Extrinsic symmetric spaces and orbits of s-representations

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1. Introduction

A submanifold M of euclidean space is called *extrinsic symmetric* if it is invariant under the reflection at each affine normal space $p+\nu_pM$, $p \in M$. In particular M with its induced metric is a Riemannian symmetric space and the orbit of a certain subgroup of euclidean motions. A simple argument shows that M is the product of a euclidean space with a compact extrinsic symmetric space which lies in a round sphere. By using a direct construction, Ferus [F1] proved in 1974 that compact extrinsic symmetric spaces are orbits of s-representations, i.e. of isotropy representations of semisimple symmetric spaces. In 1980 he gave another, very elegant proof of this using Jordan triple systems [F2]. Still another proof follows from Dadok's classification of polar representations [D] and Olmos result [O] on normal holonomy groups (see the remark after Theorem 1).

Our main purpose here is to give an elementary and simple proof of the above result of Ferus. It is in spirit close to [F1] but avoids all cumbersome computations. We also study orbits of s-representations and prove in particular that each strongly isotropy irreducible orbit is extrinsic symmetric. In fact we characterize the extrinsic symmetric orbits as those which split locally into a product of isotropy irreducible ones.

One motivation for this note came from our attempt to understand more geometrically the main result of Dadok [D] which says that any polar representation is orbit equivalent to an s- representation, i.e. it has the same orbits as an s- representation after an isometric identification of the vector spaces. An orthogonal representation is called *polar* if there exists a linear subspace (called a *section*) which meets every orbit and each time orthogonally. In this context, our result can be interpreted as a simple proof of the special case where the polar representation has an extrinsic symmetric orbit (actually, most of them do, cf. Remark 2).

It is quite obvious that an extrinsic symmetric submanifold has parallel second fundamental form, but also the converse is true for complete submanifolds. (A direct proof of this is due to Strübing [St].)

2. Construction of the s-representation

Theorem 1. *(D. Ferus) Let $M \subset \mathbb{R}^N$ be a compact extrinsic symmetric space which lies in a sphere around the origin but in no proper affine subspace. Then M is the orbit of an s-representation.*

Proof. (a) Let $V := \mathbb{R}^N$ and $K \subset O(V)$ be the group of orthogonal transformations which leave M invariant. Since under the above assumption, the affine normal spaces are linear subspaces, K contains the reflections at the normal spaces and thus acts transitively on M. On the vector space $\mathfrak{g} := \mathfrak{k} \oplus V$, (where $\mathfrak{k}$ denotes the Lie algebra of K) we extend the bracket on $\mathfrak{k}$ to one on $\mathfrak{g}$ by the requirement $[V,V] \subset \mathfrak{k}$ and by putting

$$[A,v] := -[v,A] : = Av$$

$$\langle A,[v,w]\rangle_{\mathfrak{k}} : = \langle Av,w\rangle$$

for all $A \in \mathfrak{k}$, $v,w \in V$, where $\langle\ ,\ \rangle_{\mathfrak{k}}$ denotes an $Ad(K)$-invariant inner product on $\mathfrak{k}$ which will be specified later. If $\mathfrak{g}$ satisfies the Jacobi identity then $\mathfrak{g}$ with this bracket is a Lie algebra and the inner product on $\mathfrak{g}$ extending the inner products on $\mathfrak{k}$ and V with $\mathfrak{k} \perp V$ is $Ad(G)$-invariant. Since M does not lie in a proper affine subspace, K has no fixed vector. From this and the definition of the bracket it follows that $\mathfrak{g}$ has no center and hence is semisimple. Let $G := \mathrm{Aut}(\mathfrak{g})$. Then $K \subset G$ and G/K is a semisimple symmetric space whose isotropy representation can be identified with the given action of K on V.

