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CONVERGENCE ANALYSIS OF AN ADAPTIVE INTERIOR PENALTY DISCONTINUOUS GALERKIN METHOD FOR THE BIHARMONIC PROBLEM

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Abstract. For the biharmonic problem, we study the convergence of adaptive C^0-Interior Penalty Discontinuous Galerkin (C^0-IPDG) methods of any polynomial order. We note that C^0-IPDG methods for fourth order elliptic boundary value problems have been suggested in [9], whereas a residual-type a posteriori error estimator for a quadratic C^0-IPDG method applied to the biharmonic equation has been developed and analyzed in [8]. Following the convergence analysis of adaptive IPDG methods for second order elliptic problems [6], we prove a contraction property for a weighted sum of the C^0-IPDG energy norm of the global discretization error and the estimator. The proof of the contraction property is based on the reliability of the estimator, a quasi-orthogonality result, and an estimator reduction property. Numerical results are given that illustrate the performance of the adaptive C^0-IPDG approach.

Keywords: C^0-Interior Penalty Discontinuous Galerkin method, biharmonic equation, residual type a posteriori error estimator, convergence analysis

AMS subject classification: 35J35, 65N30, 65N50

1. Introduction. For second order elliptic boundary value problems, adaptive finite element methods (AFEM) are well established numerical tools that have been intensively studied in the literature (cf., e.g., [1, 3, 4, 16, 20, 23] and the references therein). The convergence analysis of AFEM for conforming discretizations has been initiated in [14] (cf. also [19]) with the most far reaching result so far given in [13]. Nonconforming discretizations based on the lowest order Crouzeix-Raviart elements have been addressed in [11], whereas for Interior Penalty Discontinuous Galerkin (IPDG) methods we refer to [6]. However, considerably less work has been devoted to AFEM for nonconforming discretizations of fourth order elliptic boundary value problems. As far as IPDG approaches are concerned, C^0-IPDG methods have been suggested in [15] (cf. also [25]) and subsequently analyzed in [9] focusing on an a priori error analysis. An a posteriori error analysis of quadratic C^0-IPDG methods based on residual-type a posteriori error estimators has been performed in [8], however, without addressing the issue of convergence.
The purpose of this contribution is to provide a convergence analysis of C^0-IPDG methods of any polynomial order for the biharmonic problem. Following the ideas from [6], we improve on [8] by showing that for sufficiently large penalty parameter the consistency error can be controlled by the estimator (Theorem 3.1) which gives rise to a novel reliability result (Corollary 3.2). Together with standard estimator reduction for Dörfler marking (Lemma 4.1) and a quasi-orthogonality result (Theorem 5.3) this results in a contraction property for a weighted sum of the C^0-IPDG energy norm

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of the global discretization error and the estimator (Theorem 6.1). The performance of the adaptive C^0-IPDG approach is illustrated by a documentation of numerical results.

2. C^0-Interior Penalty Discontinuous Galerkin method. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\Gamma = \partial\Omega$. For a given function $f \in L^2(\Omega)$ we consider the biharmonic problem

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{2.1a}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma. \tag{2.1b}$$

We use standard notation from Lebesgue and Sobolev space theory [22]. In particular, $(\cdot,\cdot)_{0,\Omega}$ and $\|\cdot\|_{0,\Omega}$ stand for the inner product on $L^2(\Omega)$ and the associated norm. Moreover, $H^k(\Omega), k \in \mathbb{N}$, refers to the Sobolev space with norm $\|\cdot\|_{k,\Omega}$ and seminorm $|\cdot|_{k,\Omega}$, whereas $H_0^k(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the topology induced by $\|\cdot\|_{k,\Omega}$.
A weak formulation of (2.1) requires the computation of $u \in V := H_0^2(\Omega)$ such that

$$a(u,v) = (f,v)_{0,\Omega}, \quad v \in V, \tag{2.2}$$

where the bilinear form $a(\cdot,\cdot)$ is given by

$$a(v,w) = (D^2 v, D^2 w)_{0,\Omega} := \sum_{|\beta|=2} (D^\beta v, D^\beta w)_{0,\Omega}, \quad v,w \in V. \tag{2.3}$$

Let $\mathcal{T}_h$ be a geometrically conforming simplicial triangulation of Ω. We denote by $\mathcal{E}_h^\Omega$ and $\mathcal{E}_h^\Gamma$ the set of edges of $\mathcal{T}_h$ in the interior of Ω and on the boundary Γ, respectively, and set $\mathcal{E}_h := \mathcal{E}_h^\Omega \cup \mathcal{E}_h^\Gamma$. For $T \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$ we denote by h_T and h_E the diameter of T and the length of E, and we set $h := \max_{T\in\mathcal{T}_h} h_T$. For two quantities A and B we write $A \lesssim B$, if there exists a constant $C > 0$ independent of h such that $A \le CB$.
Denoting by $P_k(T), k \in \mathbb{N}$, the linear space of polynomials of degree $\le k$ on T, for $k \ge 2$ we refer to

$$V_h := \{v_h \in H_0^1(\Omega) \mid v_h|_T \in P_k(T),\ T \in \mathcal{T}_h\} \tag{2.4}$$

as the finite element space of Lagrangian finite elements of type k (cf., e.g., [7]). We refer to $\mathcal{N}_h$ as the set of nodal points such that any $v_h \in V_h$ is uniquely determined by its degrees of freedom $v_h(a), a \in \mathcal{N}_h$.
We note that $V_h \not\subset V$ and hence, V_h is a nonconforming finite element space for the approximation of the biharmonic problem (2.2). In particular, for $v_h \in V_h$ the normal derivative $\partial v_h/\partial n$ exhibits jumps across interior edges $E \in \mathcal{E}_h^\Omega$. After numbering of the elements $T \in \mathcal{T}_h$, for $E \in \mathcal{E}_h^\Omega, E = T_i \cap T_j, i > j$, we set $T_E^+ := T_i,\ T_E^- := T_j$, and for $E \in \mathcal{E}_h^\Gamma, E = T_\ell \cap \Gamma$, we set $T_E := T_\ell$. Then, for $1 \le \nu \le 2$ we define averages and jumps according to

$$\{\frac{\partial^\nu v_h}{\partial n^\nu}\}_E := \begin{cases} \frac{1}{2}\Big(\frac{\partial^\nu v_h}{\partial n^\nu}|_{E\cap T_E^+} + \frac{\partial^\nu v_h}{\partial n^\nu}|_{E\cap T_E^-}\Big) &, E \in \mathcal{E}_h^\Omega, \\ \frac{\partial^\nu v_h}{\partial n^\nu}|_{E\cap T_E} &, E \in \mathcal{E}_h^\Gamma, \end{cases} \tag{2.5a}$$

$$[\frac{\partial^\nu v_h}{\partial n^\nu}]_E := \begin{cases} \frac{\partial^\nu v_h}{\partial n^\nu}|_{E\cap T_E^+} - \frac{\partial^\nu v_h}{\partial n^\nu}|_{E\cap T_E^-} &, E \in \mathcal{E}_h^\Omega, \\ \frac{\partial^\nu v_h}{\partial n^\nu}|_{E\cap T_E} &, E \in \mathcal{E}_h^\Gamma, \end{cases} \tag{2.5b}$$

where n is the unit normal vector on E pointing in the direction from T_E^- to T_E^+ for $E \in \mathcal{E}_h^\Omega$ and the exterior normal vector for $E \in \mathcal{E}_h^\Gamma$.
We further refer to $M_h(\mathcal{T}_h;\mathbb{R}^{2\times 2})$ as the set of matrix-valued functions on $\mathcal{T}_h$ such that for $W_h \in M_h(\mathcal{T}_h;\mathbb{R}^{2\times 2})$ the restriction $W_h|_T, T \in \mathcal{T}_h$, is a 2×2 matrix with entries that are polynomials of order k.
Given a penalty parameter $\alpha > 1$, the C^0-IPDG method for the approximation of (2.2) requires the computation of $u_h \in V_h$ such that

$$a_h^{IP}(u_h, v_h) = (f, v_h)_{0,\Omega}, \quad v_h \in V_h. \tag{2.6}$$

Here, the mesh-dependent bilinear form $a_h^{IP}(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ is given according to

$$\begin{aligned} a_h^{IP}(v_h, w_h) := & \sum_{T\in\mathcal{T}_h} (D^2 v_h, D^2 w_h)_{0,T} + \sum_{E\in\mathcal{E}_h} (\{\frac{\partial^2 v_h}{\partial n^2}\}_E, [\frac{\partial w_h}{\partial n}]_E)_{0,E} \\ & + \sum_{E\in\mathcal{E}_h} ([\frac{\partial v_h}{\partial n}]_E, \{\frac{\partial^2 w_h}{\partial n^2}\}_E)_{0,E} + \alpha \sum_{E\in\mathcal{E}_h} h_E^{-1}([\frac{\partial v_h}{\partial n}]_E, [\frac{\partial w_h}{\partial n}]_E)_{0,E}. \end{aligned} \tag{2.7}$$

