) q(x-y) dy dx, \tag{40}$$

with kernel

$$q(x-y) = \max\{0, \frac{h-|x-y|}{h^2}\} \tag{41}$$

where we define the orthonormal basis

$$e_k(x) = \sqrt{\tfrac{2}{\pi}}\sin(kx), \quad \text{for } k \in \mathbb{N}. \tag{42}$$

The possibly small quantity $h > 0$ measures the correlation length of the noise. In this case the covariance matrix, i.e., $< Qe_k, e_l >_{k,l}$, is not diagonal. But for small $h > 0$ it is close to diagonal Matrix. In Figure 1, the covariance matrix is plotted for $k, l \in \{1, 2, \cdots, 100\}$ for $h = 0.1, 0.01$. Then by numerical calculation we can show that the condition on Q from (21) is satisfied for any $\rho \in (0, \frac{1}{2})$.

The stochastic evolution equation (7) reduces to

$$dX_t = \Big[\frac{\partial^2}{\partial x^2}X_t - X_t \cdot \frac{\partial}{\partial x}X_t\Big]dt + dW_t, \quad X_0(x) = \frac{6}{5}\sin(x) \tag{43}$$

with $X_t(0) = X_t(\pi) = 0$ for $t \in [0,1]$ and $x \in [0,\pi]$.

The finite dimensional SDE (6) reduces to

$$dX_t^N = \Big[\frac{\partial^2}{\partial x^2}X_t^N - P_N(X_t^N \cdot \frac{\partial}{\partial x}X_t^N)\Big]dt + P_N dW_t, \quad X_0^N(x) = \frac{6}{5}\sin(x), \tag{44}$$

with $X_t^N(0) = X_t^N(\pi) = 0$ for $t \in [0,1]$ and $x \in [0,\pi]$, and all $N \in \mathbb{N}$.
In Figure 2, $O : [0,T] \times \Omega \to C^0([0,\pi])$, the solution of the linear SPDE

$$dO_t = \Delta O_t dt + dW_t, \quad O_t|_{\partial(0,\pi)} = 0, \quad O_0 = \frac{6}{5}sin(x),$$

for $T = 1$ is plotted.

Theorem 12 yields the existence of a unique solution $X : [0,\pi] \times \Omega \to C^0([0,\pi])$ of the SPDE (43) such that

$$\sup_{0\le x\le\pi} |X_{m\Delta t}(\omega, x) - Y_m^{N,M}(\omega, x)| \le C(\omega)\left(N^{-\gamma} + (\Delta t)^{\min\{\frac{1}{4},\theta\}}\right) \tag{45}$$

for $m = 1, ..., M, \quad M = \frac{1}{\Delta t}$, such that $\gamma \in (0, \frac{1}{2})$, $\theta \in (0, \frac{1}{4})$.

By using $\Delta t = \frac{T}{N^2}$, the solutions $X_t^N(\omega, x)$ of the finite dimensional SODEs (44) converge uniformly in $t \in [0,1]$ and $x \in [0,\pi]$ to the solution $X_t(\omega, x)$ of the stochastic Burgers equation (43) with the rate $\frac{1}{2}$, as N goes to infinity for all $\omega \in \Omega$. In Figure 3 the pathwise approximation error

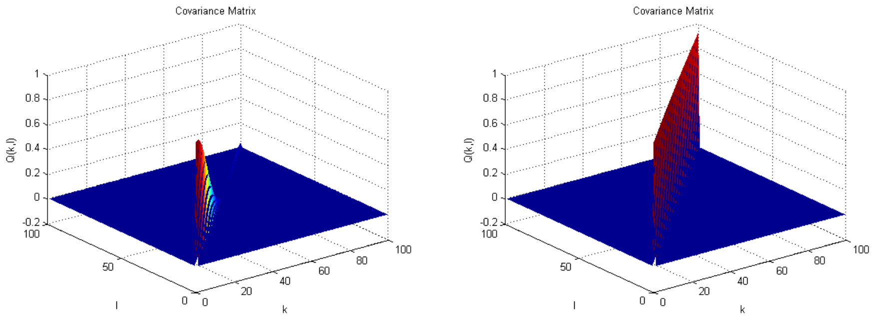
$$\sup_{0\le x\le\pi}\sup_{0\le m\le M} |X_{m\Delta t}(\omega, x) - Y_m^{N,M}(\omega, x)| \tag{46}$$

is plotted against N, for $N \in \{16, 32, \cdots, 256\}$. As a replacement for the unknown solution, we use a numerical approximation for N sufficiently large.

