# Supercurrent in ultrasmall Josephson junctions

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In ultrasmall Josephson junctions where the charging energy exceeds the Josephson coupling energy, the usual supercurrent at zero voltage is completely suppressed. However, in the presence of environmental modes, stochastic tunneling of Cooper pairs leads to a current through the junction. For any standard environmental impedances one finds a peak in the current-voltage characteristics near a small voltage proportional to the temperature and the low frequency resistance of the environment. The form of this universal peak, which is independent of the high frequency behavior of the environment, is given analytically. With increasing Josephson coupling the peak merges into the normal supercurrent of a Josephson junction.

## 1. Introduction

The behavior of an ultrasmall Josephson junction with a small capacitance $C$ such that the charging energy $E_c = 2e^2/C$ exceeds the Josephson coupling energy $E_J$ is not independent of the external circuit. Parasitic capacitances much larger than the junction capacitance, which is in the fF range or below, lead to an effective voltage bias. In addition, the tunneling particle may exchange energy with environmental modes. At temperatures much smaller than the critical temperature and voltages much smaller than the gap voltage the current through the Josephson junction is determined by stochastic tunneling of Cooper pairs. To lowest order in the Josephson coupling energy one finds for the Cooper pair current [1]

$$I(V) = \frac{\pi e E_J^2}{\hbar}\left[P(2eV) - P(-2eV)\right]. \tag{1}$$

Here,

$$P(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \mathrm{d}t \exp\left[J(t) + \frac{i}{\hbar}Et\right]$$

gives the probability that a tunneling Cooper pair transfers the energy $E$ to the environment. Since Cooper pairs have no kinetic energy, tunneling in the direction favored by the applied voltage $V$ or in the opposite direction is only possible if the environment absorbs or provides the energy $2eV$, respectively. $P(E)$ at inverse temperature $\beta = 1/k_BT$ may be calculated from

$$J(t) = 2\int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{\omega}\frac{\mathrm{Re}Z_t(\omega)}{R_Q}\frac{e^{-i\omega t} - 1}{1 - e^{-\beta\hbar\omega}} \tag{2}$$

where $R_Q = h/4e^2$ is the resistance quantum and $Z_t(\omega) = (i\omega C + 1/Z(\omega))^{-1}$ is the total impedance of the junction capacitor $C$ in parallel with the external impedance $Z(\omega)$.

## 2. Low impedance environment

Under common experimental conditions the external impedance has a finite value at zero frequency, $Z(0) = \rho R_Q$, which is much smaller than the resistance quantum, i.e. $\rho \ll 1$. To lowest order in $\rho/\beta E_c$ and for long times, the integral in (2) may be evaluated analytically yielding

$$J(t - i\hbar\beta/2) = -2\rho\left\{\ln\left[\cosh\left(\frac{\pi t}{\hbar\beta}\right)\right] + \ln\left(\frac{\beta E_c}{\pi^2\rho}\right) + \zeta\right\}. \tag{3}$$

The shift of the time argument accounts for the detailed balance symmetry $P(-E) = e^{-\beta E}P(E)$. The result (3) contains details of the frequency dependence of the total impedance only through the constant

$$\zeta = \gamma + \int_0^{\infty} \frac{\mathrm{d}\omega}{\omega}\left[\frac{\mathrm{Re}Z_t(\omega)}{\rho R_Q} - \frac{1}{1 + (\pi\rho\hbar\omega/E_c)^2}\right]$$

where $\gamma = 0.5772\ldots$ is Euler's constant. For an ohmic environment with frequency independent impedance $Z(\omega)/R_Q = \rho$ the integral vanishes. On the other hand, for an $LCR$ impedance with quality factor $Q = (L/C)^{1/2}/R$ one finds $\zeta_{\mathrm{LCR}} = \gamma - \ln(Q) - [(1-2Q^2)/(1-4Q^2)^{1/2}]\mathrm{artanh}[(1-4Q^2)^{1/2}]$.