It is worth noting that this is the symmetric interior penalty formulation (S-IPDG). It would also be possible to investigate the non-symmetric variant (N-IPDG), in which the terms involving the averages arise with different signs. But we do not follow this direction here. We note that $a_h^{IP}(\cdot,\cdot)$ is not well defined for $v, w \in V$ which can be cured in terms of a lifting operator $L : L^2(\mathcal{E}_h, \mathbb{R}^2) \to M_h(\mathcal{T}_h;\mathbb{R}^{2\times 2})$ given by

$$(L(q), W_h)_{0,\Omega} := \sum_{E\in\mathcal{E}_h} ([n\cdot q]_E, \{n \cdot W_h n\}_E)_{0,E}, \quad W_h \in M_h(\mathcal{T}_h;\mathbb{R}^{2\times 2}). \tag{2.8}$$

Then, $a_h^{IP}(\cdot,\cdot)$ can be extended to $V + V_h$ by means of

$$\begin{aligned} a_h^{IP}(v, w) := & \sum_{T\in\mathcal{T}_h} (D^2 v, D^2 w)_{0,T} + \sum_{T\in\mathcal{T}_h} (L(\nabla w), D^2 v)_{0,T} \\ & + \sum_{T\in\mathcal{T}_h} (L(\nabla v), D^2 w)_{0,T} + \alpha \sum_{E\in\mathcal{E}_h} h_E^{-1}([\frac{\partial v}{\partial n}]_E, [\frac{\partial w}{\partial n}]_E)_{0,E}, \end{aligned} \tag{2.9}$$

where with a slight abuse of notation we have also used $a_h^{IP}(\cdot,\cdot)$ for that extension.
The lifting operator satisfies the following stability estimate:

Theorem 2.1. *Let $L : L^2(\mathcal{E}_h, \mathbb{R}^2) \to M_h(\mathcal{T}_h;\mathbb{R}^{2\times 2})$ be the lifting operator as given by* (2.8). *Then, there exists a positive constant C_L, depending only on the local geometry of the triangulation and on the polynomial order k, such that there holds*

$$\|L(q)\|_{0,\Omega}^2 \le C_L \sum_{E\in\mathcal{E}_h} h_E^{-1} \, \|[n\cdot q]_E\|_{0,E}^2, \quad q \in L^2(\mathcal{E}_h, \mathbb{R}^2). \tag{2.10}$$

Proof. For $q \in L^2(\mathcal{E}_h, \mathbb{R}^2)$ and $W_h \in M_h(\mathcal{T}_h;\mathbb{R}^{2\times 2})$ we have

$$\|L(q)\|_{0,\Omega} = \sup_{\|W_h\|_{0,\Omega}\le 1} |(L(q), W_h)_{0,\Omega}|.$$

In view of (2.8) we find

$$|(L(q), W_h)_{0,\Omega}| \le \Big(\sum_{E\in\mathcal{E}_h} \|[n\cdot q]_E\|_{0,E}^2 \Big)^{1/2} \Big(\sum_{T\in\mathcal{T}_h} \|n_{\partial T} \cdot W_h n_{\partial T}\|_{0,\partial T}^2 \Big)^{1/2},$$

where $n_{\partial T}$ is the exterior unit normal on ∂T. Then, the trace inequality (cf., e.g., [24])

$$\|n_{\partial T} \cdot W_h n_{\partial T}\|_{0,\partial T} \lesssim k\ h_T^{-1/2}\ \|W_h\|_{0,T}, \quad T \in \mathcal{T}_h,$$

gives the assertion. □

On $V + V_h$ we introduce the mesh-dependent C^0-IPDG norm

$$\|v\|_{2,h,\Omega} := \sum_{T\in\mathcal{T}_h} |v|_{2,T}^2 + \sum_{E\in\mathcal{E}_h} \frac{\alpha}{h_E}\ \|[\frac{\partial v}{\partial n}]_E\|_{0,E}^2, \quad v \in V + V_h, \tag{2.11}$$

where $|\cdot|_{2,T}^2$ stands for

$$|\cdot|_{2,T}^2 := \sum_{|\beta|=2} \|D^\beta \cdot\|_{0,T}^2, \quad T \in \mathcal{T}_h. \tag{2.12}$$

It has been shown in [9] that for sufficiently large penalty parameter α there exists a positive constant $\gamma < 1$ such that

$$a_h^{IP}(v,v) \geq \gamma\ \|v\|_{2,h,\Omega}^2, \quad v \in V + V_h, \tag{2.13}$$

whereas for any $\alpha \geq 1$ there exists a constant $C_1 > 1$ such that

$$a_h^{IP}(v,v) \leq C_1\ \|v\|_{2,h,\Omega}^2, \quad v \in V + V_h. \tag{2.14}$$

In particular, it follows from (2.13) and (2.14) that (2.6) admits a unique solution $u_h \in V_h$.

3. Residual-type a posteriori error estimator and its reliability. For adaptive mesh refinement we consider the residual-type a posteriori error estimator

$$\eta_h := \Big(\sum_{T\in\mathcal{T}_h} \eta_T^2 + \sum_{E\in\mathcal{E}_h^\Omega} \eta_E^2\Big)^{1/2}, \tag{3.1}$$

where the element residuals $\eta_T, T \in \mathcal{T}_h$, and the edge residuals $\eta_E, E \in \mathcal{E}_h$, are given by

$$\eta_T^2 := h_T^4\ \|f - \Delta^2 u_h\|_{0,T}^2, \quad T \in \mathcal{T}_h, \tag{3.2a}$$

$$\eta_E^2 := h_E\ \|[\frac{\partial^2 u_h}{\partial n^2}]_E\|_{0,E}^2 + h_E^3\ \|[\frac{\partial}{\partial n}\Delta u_h]_E\|_{0,E}^2, \quad E \in \mathcal{E}_h^\Omega. \tag{3.2b}$$

We further introduce

$$\eta_{h,c} := \Big(\sum_{E\in\mathcal{E}_h} \eta_{E,c}^2\Big)^{1/2}, \quad \eta_{E,c}^2 := h_E^{-1}\|[\frac{\partial u_h}{\partial n}]_E\|_{0,E}^2. \tag{3.3}$$

The term $\eta_{h,c}$ represents an upper bound for the consistency error

$$\inf_{v_h\in V_h^c} a_h^{IP}(u_h - v_h, u_h - v_h),$$

where $V_h^c \subset H_0^2(\Omega)$ stands for the C^1 conforming finite element space generated by the Argyris elements of the so-called TUBA family [2]. In fact, denoting by $E_h : V_h \to V_h^c$

the enriching operator from [9], it follows from the results in [9] (cf. also [8]) that there exists a constant $C_{nc} > 0$, depending only on the local geometry of $\mathcal{T}_h$, such that

$$\inf_{v_h \in V_h^c} a_h^{IP}(u_h - v_h, u_h - v_h) \le a_h^{IP}(u_h - E_h(u_h), u_h - E_h(u_h)) \le C_{nc}\ \eta_{h,c}^2. \quad (3.4)$$