Figure 3 confirms that, as we expected from Theorem 12, the order of convergence is $\frac{1}{2}$. Obviously, these are only two examples, but all calculated examples and even their mean behave similarly.

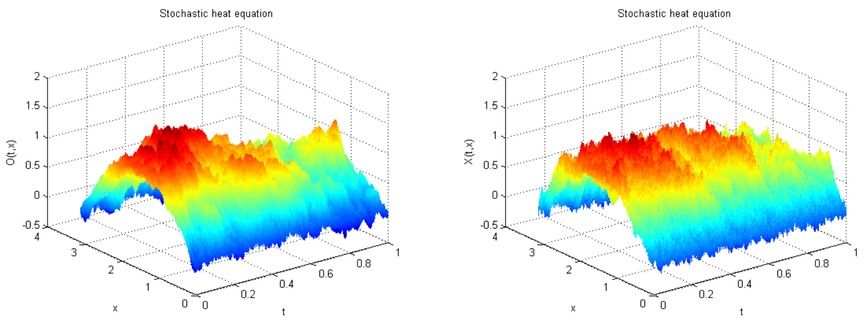
Finally, as an example in Figure 4, $X_t(\omega)$, $x \in [0,\pi]$, is plotted for $t \in \{0, \frac{3}{200}, 0.2, 1\}$, for $h = 0.01, 0.1$.

Acknowledgment.
The authors of Tarbiat Modares University would like to thank the Department of Mathematics of University of Augsburg for its supports during the second author's visit and also for providing an opportunity of joint research collaborations.



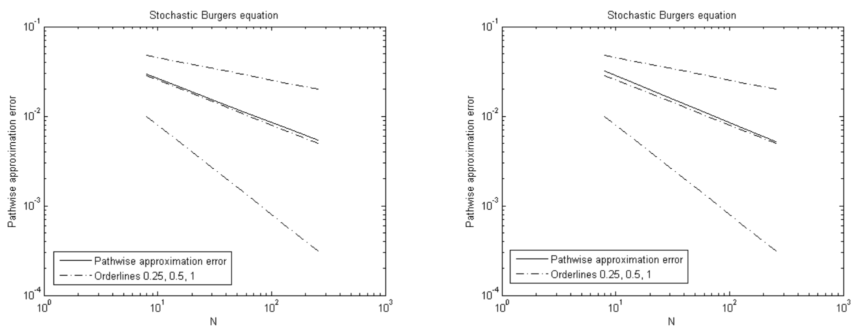
(a) (b)

Figure 1: Covariance Matrix $< Qe_k, e_l >_{k,l}$ for $k, l \in \{1, 2, \cdots, 100\}$, for (a) $h = 0.1$ and (b) $h = 0.01$.



