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Peter Quast

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Universität Augsburg

Institut für
Mathematik

Peter Quast

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Institut für Mathematik, Universitätsstraße, D-86135 Augsburg

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Peter Quast

Institut für Mathematik

Universität Augsburg

86135 Augsburg

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REAL FORMS OF HERMITIAN SYMMETRIC SPACES, REVISITED

PETER QUAST

ABSTRACT. In 1984 Masaru Takeuchi showed that every real form of a hermitian symmetric space of compact type is a symmetric R -space. In this paper we present a geometric proof of Takeuchi's result.

1. INTRODUCTION

Symmetric R -spaces are compact Riemannian symmetric spaces that are also R -spaces (generalized flag manifolds) in the sense of Takeuchi [T65], that is quotients of connected center-free semi-simple Lie groups by parabolic subgroups. Symmetric R -spaces can be realized as s -orbits of extrinsically symmetric elements (see [N65, Ko68, KT68, Ke71, Ke72, F74a] and Section 2). From this point of view they are the building blocks of extrinsically symmetric submanifolds in Euclidean space (see [F74b, F80, EH95]). Indecomposable symmetric R -spaces split into two types:

- (i) irreducible hermitian symmetric spaces of compact type;
- (ii) indecomposable symmetric R -spaces of non-hermitian type.

The (local) classification of indecomposable symmetric R -spaces is due to Kobayashi and Nagano [KN64, KN65] (see also [BCO03, p. 310f]). In this note we give a *geometric* proof of Takeuchi's result:

Theorem 1 (Masaru Takeuchi [T84]). *Every symmetric R -space can be realized as a real form of a hermitian symmetric space of compact type. Vice-versa every real form of a hermitian symmetric space of compact type is a symmetric R -space.*

Our main tool is the extension of isometries of hermitian symmetric spaces of compact type presented in [EQT13].

A classification of real forms of hermitian symmetric spaces can also be found in [Le79].

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2. PRELIMINARIES

2.1. Symmetric R -space as s -orbits. Every symmetric R -space arises in the following way (see [N65, Ko68, KT68, Ke71, Ke72, F74a] and also [BCO03, pp. 70-72]):

Let S be a symmetric space of compact type (we always assume symmetric spaces to be connected) with a chosen base point $o \in S$, and L the identity component of the isometry group of S . For the classical facts about symmetric spaces mentioned below we refer to the standard literature like Helgason's famous monograph [H78] or [W84, Part IV]. The geodesic symmetry s_o of S at the base point o gives rise to an involutive automorphism

$$\sigma : L \rightarrow L, \quad l \mapsto s_o \circ l \circ s_o.$$

The differential σ_* of σ at the identity is therefore an involutive automorphism of the Lie algebra \mathfrak{l} of L , called the *Cartan involution* of (S, o) . We denote by \mathfrak{h} the fixed point set and by \mathfrak{s} the (-1) -eigenspace of σ_* . The decomposition

$$\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{s},$$

called *Cartan decomposition* of \mathfrak{l} corresponding to (S, o) , is orthogonal w.r.t. the Cartan-Killing form $B_{\mathfrak{l}}$ of \mathfrak{l} . This decomposition satisfies the *Cartan relations*, namely

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{s}] \subset \mathfrak{s} \text{ and } [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{h}.$$

The Lie subalgebra \mathfrak{h} is the Lie algebra of the identity component H of the isotropy group of o in G . Moreover, \mathfrak{s} can be identified with the tangent space $T_o S$ by the restriction of the differential at the identity of the principal bundle $L \rightarrow S$, $l \mapsto l.o$. Here $l.o$ denotes the action of the isometry l of S on the point $o \in S$. Using the above identification $\mathfrak{s} \cong T_o S$, the linear isotropy action of H on $T_o S$, also known as *s -representation*, becomes the restriction of the adjoint action:

$$H \times \mathfrak{s} \rightarrow \mathfrak{s}, \quad (h, X) \mapsto \text{Ad}_L(h)X.$$

A non-zero element $\xi \in \mathfrak{s}$ is called *extrinsically symmetric* (or *minuscule coweight*), if

$$\text{ad}_{\mathfrak{l}}(\xi)^3 = -\text{ad}_{\mathfrak{l}}(\xi),$$

or equivalently, if the eigenvalue spectrum of $\text{ad}(\xi)$ equals $\{-i, 0, i\}$. For a description of extrinsically symmetric elements in terms of roots we refer to [MQ12, Lem. 2.1] and also to [KN64].

