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Abstract We propose axioms for Kleene modules (KM). These structures have a Kleene algebra K and a Boolean algebra B as sorts. The scalar products are mappings $K \times B \to B$; they arise as algebraic abstractions of relational image and preimage operations. KM is the basis of algebraic variants of dynamic logics. We develop a calculus for KM and discuss its relation to Kleene algebra with domain and to dynamic and test algebras. As an example, we apply KM to the reachability analysis in directed graphs.

Keywords: Idempotent semirings, Kleene algebra, propositional dynamic logic, dynamic and test algebra, image and preimage operation, state transition systems, program development and analysis, graph algorithms.

1 Introduction

Programs and state transition systems can be described in a bipartite world in which propositions model static properties, and actions or events model the dynamics. Propositions live in a Boolean algebra and actions in a Kleene algebra with the regular operations of sequential composition, non-deterministic choice and reflexive transitive closure. Propositions and actions cooperate via modal operators that view actions as mappings on propositions in order to describe state-change and via test operators that embed propositions into actions in order to describe observations on states and to model the usual program constructs.

Most previous approaches show an asymmetric treatment of propositions and actions. On the one hand, propositional dynamic logic (PDL) [11] and its algebraic relatives dynamic algebras (DA) [14,18,20] and test algebras (TA) [18,20,24] are proposition-based. DA has only modalities, TA has also tests. Most axiomatizations do not even contain explicit axioms for actions: their algebra is only implicitly imposed via the definition of modalities. On the other hand, Kleene algebra with tests (KAT) [16] — Kleene algebra with an embedded Boolean algebra — is action-based and, complementarily to DA, has only tests. Therefore, action-based reasoning in DA and TA and proposition-based reasoning in KAT is indirect and restricted. In order to overcome these rather artificial asymmetries

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and limitations, KAT has recently been extended to Kleene algebra with domain (KAD) with equational axioms for abstract domain and codomain operations [9]. This alternative to PDL supports both proposition- and action-based reasoning and admits both tests and modalities. The defining axioms of KAD, however, are quite different from those of DA and TA. Therefore, what is the precise relation between KAD and PDL and its algebraic relatives? Moreover, is the asymmetry and the implicitness of the algebra of actions in DA and TA substantial?

We answer these two questions by extending the above picture with the further intermediate structure KM of *Kleene modules* (cf. Figure 1).

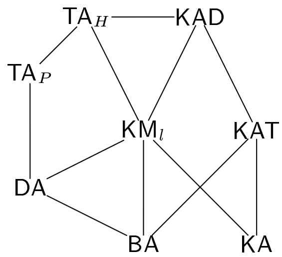


Figure1. Relations between algebraic systems

As already observed by Pratt [20], the definition of DA resembles that of a module in algebra, up to implicitness of the algebra of actions, in which the scalar products defines the modalities. When DA was presented, this was reasonable, since there was no satisfactory axiomatization of Kleene algebra. So Pratt could only conjecture that a KM with a Kleene algebra as scalar sort and a Boolean algebra as the other one would yield a more natural and convenient axiomatization of DA. Depending on more recent developments in Kleene algebra, our axiomatization of KM verifies Pratt's conjecture and shows that the implicitness of Kleene algebra in DA is in fact unnecessary. KM is also used as a key for answering the first question and establishing KAD as a natural extension of previous approaches.

Our contributions. First, we axiomatize and motivate the class KM as a straightforward adaptation of the usual modules from algebra [13]. We show that the scalar products abstractly characterize relational image and preimage operations. We outline a calculus for KM, including a duality between left and right scalar products in terms of a converse operation and a discussion of separability, that is, when actions are completely determined by their effects on states. We provide several examples of KMs. We also relate our approach to a previous one based on a second-order axiomatization of the star [14].

Second, we relate KM and DA. We show that KM subsumes DA and, using a result of [20], that the equational classes of separable KM and separable DA coincide. This answers Pratt's conjecture. Consequently, the axioms of separable KM are complete with respect to the equational theory of finite Kripke structures.

Third, we relate KAD to KM and TA. We identify KAD with a subclass of TA, but obtain a considerably more economic axiomatization of that class. We show that the equational classes of separable KADs and separable TAs coincide, improving on a previous related result [12]. Consequently, the axioms of separable KAD are complete for the equational theory of finite Kripke (test) structures; the equational theory of separable KAD is EXPTIME-complete.

Fourth, we present extensions of KM that subsume TA, its above-mentioned subclass and KAD. This clarifies some points in a related axiomatization [12].

Fifth, we demonstrate the expressibility gap between KM and KAD by defining a basic tool-kit for dynamic reachability analysis in directed graphs with interesting applications in the development and analysis of (graph and pointer) algorithms.

More generally, our technical comparison establishes KAD as a versatile alternative to PDL. Its uniform treatment of modal, scalar product and domain operators supports the inter-operability of different traditional approaches to program analysis and development, an integration of action- and proposition-based views and a unification of techniques and results from these approaches.

Related Work. We can only briefly mention some closely related work. Our semiring-based variants of Kleene algebra and KAT are due to Kozen [15,16]. DA has been proposed by Pratt [20] and Kozen [14] and further investigated, for instance, in [18,19]. TA has been proposed by Pratt [20] and further investigated in [18,24]. With the exception of [14], these approaches implicitly axiomatize the algebra of actions; the explicit Kleene algebra axioms for DA in [14] contain a second-order axiom for the star. More recently, Hollenberg [12] has proposed TA with explicit Kleene algebra axioms. This approach is similar to, but less economic than ours. The related class of Kleenean semimodules has recently been introduced by Leiß [17] in applications to formal language theory, with our Boolean algebra weakened to a semilattice. Earlier on, Brink [3] has presented Boolean modules, using a relation algebra instead of a Kleene algebra. A particular matrix-model of KM has been implicitly used by Clenaghan [6] for calculating path algorithms. In the context of reachability analysis, concrete models of Kleene algebras or relational approaches have also be used, for instance, by Backhouse, van den Eijnde and van Gasteren [2], by Brunn, Möller and Russling [4], by Ravelo [22] and by Berghammer, von Karger and Wolf [21]. Ehm [10] uses an extension of KM for analyzing pointer structures.

Survey. The remainder of this paper is organized as follows. Section 2 collects some basic properties of KA and KAT. Section 3 introduces KM, Section 4 discusses the two most important example structures, Section 5 presents the basic properties of the class. Section 6 compare KM with some related structures, among them Boolean algebras with operators. Section 7 introduces the concept of extensionality or separability. Section 8 to Section 11 relate KM to KAD, DA and TA. Section 12 discusses the previous results. Section 13 presents a main application of KM and KAD, namely reachability analysis in graphs and state transition systems. Section 14 contains a conclusion.

2 Kleene Algebra

In this section, we provide some preliminary definitions related to Kleene algebra, Boolean algebra and Kleene algebra with tests. We use a semiring-based variant of Kleene algebra as opposed to lattice-based ones. We also use a finitary first-order axiomatization in opposition to infinitary or second-order ones.

A *Kleene algebra* [15] is a structure $(K, +, \cdot, ^*, 0, 1)$ such that $(K, +, \cdot, 0, 1)$ is an (additively) idempotent semiring and *, the *star*, is a unary operation defined by the identities

$$1 + aa^* \leq a^*, \tag{$*$-1}$$

$$1 + a^*a \leq a^*, \tag{$*$-2}$$

and the quasi-identities

$$b + ac \leq c \Rightarrow a^*b \leq c, \tag{$*$-3}$$

$$b + ca \leq c \Rightarrow ba^* \leq c, \tag{$*$-4}$$

for all $a, b, c \in K$ (the operation $\cdot$ is omitted here and in the sequel). The relation $\leq$ is the *natural ordering* on K defined by $a \leq b$ iff $a + b = b$. We call ($*$-1) and ($*$-2) the *star unfold* laws and ($*$-3) and ($*$-4) the *star induction* laws. KA denotes the class of Kleene algebras.

In calculations, we often appeal to the *principle of indirect inequality* from order theory. Instead of $a \leq b$ we show $c \leq a \Rightarrow c \leq b$ or $b \leq c \Rightarrow a \leq c$ for some fixed arbitrary c.

Models of Kleene algebra are for instance the set-theoretic relations under set union, relational composition and reflexive transitive closure (the *relational Kleene algebra*), and the set of regular languages (regular events) over some finite alphabet (the *language Kleene algebra*).

The structure $(K, \leq)$ is an upper semilattice. Moreover, the operations of addition, multiplication and star are monotonic with respect to $\leq$. The algebra of regular languages over an alphabet A is the free Kleene algebra generated by A [15]; its equational theory coincides with the free equational theory of KA. Besides the semiring laws, we can therefore freely use the well-known regular identities in KA. The following lemma collects some of these together with some quasi-identities.