The following result shows that $\eta_h^2 + \alpha\eta_{h,c}^2$ provides an upper bound for the IPDG energy norm of the discretization error $u - u_h$. As an essential tool we will use Clément's quasi-interpolation operator $\Pi_C : H_0^2(\Omega) \to V_h^c$ which enjoys the local approximation properties (cf. section 3.7 in [23]):

$$\|D^\ell(v - \Pi_C v)\|_{0,T} \lesssim h_T^{2-\ell} |v|_{2,\tilde{\omega}_T}, \quad T \in \mathcal{T}_h, \quad (3.5a)$$

$$\|v - \Pi_C v\|_{0,E} + h_E \|\frac{\partial}{\partial n}(v - \Pi_C v)\|_{0,E} \lesssim h_E^{3/2}\ |v|_{2,\tilde{\omega}_E}, \quad E \in \mathcal{E}_h. \quad (3.5b)$$

Here, $\tilde{\omega}_T$ and $\tilde{\omega}_E$ stand for the patches

$$\tilde{\omega}_T := \bigcup\{T' \in \mathcal{T}_h \mid \mathcal{N}_h(T) \cap \mathcal{N}_h(T') \ne \emptyset\},$$
$$\tilde{\omega}_E := \bigcup\{T' \in \mathcal{T}_h \mid \mathcal{N}_h(E) \cap \mathcal{N}_h(T') \ne \emptyset\}.$$

THEOREM 3.1. *Let $u \in V$ and $u_h \in V_h$ be the unique solution of* (2.2) *and* (2.6), *and let η_h and $\eta_{h,c}$ be given by* (3.1)-(3.3). *Then, there exists a constant $C_r > 0$, depending only on the local geometry of $\mathcal{T}_h$, such that*

$$a_h^{IP}(u - u_h, u - u_h) \le C_r\Big(\eta_h^2 + \alpha\eta_{h,c}^2\Big). \quad (3.6)$$

Proof. We split $u_h \in V_h$ according to $u_h = u_h^0 + u_h^\perp, u_h^0 \in V_h^c, u_h^\perp \in V_h^\perp$, where $V_h^\perp$ is the orthogonal complement of V_h^c with respect to $a_h^{IP}(\cdot,\cdot)$. We further set $e_h := u - u_h$ and $e_h^0 := u - u_h^0$. Due to Galerkin orthogonality $a_h^{IP}(e_h, \Pi_C e_h^0) = 0$, we have

$$\begin{aligned} a_h^{IP}(e_h, e_h) = a_h^{IP}(e_h, e_h^0 - \Pi_C e_h^0) - a_h^{IP}(e_h, u_h^\perp)\ = \\ (f, e_h^0 - \Pi_C e_h^0)_{0,\Omega} - a_h^{IP}(u_h, e_h^0 - \Pi_C e_h^0) - a_h^{IP}(e_h, u_h^\perp). \end{aligned} \quad (3.7)$$

Since $[\frac{\partial}{\partial n}(e_h^0 - \Pi_C e_h^0)]_E = 0, E \in \mathcal{E}_h$, for the second term on the right-hand side in (3.7) we find

$$\begin{aligned} a_h^{IP}(u_h, e_h^0 - \Pi_C e_h^0) = &\sum_{T \in \mathcal{T}_h} (D^2 u_h, D^2(e_h^0 - \Pi_C e_h^0))_{0,T} \\ &+ \sum_{T \in \mathcal{T}_h} (L(\nabla u_h), D^2(e_h^0 - \Pi_C e_h^0))_{0,T}. \end{aligned} \quad (3.8)$$

Elementwise integration by parts yields

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} (D^2 u_h, D^2(e_h^0 - \Pi_C e_h^0))_{0,T} = \sum_{T \in \mathcal{T}_h} (\Delta^2 u_h, e_h^0 - \Pi_C e_h^0)_{0,T} \\ - \sum_{E \in \mathcal{E}_h^\Omega} ([\frac{\partial^2 u_h}{\partial n^2}]_E, \frac{\partial}{\partial n}(e_h^0 - \Pi_C e_h^0))_{0,E} + \sum_{E \in \mathcal{E}_h^\Omega} ([\frac{\partial}{\partial n}\Delta u_h]_E, e_h^0 - \Pi_C e_h^0)_{0,E}. \end{aligned} \quad (3.9)$$

Using (3.8),(3.9) in (3.7) and taking advantage of (2.10),(3.5a),(3.5b), for the first two terms on the right-hand side in (3.7) we obtain

$$
\begin{aligned}
&|(f, e_h^0 - \Pi_C e_h^0)_{0,\Omega} - a_h^{IP}(u_h, e_h^0 - \Pi_C e_h^0)| \qquad (3.10)\\
&\lesssim \sum_{T\in\mathcal{T}_h} \|f - \Delta^2 u_h\|_{0,T}\|e_h^0 - \Pi_C e_h^0\|_{0,T} + \sum_{T\in\mathcal{T}_h} \|L(\nabla u_h)\|_{0,T}\|D^2(e_h^0 - \Pi_C e_h^0)\|_{0,T}\\
&+ \sum_{E\in\mathcal{E}_h^\Omega} \|[\frac{\partial^2 u_h}{\partial n^2}]_E\|_{0,E}\|\frac{\partial}{\partial n}(e_h^0 - \Pi_C e_h^0)\|_{0,E} + \sum_{E\in\mathcal{E}_h^\Omega} \|[\frac{\partial}{\partial n}\Delta u_h]_E\|_{0,E}\|e_h^0 - \Pi_C e_h^0\|_{0,E}\\
&\lesssim \sum_{T\in\mathcal{T}_h} \Big(\eta_T + \|L(\nabla u_h)\|_{0,T}\Big)|e_h^0|_{2,\tilde\omega_T} + \sum_{E\in\mathcal{E}_h^\Omega} \Big(h_E^{1/2}\|[\frac{\partial^2 u_h}{\partial n^2}]_E\|_{0,E}\\
&\quad + h_E^{3/2}\|[\frac{\partial}{\partial n}\Delta u_h]_E\|_{0,E}\Big)|e_h^0|_{2,\tilde\omega_E} \lesssim (\eta_h^2 + \alpha\eta_{h,c}^2)^{1/2} a_h^{IP}(e_h^0, e_h^0)^{1/2},
\end{aligned}
$$

where we have observed $\alpha \geq 1$ as well.
On the other hand, for the last term on the right-hand side in (3.7) there holds

$$
\begin{aligned}
&a_h^{IP}(e_h, u_h^\perp) \leq a_h^{IP}(e_h, e_h)^{1/2} a_h^{IP}(u_h^\perp, u_h^\perp)^{1/2} \qquad (3.11)\\
&\lesssim a_h^{IP}(e_h, e_h)^{1/2}\Big(\sum_{E\in\mathcal{E}_h} \alpha h_E^{-1}\|[\frac{\partial u_h}{\partial n}]_E\|_{0,E}^2\Big)^{1/2} = \alpha^{1/2}\eta_{h,c} a_h^{IP}(e_h, e_h)^{1/2}.
\end{aligned}
$$

Combining (3.10),(3.11) and using Young's inequality gives (3.6). □

We will improve on (3.6) by showing that similar to the case of adaptive IPDG methods for linear second order elliptic boundary value problems [6] the second term on the right-hand side in (3.6) can be controlled by the first one, provided the penalty parameter α is chosen sufficiently large.