A *symmetric R -space* is an isotropy orbit (s -orbit)

$$M = \text{Ad}_L(H)\xi \subset \mathfrak{s},$$

of S where $\xi \in \mathfrak{s}$ is an extrinsically symmetric element.

If S is an irreducible symmetric space of compact type, we call M *indecomposable*. If S is an irreducible symmetric space of compact

type, but not a compact simple Lie group, M is an indecomposable symmetric R -space of non-hermitian type (see e.g. [BCO03, p. 310f.]).

2.2. Hermitian symmetric spaces of compact type as R -spaces.

If S is a compact connected semi-simple center-free Lie group G , then L is isomorphic to $G \times G$. (see [H78, Chap. IV, §6]). The linear isotropy representation on the tangent space $T_e G$ is isomorphic to the adjoint representation of G on \mathfrak{g} .

Let $\xi \in \mathfrak{g}$ be extrinsically symmetric and assume that the projections of ξ onto the simple factors of \mathfrak{g} never vanish. It is well-known (see [Li58, pp. 165 ff.] and [Hi70]) that $P := \text{Ad}(G)\xi \subset \mathfrak{g}$, endowed with the Riemannian metric induced by the scalar product $-B_{\mathfrak{g}}$ on \mathfrak{g} , is a hermitian symmetric space of compact type. Let $X \in P$, then $\text{Ad}(\exp(\pi/2 \cdot X))$ and $\text{ad}(X)$ coincide on $T_X P \subset \mathfrak{g}$ and they define a Kähler structure J_X of P at the point X , that is

$$(1) \quad J_X = \text{Ad}(\exp(\pi/2 \cdot X))|_{T_X P} = \text{ad}(X)|_{T_X P},$$

which turns P into a hermitian symmetric space.

The geodesic symmetry s_X of P at the point X extends to the reflection ρ_X of \mathfrak{g} along the normal space $N_X P = \{Y \in \mathfrak{g} : \text{ad}(X)Y = 0\}$ given by the involutive automorphism

$$(2) \quad \rho_X := \text{Ad}(\exp(\pi X))$$

of \mathfrak{g} . Finally, G can be identified with the identity component of the isometry group of P .

Conversely every hermitian symmetric space P of compact type can be realized as such an orbit ([Li58, pp. 165 ff.] and [Hi70]) in the Lie algebra of its infinitesimal isometries. If we endow this Lie algebra with a scalar product that coincides on each irreducible factor with the Cartan-Killing form up to a suitable negative constant, this embedding is isometric. We call this the *standard embedding* of a hermitian symmetric space of compact type.

2.3. Real forms of hermitian symmetric spaces.

Following Takeuchi [T84], a *real form* of a hermitian symmetric space P is a connected component of the fixed point set of some involutive and anti-holomorphic isometry f of P . Therefore real forms are in particular totally geodesic real submanifolds of P .

3. THE PROOF

In this section we present a geometric proof of Takeuchi's result, Theorem 1 (see [T84]). As a major tool we use in Paragraph 3.2 the results of Eschenburg, Tanaka and the author on the extension of isometries of hermitian symmetric spaces published in [EQT13].

3.1. The proof of the first implication. The arguments presented in this paragraph are classical and straight forward. They may also be adapted to more general situations.

Let S be a symmetric space of compact type, $o \in S$ a base point, σ_* the corresponding Cartan involution and $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{s}$ the induced Cartan decomposition of the semi-simple Lie algebra \mathfrak{l} of infinitesimal isometries of S . Let $\xi \in \mathfrak{s}$ be an extrinsic symmetric element and $M := \text{Ad}_L(H)\xi$ a symmetric R -space. We may assume that no projection of ξ onto a simple factor of \mathfrak{l} vanishes.