(b) Since the bracket on $\mathfrak{g}$ is equivariant with respect to the action of K on $\mathfrak{g}$ it follows by differentiating that it satisfies the Jacobi identity whenever at least one element lies in $\mathfrak{k}$. Thus it remains to show

$$\mathrm{Jac}(u,v,w) := [u,v]w + [v,w]u + [w,u]v = 0$$

for all $u,v,w \in V$.

We fix some $x \in M$ and let $\tau = \tau_x$ be the tangent space and $\nu = \nu_x$ the normal space of M at x . We have $V = \tau \oplus \nu$. Moreover, let $\mathfrak{k} = \mathfrak{k}_x + \mathfrak{p}_x$ be the Cartan decomposition of $\mathfrak{k}$ with respect to the geodesic symmetry at x . The mapping $\mathfrak{p}_x \to \tau$, $A \mapsto Ax$, is a linear isomorphism whose inverse mapping we denote by T . Thus for each $v \in \tau$, T_v is the so called infinitesimal transvection in the direction of v . The 1-parameter group $\exp t \cdot T_v$ acts by parallel translation in the tangent as well as in the normal bundle along the geodesic $(\exp t \cdot T_v).x$. Thus we get for any $w \in \tau$ and $\xi \in \nu$ by differentiating $w(t) := (\exp t \cdot T_v).w$ and $\xi(t) := (\exp t \cdot T_v).\xi$ respectively,

$$T_v w = w'(0) = \alpha(v,w) \tag{1}$$

$$T_v \xi = \xi'(0) = -A_\xi v \tag{2}$$

where α denotes the second fundamental form and A_ξ the shape operator in the direction of ξ . In particular the action of elements of $\mathfrak{p}_x$ interchange the subspaces τ and ν whereas those of $\mathfrak{k}_x$ leave them invariant, of course. Since the geodesic symmetry at x lies in K and acts on $\mathfrak{g}$ with eigenspaces $\mathfrak{k}_x$ and $\mathfrak{p}_x$ we have $\mathfrak{k}_x \perp \mathfrak{p}_x$ and thus

$$[\tau,\tau] + [\nu,\nu] \subset \mathfrak{k}_x \tag{3}$$

$$[\tau,\nu] \subset \mathfrak{p}_x \tag{4}$$

The mappings $\tau \to \mathfrak{p}_x : v \mapsto T_v$ and $\nu \to S(\tau) : \xi \mapsto A_\xi$ (where $S(\tau)$ is the space of self adjoint endomorphisms of τ) are equivariant with respect to the isotropy group K_x. Hence we have for all $B \in \mathfrak{k}_x$

$$[B, T_v] = T_{Bv} \tag{5}$$

$$[B, A_\xi] = A_{B\xi} \tag{6}$$

(c) Now we choose the inner product on $\mathfrak{k}$ as follows. Since $\mathfrak{k} = \mathfrak{k}_x + \mathfrak{p}_x$ is a Cartan decomposition any $Ad(K_x)$-invariant inner product on $\mathfrak{p}_x$ extends uniquely to an $Ad(K)$-invariant inner product on $\mathfrak{k}$. This is clear if K_x acts irreducibly on $\mathfrak{p}_x$ and follows in general by decomposing $\mathfrak{p}_x$ into irreducible summands. Hence we may choose the inner product on $\mathfrak{k}$ such that the canonical K_x-equivariant isomorphism $\mathfrak{p}_x \to \tau\ :\ v \mapsto T_v$, becomes an isometry.