THEOREM 3.2. *Let η_h and $\eta_{h,c}$ be given by (3.1)-(3.3). Then, there exists a constant $C_J > 0$ depending only on the shape regularity of $\mathcal{T}_h$ such that for $\alpha \geq 2C_J/\gamma$ there holds*

$$
\alpha\eta_{h,c}^2 \leq 2\ \frac{C_J}{\gamma}\ \eta_h^2. \qquad (3.12)
$$

Proof. In view of (2.11),(2.13), and (3.3) we have

$$
\alpha\eta_{h,c}^2 \leq \|u_h - u_h^c\|_{2,h,\Omega}^2 \leq \frac{1}{\gamma}\ a_h^{IP}(u_h - u_h^c, u_h - u_h^c), \qquad (3.13)
$$

where $u_h^c := E_h(u_h)$. Since u_h satisfies (2.6), for $v_h = u_h - u_h^c$ it follows that

$$
a_h^{IP}(u_h - u_h^c, u_h - u_h^c) = (f, u_h - u_h^c)_{0,\Omega} - a_h^{IP}(u_h^c, u_h - u_h^c). \qquad (3.14)
$$

For the last term on the right-hand side in (3.14) we find

$$
\begin{aligned}
&a_h^{IP}(u_h^c, u_h - u_h^c) = \sum_{T\in\mathcal{T}_h} (D^2 u_h, D^2(u_h - u_h^c))_{0,T} \qquad (3.15)\\
&- \sum_{T\in\mathcal{T}_h} \|D^2(u_h - u_h^c)\|_{0,T}^2 + \sum_{E\in\mathcal{E}_h} (\{\frac{\partial^2 u_h^c}{\partial n^2}\}_E, [\frac{\partial u_h}{\partial n}]_E)_{0,E}.
\end{aligned}
$$

Using the integration by parts formulas

$$\sum_{T\in\mathcal{T}_h}\int_T D^2u_h : D^2v_h\ dx = -\sum_{T\in\mathcal{T}_h}\int_T \nabla\Delta u_h\cdot\nabla v_h\ dx \tag{3.16a}$$
$$+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\Big(\frac{\partial^2 u_h}{\partial n_{\partial T}^2}\ \frac{\partial v_h}{\partial n_{\partial T}}+\frac{\partial^2 u_h}{\partial t_{\partial T}\partial n_{\partial T}}\ \frac{\partial v_h}{\partial t_{\partial T}}\Big)\ ds,\quad u_h,v_h\in V_h,$$
$$\sum_{T\in\mathcal{T}_h}\int_T \nabla\Delta u_h\cdot\nabla v_h\ dx\ = \tag{3.16b}$$
$$-\sum_{T\in\mathcal{T}_h}\int_T \Delta^2 u_h v_h\ dx+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial}{\partial n_{\partial T}}\Delta u_h v_h ds,\quad u_h,v_h\in V_h,$$

where $t_{\partial T}$ stands for the unit tangential vector on ∂T, and observing $(u_h-u_h^c)(a)=0, a\in\mathcal{N}_h$, for the first term on the right-hand side in (3.15) we obtain

$$\sum_{T\in\mathcal{T}_h}(D^2u_h,D^2(u_h-u_h^c))_{0,T}=\sum_{T\in\mathcal{T}_h}(\Delta^2u_h,u_h-u_h^c)_{0,T} \tag{3.17}$$
$$+\sum_{T\in\mathcal{T}_h}\Big((\frac{\partial^2u_h}{\partial n_{\partial T}^2},\frac{\partial(u_h-u_h^c)}{\partial n_{\partial T}})_{0,\partial T}-(\frac{\partial}{\partial n_{\partial T}}\Delta u_h,u_h-u_h^c)_{0,\partial T}\Big).$$

Taking advantage of (3.17) in (3.15), from (3.14) we get

$$a_h^{IP}(u_h-u_h^c,u_h-u_h^c)=\sum_{T\in\mathcal{T}_h}(f-\Delta^2u_h,u_h-u_h^c)_{0,T}+\sum_{T\in\mathcal{T}_h}\|D^2(u_h-u_h^c)\|_{0,T}^2$$
$$+\sum_{E\in\mathcal{E}_h^\Omega}\Big((\{\frac{\partial(u_h-u_h^c)}{\partial n}\}_E,[\frac{\partial^2u_h}{\partial n^2}]_E)_{0,E}-([\frac{\partial}{\partial n}\Delta u_h]_E,\{u_h-u_h^c\}_E)_{0,E}\Big)$$
$$+\sum_{E\in\mathcal{E}_h}(\{\frac{\partial^2(u_h-u_h^c)}{\partial n^2}\}_E,[\frac{\partial u_h}{\partial n}]_E)_{0,E}.$$

Straightforward estimation yields

$$a_h^{IP}(u_h-u_h^c,u_h-u_h^c)\le\frac{1}{2}\Big(\sum_{T\in\mathcal{T}_h}\Big(h_T^4\ \|f-\Delta^2u_h\|_{0,T}^2+h_T^{-4}\ \|u_h-u_h^c\|_{0,T}^2\Big) \tag{3.18}$$
$$+\sum_{E\in\mathcal{E}_h^\Omega}\Big(h_E\|[\frac{\partial^2u_h}{\partial n^2}]_E\|_{0,E}^2+h_E^{-1}\|\{\frac{\partial(u_h-u_h^c)}{\partial n}\}_E\|_{0,E}^2+h_E^3\|[\frac{\partial}{\partial n}\Delta u_h]_E\|_{0,E}^2$$
$$+h_E^{-3}\|\{u_h-u_h^c\}_E\|_{0,E}^2\Big)+\sum_{E\in\mathcal{E}_h}\Big(h_E^{-1}\|[\frac{\partial u_h}{\partial n}]_E\|_{0,E}^2+h_E\|\{\frac{\partial^2(u_h-u_h^c)}{\partial n^2}\}_E\|_{0,E}^2\Big)$$
$$+\sum_{T\in\mathcal{T}_h}\|D^2(u_h-u_h^c)\|_{0,T}^2.$$

Now, using the trace inequality

$$\|\frac{\partial^2(u_h-u_h^c)}{\partial n_{\partial T}^2}\|_{0,\partial T}^2\lesssim h_T^{-1}|u_h-u_h^c|_{2,T}^2$$

and the interpolation estimates

$$\|u_h - u_h^c\|_{0,T}^2 = \|u_h - u_h^c - \Pi_h(u_h - u_h^c)\|_{0,T}^2 \lesssim h_T^4 \ |u_h - u_h^c|_{2,T}^2,$$
$$\|u_h - u_h^c\|_{0,\partial T}^2 = \|u_h - u_h^c - \Pi_h(u_h - u_h^c)\|_{0,\partial T}^2 \lesssim h_T^3 |u_h - u_h^c|_{2,T}^2,$$
$$\|\frac{\partial(u_h - u_h^c)}{\partial n_{\partial T}}\|_{0,\partial T}^2 = \|\frac{\partial(u_h - u_h^c - \Pi_h(u_h - u_h^c))}{\partial n_{\partial T}}\|_{0,\partial T}^2 \lesssim h_T \ |u_h - u_h^c|_{2,T}^2,$$

where $\Pi_h : H_0^1(\Omega) \cap H^s(\Omega) \to V_h, s > 1$, is the Lagrangian nodal interpolation operator, as well as (3.4), it follows from (3.13),(3.18) that there exists $C_J > 0$, depending only on the local geometry of $\mathcal{T}_h$, such that

$$\alpha \ \eta_{h,c}^2 \le \frac{C_J}{\gamma} \ \Big(\eta_h^2 + \eta_{h,c}^2\Big),$$

from which one easily deduces (3.12). □

Combining (3.6) and (3.12) results in the reliability of the estimator η_h.

COROLLARY 3.3. *Let $u \in V$ and $u_h \in V_h$ be the unique solution of* (2.2) *and* (2.6)*, and let η_h be the residual error estimator as given by* (3.1)*. Then, there holds*

$$a_h^{IP}(u - u_h, u - u_h) \le C_R \ \eta_h^2, \tag{3.19}$$

where $C_R := C_r(1 + 2\gamma^{-1}C_J)$.