Lemma 1. *Let $K \in \mathsf{KA}$. For all $a, b, c, p \in K$,*

(i) $p \leq 1 \Rightarrow p^* = 1$,
(ii) $1 \leq a^*$,
(iii) $a^*a^* = a^*$,
(iv) $a \leq a^*$,
(v) $a^{**} = a^*$,
(vi) $(ab)^*a = a(ba)^*$,
(vii) $(a + b)^* = a^*(ba^*)^*$,
(viii) $1 + aa^* = a^*$,

(ix) $1 + a^*a = a^*$,
(x) $ac \leq cb \Rightarrow a^*c \leq cb^*$,
(xi) $ca \leq bc \Rightarrow ca^* \leq b^*c$.

A Kleene algebra is **-continous*, if

$$ab^*c = \sup(\{ab^nc \mid n \in \mathbb{N}\}) \tag{1}$$

holds for all $a, b, c \in K$; the powers of a are defined as $a^0 = 1$ and $a^{n+1} = aa^n$ for all $a \in K$ and $n \in \mathbb{N}$. Continuity is a second-order (or at least an infinitary first-order) property. In presence of (1), the axioms ($*$-1) to ($*$-4) are redundant. KA^* denotes the class of *-continuous KA. It is a strict subclass of KA.

Kleene algebra provides an algebra of actions with operations of non-deterministic choice, sequential composition and iteration. It can be enriched by a Boolean algebra to a two-sorted structure that incorporates both actions and proposition.

A *Boolean algebra* is a complemented distributive lattice. By overloading, we usually write $+$ and $\cdot$ also for the Boolean join and meet operation and use 0 and 1 for the least and greatest elements of the lattice. $'$ denotes the operation of complementation, $-$ denotes the operation of sectional complementation. It can either be defined by $p - q = pq'$ or by the Galois connection

$$p - q \leq r \Leftrightarrow p \leq q + r. \tag{2}$$

BA denotes the class of Boolean algebras. We will consistently use the letters $a, b, c \ldots$ for Kleenean elements and $p, q, r, \ldots$ for Boolean elements. We will freely use the standard laws of Boolean algebra in calculations.

A first integration of actions and propositions is given by a *Kleene algebra with tests* [16], which is a two-sorted structure (K, B), where $K \in \mathsf{KA}$ and $B \in \mathsf{BA}$ satisfies $B \subseteq K$ and has least element 0 and greatest element 1. In general, B is only a subalgebra of the subalgebra of all elements below 1 in K, since elements of the latter need not be multiplicatively idempotent. We call elements of B *tests* and write $\mathsf{test}(K)$ instead of B. KAT denotes the class of Kleene algebras with tests.

3 Definition of Kleene Modules

In this section we define the class of Kleene modules. These arise as natural variants of the usual modules in algebra [13]. Modules are two-sorted structures consisting of a ring and an Abelian group that interact via a scalar product; a mapping from the ring and the Abelian group into the Abelian group. To distinguish them from Kleene modules, we call them *standard* modules. We replace the ring by a Kleene algebra and the Abelian group by a Boolean algebra.

Definition 1. *A* Kleene left-module *is a two-sorted algebra* $(K, B, :)$*, where* $K \in \mathsf{KA}$ *and* $B \in \mathsf{BA}$ *and where the* left scalar product $:$ *is a mapping* $K \times B \to B$ *such that for all* $a, b \in K$ *and* $p, q \in B$*,*

$$a : (p + q) = a : p + a : q, \tag{km1}$$
$$(a + b) : p = a : p + b : p, \tag{km2}$$
$$(ab) : p = a : (b : p), \tag{km3}$$
$$1 : p = p, \tag{km4}$$
$$0 : p = 0, \tag{km5}$$
$$p + a : q \leq q \Rightarrow a^* : p \leq q. \tag{km6}$$

We do not distinguish notationally between the zeros and ones of the Kleene algebra and the Boolean algebra. In accordance with the relation-algebraic tradition, we call the scalar product of a Kleene left-module also a *Peirce product*. We assign the following priorities. Complements bind stronger than products and Peirce products, which again bind stronger than addition and sectional complement. We denote the class of Kleene left-modules by KM_l.

Let us first discuss these axioms. Axioms of the form (km1), (km2) and (km3) are well-known from the definitions of standard left-modules. An axiom of the form (km4) defines the class of unitary standard left-modules. This presupposes that the underlying ring has a unit. For standard modules, an axiom of the form (km5) is redundant. For semi-rings, that is in absence of ring-inverses, this is not the case. Axiom (km6) is of course beyond ring theory. It is the star induction rule ($*$-3) with the semiring product replaced by the Peirce product and the sorts of elements adjusted, that is b and c replaced by Boolean elements. We call such a transformation of a KA-expression to a KM-expression a *peircing*.

As usual in algebra, we define *Kleene right-modules* as Kleene left-modules over the opposite semiring. Here, the *opposite* of a semiring $(A, +, \cdot, 0, 1)$ is the structure $A^{op} = (A, +, \check{\cdot}, 0, 1)$, where $a \mathbin{\check{\cdot}} b = b \cdot a$. We write $p : a$ for right scalar products. The class of Kleene right-modules is denoted by KM_r. A *Kleene bimodule* is a Kleene left-module that is also a Kleene right-module. The class of Kleene bimodules is denoted by KM_b. Note that left and right scalar products can always be uniquely determined by bracketing. While standard bimodules have the additional axiom $(a : p) : b = a : (p : b)$ we do not require this for Kleene bimodules, for reasons discussed below.

Opposing induces a duality between left- and right-modules. We can therefore restrict our attention entirely to left-modules. The same duality can also be described in terms of a converse operation, as we will see Section 5. This yields another automatic translation between theorems in both structures. Kleene right-modules have interesting applications in reachability analysis for directed graphs, as we will see in Section 13.

4 Example Structures

We now discuss the two models of Kleene modules that are most important for our purposes, namely relational Kleene modules and Kripke structures.

Example 1. (**Relational Kleene modules**) Consider the relational Kleene algebra $\mathsf{REL}(A) = (2^{A\times A}, \cup, \circ, \emptyset, \Delta, {}^*)$, where A is a set, $2^{A\times A}$ denotes the set of binary relations over A, $\cup$ denotes set union, $\circ$ denotes relational product, $\emptyset$ denotes the empty relation, Δ denotes the identity relation and for all $R \in \mathsf{REL}(A)$ the expression R^* denotes the reflexive transitive closure of R, that is, $R^* = \bigcup_{i\geq 0} R^i$, where $R^0 = \Delta$ and $R^{i+1} = R \circ R^i$.

Of course, $\mathsf{REL}(A)$ is even in KAT^* with $\mathsf{test}(\mathsf{REL}(A))$ consisting of the set of all subrelations of Δ. This is so, since $\mathsf{test}(\mathsf{REL}(A))$ is a field of sets, whence a Boolean algebra, with $P\cap Q = P\circ Q$ and $P' = \Delta - P$, the minus now denoting set difference. Moreover, $\mathsf{test}(\mathsf{REL}(A))$ is isomorphic with the field of sets 2^A under the homomorphic extension of the mapping that sends B to $\{(b,b) \mid b \in B\}$ for all $B \subseteq A$.

The *preimage* and *image* of a set $B \subseteq A$ under a relation $R \subseteq A \times A$ are defined as

$$R : B = \{x \in A \mid \exists y \in B.(x,y) \in R\}, \tag{3}$$

$$B : R = \{y \in A \mid \exists x \in B.(x,y) \in R.\}. \tag{4}$$

It is easy to verify that $(\mathsf{REL}(A), 2^A), :)$ with $:$ given by (3), is in KM_l. Therefore the KM_l axioms abstractly model binary relations with a preimage operation. Dually, $(\mathsf{REL}(A), 2^A, :)$ with $:$ given by (4), is in KM_l and the structure equipped with both scalar products is in KM_b. □

Therefore, Kleene modules are algebraic abstractions of set-theoretic and relational structures. They provide a particularly interesting class of Kleene modules in which the Boolean algebra is embedded into the Kleene algebra. We will return to this class in Section 11.

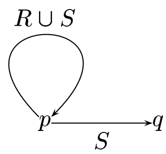


Figure2. Counterexample to bimodule equation

We will now show that, in opposition to standard bimodules, the identity $(a : p) : b = a : (p : b)$ does not hold in the relational Kleene bimodule. Consider the set $\{p,q\}$ and let $R = \{(p,p)\}$ and $S = \{(p,p),(p,q)\}$. This is shown in Figure 2. Obviously, $(R : \{p\}) : S = \{p,q\}$ and $R : (\{p\} : S) = \{p\}$. We did not use this identity in the definition of a Kleene bimodule, since we do not want to lose the relational Kleene bimodule.

Example 2. (**Kripke Structure**) As already mentioned in Example 1, there is an isomorphism between the subsets of a set A and the set of subrelations of the identity relation $\Delta \subseteq A \times A$. A *Kripke structure* on a set A is a pair (B, K), where B is a field of sets over A (whence a Boolean algebra) and K is an algebra of binary relations on A under the operations of union, relational composition and reflexive transitive closure. Finally, a preimage operation on (B, K) is defined by (3).

Every Kripke structure contains the identity relation, since it is presumed in the definition of the reflexive transitive closure operation. However, it need not contain the empty relation. Therefore, not every Kripke structure is a Kleene left-module, but every Kripke structure with the empty relation is. We will return to this fact in Section 8.