The inclusion $H \hookrightarrow L$ of the identity component H of the isotropy group of o into the identity component L of the full isometry group of S provides a natural inclusion

$$\mathfrak{s} \supset M = \text{Ad}_L(H)\xi \hookrightarrow \text{Ad}_L(L)\xi =: P \subset \mathfrak{l}$$

of the symmetric R -space M into the hermitian symmetric space P .

The linear endomorphism $F := -\sigma_*$ of \mathfrak{l} preserves the scalar product on \mathfrak{l} and maps adjoint orbits onto adjoint orbits. Since ξ lies in \mathfrak{s} , the (-1) -eigenspace of σ_* , ξ is a fixed point of F . Thus F leaves P invariant and $f := F|_P$ is an involutive isometry of P . Let f_* denote the differential of f at the fixed point ξ . To show that f is anti-holomorphic, it is sufficient to verify that $f_*(J_\xi X) = -J_\xi f_*(X)$ for all $X \in T_\xi P$, because the complex structure J of P is parallel. Equation 1 implies

$$\begin{aligned} f_*(J_\xi X) &= F[\xi, X] = -\sigma_*[\xi, X] = -[\sigma_*\xi, \sigma_*X] \\ &= [\xi, \sigma_*X] = -[\xi, FX] = -J_\xi f_*(X). \end{aligned}$$

Recall that $T_\xi P$ is the orthogonal complement in \mathfrak{l} of $\{X \in \mathfrak{l} : [X, \xi] = 0\}$ and $T_\xi M$ is the orthogonal complement in \mathfrak{s} of $\{X \in \mathfrak{s} : [X, \xi] = 0\}$ (see e.g. [BCO03, p. 71]). Thus $T_\xi M = T_\xi P \cap \mathfrak{s}$ and M is a connected component of the fixed point set of f . This shows that M is a real form of P .

3.2. The proof of the converse implication. We now show the converse, namely that every real form of a hermitian symmetric space P of compact type is a symmetric R -space.

Since P is simply connected (see e.g. [H78, Ch. VIII, Thm. 4.6]), P is a product of its irreducible de Rham factors

$$P = P_1 \times \dots \times P_k,$$

where each factor is an irreducible hermitian symmetric space of compact type (see [W84, Cor. 8.7.11]). An involutive anti-holomorphic isometry f either respects a de Rham factor or permutes isometric de Rham factors pairwise. Thus it is sufficient to only consider the following two cases:

- (I) P is the Riemannian product of two equal irreducible hermitian symmetric spaces Q of compact type, that is $P = Q \times Q$, and f interchanges both factors.

(II) P is irreducible.

We start by investigating the first case. Let ι denote the isometry of $P = Q \times Q$ that just interchanges both factors, that is $\iota(x, y) = (y, x)$ for all $x, y \in Q$. Then f has the form $f = (f_1 \times f_2) \circ \tau$ where $f_{1,2}$ are isometries of Q . Since f is involutive we get $f_2 = f_1^{-1}$, that is $f = (f_1 \times f_1^{-1}) \circ \tau$. The fixed point set of f ,

$$\{(x, y) \in P : f(x, y) = (x, y)\} = \{(x, f_1^{-1}(x)) : x \in Q\},$$

is isomorphic to Q and hence a symmetric R -space.

To treat the second case we actually prove the following sharpened formulation of Theorem 1:

Proposition 2. *Every real form of an irreducible hermitian symmetric space P of compact type is an indecomposable symmetric R -space of non-hermitian type.*

If $P = \text{Ad}(G)\xi \subset \mathfrak{g}$ is a standardly embedded irreducible hermitian symmetric space of compact type, then the Lie algebra \mathfrak{g} of its infinitesimal isometries is simple. We consider \mathfrak{g} endowed with the scalar product that coincides with $B_{\mathfrak{g}}$ up to a negative factor. The Cartan involution corresponding to (P, ξ) is ρ_{ξ} given in Equation 2. The induced Cartan decomposition is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where \mathfrak{k} is the fixed point set of ρ_{ξ} .

Let f be an involutive anti-holomorphic isometry of P and assume that f fixes $\xi \in P$. We denote by M the connected component of the fixed point set of f that contains ξ . Then M is a real form of P and we must only show that M is an indecomposable symmetric R -space of non-hermitian type.