Proof. The assertion follows readily from (3.6) and (3.12). □

4. Refinement strategy and estimator reduction. As a marking strategy for adaptive refinement we use Dörfler marking [14], i.e., given a constant $0 < \Theta < 1$, we compute sets of elements $\mathcal{M}_1$ of elements $T \in \mathcal{T}_h(\Omega)$ and $\mathcal{M}_2$ of edges $E \in \mathcal{E}_h(\Omega)$ such that

$$\Theta \ \eta_h^2 \le \sum_{T \in \mathcal{M}_1} \eta_T^2 + \sum_{E \in \mathcal{M}_2} \eta_E^2. \tag{4.1}$$

After having determined the sets $\mathcal{M}_i, 1 \le i \le 2$, a refined triangulation is generated by a recursive application of newest vertex bisection (cf. [13] and the references therein). This choice yields optimal complexity of the refinement process as has been established in [5] for the two dimensional case and a conforming initial triangulation and in [13] for higher dimensions requiring the initial triangulation additionally to satisfy a certain labeling condition (cf. section 4 in [13]). In particular, there exist constants $0 < \beta_1 < \beta_2$, depending only on the initial triangulation, such that for each triangle T of refinement level ℓ it holds $\beta_1 2^{-\ell/2} \le h_T \le \beta_2 2^{-\ell/2}$. Hence, if $\mathcal{T}_h$ is obtained from $\mathcal{T}_H$ by newest vertex bisection, for $T \in \mathcal{T}_H$ and $T' \in \mathcal{T}_h$we have

$$\kappa_1 h_{T'} \le H_T \le \kappa_2 h_{T'}, \tag{4.2}$$

where $\kappa_1 := 2^{1/2}\beta_1/\beta_2$ and $\kappa_2 := 2^{1/2}\beta_2/\beta_1$.
As in [13] (cf. also [6]), we can prove the following estimator reduction property:

LEMMA 4.1. *Let $\mathcal{T}_h$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H$, let $u_h \in V_h, u_H \in V_H$, and η_h, η_H be the associated C^0-IPDG solutions and error estimators, respectively, and let $\Theta > 0$ be the universal constant from* (4.1)*. Then,* *for any $\tau > 0$ there exists a constant $C_\tau > 1$, depending only on the local geometry of the triangulations, such that for $\kappa(\Theta) := (1+\tau)(1-2^{-1/2})\Theta$ there holds*

$$\eta_h^2 \le \kappa(\Theta)\ \eta_H^2 + C_\tau\ \|u_h - u_H\|_{2,h,\Omega}^2. \tag{4.3}$$

Proof. By definition of η_h and taking into account the inverse estimates

$$\begin{aligned}
\|\Delta^2(u_h - u_H)\|_{0,T} &\le C_{\rm inv}^{(1)}\ h_T^{-2}\ \|D^2(u_h - u_H)\|_{0,T}, \quad T \in \mathcal{T}_h,\\
\|\frac{\partial^2(u_h - u_H)}{\partial n_{E\cap\partial T}^2}\|_{0,E} &\le C_{\rm inv}^{(2)}\ h_E^{-1}\ \|\frac{\partial(u_h - u_H)}{\partial n_{E\cap\partial T}}\|_{0,E}, \quad E \in \mathcal{E}_h^T,\\
\|\frac{\partial}{\partial n_{E\cap\partial T}}\Delta(u_h - u_H)\|_{0,E} &\le C_{\rm inv}^{(3)}\ h_E^{-2}\ \|\frac{\partial(u_h - u_H)}{\partial n_{E\cap\partial T}}\|_{0,E}, \quad E \in \mathcal{E}_h^T,
\end{aligned}$$

where $C_{\rm inv}^{(i)}, 1 \le i \le 3$, are positive constants, depending only on the local geometry of the triangulations, we have

$$h_T^2\ \|f - \Delta^2 u_h\|_{0,T} \le h_T^2\ \Big(\|f - \Delta^2 u_H\|_{0,T} + C_{\rm inv}^{(1)}\ h_T^{-2}\ \|D^2(u_h - u_H)\|_{0,T}\Big), \tag{4.4a}$$

$$h_E^{1/2}\ \|\frac{\partial^2 u_h}{\partial n_{E\cap\partial T}^2}\|_{0,E} \le h_E^{1/2}\ \|\frac{\partial^2 u_H}{\partial n_{E\cap\partial T}^2}\|_{0,E} + C_{\rm inv}^{(2)}\ h_E^{-1/2}\ \|\frac{\partial(u_h - u_H)}{\partial n_{E\cap\partial T}}\|_{0,E}, \tag{4.4b}$$

$$h_E^{3/2}\ \|\frac{\partial}{\partial n_{E\cap\partial T}}\Delta u_h\|_{0,E} \le h_E^{3/2}\ \|\frac{\partial}{\partial n_{E\cap\partial T}}\Delta u_H\|_{0,E} + C_{\rm inv}^{(3)}\ h_E^{-1/2}\ \|\frac{\partial(u_h - u_H)}{\partial n_{E\cap\partial T}}\|_{0,E}. \tag{4.4c}$$

By an application of Young's inequality, in view of (2.11),(4.2) and observing $\alpha \ge 1$, from (4.4a),(4.4b) and the marking and refinement strategy we deduce the existence of $C_{\rm er} > 1$, depending only on the local geometry of the triangulations, such that for $\tau > 0$ there holds

$$\begin{aligned}
\eta_h^2 \le (1+\tau)(1-2^{-1/2})\Theta\Big(\sum_{T\in\mathcal{T}_H} H_T^4\|f - \Delta^2 u_H\|_{0,T}^2 + \qquad\qquad (4.5)\\
\sum_{E\in\mathcal{E}_H^\Omega}\Big(H_E\|[\frac{\partial^2 u_H}{\partial n}]_E\|_{0,E}^2 + H_E^3\|[\frac{\partial}{\partial n_E}\Delta u_H]_E\|_{0,E}^2\Big)\Big) + (1+\tau^{-1})\ C_{\rm er}\ \|u_h - u_H\|_{2,h,\Omega}^2,
\end{aligned}$$

which gives the assertion with $C_\tau := (1+\tau^{-1})\ C_{\rm er}$. □

5. Quasi-Orthogonality. As a further significant ingredient of the convergence analysis, in this section we prove quasi-orthogonality of the C^0-IPDG approach. We first provide a mesh perturbation result in subsection 5.1 and then establish quasi-orthogonality in subsection 5.2.

5.1. Mesh perturbation result. In the convergence analysis of IPDG methods for second order elliptic boundary value problems, mesh perturbation results estimating the coarse mesh error in the fine mesh energy norm from above by its coarse mesh energy norm have played a central role in the convergence analysis as a prerequisite for establishing a quasi-orthogonality result (cf., e.g., [6, 17, 18]). Here, we provide the following mesh perturbation result:

LEMMA 5.1. *Let $\mathcal{T}_h$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H$. Then, there exists a constant $C_P > 0$ such that for any $\varepsilon > 0$ and $v \in V + V_H$ there holds*

$$a_h^{IP}(v,v) \le (1+\varepsilon)\ a_H^{IP}(v,v) + \frac{C_P}{\gamma\varepsilon}\ \Big(\eta_{h,c}^2 + \eta_{H,c}^2\Big). \tag{5.1}$$

Proof. For $v \in V + V_H$ we have

$$\begin{aligned} a_h^{IP}(v,v) = & \sum_{T\in\mathcal{T}_h} \|D^2 v\|_{0,T}^2 + \sum_{E\in\mathcal{E}_h} \frac{\alpha}{h_E}\ \|[\frac{\partial v}{\partial n}]_E\|_{0,E}^2 \\ & +\ 2 \sum_{T\in\mathcal{T}_h} (L(\nabla v), D^2 v)_{0,T}. \end{aligned} \tag{5.2}$$