A *Kripke test structure* on A is a Kripke structure endowed with the additional operation

$$?p = \{(x, x) \mid x \in p\}$$

for all $p \in B$. We will return to these structures in Section 10. We denote the class of Kripke structures by Kri and the class of Kripke test structures by KriT.

The Kripke structure $(2^A, 2^{A \times A})$ is called the *full* Kripke structure on A. Thus every Kripke structure on A is a subalgebra of the full Kripke structure on A. The full Kripke structure on A is isomorphic to $\mathsf{REL}(A)$. □

More example structures can be found in [9]. Those examples are based on Kleene algebra with domain. But by the correspondence shown in Section 9, they can easily be transferred to Kleene modules.

5 Calculus of Kleene Modules

In this section, we prove some properties of Kleene modules that are helpful in an elementary calculus. These properties are also needed in the syntactic comparison and subsumption analysis of Kleene modules and related structures later in this paper.

We consider only left-modules. Analogous properties of right-modules hold automatically via opposing. We also show an alternative automatic translation between left- and right-modules that holds in presence of an operation of converse.

For the sake of elegance, we often use $p - q$ instead of pq', in particular using the associated Galois that gives us theorems for free (cf. [1]).

The first lemma provide some properties that do not mention the star.

Lemma 2. *Let $(K, B, :) \in \mathsf{KM}_l$. The scalar product has the following properties.*

(i) It is right-strict, that is $a : 0 = 0$ for all $a \in K$.

(ii) It is left- and right- monotonic, that is for all $a, b \in K$ and $p, q \in B$,

$$a \leq b \wedge p \leq q \Rightarrow a : p \leq b : q.$$

(iii) It is sub-multiplicative, that is $a : (pq) \leq (a : p)(a : q)$ for all $a \in K$ and $p, q \in B$.

(iv) $a : p - a : q \leq a : (p - q)$ for all $a \in K$ and $p, q \in B$.

Proof. (i) We calculate

$$a : 0 = a : (0 : p) = (a0) : p = 0 : p = 0.$$

The first step uses (km5), replacing equals by equals. The second step uses (km3). The third step uses Kleene algebra (0 is a right-annihilator). The fourth step uses (km5).

(ii) It is well-known that every function that distributes over suprema is monotonic.

(iii) It is well-known that every monotonic function is sub-distributive over infima.

(iv) The Galois connection (2) implies the cancellation law $p \leq q + (p - q)$. Using this together with right-monotonicity and (km1), we calculate

$$a : p \leq a : ((p - q) + q) = a : (p - q) + a : q,$$

whence $a : p - a : q \leq a : (p - q)$ by (2). □

The next lemma provides some properties in an extension of Kleene algebra with converse. In the relational semiring it is evident that the preimage of a relation under a set is the image of the converse relation under this set. We now investigate an abstract notion of converse that induces a further duality between left- and right-modules.

A *Kleene algebra with (weak) converse* is a structure $(K, ^\circ)$ such that K is a Kleene algebra and $^\circ$ a unary operation that satisfies the following equations. For all $a, b, p \in K$, $p \leq 1$,

$$a^{\circ\circ} = a, \tag{c1}$$

$$(a + b)^\circ = a^\circ + b^\circ, \tag{c2}$$

$$(ab)^\circ = b^\circ a^\circ, \tag{c3}$$

$$a^{*\circ} = a^{\circ *}, \tag{c4}$$

$$p^\circ \leq p. \tag{c5}$$

A Kleene algebra with converse is then a Kleene algebra with weak converse that satisfies also $a \leq aa^\circ a$ [8].

It is easy to show that $1^\circ = 1$, $0^\circ = 0$, $p^\circ = p$ and $a \leq b \Leftrightarrow a^\circ \leq b^\circ$ hold in KA with weak converse. Kleene modules with converse are interesting in the context of dynamic logic for which variants with program conversion exist. This allows one, for instance, to model backtracking.

Proposition 1. *Let* $(K, B, :) \in \mathsf{KM}_l$*. Then for all* $a \in K$ *and* $p \in B$*, then the operation* $:$ *of type* $B \times A \to B$ *defined by the equation*

$$p : a = a^\circ : p, \tag{5}$$

is a right-scalar product that turns $(K, B, :)$ *into a Kleene right-module.*

Proof. First we note that (c5) and (c1) imply $p^\circ = p$ for $p \leq 1$.

(Dual of (km1)) Using (5) and (km1), we calculate

$$(p + q) : a = a^\circ : (p + q) = a^\circ : p + a^\circ : q = p : a + q : a.$$

(Dual of (km2)) Using (5), (c2) and (km2), we calculate

$$p : (a + b) = (a + b)^\circ : p = a^\circ : p + b^\circ : p = p : a + p : b.$$

(Dual of (km3)) Using (5), (c3) and (km3), we calculate

$$p : (ab) = (ab)^\circ : p = b^\circ : (a^\circ : p) = (a^\circ : p) : b = (p : a) : b.$$

(Dual of (km4)) Using (5) and (km4), we calculate $p : 1 = 1^\circ : p = 1 : p = p$.

(Dual of (km5)) Using (5) and (km5), we calculate $p : 0 = 0^\circ : p = 0 : p = 0$.

(Dual of (km6)) Let $p + q : a \leq q$, whence $p + a^\circ : q \leq q$. Then $a^{\circ *} : p \leq q$ by (km6), which is equivalent to $p : a^* \leq q$ by (c4) and (5). □

Proposition 1 gives us an algorithm for automatically translating statements about KM_l into those about KM_r and vice versa.

The following statements deal with peirced variants of the well-known star rules in Kleene algebra. The first proposition explains why there is no peirced variant of ($*$-1) in the axioms for Kleene modules.

Proposition 2. *Let* $(K, B, :) \in \mathsf{KM}_l$*. Let* $a \in K$ *and* $p \in B$*.*

(i) $p + a : (a^* : p) = a^* : p$,
(ii) $p + a^* : (a : p) = a^* : p$.

Proof. (i) We calculate, using (km3), (km4), (km2) and Kleene algebra,

$$p + a : (a^* : p) = 1 : p + (aa^*) : p = (1 + aa^*) : p = a^* : p.$$

The proof of (ii) is similar. □

Corollary 1. *Let* $(K, B, :) \in \mathsf{KM}_l$*. Let* $a \in K$ *and* $p \in B$*.*

(i) $p \leq a^* : p$.
(ii) $a : (a^* : p) \leq a^* : p$.
(iii) $a^* : (a : p) \leq a^* : p$.
(iv) $a^* : 1 = 1$.
(v) $a \leq 1 \Rightarrow a^* : p = p$.
(vi) $a : p \leq a^* : p$.

(vii) $a^* : p = a^* : (a^* : p)$.
(viii) $a^* : p = a^{**} : p$.

Proof. (i)–(iii) are immediate from Proposition 2.
(iv) By (i), $1 \leq a^* : 1$. But 1 is the greatest element of B.
(v)–(viii) follow immediately from Kleene algebra. □

The following statement shows that the defining quasi-identity (km6), although quite natural as a peirced version of the Kleene algebra axiom (∗-3) is overly complex and can be replaced by an identity.

Proposition 3. *Let $(K, B, :) \in \mathsf{KM}_l$ Then the quasi-identity (km6) and the following identity are equivalent.*

$$a^* : p \leq p + a^* : (a : p - p). \tag{6}$$

Proof. The Galois connection (2) implies

$$p \leq q \Leftrightarrow p - q \leq 0$$

and the cancellation law

$$p \leq q + (p - q)\ .$$

(km6) implies (6). By (km6) it suffices to show that

$$p + a : (p + a^* : (a : p - p)) \leq p + a^* : (a : p - p).$$

We calculate

$$\begin{aligned} p + a : (p + a^* : (a : p - p)) &= p + a : p + a : (a^* : (a : p - p)) \\ &\leq p + ((a : p) - p) + a : (a^* : (a : p - p)) \\ &= p + a^* : (a : p - p), \end{aligned}$$

using the above cancellation law in the second step and Proposition 2 in the third step.
(6) implies (km6). Let $a : q + p \leq q$, whence $a : q \leq q$ and $p \leq q$ and therefore $a : q - q \leq 0$. We calculate, using right monotonicity and (km5),

$$a^* : p \leq a^* : q \leq q + a^* : (a : q - q) = q + a^* : 0 = q.$$

□

Lemma 3. *Let $(K, B, :) \in \mathsf{KM}_l$. Then (km6) (and (6)) and the following quasi-identity are equivalent.*

$$a : p \leq p \Rightarrow a^* : p \leq p. \tag{7}$$

Proof. (km6) implies (7). Set $p = q$ in (km6).
(7) implies (km6). Let $a : q + p \leq q$. This is the case iff $a : q \leq q$ and $p \leq q$. Then $a^* : q \leq q$ follows from (7). Hence also $a^* : p \leq q$ by right-monotonicity. □

(7) is a peirced variant of the quasi-identity

$$ac \leq c \Rightarrow a^*c \leq c$$

that is equivalent to (∗-3) in KA. Note that there is no identity corresponding to (6) in KA, which is not a finitely based variety.

Finally, we show that (6) can be strengthened to an equality.