The differential f_* of f at ξ is an involutive linear automorphism of $\mathfrak{p} \cong T_{\xi}P$. The fixed point set \mathfrak{m} of f_* is canonically identified with the tangent space $T_{\xi}M$.

Following the reasoning in [EQT13, Sect. 3] we consider the Lie group automorphism

$$\phi : G \rightarrow G, \quad g \mapsto f \circ g \circ f$$

of the identity component G of the full isometry group of P . Since ϕ leaves the stabilizer K of ξ in G invariant, its differential ϕ_* at the identity induces an automorphism of \mathfrak{k} . We conclude (see [EQT13, Lemma 3.1]) that

$$\phi_*(\xi) \in \{\pm\xi\}.$$

Lemma 3. $\phi_*(\xi) = -\xi$.

Proof. Assume that $\phi_*(\xi) = \xi$. Then the derivative of the one-parameter family

$$\mathbb{R} \rightarrow G, \quad s \mapsto \phi(\exp(s \cdot \xi)) = f \circ \exp(s \cdot \xi) \circ f$$

at $s = 0$ is $\phi_*(\xi) = \xi$. Hence

$$\exp(s \cdot \xi) = f \circ \exp(s \cdot \xi) \circ f \quad \text{for all } s \in \mathbb{R}.$$

The geodesic γ in M that starts at $\xi \in M$ in direction $X \in \mathfrak{m} \setminus \{0\}$ is given by $\gamma(t) = \text{Ad}(\exp(tX))\xi = \exp(tX).\xi$. Taking $s = \frac{\pi}{2}$ we get

$$\begin{aligned} \exp\left(\frac{\pi}{2} \cdot \xi\right) \cdot \gamma(t) &= \left(f \circ \exp\left(\frac{\pi}{2} \cdot \xi\right) \circ f\right) \cdot \gamma(t) \\ &= \left(f \circ \exp\left(\frac{\pi}{2} \cdot \xi\right)\right) \cdot \gamma(t). \end{aligned}$$

The derivative at $t = 0$ yields

$$\begin{aligned} \left(f_* \circ d\left(\exp\left(\frac{\pi}{2} \cdot \xi\right)\right)_\xi\right) X &= f_*\left(\text{Ad}\left(\exp\left(\frac{\pi}{2} \cdot \xi\right)\right) X\right) \\ &= f_*(J_\xi X) = d\left(\exp\left(\frac{\pi}{2} \cdot \xi\right)\right)_\xi X = \text{Ad}\left(\exp\left(\frac{\pi}{2} \cdot \xi\right)\right) X = J_\xi X \end{aligned}$$

(see Equation 1). The equation $f_*(J_\xi X) = J_\xi X = J_\xi f_*(X)$ for a nonzero $X \in \mathfrak{m} \cong T_\xi M$ contradicts the fact that f is anti-holomorphic. \square

The proof of Lemma 3.1 in [EQT13] shows that

$$F := -\phi_* \circ \rho_\xi : \mathfrak{g} \rightarrow \mathfrak{g}$$

extends the isometry f of $P \subset \mathfrak{g}$ to a linear isometry of the ambient space \mathfrak{g} .

Lemma 4. $\tau := -F = \phi_* \circ \rho_\xi$ is an involutive automorphism of \mathfrak{g} that commutes with ρ_ξ .

Proof. As a composition of two automorphisms, τ is obviously an automorphism of \mathfrak{g} . Recall that ϕ_* preserves \mathfrak{k} and therefore also \mathfrak{p} . Notice further that $\rho_\xi = \text{Ad}(\exp(\pi\xi))$ is the identity on \mathfrak{k} and $-\text{Id}$ on \mathfrak{p} . This implies

$$\phi_* \circ \rho_\xi = \rho_\xi \circ \phi_*.$$

Hence $\tau^2 = \phi_*^2 \circ \rho_\xi^2 = \text{Id}$ and $\tau \circ \rho_\xi = \phi_* = \rho_\xi \circ \tau$. \square

Thus (\mathfrak{g}, τ) is an orthogonal involutive Lie algebra (see e.g. [W84, Ch. 8]). Let \mathfrak{h} be the fixed point set of τ and \mathfrak{s} the fixed point set of F . Then the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$$

is the Cartan decomposition of some irreducible pointed symmetric space S of compact type (see e.g. [W84, Sect. 8.3]), which is not a compact Lie group (see e.g. [H78, p. 379]).