Obviously, there holds

$$\sum_{T\in\mathcal{T}_h} \|D^2 v\|_{0,T}^2 = \sum_{T\in\mathcal{T}_H} \|D^2 v\|_{0,T}^2. \tag{5.3}$$

Moreover, in view of (4.2) we have

$$\sum_{E\in\mathcal{E}_h} \frac{\alpha}{h_E}\ \|[\frac{\partial v}{\partial n}]_E\|_{0,E,h}^2 \le \kappa_2^{-1} \sum_{E\in\mathcal{E}_H} \frac{\alpha}{H_E}\ \|[\frac{\partial v}{\partial n}]_E\|_{0,E,H}^2. \tag{5.4}$$

Using (5.3) in (5.2), we find

$$\begin{aligned} a_h^{IP}(v,v) = a_H^{IP}(v,v) + & \sum_{E\in\mathcal{E}_h} \frac{\alpha}{h_E}\ \|[\frac{\partial v}{\partial n}]_E\|_{0,E}^2 - \sum_{E\in\mathcal{E}_H} \frac{\alpha}{H_E}\ \|[\frac{\partial v}{\partial n}]_E\|_{0,E}^2 \\ +\ 2 & \sum_{T\in\mathcal{T}_h} (L(\nabla v), D^2 v)_{0,T} - 2 \sum_{T\in\mathcal{T}_H} (L(\nabla v), D^2 v)_{0,T}. \end{aligned} \tag{5.5}$$

The assertion follows by using Young's inequality in (5.5) and taking (2.10),(2.13), and (5.3),(5.4) into account. □

5.2. Quasi-Orthogonality. The quasi-orthogonality result can be derived from the following property of the conforming approximations $u_H^c \in V_H^c, u_h^c \in V_h^c$ of (2.2) which are given as the unique solutions of

$$\begin{aligned} a(u_H^c, v_H^c) &= (f, v_H^c), \quad v_H^c \in V_H^c, \\ a(u_h^c, v_h^c) &= (f, v_h^c), \quad v_h^c \in V_h^c. \end{aligned} \tag{5.6}$$

LEMMA 5.2. *Let $\mathcal{T}_h$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H$, and let $u_h \in V_h$, $u_H \in V_H$ and η_h, η_H be the C^0-IPDG solutions of (2.6) and error estimators, respectively. Moreover, let $u_h^c \in V_h^c$ and $u_H^c \in V_H^c$ be the conforming approximations of (2.2) according to (5.6). Then, for $u_h^{nc} := u_h - u_h^c$ and $u_H^{nc} := u_H - u_H^c$ there holds*

$$\|u_h^{nc} - u_H^{nc}\|_{2,h,\Omega}^2 \le \frac{4C_J C_{nc}}{\gamma^2\alpha}\ \Big(\eta_h^2 + \eta_H^2\Big), \tag{5.7}$$

where C_{nc} and C_J are the constants from (3.4) and (3.12).

Proof. Due to (2.13) we have

$$\|u_h^{nc} - u_H^{nc}\|_{2,h,\Omega}^2 \le 2\ \Big(\|u_h^{nc}\|_{2,h,\Omega}^2 + \|u_H^{nc}\|_{2,h,\Omega}^2\Big) \qquad (5.8)$$
$$\le \frac{2}{\gamma}\ \Big(a_h^{IP}(u_h^{nc}, u_h^{nc}) + a_h^{IP}(u_H^{nc}, u_H^{nc})\Big).$$

On the other hand, in view of (3.4) and (3.12) there holds

$$\frac{2}{\gamma}\ a_h^{IP}(u_h^{nc}, u_h^{nc}) \le \frac{2C_{nc}}{\gamma\alpha}\ \alpha\ \eta_{h,c}^2 \le \frac{4C_J C_{nc}}{\gamma^2\alpha}\ \eta_h^2. \qquad (5.9)$$

Likewise, taking

$$\sum_{E\in\mathcal{E}_h} \frac{1}{h_E}\ \|[\frac{\partial u_H^{nc}}{\partial n}]_E\|_{0,E}^2 \le \kappa_2^{-1}\ \sum_{E\in\mathcal{E}_H} \frac{1}{H_E}\ \|[\frac{\partial u_H^{nc}}{\partial n}]_E\|_{0,E}^2$$

into account, we find

$$\frac{2}{\gamma}\ a_h^{IP}(u_H^{nc}, u_H^{nc}) \le \frac{4\kappa_2^{-1} C_J C_{nc}}{\gamma^2\alpha}\ \eta_H^2. \qquad (5.10)$$

Noting that $\kappa_2^{-1} < 1$, we conclude by using (5.9) and (5.10) in (5.8). □

The quasi-orthogonality result reads as follows:

THEOREM 5.3. *Let $\mathcal{T}_h$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H$, and let $u_h \in V_h$, $u_H \in V_H$ and η_h, η_H be the associated C^0-IPDG solutions of* (2.6) *and error estimators, respectively. Further, let $e_h := u - u_h$ and $e_H := u - u_H$ be the fine and coarse mesh errors. Then, for any $0 < \varepsilon < 1$ there exists a constant $C_Q > 0$, depending on $\gamma, C_1, C_J, C_{nc}, C_P$, such that there holds*

$$a_h^{IP}(e_h, e_h) \le \qquad (5.11)$$
$$(1+\varepsilon)\ a_H^{IP}(e_H, e_H) - \frac{\gamma}{2}\ \|u_h - u_H\|_{2,h,\Omega}^2 + \frac{C_Q}{\alpha\varepsilon}\ \Big(\eta_h^2 + \eta_H^2\Big).$$

Proof. In view of the partial Galerkin orthogonality

$$a_h^{IP}(e_h, u_h^c - u_H^c) = 0,$$

we get

$$a_h^{IP}(e_h, e_h) = a_h^{IP}(e_h + u_h^c - u_H^c, e_h + u_h^c - u_H^c) - \qquad (5.12)$$
$$a_h^{IP}(u_h^c - u_H^c, u_h^c - u_H^c).$$

Using $u_h + u_H^c - u_h^c = u_H - u_H^{nc} + u_h^{nc}$, (2.14), and Young's inequality, for the first term on the right-hand side of (5.12) we find that for some $0 < \varepsilon_1 < 1$ there holds

$$a_h^{IP}(e_h + u_h^c - u_H^c, e_h + u_h^c - u_H^c) = a_h^{IP}(e_H - u_h^{nc} + u_H^{nc}, e_H - u_h^{nc} + u_H^{nc}) \qquad (5.13)$$
$$\le a_h^{IP}(e_H, e_H) + C_1\ \|u_h^{nc} - u_H^{nc}\|_{2,h,\Omega}^2 + 2C_1^{1/2}\ a_h^{IP}(e_H, e_H)^{1/2}\ \|u_h^{nc} - u_H^{nc}\|_{2,h,\Omega}$$
$$\le (1+\varepsilon_1)\ a_h^{IP}(e_H, e_H) + C_1(1 + \frac{1}{\varepsilon_1})\ \|u_h^{nc} - u_H^{nc}\|_{2,h,\Omega}^2.$$

Moreover, in view of Lemma 5.1 and (3.12), for some $0 < \varepsilon_2 < 1$ we have

$$\begin{aligned}
&(1+\varepsilon_1)\ a_h^{IP}(e_H,e_H) \\
&\le\ (1+\varepsilon_1)\Big((1+\varepsilon_2)\ a_H^{IP}(e_H,e_H) + \frac{2C_JC_P}{\gamma^2\alpha\varepsilon_2}\ (\eta_h^2+\eta_H^2)\Big).
\end{aligned} \tag{5.14}$$