Lemma 4. *Let* $(K, B, :) \in \mathsf{KM}_l$*. Then for all* $a \in K$ *and* $p \in B$*,*

$$a^* : p = p + a^* : (a : p - p).$$

Proof. By Proposition 3, it suffices to show that $p + a^* : (a : p - p) \leq a^* : p$. We calculate

$$p + a^* : (a : p - p) \leq p + a^* : (a : p) = a^* : p.$$

The first step uses Boolean algebra and right-monotonicity. The second step uses Proposition 2(ii). □

As we will see, most of the statements of this section can easily be translated into theorems or derived inference rules of propositional dynamic logic.

6 Related Structures

We now discuss some related structures. However, the most important relatives, namely dynamic algebras, test algebras and Kleene algebras with domain are discussed in separate sections.

We obtain the class KS_l of *Kleenean left-semimodules* [17] from Definition 1 by requiring a semilattice B instead of a Boolean algebra. This reduction is possible, since the Kleene module axioms mention neither the Boolean meet nor the Boolean complement.

Lemma 5. $\mathsf{KM}_l \subseteq \mathsf{KS}_l$.

We obtain the class KM_l^* of **-continuous Kleene left-modules* from Definition 1 by requiring $K \in \mathsf{KA}^*$ instead of KA and replacing (km6) by the peirced variant

$$a^* : p = \sup(\{a^n : p \,|\, n \in \mathbb{N}\}), \tag{8}$$

for all $a \in K$ and $p \in B$, of (1). KM_l^* has first been studied in [14] under the name *dynamic algebra* as an algebraic analog to propositional dynamic logics.

Lemma 6. $\mathsf{KM}_l^* \subseteq \mathsf{KM}_l$.

Proof. We must show that (8) implies (km6). We verify (7) instead, which is equivalent to (km6) by Lemma 3. Let $a : p \leq p$. Then $a^n : p \leq p$ for all $n \in \mathbb{N}$ by a simple induction and therefore $a^* : p = \sup(\{a^n : p \,|\, n \in \mathbb{N}\}) \leq p$ by definition of the supremum. □

We obtain the class BM_l of *Boolean (left-)modules* [3] from Definition 1 by requiring a relation algebra [23] K instead of a Kleene algebra and adding the axiom

$$a^\circ : (a : p)' \leq p', \tag{9}$$

for all $a \in K$ and $p \in B$.

Lemma 7. $\mathsf{BM}_l \subseteq \mathsf{KM}_l^*$.

Proof. Define the star in relation algebra as a reflexive transitive closure operation. ☐

Comparing KM_l with these structures, we see the following benefits. On the one hand, KS is too poor for our intended application, that is for modeling an algebra of propositions. Statements like Proposition 3 cannot even be expressed. KM^* and BM, on the other hand, are generalized by KM. In particular, the full relation algebra in BM makes this structure overly rich for programming applications. For instance, complements of programs can be modeled, although this may be irrelevant in practice.

The following examples establish the connection between Kleene modules, modal algebras and predicated transformer algebras.

A *Boolean algebra with operators* is a structure $(B, \{f_i : i \in I\})$, where B is a Boolean algebra endowed with a family $\{f_i : i \in I\}$ of strict additive endofunctions (also called *hemimorphisms*), that is $f(0) = 0$ and $f(p + q) = f(p) + f(q)$. These structures are a starting point for the investigation of modal logics and algebras (cf. [5]). We denote the class by BAO.

Lemma 8. $\mathsf{KM}_l \subseteq \mathsf{BAO}$.

Proof. For every $(B, K, :) \in \mathsf{KM}_l$, the mappings $f_a = \lambda x.a : x$, with indices $a \in K$ are hemimorphisms on B. They are additive by axiom (km1) and strict by Lemma 2 (i). ☐

An expression $a : p$ can therefore be written as a (multi)modal diamond operator $\langle a \rangle p$; the dual box operators $[a]p$ are given by $(a : p')'$. KM_l therefore is a class of *modal algebras*. By this translation, all statements from Section 5 correspond to valid expressions in propositional dynamic logic (cf. [11]).

In the context of BAO, the axioms (km2) and (km3) express compositionality of hemimorphisms with respect to the index algebra: $f_{a+b} = f_a + f_b$ and $f_{ab} = f_a f_b$.

An *algebra of monotonic predicate transformers* is a structure $(B, \{f_i : i \in I\})$, where B is a Boolean algebra endowed with a family $\{f_i : i \in I\}$ of endofunctions that satisfy $p \leq q \Rightarrow f_i(p) \leq f_i(q)$. We denote the class by MPT.

Lemma 9. $\mathsf{BAO} \subseteq \mathsf{MPT}$.

Proof. By the proof of Lemma 2 (ii), additivity implies right-monotonicity. ☐

The results of this section are summed up in Figure 3.

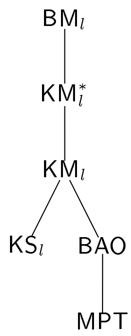


Figure3. From predicate transformer algebras to bimodules

7 Extensionality

In Kleene modules, the algebra of actions and that of propositions are only weakly coupled. The finer the algebra of propositions, the more precisely can we observe properties of actions. In general, actions are *intensional*, that is, their behavior is not completely determined by observations on states. Set-theoretic relations, however, are *extensional*, simply because they are sets and sets are completely determined by their members: Let A and B be sets. Then $A = B$ if $a \in A \Leftrightarrow a \in B$ holds for all a.

This extensionality property can be lifted to Kleene modules. In analogy to dynamic algebra [20,14], we call $(K, B, :) \in \mathsf{KM}$ *(left)-separable*, if for all $a, b \in K$

$$\forall p \in B.(a : p \leq b : p) \Rightarrow a \leq b. \tag{10}$$

For every algebraic class V with appropriate signature, we denote the separable subclass by SV.

Lemma 10. *Let* $(K, B, :) \in \mathsf{KM}_l$*. Then (10) and the following quasi-identity are equivalent.*

$$\forall p \in B.(a : p = b : p) \Rightarrow a = b. \tag{11}$$

Proof. (10) implies (11). Let $a : p = b : p$ for all $p \in B$, thus also $a : p \leq b : p$ for all $b \in B$ and $b : p \leq a : p$ for all $b \in B$. Consequently, $a \leq b$ and $b \leq a$, whence $a = b$ by (10).

(11) implies (10). Using (km2), we calculate

$$\begin{aligned} \forall p\,.\,(a : p \leq b : p) &\Leftrightarrow \forall p\,.\,(a : p + b : p = b : p) \\ &\Leftrightarrow \forall p\,.\,((a + b) : p = b : p) \\ &\Rightarrow a + b = b \\ &\Leftrightarrow a \leq b. \end{aligned}$$

□

The following corollary is immediate from left-monotonicity.

Corollary 2. *Let* $(K, B, :) \in \mathsf{SKM}_l$*. Let* $a, b \in K$*.*

(i) $\forall p \in B.(a : p \leq b : p) \Leftrightarrow a \leq b$*.*
(ii) $\forall p \in B.(a : p = b : p) \Leftrightarrow a = b$*.*

The term *separability* may perhaps better be motivated by the following property that is equivalent to (11): Let $a \neq b$, $a, b \in K$. Then $a : p \neq b : p$ for some $p \in B$. Thus this witness p allows us to separate action a from action b.

Lemma 11. *Separability is independent in* KM_l*.*

Proof. Consider the structure $(K, B, :) = (\{a, 0, 1\}, \{0, 1\} :)$ with addition, multiplication and scalar multiplication tables

$+$	0	a	1
0	0	a	1
a	a	a	a
1	1	a	1

$\cdot$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

$:$	0	1
0	0	0
a	0	1
1	0	1

It has been shown in [7] that K is an idempotent semiring with natural ordering defined by $0 \leq 1 \leq a$, which can be uniquely extended to a Kleene algebra by setting $0^* = 1^* = 1$ and $a^* = a$. Moreover, $B = \mathsf{test}(K)$. By the scalar multiplication, $x : p$ is mapped to 0 if one of x and p is 0 and to 1 else. It is then easily verified that $(K, B, :) \in \mathsf{KM}_l$.

However, $1 : p = a : p$ holds for $p = 0, 1$, but $1 < a$. Thus separability fails. □

Besides this relational motivation, separability can also be introduced algebraically via a (pre)congruence. This is interesting by itself, since it introduces a notion of observational equivalence.

Consider the relation $\preceq$ on $(K, A, :) \in \mathsf{KM}_l$ defined by

$$a \preceq b \Leftrightarrow \forall p \in B. a : p \leq b : p, \tag{12}$$

for all $a, b \in K$.