Moreover, since τ and ρ_ξ commute, we get a common eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k}_+ \oplus \mathfrak{k}_- \oplus \mathfrak{p}_- \oplus \mathfrak{p}_+,$$

where $\mathfrak{k} = \mathfrak{k}_+ \oplus \mathfrak{k}_-$, $\mathfrak{p} = \mathfrak{p}_- \oplus \mathfrak{p}_+$, $\mathfrak{h} = \mathfrak{k}_+ \oplus \mathfrak{p}_+$ and $\mathfrak{s} = \mathfrak{k}_- \oplus \mathfrak{p}_-$. Notice that $\xi \in \mathfrak{k}_- \subset \mathfrak{s}$ and that $\mathfrak{m} = \mathfrak{p} \cap \mathfrak{s} = \mathfrak{p}_-$.

We observe that M is the connected component of $P \cap \mathfrak{s}$ that contains ξ . Let H be the identity component of the closed subgroup of G formed

by all elements $g \in G$ enjoying the property $\text{Ad}_G(g)\mathfrak{s} = \mathfrak{s}$. Since the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ is orthogonal, we get $\text{Ad}_G(h)\mathfrak{h} = \mathfrak{h}$ for all $h \in H$. One easily checks that \mathfrak{h} is the Lie algebra of H .

Since the representation $\text{Ad}_G(H)|_{\mathfrak{s}}$ is isomorphic to the s -representation of some irreducible symmetric space of compact type, which is not a compact Lie group, the following Lemma implies Theorem 2:

Lemma 5. $M = \text{Ad}_G(H)\xi$.

Proof. The inclusion $\text{Ad}_G(H)\xi \subset M$ is evident. Since both M and $\text{Ad}_G(H)\xi$ are compact submanifolds of P without boundary, it now suffices to show that the dimensions of M and $\text{Ad}_G(H)\xi$ coincide.

The Lie algebra of the stabilizer of ξ in H is $\mathfrak{k}_+ = \{X \in \mathfrak{h} : \text{ad}(X)\xi = 0\}$ and therefore $\dim(\text{Ad}_G(H)\xi) = \dim(\mathfrak{p}_+)$. On the other hand we have $\dim(M) = \dim(\mathfrak{m}) = \dim(\mathfrak{p}_-)$. The automorphism $\text{Ad}(\exp(\pi/2 \cdot \xi))$ of \mathfrak{g} , which coincides on \mathfrak{p} with J_ξ (see Equation 1), identifies \mathfrak{p}_- with \mathfrak{p}_+ . Indeed for $X \in \mathfrak{p}_\pm$ we get:

$$\begin{aligned} \tau \left(\text{Ad} \left(\exp \left(\frac{\pi}{2} \cdot \xi \right) \right) X \right) &= \text{Ad} \left(\exp \left(\frac{\pi}{2} \cdot \tau(\xi) \right) \right) \tau(X) \\ &= \pm \text{Ad} \left(\exp \left(-\frac{\pi}{2} \cdot \xi \right) \right) X \\ &= \pm \text{Ad} \left(\exp \left(\frac{\pi}{2} \cdot \xi \right) \right) (\text{Ad}(\exp(-\pi \cdot \xi))X) \\ &= \pm \text{Ad} \left(\exp \left(\frac{\pi}{2} \cdot \xi \right) \right) (\text{Ad}(\exp(\pi \cdot \xi))X) \\ &= \mp \text{Ad} \left(\exp \left(\frac{\pi}{2} \cdot \xi \right) \right) X. \end{aligned}$$

In the last equality used Equation 2. □

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(Peter Quast) INSTITUT OF MATHEMATICS, UNIVERSITY OF AUGSBURG, 86135
AUGSBURG, GERMANY

E-mail address: `peter.quast@math.uni-augsburg.de`