We choose $\varepsilon_1 = \varepsilon_2 = \varepsilon/4,\ 0 < \varepsilon < 1$, and obtain from (5.13), (5.14)

$$\begin{aligned}
&a_h^{IP}(e_h + u_h^c - u_H^c, e_h + u_h^c - u_H^c) \\
&\le (1+\varepsilon)\ a_H^{IP}(e_H,e_H) + \frac{16C_JC_P}{\gamma^2\alpha\varepsilon}(\eta_h^2+\eta_H^2) + C_1(1+\frac{4}{\varepsilon})\ \|u_h^{nc}-u_H^{nc}\|_{2,h,\Omega}^2.
\end{aligned} \tag{5.15}$$

On the other hand, due to (2.13), $u_H^c = u_H - u_H^{nc}, u_h^c = u_h - u_h^{nc}$, and Young's inequality, for the second term on the right-hand side of (5.12) we obtain

$$\begin{aligned}
&a_h^{IP}(u_h^c - u_H^c, u_h^c - u_H^c) \ge \gamma\ \|u_h^c - u_H^c\|_{2,h,\Omega}^2\ \ge \\
&\gamma\ \Big(\|u_h - u_H\|_{2,h,\Omega} - \|u_h^{nc}-u_H^{nc}\|_{2,h,\Omega}\Big)^2 \ge \frac{\gamma}{2}\|u_h-u_H\|_{2,h,\Omega}^2 - \gamma\ \|u_h^{nc}-u_H^{nc}\|_{2,h,\Omega}^2.
\end{aligned} \tag{5.16}$$

Taking advantage of the estimates (5.15) and (5.16) in (5.12), it follows that

$$\begin{aligned}
&a_h^{IP}(e_h,e_h) \le (1+\varepsilon)\ a_H^{IP}(e_H,e_H) - \frac{\gamma}{2}\|u_h-u_H\|_{2,h,\Omega}^2 + \\
&+ \frac{16C_JC_P}{\gamma^2\alpha\varepsilon}(\eta_h^2+\eta_H^2) + (C_1(1+\frac{4}{\varepsilon})+\gamma)\ \|u_h^{nc}-u_H^{nc}\|_{2,h,\Omega}^2.
\end{aligned} \tag{5.17}$$

The assertion now follows from Lemma 5.2. $\square$

6. Contraction property. We now use the error reduction property (4.3), the quasi-orthogonality (5.11), and the reliability (3.19) to prove the following contraction property:

THEOREM 6.1. *Let $u \in H_0^2(\Omega)$ be the unique solution of (2.2). Further, let $\mathcal{T}_h(\Omega)$ be a simplicial triangulation obtained by refinement from $\mathcal{T}_H(\Omega)$, and let $u_h \in V_h,\ u_H \in V_H$ and η_h, η_H be the C^0-IPDG solutions of (2.6) and error estimators, respectively. Then, there exist constants $0 < \delta < 1$ and $\rho > 0$, depending only on the local geometry of the triangulations and the parameter Θ from the Dörfler marking, such that for sufficiently large penalty parameter α the fine mesh and coarse mesh discretization errors $e_h := u - u_h$ and $e_H = u - u_H$ satisfy*

$$a_h^{IP}(e_h,e_h) + \rho\ \eta_h^2 \le \delta\ \Big(a_H^{IP}(e_H,e_H) + \rho\ \eta_H^2\Big). \tag{6.1}$$

Proof. Multiplying the estimator reduction property (4.3) by $\gamma/(2C_\tau)$ and substituting the result into the quasi-orthogonality (5.11), we obtain

$$\begin{aligned}
a_h^{IP}(e_h,e_h) + \rho\ \eta_h^2 &\le (1+\varepsilon)\ a_H^{IP}(e_H,e_H) \\
&\quad + \ \Big(\frac{C_Q}{\alpha\varepsilon} - \frac{\gamma}{2C_\tau} + \rho\Big)\ \eta_h^2 + \Big(\frac{C_Q}{\alpha\varepsilon} + \frac{\gamma\kappa(\Theta)}{2C_\tau}\Big)\ \eta_H^2.
\end{aligned} \tag{6.2}$$

If we choose $\alpha > (2C_QC_\tau)/(\gamma\varepsilon)$, we have $\rho := \gamma/(2C_\tau) - C_Q/(\alpha\varepsilon) > 0$, and it follows from (6.2) that

$$a_h^{IP}(e_h,e_h) + \rho\ \eta_h^2 \le (1+\varepsilon)\ a_H^{IP}(e_H,e_H) + \Big(\frac{C_Q}{\alpha\varepsilon} + \frac{\gamma\kappa(\Theta)}{2C_\tau}\Big)\ \eta_H^2. \tag{6.3}$$

Now, taking advantage of the reliability result

$$a_H^{IP}(e_H, e_H) \le C_R \, \eta_H^2$$

(cf. (3.19)), for $0 < \delta < 1$ we obtain

$$\begin{aligned} & a_h^{IP}(e_h, e_h) + \rho \, \eta_h^2 \le \\ & \delta \, a_H^{IP}(e_H, e_H) + \delta^{-1} \, \Big(C_R(1 + \varepsilon - \delta) + \big(\frac{C_Q}{\alpha\varepsilon} + \frac{\gamma\kappa(\Theta)}{2C_\tau}\big)\Big) \, \eta_H^2. \end{aligned} \tag{6.4}$$

We choose δ such that

$$\rho = \frac{\gamma}{2C_\tau} - \frac{C_Q}{\alpha\varepsilon} = \delta^{-1} \, \Big(C_R(1 + \varepsilon - \delta) + \big(\frac{C_Q}{\alpha\varepsilon} + \frac{\gamma\kappa(\Theta)}{2C_\tau}\big)\Big). \tag{6.5}$$

Solving for δ, we obtain

$$\delta = \frac{C_R\Big(1 + \varepsilon\Big) + \frac{C_Q}{\alpha\varepsilon} + \frac{\gamma\kappa(\Theta)}{2C_\tau}}{C_R + \frac{\gamma}{2C_\tau} - \frac{C_Q}{\alpha\varepsilon}}. \tag{6.6}$$

For instance, if we choose $\tau = \tau^* := 2^{-1/2}$ and $\varepsilon := \gamma/(8C_R C_{\tau^*})$, we have $\varepsilon < 1$ (due to $\gamma < 1,\ C_R > 1,\ C_{\tau^*} > 1$), and it follows that

$$\delta = \frac{C_R + \frac{\gamma}{8C_{\tau^*}} + \frac{8C_Q C_R C_{\tau^*}}{\alpha\gamma} + \frac{\gamma\Theta}{4C_{\tau^*}}}{C_R + \frac{\gamma}{2C_{\tau*}} - \frac{8C_Q C_R C_{\tau^*}}{\alpha\gamma}}. \tag{6.7}$$

Looking for α such that

$$\frac{\gamma}{8C_{\tau^*}} + \frac{8C_Q C_R C_{\tau^*}}{\alpha\gamma} + \frac{\gamma\Theta}{4C_{\tau^*}} < \frac{\gamma}{2C_{\tau*}} - \frac{8C_Q C_R C_{\tau^*}}{\alpha\gamma},$$

we find that $0 < \delta < 1$ for

$$\alpha > \frac{128 C_Q C_R C_{\tau^*}^2}{(3 - 2\Theta)\gamma^2}. \tag{6.8}$$

This concludes the proof of the contraction property. □

7. Numerical results. We provide a detailed documentation of the performance of the adaptive C^0-IPDG method for an illustrative example taken from [8].