Lemma 12. *The relation* $\preceq$ *is a precongruence on* KM_l*.*

Proof. First, we show that $a \preceq b$ implies $a + c \preceq b + c$. Using (km2) we calculate

$$(a + c) : p = a : p + c : p \leq b : p + c : p = (b + c) : p.$$

Second, we show that $a \preceq b$ implies $ca \preceq cb$. Using (km3) and right-monotonicity we calculate

$$(ca) : p = c : (a : p) \leq c : (b : p) = (cb) : p$$

Third, we show that $a \preceq b$ implies $ac \preceq bc$. Using (km3) we calculate

$$(ac) : p = a : (c : p) \leq b : (c : p) = (bc) : p.$$

Fourth, we show that $a \preceq b$ implies $a^* \preceq b^*$. Let $a : p \leq b : p$. Using Lemma 2 (i) we calculate

$$b^* : p = p + b : (b^* : p) \geq p + a : (b^* : p).$$

Thus $a^* : p \leq b^* : p$ by (km6). $\square$

Corollary 3. *The relation $\sim$ on $(K, B, :) \in \mathsf{BM}_l$ defined by*

$$a \sim b \Leftrightarrow \forall p \in B. a : p = b : p \tag{13}$$

for all $a, b \in K$ is a congruence on KM_l.

Corollary 4. *A Kleene module is separable iff $\sim$ is the identity relation.*

For a set A, the preimage $R : \{p\}$ of a relation $R \subset A \times A$ under a singleton set $\{p\} \in A$ is the set of all $q \in A$ with $(q, p) \in R$. Intuitively, $R : \{p\}$ scans R point-wise for its input-output behavior. Since relations are extensional, they are completely determined by this scanning. In intensional models, one can distinguish between observable and hidden intrinsic behavior. The congruence $\sim$ then identifies two actions up to intrinsic behavior and therefore via observational equivalence. The freedom of choosing the algebra of propositions in KM with arbitrary coarseness fits very well with this idea of measuring and identifying actions in a more or less precise way.

A deeper investigation of these concepts is beyond the scope of this paper.

8 Kleene Modules and Dynamic Algebra

In the remaining sections of this text, we position the class KM_l within the context of Kleene algebra with domain and algebraic variants of propositional dynamic logic. Most of the results are subsumption results. Most of our arguments are purely syntactic. We show that the axioms of the subsumed class are theorems of the subsuming class. Some of our statements go beyond a purely syntactic analysis. But these rely on previous semantic work of others.

In this section, we compare KM_l with dynamic algebra [20,18,24].

We obtain the class DA of *dynamic algebras* from Definition 1 by requiring an absolutely free algebra of Kleene algebra signature K (without 0 and 1) instead of a Kleene algebra, such that, for all $a, b \in K$ and $p, q \in B$,

$$a : (p + q) = a : p + a : q, \tag{km1}$$

$$(a + b) : p = a : p + b : p, \tag{km2}$$

$$(ab) : p = a : (b : p), \tag{km3}$$

$$a : 0 = 0, \tag{14}$$

$$p + a : (a^* : p) \leq a^* : p, \tag{15}$$

$$a^* : p \leq p + a^* : (a : p - p). \tag{6}$$

Proposition 4. $\mathsf{KM}_l \subseteq \mathsf{DA}$.

Proof. We have to show that the axioms (14), (15) and (6) of DA are theorems of KM_l. (14) has been shown in Lemma 2 (i). (15) has been shown in Proposition 2 (i). (6) has been shown in Proposition 3. ☐

As usual, we now write $HSP(\mathsf{V})$ for the equational class or variety generated by a class V of algebras. This is the class of homomorphic images of subalgebras of products of algebras in V, according to Birkhoff's theorem.

The following result is due to Pratt (Theorem 6.4. of [20]).

Theorem 1 ([20]). $HSP(\mathsf{SDA}) = HSP(\mathsf{Kri})$.

Based on this result, Pratt conjectures that $HSP(\mathsf{SDA})$ *may be defined axiomatically by the dynamic algebra axioms [...] together with an appropriate set of axioms for binary relations.* In the late 1970ies, when Pratt wrote the first version of his paper, the axomatization of KA presented in Section 2 did not yet exist. The following corollaries of Theorem 1 and Proposition 4 verify Pratt's conjecture.

Note, however, that in SDA, the existence of a Kleenean zero or a one is not assumed; in Kri, there need not be a zero relation (c.f Section 4). Let now SDA_{01} be the class of separable dynamic algebras with additional Kleenean constants 0 and 1 satisfying the axioms (km4) and (km5). Let Kri_0 be the class of Kripke structures that contain the empty relation and the identity relation. Inspection of the proof of Theorem 6.4. in [20] shows that Theorem 1 can be adapted as follows.

Corollary 5. $HSP(\mathsf{SDA}_{01}) = HSP(\mathsf{Kri}_1)$.

As a consequence, we obtain the following relation between the equational theories of separable Kleene modules and separable dynamic algebras with one and zero.

Corollary 6. $HSP(\mathsf{SDA}_{01}) = HSP(\mathsf{SKM}_l)$.

Proof. By Corollary 5, $HSP(\mathsf{SDA}_{01}) = HSP(\mathsf{Kri}_0)$. By Proposition 4 $\mathsf{KM}_l \subseteq \mathsf{DA}_{01}$ and therefore $\mathsf{SKM}_l \subseteq \mathsf{SDA}_{01}$. This result specializes to identities and consequently $HSP(\mathsf{SKM}_l) \subseteq HSP(\mathsf{SDA}_{01})$. Since Kripke structures with zero and one are models of SKM_l, we also have $HSP(\mathsf{Kri}_0) = HSP(\mathsf{SKM}_l)$ and therefore $HSP(\mathsf{SDA}_{01}) = HSP(\mathsf{KM}_l)$. ☐

Lemma 4 and Corollary 6 show that Kleene modules provide a natural alternative to dynamic algebra.

9 Kleene Algebras with Domain Subsume Kleene Modules

In this section we show that Kleene algebra with domain is a more flexible tool than Kleene modules. A simple example is that pa where p is a proposition and

a is an action is well-formed in Kleene algebra with domain, whereas it is not in KM_l. Consequently, program constructs like conditional or while-loops are beyond the expressibility of KM_l.

A *Kleene algebra with domain* [9] is a structure (K, δ), where $K \in \mathsf{KAT}$ and the *domain operation* $\delta : K \to \mathsf{test}(K)$ satisfies for all $a, b \in K$ and $p \in \mathsf{test}(K)$

$$a \leq \delta(a)a, \tag{d1}$$

$$\delta(pa) \leq p, \tag{d2}$$

$$\delta(a\delta(b)) \leq \delta(ab). \tag{d3}$$

The class of Kleene algebras with domain is denoted by KAD. The impact of (d1), (d2) and (d3) can be motivated as follows. (d1) is equivalent to one implication in each of the statements

$$\delta(a) \leq p \Leftrightarrow a \leq pa, \tag{llp}$$

$$\delta(a) \leq p \Leftrightarrow p'a \leq 0, \tag{gla}$$

which constitute elimination laws for δ. (d2) is equivalent to the other implications, respectively. (llp) says that $\delta(a)$ is the least left preserver of a. (gla) says that $\delta(a)'$ is the greatest left annihilator of a. Both properties obviously characterize domain in set-theoretic relations. (d3) states that the domain of ab is not determined by the inner structure of b or its codomain; information about $\delta(b)$ in interaction with a suffices. All three axioms hold in relational Kleene algebra. Note that in opposition to KM_l, there is no particular axiom for the star. As Lemma 13 (vii) below shows, a variant of the star induction law is a theorem of KAD. (d1) and (d2) suffice for many applications, but here, (d3) is essential.

Like for Kleene modules, a codomain operation can be defined in the opposite Kleene algebra. Moreover, the following properties of domain follow from the domain axioms.

Lemma 13 ([9]). *Let $K \in \mathsf{KAD}$. For all $a \in K$ and $p \in \mathsf{test}(A)$, the domain operation satisfies the following laws.*

(i) Strictness, $\delta(a) = 0 \Leftrightarrow a = 0$.
(ii) Additivity, $\delta(a + b) = \delta(a) + \delta(b)$.
(iii) Monotonicity, $a \leq b \Rightarrow \delta(a) \leq \delta(b)$.
(iv) Locality, $\delta(ab) = \delta(a\delta(b))$.
(v) Import/Export, $\delta(pa) = p\delta(a)$.
(vi) Stability, $\delta(p) = p$.
(vii) Induction, $\delta(ap) \leq p \Rightarrow \delta(a^*p) \leq p$.

Proof. Because of their particular interest, we give proofs of (ii) and (vii).

(ii) Using (gla), we calculate

$$\begin{aligned}\delta(a + b) \leq p &\Leftrightarrow p'(a + b) \leq 0\\ &\Leftrightarrow p'a \leq 0 \wedge p'b \leq 0\\ &\Leftrightarrow \delta(a) \leq p \wedge \delta(b) \leq p\\ &\Leftrightarrow \delta(a) + \delta(b) \leq p.\end{aligned}$$

Then $\delta(a+b) = \delta(a) + \delta(b)$ by the principle of indirect inequality.

(vii) Using (llp) and Kleene algebra, we calculate

$$\delta(ap) \leq p \Leftrightarrow ap \leq pap \Rightarrow ap \leq pa \Rightarrow a^*p \leq pa^* \Leftrightarrow \delta(a^*p) \leq p.$$

□

Of course, the relational preimage can also be defined using domain. $\delta(RP)$ yields the preimage of relation R under the set P. The image is defined similarly using codomain. Abstractly, we define

$$a : p = \delta(ap) \tag{16}$$

in KAD. Of course, domain reasoning can also be performed in KM_l via

$$\delta(a) = a : 1. \tag{17}$$

These two translation laws are the key to further subsumption analysis.