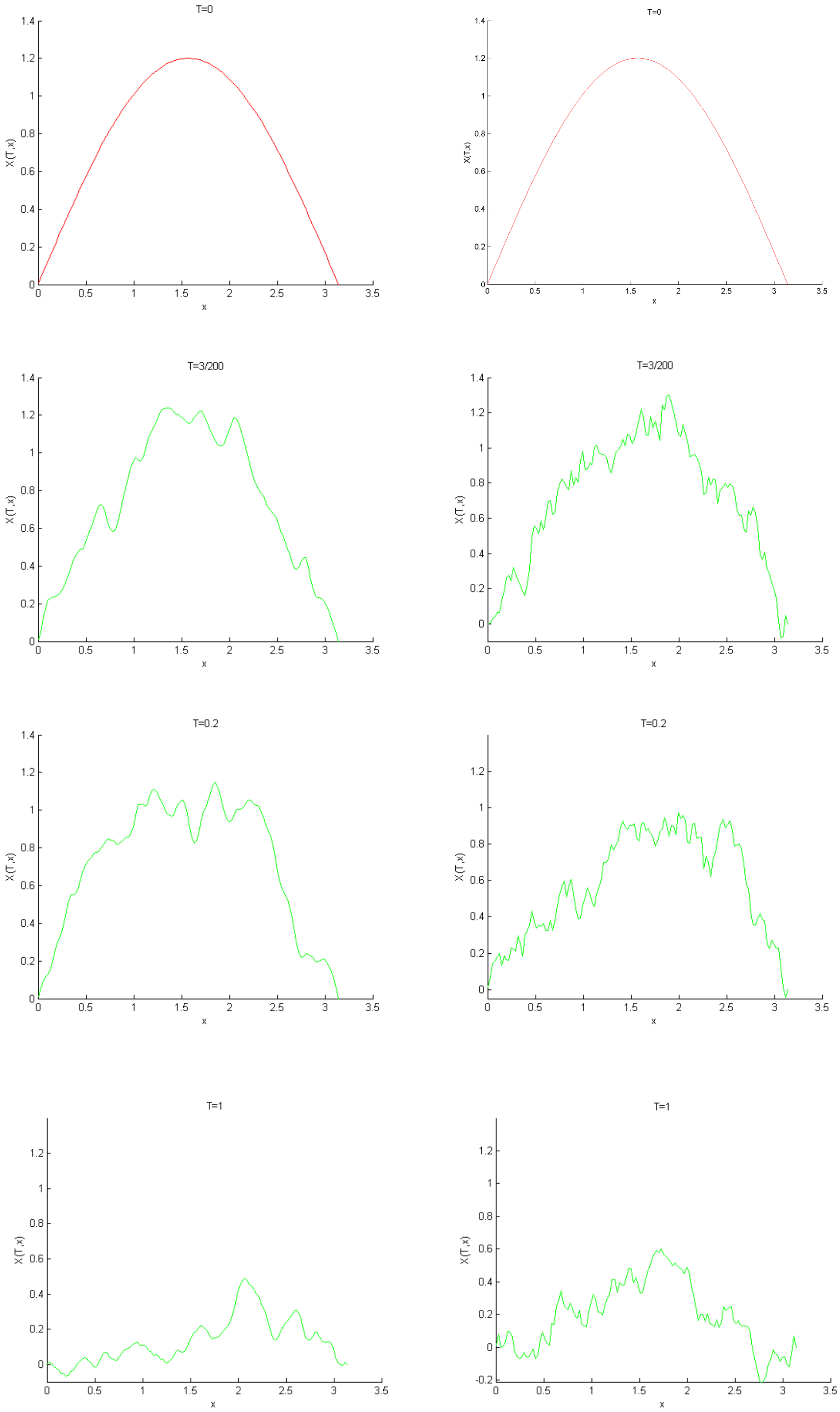
(a) (b)

Figure 2: $O_t(\omega, x)$, $x \in [0, \pi]$, $t \in [0, 1]$ and one random $\omega \in \Omega$, for (a) $h = 0.1$ and (b) $h = 0.01$.



(a) (b)

Figure 3: Pathwise approximation error (46) against N for $N \in \{16, 32, ..., 256\}$ for two random $\omega \in \Omega$, with $h = 0.1$. These are only two examples, but all other calculated trajectories behave similarly.



(a) (b)

Figure 4: Stochastic Burgers equation $X_t(\omega, x)$, $x \in [0, \pi]$, $t \in \{0, 3/200, 0.2, 1\}$, given by (43) for (a) $h = 0.1$ and (b) $h = 0.01$, for one random $\omega \in \Omega$.

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