Example: We choose Ω as the L-shaped domain $\Omega := (-1, +1)^2 \setminus ([0, 1) \times (-1, 0])$ and choose f in (2.1a) such that

$$u(r, \varphi) = \Big(r^2 \, \cos^2\varphi - 1\Big)^2 \Big(r^2 \, \sin^2\varphi - 1\Big)^2 \, r^{1+z} \, g(\varphi) \tag{7.1}$$

is the exact solution of (2.1a),(2.1b), where

$$\begin{aligned} & g(\varphi) := \\ & \Big(\frac{1}{z-1} \, \sin(\frac{3(z-1)\pi}{2}) - \frac{1}{z+1} \, \sin(\frac{3(z+1)\pi}{2})\Big)\Big(\cos((z-1)\varphi) - \cos((z+1)\varphi)\Big) - \\ & \Big(\frac{1}{z-1} \, \sin((z-1)\varphi) - \frac{1}{z+1} \, \sin((z+1)\varphi)\Big)\Big(\cos(\frac{3(z-1)\pi}{2}) - \cos(\frac{3(z-1)\pi}{2})\Big) \end{aligned}$$

and $z \approx 0.54448$ is a non-characteristic root of $\sin^2(\frac{3z\pi}{2}) = z^2 \sin^2(\frac{3\pi}{2})$.
For the documentation of the performance of the adaptive C^0-IPDG scheme, we have run simulations for polynomial degrees $2 \leq k \leq 6$ with penalty parameter $\alpha = 2.5(k+1)^2$. For $k = 2, k = 4$, and $k = 6$, Figures 7.1-7.6 show the adaptively refined meshes after 10 adaptive cycles (top left), the convergence histories in terms of the broken C^0-IPDG energy norm of the error $a_h^{IP}(u - u_h, u - u_h)$ as a function of the total number of degrees of freedom (DOF) on a logarithmic scale (top right), the decrease of the estimator as a function of the DOF (bottom left), as well as the computed effectivity indices $\eta_h / a_h^{IP}(u - u_h, u - u_h)^{1/2}$ (bottom right) for uniform refinement and adaptive refinement with $\Theta = 0.7$ and $\Theta = 0.3$ in the Dörfler marking. As far as the convergence rates and the estimator reduction are concerned, the benefits of adaptive versus uniform refinement can be clearly recognized, in particular for increasing polynomial degree. Moreover, as in case of IPDG methods for second order elliptic boundary value problems [17] and H-IPDG methods for Maxwell's equations [12] we observe a different convergence behavior depending on the choice of Θ in the Dörfler marking. The effectivity indices show a clear dependence on the polynomial degree k.

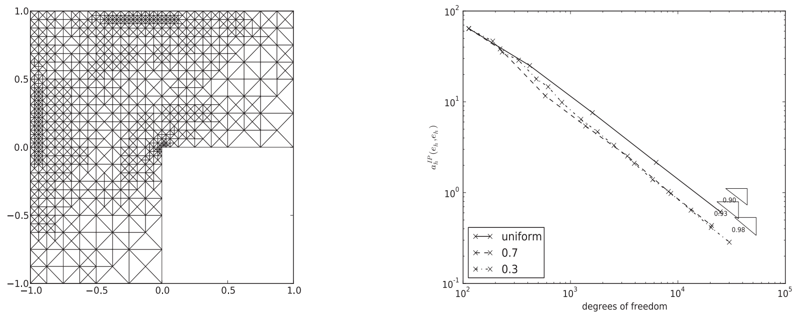

FIG. 7.1. $k = 2$: *Refined mesh after 10 adaptive cycles (left) and convergence history (right).*

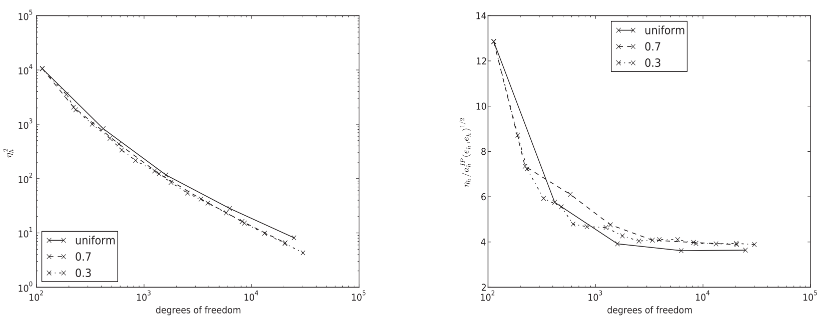

FIG. 7.2. $k = 2$: *Estimator reduction (left) and effectivity indices (right).*

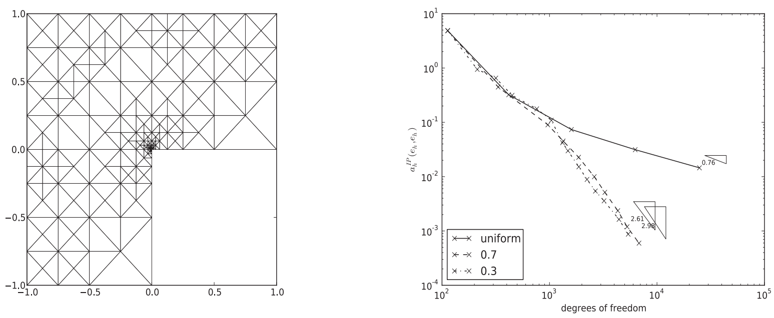

FIG. 7.3. $k = 4$*: Refined mesh after 10 adaptive cycles (left) and convergence history (right).*

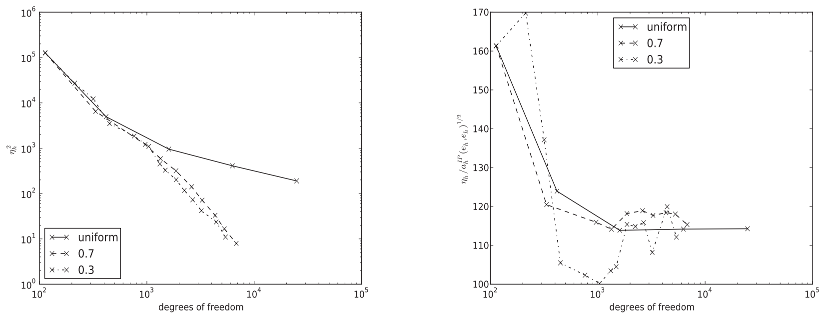

FIG. 7.4. $k = 4$*: Estimator reduction (left) and effectivity indices (right).*

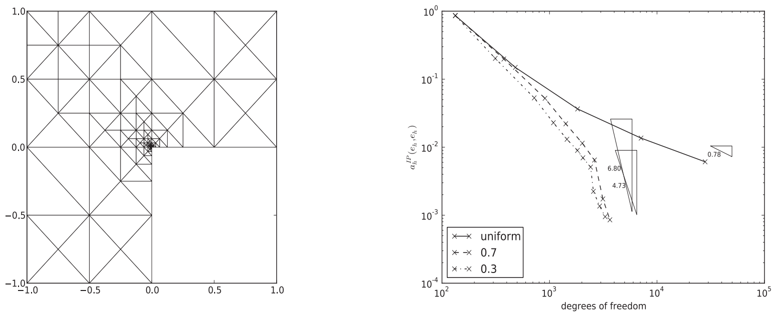

FIG. 7.5. $k = 6$*: Refined mesh after 10 adaptive cycles (left) and convergence history (right).*

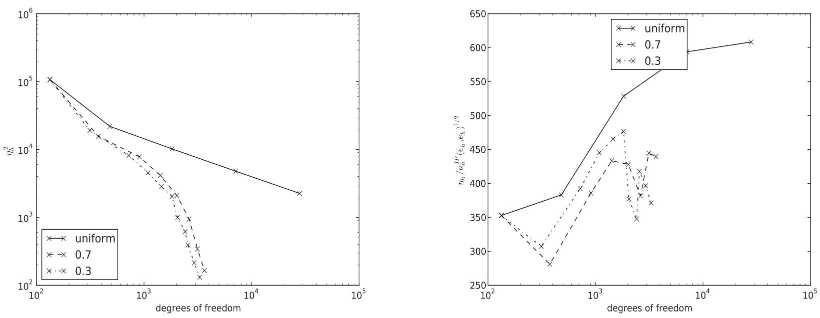

FIG. 7.6. $k = 6$*: Estimator reduction (left) and effectivity indices (right).*

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