Proposition 5. $\mathsf{KAD} \subseteq \mathsf{KM}_l$.

Proof. We show that the KM_l axioms are theorems in KAD, using (16) and the results of Lemma 13.

(km1) Using additivity of domain (Lemma 13 (ii)), we calculate

$$a : (p+q) = \delta(a(p+q)) = \delta(ap+aq) = \delta(ap) + \delta(aq) = a : p + a : q.$$

(km2) Using again additivity of domain, we calculate

$$(a+b) : p = \delta((a+b)p) = \delta(ap+bp) = \delta(ap) + \delta(bp) = a : p + b : p.$$

(km3) Using locality of domain (Lemma 13 (iv)), we calculate

$$(ab) : p = \delta(abp) = \delta(a\delta(bp)) = a : (b : p).$$

(km4) Using Kleene algebra and stability of domain (Lemma 13 (vi)), we calculate

$$1 : p = \delta(1p) = \delta(p) = p.$$

(km5) Using Kleene algebra and strictness of domain (Lemma 13 (i)), we calculate

$$0 : p = \delta(0p) = \delta(0) = 0.$$

(km6). We show (7) instead, which is equivalent to (km6) by Lemma 3. Let $a : p \leq p$, thus $\delta(ap) \leq p$. Then $\delta(a^*p) \leq p$ and therefore $a^* : p \leq p$ follows from domain induction (Lemma 13 (vii)). Thus (7) holds. □

By Proposition 4 and Proposition 5, we obtain the following corollary.

Corollary 7. $\mathsf{KAD} \subseteq \mathsf{DA}$.

We have thus shown that Kleene algebra provides a much more compact way for representing Kleene modules and dynamic algebra. It is also more expressive, since programming constructs using expressions like pa cannot be written in KM_l or DA.

10 Kleene Algebra with Domain and Test Algebra

We now compare Kleene algebra with domain and test algebras. We distinguish two axiomatizations. First, Pratt [20] extends the signature of dynamic algebra with a test operator ? of type $B \to K$ and he adds the axiom

$$p? : q = pq \tag{18}$$

to the axioms of dynamic algebra. We denote the class of test algebras à la Pratt by TA_P.

Proposition 6. $\mathsf{KAD} \subseteq \mathsf{TA}_P$.

Proof. According to Corollary 7, $\mathsf{KAD} \subseteq \mathsf{DA}$. It therefore suffices to show that the axiom (18) of TA_P is a theorem of KAD. Note that the test operator vanishes in KAD, since the Boolean algebra is implicitly embedded into the Kleene algebra. By (16) we must show that $\delta(pq) = pq$ holds in KAD. This is immediate from stability of domain (Lemma 13 (vi)). □

The second axiomatization has been given by Hollenberg [12]. He uses a two-sorted structure $(K, B, :)$ with $K \in \mathsf{KA}$, $B \in \mathsf{BA}$ and the axioms

$$\begin{aligned}
(a+b) : p &= a : p + b : p, && \text{(km2)}\\
(ab) : p &= a : (b : p), && \text{(km3)}\\
a^* : p &\leq p + a^* : (a : p - p), && (6)\\
p? : q &= pq, && (18)\\
0? &= 0, && (19)\\
(p+q)? &= p? + q?, && (20)\\
(pq)? &= (p?)(q?), && (21)\\
(a : 1)?a &= a. && (22)
\end{aligned}$$

We denote the class of test algebras à la Hollenberg by TA_H. We first discuss his axioms.

Lemma 14. *Let* $(K, B, :, ?) \in \mathsf{TA}_H$. *Then ? is an embedding.*

Proof. (18)–(21) establish that ? is a homomorphism from the distributive sublattice with zero of B into a distributive sublattice with zero of K. In [12] it has been shown that ? preserves 1. It is also easy to show that $(p')? = (p?)'$. Thus ? is a homomorphism from B into a Boolean subalgebra of K that identifies ones and zeros.

It remains to show that ? is injective. Let $p? = q?$. Then $p? : 1 = q? : 1$ and therefore $p1 = q1$, whence $p = q$ by (18). □

According to Lemma 14, we can again make ? implicit and restrict our attention to the structure $(K, \mathsf{test}(K), :)$, discard the axioms (19)–(21) and the ? symbol in the axioms (18) and (22).

Proposition 7.

(i) $\mathsf{TA}_H \subseteq \mathsf{KM}_l$.
(ii) $\mathsf{TA}_H \subseteq \mathsf{DA}$.
(iii) $\mathsf{TA}_H \subseteq \mathsf{TA}_P$.

Proof. (i) In [12], it has been shown that the axioms (km1), (km4) and (km6) of KM_l are theorems of TA_H. The axioms (km2) and (km3) of KM_l are also axioms of TA_H. (6) is equivalent to (km6) by Proposition 3.

(ii) By Proposition 4, $\mathsf{KM}_L \subseteq \mathsf{DA}$. Now use (i).

(iii) By (ii), $\mathsf{TA}_H \subseteq \mathsf{DA}$. TA_P is DA plus the axiom (18), which is also an axiom of TA_H. □

Proposition 8. $\mathsf{KAD} = \mathsf{TA}_H$.

Proof. We first show that $\mathsf{KAD} \subseteq \mathsf{TA}_H$. By Proposition 5 and Proposition 6, all axioms of TA_H but (22) are theorems of KAD. According to Lemma 14, (22) can be written in the form $(a:1)a = a$, whence as $\delta(a)a = a$ by (16). $a \leq \delta(a)a$ is axiom (d1) of KAD. The converse direction holds, since $\delta(a) \leq 1$. □

We now show that $\mathsf{TA}_H \subseteq \mathsf{KAD}$. By the previous part of the proof it remains to show that axioms (d2) and (d3) are theorems of TA_H.

For (d2), we must show that $(pa) : 1 \leq p$ by (17). Using (km3) and (18), which are axioms of TA_H, and Boolean algebra, we calculate

$$(pa) : 1 = p : (a : 1) = p(a : 1) \leq p.$$

For (d3), we must show that $(a(b:1)) : 1 = (ab) : 1$. We calculate

$$(a(b:1)) : 1 = a : ((b:1) : 1) = a : ((b:1)1) = a : (b:1) = (ab) : 1.$$

The first step uses (km3). The second step uses (18), the third step uses Boolean algebra, the fourth step uses again (km3). □

This result shows that from an axiomatic point of view, KAD is a considerable improvement over TA_H.

Corollary 8. *The axioms (km2) and (km6) are redundant in* TA_H*. The axioms (19)–(21) can be made implicit, using* KAT *for axiomatizing* TA_H*.*

The following theorem is a straightforward adaptation of a semantic statement (Corollary 1) from [24]

Theorem 2 ([12]). $HSP(\mathsf{STA}_H) = HSP(\mathsf{KriT})$.

This and Proposition 8 immediately yield the following corollary.

Corollary 9. $HSP(\mathsf{SKAD}) = HSP(\mathsf{KriT})$.

11 Extending Kleene Modules to Kleene Algebra with Domain

In this section, we consider extensions of KM_l that subsume KAD and TA_P. In particular, these extensions clarify the appearance of some axioms in TA_H.

Proposition 9.

(i) $\mathsf{KM}_b \subseteq \mathsf{TA}_P$.
(ii) $\mathsf{SKM}_b \subseteq \mathsf{STA}_H$.
(iii) $\mathsf{SKM}_b \subseteq \mathsf{SKAD}$.

Proof. (i) $\mathsf{KM}_l \subseteq \mathsf{DA}$ by Proposition 4. Since Kleene bimodules are expansions of Kleene left-modules, and TA_P is DA with the additional axiom (18), it suffices to show that (18) is a theorem of KM_b. We calculate

$$pq = 1 : (pq) = (1 : p) : q = p : q.$$

The first step uses (km4). The second step uses the dual of (km3). The third step uses again (km4).

(ii) Because of the redundancies in TA_H, it suffices to show that (22) is a theorem of SKM_b. We calculate

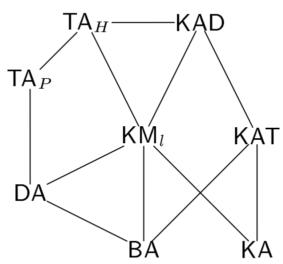
$$a : p = (a : p)(a : p) \leq (a : 1)(a : p) = (a : 1) : (a : p) = ((a : 1)a) : p.$$

The second step uses right-montonicity. The third step uses (18), which is a theorem of KM_b. The fourth step uses (km3). Separability now implies that $a \leq (a : 1)a$. The converse inequality follows by monotonicity and $a : 1 \leq 1$.

(iii) By Proposition 8 and (ii). □

12 Discussion

The subsumption results of the previous sections have already been summed up in Figure 1 that we repeat here for convenience.