### 3. Cooper pair current

From (3) which is valid for long times, one may calculate $P(E)$ for energies smaller than $\hbar\omega_Z$, where $\omega_Z$ is a characteristic frequency above which $Z_t(\omega)$ deviates significantly from $Z(0)$. Using (1) one then obtains for the current-voltage characteristics at voltages smaller than $\hbar\omega_Z/e$

$$I(V) = \frac{\pi e}{\hbar}\frac{E_J^2}{E_c}\rho^{2\rho}\left(\frac{\beta E_c}{2\pi^2}\right)^{1-2\rho}\exp[-2\zeta\rho] \times \frac{|\Gamma(\rho - i\frac{\beta eV}{\pi})|^2}{\Gamma(2\rho)}\sinh(\beta eV). \qquad (5)$$

This general result chiefly depends on the low frequency impedance $Z(0) = \rho R_Q$ while the details of $Z(\omega)$ enter only via the exponential factor containing the constant $\zeta$. We note, that the approximations made in deriving (3) imply an upper bound on temperature, $\beta E_c/\rho \gg \pi$, while the perturbation theory leading to (1) yields a lower bound $\beta E_J \ll \rho$ since $P(E)$ should satisfy $P_{\mathrm{max}}E_J \ll 1$. Hence, for ultrasmall Josephson junctions with $E_J \ll E_c$ there is a large range of temperatures for which the result (5) is valid.

For small environmental impedance $\rho \ll 1$ the result (5) describes a peak in the current-voltage characteristic which becomes more pronounced with decreasing temperature as shown in Fig. 1. To leading order in $\rho$ the position of the peak $V_{\mathrm{max}} = \pi\rho/e\beta$ is proportional to the temperature as well as to the low frequency impedance. Inserting $V_{\mathrm{max}}$ into (5) one sees that the maximum current scales like $T^{-1+2\rho}$. For $\rho \to 0$ it is given by $I_{\mathrm{max}} = (eE_J^2/2\hbar)\beta$. According to the discussion above, the theory ceases to hold when $I_{\mathrm{max}}$ becomes comparable to $\rho I_c$ where $I_c = 2eE_J/\hbar$ is the usual critical current.

Another quantity of interest is the zero bias differential conductance $\mathrm{d}I/\mathrm{d}V|_{V=0}$ which scales like $1/\rho$ for small damping and like $T^{-2+2\rho}$ for small temperatures. In the common case where

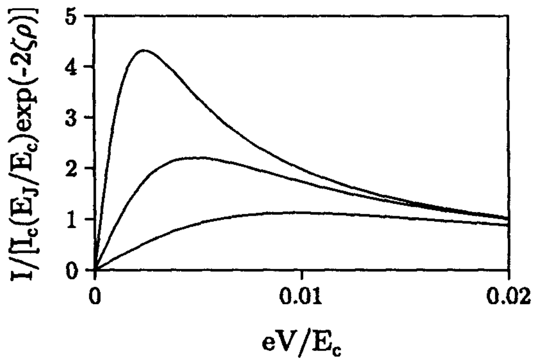


Figure 1. Cooper pair current-voltage characteristics, eq. (7), for $Z(0) = 100\Omega$ and temperatures $k_BT/E_c = 0.05, 0.1, 0.2$ increasing from the upper curve to the lower curve.

$\rho$ is much smaller than one, the zero bias differential conductance diverges for decreasing temperature. This is an indication of the zero bias anomaly of the current-voltage characteristics at zero temperature where $I \sim V^{2\rho-1}$ [1].

For small impedances $Z(0)$ and large temperatures with $\beta eV \ll 1$, (5) reduces to

$$I(V) = \frac{1}{2}I_c^2\frac{RV}{V^2 + (2eRk_BT/\hbar)^2} \qquad (6)$$

This result can also be found from classical phase diffusion of a voltage biased overdamped Josephson junction. The initial slope of the current-voltage characteristic (6) is related to the phase diffusion resistance observed for current biased Josephson junctions. Hence, the supercurrent peak (5) can be looked upon as a remnant of the usual supercurrent for ultrasmall junctions. The peak can be shown to merge into the latter with increasing $E_J$.

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### References


1. G.-L. Ingold and Yu. V. Nazarov, in: *Single Charge Tunneling*, ed. by H. Grabert and M. H. Devoret, NATO ASI Series B, Vol. 294 (Plenum, 1991) and references therein.