At the bottom of the picture we find BA and KA, which are pure algebras of propositions or actions. KAT provides tests, but no modalities, KM_l provides modalities, but no tests. KAT supports reasoning at the side of actions, whereas KM_l provides reasoning at the side of propositions. It is evident, however, by (llp) and (gla), that some modal expressions can be translated to KAT expressions and vice versa. Therefore KAT supports indirect and restricted modal reasoning. Also, in presence of separability, reasoning with actions can be simulated by reasoning with propositions in KM_l. In this sense, the two classes are complementary. DA is a companion to KM_l in which the Kleene algebra is implicitly axiomatized, whereas it is explicit in KM_l. KM_l and KAT are combined into KAD. This class provides both tests and modalities and supports action- and proposition-based reasoning. KAD is the same class as TA_H, but apart from the Kleenean and the Boolean axioms, it has only three axioms instead of eight. The use of KAT instead of an explicit embedding of Boolean elements into Kleenenan elements leads to additional economy of expression. Moreover, the axioms of KAD have a natural motivation as abstractions of set-theoretic domain operations, whereas the axiom (22) is not motivated in [12]. Again, TA_P can be seen as a companion to TA_H and KAD with implicitly axiomatized Kleene algebra.

The results of this paper, in particular Corollary 6, Proposition 8 and Corollary 9, allow us to carry over previous results about TA to KAD.

We obtain, for instance, completeness results for KAD from completeness results for TA_H (Theorem 3.2 and Theorem 3.14 in [12]).

Proposition 10. *The axioms of* SKAD *are complete with respect to the valid equations in* KriT*, both of Boolean and of Kleenean sort.*

We also obtain complexity results.

Corollary 10. *HSP*(SKAD) *is EXPTIME-complete.*

Proof. Immediate from the linear translation of KAD identities to equivalences in propositional dynamic logic, whose validity is EXPTIME-complete [11]. □

A transfer of similar results between DA and KM_l is also possible.

Proposition 11. *The axioms of* SKAD *are complete with respect to the valid equations in* Kri.

Such completeness result are the basis of the representation theory for KM_l and KAD. A deeper investigation of these semantic issues is, however, out of the scope of the syntactic analysis of this paper.

13 Reachability Analysis in Directed Graphs

We now define some basic operations and predicates on KM and KAD that are appropriate for the abstract analysis of directed graphs, interpreted as finite relations. That is, we motivate our considerations through the relational model. Our tool-kit has many interesting applications in the development and analysis of programs and state transition systems, in the analysis of pointer and object structures and in garbage collection algorithms. The main idea is to work in abstract Kleene algebra as long as possible and to descend to the particular structures, for instance matrices, only when needed. Properties in the particular models can then be formulated as *bridge lemmas* and used as hypotheses in Kleene algebra.

Here, we do not consider the classical modal properties that are expressible in KM and KAD due to the BAO-connection. We restrict our attention to operations and properties related to reachability. Also an analysis of concrete algorithms, like for instance, cycle detection, (topological) sorting or shortest path algorithms are beyond the scope of this paper.

In KM, we represent edge sets of a graph by Kleenean elements and sets of nodes by Boolean elements. The following operation collects the set of nodes that are *reachable* from a set p via a.

$$\mathsf{reach}(p,a) = p : a^*. \tag{23}$$

Similarly, we can collect the set of nodes that are *non-reachable* from p via a.

$$\mathsf{nreach}(p,a) = \mathsf{reach}(p,a)'. \tag{24}$$

We now define two predicates that express the property that some set q is reachable and non-reachable from a set p via a. Intuitively,

$$\mathsf{reach\text{-}p}(p,a,q) \Leftrightarrow q \le \mathsf{reach}(p,a), \tag{25}$$

$$\mathsf{nreach\text{-}p}(p,a,q) \Leftrightarrow q \le \mathsf{nreach}(p,a). \tag{26}$$

Interestingly, nreach-p can be expressed already in KAT.

Lemma 15. *Let $K \in \mathsf{KAD}$. Then for all $a \in K$ and $p, q \in \mathsf{test}(K)$,*

$$\textit{nreach-p}(p,a,q) \Leftrightarrow pa^*q \le 0.$$

Proof. We calculate

$$\begin{aligned} \mathsf{nreach\text{-}p}(p,a,q) &\Leftrightarrow q \le \mathsf{nreach}(p,a) \\ &\Leftrightarrow q \le \mathsf{reach}(p,a)' \\ &\Leftrightarrow \mathsf{reach}(p,a) \le q' \\ &\Leftrightarrow \delta(pa^*) \le q' \\ &\Leftrightarrow pa^*q \le 0. \end{aligned}$$

The last step uses (gla). ☐

Note that there is no similar fact for reach-p, nor does Lemma 15 hold in KM. Moroever $\mathsf{nreach\text{-}p}(p,a,q)$ and $\neg\mathsf{reach}(p,a,q)$ are not logically equivalent. $\neg\mathsf{reach}(p,a,q)$ holds if q contains some element that is not reachable from p via a, whereas $\mathsf{nreach}(p,a,q)$ holds if all elements of q are not reachable from p via a.

Another interesting set are the nodes of a that are not reachable via a from p. The set can be defined in KM as follows.

$$\mathsf{snreach}(p,a) = \delta(a)\mathsf{nreach}(p,a). \tag{27}$$

Using reach, we can also characterize the final nodes with respect to reachability from p.

$$\mathsf{final}(p,a) = \mathsf{reach}(p,a)\delta(a)'. \tag{28}$$

When reach describes the run of the main loop of some program a from some set of initial states, final represents the set of final or terminal states of a. Similarly,

$$\mathsf{nfinal}(p,a) = \mathsf{reach}(p,a)\delta(a) \tag{29}$$

characterizes the set of unfinished computations of some program. The expression $\mathsf{nreach}(p,a)\delta(a)'$ is of no particular interest to us. It characterizes the unreachable states in a state-space which do not belong to a. Obviously,

$$\begin{aligned}\mathsf{reach}(p,a) &= \mathsf{final}(p,a) + \mathsf{nfinal}(p,a),\\ 0 &= \mathsf{final}(p,a)\mathsf{nfinal}(p,a),\\ 0 &= \mathsf{snreach}(p,a)\mathsf{nfinal}(p,a).\end{aligned}$$

For many applications, for instance the analysis of dynamic graph algorithms, updates, that is insertion and deletion of edges in a graph is important. These properties must be modeled in KAD instead of KM.

$$\mathsf{del}(a,b) = \delta(a)'b, \tag{30}$$

$$\mathsf{ins}(a,b) = a + \mathsf{del}(a,b). \tag{31}$$

Note that del behaves as suggested by its name only when a represents a single edge. Then it deletes a from the set of edges b. In general, it deletes all edges of b whose starting point lies in $\delta(a)$. ins behaves as expected also when a represents a set of edges. If $\delta(a)\delta(b) = 0$, $\mathsf{ins}(a,b)$ simply inserts new elements into b. Otherwise, the old elements of b are overwritten by a.

We now present another graph property that cannot be characterized in KM. We say that graph b is the *span* of graph a with respect to the set of nodes p, if b is the subgraph of a whose nodes are reachable from p. This can be defined in KAD as

$$\mathsf{span}(p,a) = \mathsf{reach}(p,a)a. \tag{32}$$

We now collect some basic properties of these operations. The first set of properties deals with unfolding reach.

Lemma 16. *Let* $(K,B,:) \in \mathsf{KM}$. *Let* $a \in K$ *and* $p,q \in B$.

(i) $\mathsf{reach}(p,a) = p + \mathsf{reach}((p:a) - p,\ a)$.
(ii) $\mathsf{reach}(p,a) = p + \mathsf{reach}(p,a) : a$.
(iii) $\mathsf{reach}(p,a) = p + \mathsf{reach}(p:a,a)$.
(iv) $\mathsf{reach}(p,a) = p + \mathsf{reach}(p:a,\ p'a)$ *for* $(K,B,:) \in \mathsf{KAD}$.

Proof. (i) Dualize the identity shown in Lemma 4.

(ii) Dualize the peirced star unfold rule from Lemma 2 (i).

(iii) Dualize the peirced star unfold rule from Lemma 2 (ii).

(iv) Dualize an identity shown in [9]. ☐

We now present elimination rules for reach.

Lemma 17. *Let* $(K, B, :) \in \mathsf{KM}$. *Let* $a \in K$ *and* $p, q \in B$.

(i) $p : a \leq p \Rightarrow \mathsf{reach}(p, a) \leq p$.
(ii) $q : a + p \leq q \Rightarrow \mathsf{reach}(p, a) \leq q$.

Proof. Dualize (7) and (km6). ☐

The above lemmas immediately imply similar properties for nreach, snreach, final, nfinal and span. We now show further properties of reach that immediately transfer to the other operations.

Lemma 18. *Let* $(K, B, :) \in \mathsf{KM}$. *Let* $a, b \in K$ *and* $p, q \in B$.

(i) $\mathsf{reach}(p + q, a) = \mathsf{reach}(p, a) + \mathsf{reach}(q, a)$.
(ii) $p \leq q \wedge a \leq b \Rightarrow \mathsf{reach}(p, a) \leq \mathsf{reach}(q, b)$.
(iii) $\mathsf{reach}(p, a) \leq \mathsf{reach}(p, a + b)$.
(iv) $\mathsf{reach}(\mathsf{reach}(p, a), a) = \mathsf{reach}(p, a)$.

Proof. (i) Immediate from (km1).

(ii) Immediate from left and right monotonicity of the scalar product and from and monotonicity of the star.

(iii) Immediate from (ii).

(iv) We calculate

$$\mathsf{reach}(\mathsf{reach}(p, a), a) = (p : a^*) : a^* = p : (a^* a^*) = p : a^* = \mathsf{reach}(p, a).$$

☐

Lemma 19. *Let* $K \in \mathsf{KAD}$. *For every* $a, b \in K$ *and* $p \in \mathsf{test}(K)$,

$$\textit{nreach-p}(p, a, \delta(b)) \Rightarrow \mathsf{reach}(p, a + b) \leq \mathsf{reach}(p, a).$$

Proof. First observe that by Lemma 15, $\mathsf{nreach\text{-}p}(p, a\delta(b))$ is equivalent to

$$\mathsf{reach}(p, a)\delta(b) \leq 0.$$

By Lemma 17 (ii), the dual of (km2) and Lemma 16) (ii), it suffices to show that

$$\begin{aligned} \mathsf{reach}(p, a) &\geq p + \mathsf{reach}(p, a) : (a + b) \\ &= p + \mathsf{reach}(p, a) : a + \mathsf{reach}(p, a) : b \\ &= \mathsf{reach}(p, a) + \mathsf{reach}(p, a) : b. \end{aligned}$$

But $\mathsf{reach}(p,a) : b$ vanishes by assumption, (d1), (km3) and (18), since

$$\begin{aligned}\mathsf{reach}(p,a) : b &= \mathsf{reach}(p,a) : (\delta(b)b)\\ &= (\mathsf{reach}(p:a) : \delta(b)) : b\\ &= (\mathsf{reach}(p:a)\delta(b)) : b\\ &= 0.\end{aligned}$$

□

The next lemma relates reach and span.

Lemma 20. *Let $K \in \mathsf{KAD}$. For all $a \in K$ and $p \in \mathsf{test}(K)$,*

$$\mathsf{reach}(p,a) = p + 1 : \mathsf{span}(p,a).$$

Proof. Using Lemma 16 (ii) and (18), we calculate

$$\begin{aligned}p + 1 : \mathsf{span}(p,a) &= p + 1 : (\mathsf{reach}(p,a)a)\\ &= p + (1 : (\mathsf{reach}(p,a)) : a\\ &= p + (1(\mathsf{reach}(p,a)) : a\\ &= \mathsf{reach}(p,a).\end{aligned}$$

□

Lemma 21. *Let $K \in \mathsf{KAD}$. Let $a, b \in K$ and $p, q \in \mathsf{test}(K)$.*

(i) $\mathsf{span}(p,a) = \mathsf{span}(p,b) \Rightarrow \mathsf{reach}(p,a) = \mathsf{reach}(p,b)$.
(ii) $\mathsf{span}(p, \mathsf{span}(p,a)) = \mathsf{span}(p,a)$.

Proof. (i) Immediate from Lemma 20.
(ii) Let $b = \mathsf{span}(p,a) = \mathsf{reach}(p,a)a$. If we can show that

$$\mathsf{reach}(p,a) = \mathsf{reach}(p,b), \tag{33}$$

we are done, since using (33) we can calculate

$$\mathsf{span}(p,b) = \mathsf{reach}(p,b)b = \mathsf{reach}(p,b)\mathsf{reach}(p,a)a = \mathsf{reach}(p,a)a = \mathsf{span}(p,a).$$

For claim (33), first note that $\mathsf{reach}(p,b) \leq \mathsf{reach}(p,a)$, since $b \leq a$ and by monotonicity of reach. For the converse direction, it suffices by Lemma 17 (ii) to show that $p + \mathsf{reach}(p,b) : a \leq \mathsf{reach}(p,b)$. We calculate

$$\begin{aligned}&p + \mathsf{reach}(p,b) : a\\ &= p + (\mathsf{reach}(p,b)(\mathsf{reach}(p,a) + \mathsf{reach}(p,a)')) : a\\ &= p + (\mathsf{reach}(p,b)\mathsf{reach}(p,a)) : a + (\mathsf{reach}(p,b)\mathsf{reach}(p,a)') : a\\ &\leq p + (\mathsf{reach}(p,b)\mathsf{reach}(p,a)) : a + (\mathsf{reach}(p,a)\mathsf{reach}(p,a)') : a\\ &= p + (\mathsf{reach}(p,b) : \mathsf{reach}(p,a)) : a + 0 : a\\ &= p + \mathsf{reach}(p,b) : (\mathsf{reach}(p,a)a)\\ &= p + \mathsf{reach}(p,b) : b\\ &= \mathsf{reach}(p,b).\end{aligned}$$

The first step uses the definition of complement. The second step uses (km1). The third step uses $\mathsf{reach}(p, b) \leq \mathsf{reach}(p, a)$ and monotonicity. The fourth step uses again the definition of complement and (18), which is a theorem of KM. The fifth step uses strictness of scalar products and (km3). The sixth step uses the definition of b. The seventh step uses Lemma 16 (ii). □

The following lemma is analogous to Lemma 19.

Lemma 22. *Let $K \in \mathsf{KAD}$. For every $a, b \in K$ and $p \in \mathsf{test}(K)$,*

$$\textit{nreach-p}(p, a, \delta(b)) \Rightarrow \mathsf{span}(p, a + b) \leq \mathsf{span}(p, a).$$

Proof. Using Lemma 19, and in particular the initial observation in its proof, we calculate

$$\begin{aligned}\mathsf{span}(p, a + b) &= \mathsf{reach}(p, a + b)(a + b)\\ &\leq \mathsf{reach}(p, a)a + \mathsf{reach}(p, a)b\\ &= \mathsf{span}(p, a) + \mathsf{reach}(p, a)\delta(b)b\\ &= \mathsf{span}(p, a).\end{aligned}$$

□

The next lemma collects some properties of ins.

Lemma 23. *Let $K \in \mathsf{KAD}$. Let $a, b, c \in K$.*

(i) $a \leq \mathsf{ins}(a, b)$.
(ii) $a = \delta(a)\mathsf{ins}(a, b)$.
(iii) $a = \mathsf{ins}(a, a)$.
(iv) $\mathsf{ins}(a, b + c) = \mathsf{ins}(a, b) + \mathsf{ins}(a, c)$.
(v) $\delta(\mathsf{ins}(a, b)) = \delta(a) + \delta(b)$.

The next lemma relates snreach to span.

Lemma 24. *Let $K \in \mathsf{KAD}$. For every $a \in K$ and $p \in \mathsf{test}(K)$*

$$\mathsf{snreach}(p, a) = \delta(a)\delta(\mathsf{span}(p, a))'.$$

Proof.

$$\begin{aligned}\delta(a)\delta(\mathsf{span}(p, a))' &= \delta(a)\delta(\mathsf{reach}(p, a)a)'\\ &= \delta(a)(\mathsf{reach}(p, a)\delta(a))'\\ &= \delta(a)\mathsf{reach}(p, a)' + \delta(a)\delta(a)'\\ &= \delta(a)\mathsf{reach}(p, a)'\\ &= \delta(a)\mathsf{nreach}(p, a)\\ &= \mathsf{snreach}(p, a)\end{aligned}$$

□

The next lemma relates reach, span and ins.

Lemma 25. *Let $K \in \mathsf{KAD}$. For every $a, b \in K$ and $p \in \mathsf{test}(K)$,*

(i) $\mathit{nreach\text{-}p}(p, a, \delta(b)) \Rightarrow \mathsf{reach}(p, \mathsf{ins}(b, a)) = \mathsf{reach}(p, a)$.
(ii) $\mathit{nreach\text{-}p}(p, a, \delta(b)) \Rightarrow \mathsf{span}(p, \mathsf{ins}(b, a)) = \mathsf{span}(p, a)$.

Proof. (i) $\mathsf{reach}(p, \mathsf{ins}(b, a)) \geq \mathsf{reach}(p, a)$ holds by Lemma 18 (iii). For the converse direction, we calculate, using Lemma 19 and the assumption

$$\mathsf{reach}(p, \mathsf{ins}(b, a)) = \mathsf{reach}(p, a + \delta(a)'b) \leq \mathsf{reach}(p, a + b) \leq \mathsf{reach}(p, a).$$

(ii) Similar to (i), using Lemma 22. □

14 Conclusion

We have presented an axiomatization of Kleene modules as a complementation to Kleene algebra with domain. This allows a fine-grained comparison with algebras related to propositional dynamic logic. Our results support a transfer between concepts and techniques from set- and relation-based program development methods and those based on modal logics. It encompasses the state-based and event-based view. Although the striking correspondence between scalar products, relational preimage operations and modal operators is not entirely new, we find it still surprising. On the theoretical side, our results are only first steps of the representation theory for KM_l and KAD. A deeper investigation of these semantic issues is beyond the syntactic analysis of this paper. On the practical side, we have already started considering applications in the development of graph, pointer and greedy algorithms.

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