(d) Claim: for all $v, w \in \tau$ and $\xi, \eta \in \nu$,

$$[v,w] = [T_v, T_w], \tag{7}$$

$$[v,\xi] = T_{A_\xi v}, \tag{8}$$

$$[\xi,\eta]v = -[A_\xi, A_\eta]v \tag{9}$$

In fact, for all $B \in \mathfrak{k}_x$ we have

$$\langle B, [T_v, T_w]\rangle = \langle [B, T_v], T_w\rangle = \langle T_{Bv}, T_w\rangle = \langle Bv, w\rangle = \langle B, [v,w]\rangle.$$

This proves (7). From

$$\langle T_w, [v,\xi]\rangle = \langle T_w v, \xi\rangle = \langle \alpha(w,v), \xi\rangle = \langle w, A_\xi v\rangle = \langle T_w, T_{A_\xi v}\rangle$$

we get (8). Finally, we have

$$\langle [\xi,\eta]v, w\rangle = \langle [\xi,\eta], [v,w]\rangle = \langle [v,w]\xi, \eta\rangle = -\langle [A_\xi, A_\eta]v, w\rangle$$

since

$$[v,w]\xi = T_v T_w \xi - T_w T_v \xi = -\alpha(v, A_\xi w) + \alpha(w, A_\xi v).$$

(e) Now we can compute triple products in order to prove the Jacobi identity. By (7) we get $[v,w]u = [T_v, T_w]u$ for any $v, w, u \in \tau$. Since $B := [T_v, T_w] \in \mathfrak{k}_x$, we may apply (5) and obtain

$$T_{[v,w]u} = [[T_v, T_w], T_u]$$

which proves $\mathrm{Jac}(v,w,u) = 0$. Similarly, we have for three normal vectors

$$A_{[\xi,\eta]\zeta} = [[\xi,\eta], A_\zeta] = -[[A_\xi, A_\eta], A_\zeta].$$

by (3), (6) and (9), and hence $A_\omega = 0$ where $\omega := \mathrm{Jac}(\xi,\eta,\zeta)$. Thus (by (2)), $T_v\omega = 0$ for all $v \in \tau$ which shows $\mathfrak{p}_x.\omega = 0$. From (9) we get $[\omega,\vartheta]v = 0$ for all $v \in \tau$, $\vartheta \in \nu$ and thus $[\omega,\vartheta] = 0$ since any isometry of V which fixes M is the identity. Therefore $0 = \langle \mathfrak{k}_x, [\omega,\vartheta]\rangle = \langle \mathfrak{k}_x.\omega, \vartheta\rangle$, hence $\mathfrak{k}_x.\omega = 0$ and therefore $\mathfrak{k}.\omega = 0$. Thus $\mathrm{Jac}(\xi,\eta,\zeta) = \omega = 0$.

From (7), (8), (1), the symmetry of α and (2) we get

$$\mathrm{Jac}(v,w,\xi) = [T_v, T_w]\xi + T_v A_\xi w - T_w A_\xi v = 0,$$

and from (9), (8) and (2),

$$\mathrm{Jac}(\xi,\eta,v) = -[A_\xi, A_\eta]v + A_\xi A_\eta v - A_\eta A_\xi v = 0.$$

This completes the construction of the Lie algebra structure on $\mathfrak{g} = \mathfrak{k} \oplus V$ and finishes the proof. □

Remark A proof of Theorem 1 using results of Dadok [D] and Olmos [O] as mentioned in the introduction could be given as follows. By Olmos [O], the action of the isotropy group at $x \in M$ is polar on the normal space $\nu_x M$. Let $\Sigma \subset \nu_x M$ be a section for this representation. Then Σ is a section for the K-action on V as well. In fact, every K-orbit meets $\nu_x M$ and thus Σ. Let $y = x + \xi \in \Sigma$. Since Σ is a section for K_x, we have $\mathfrak{k}_x.y \perp \Sigma$. Consider the Cartan decomposition $\mathfrak{k} = \mathfrak{k}_x + \mathfrak{p}_x$ where $\mathfrak{p}_x$ is the space of infinitesimal transvections at x, i.e. for every $T \in \mathfrak{p}_x$, the group element $\exp t \cdot T$ acts by parallel transport in the tangent and the normal bundle along the geodesic $(\exp t \cdot T).x$. This shows that $Ty = Tx + T\xi$ is perpendicular to $\nu_x M$ and hence to Σ. Thus also $\mathfrak{p}_x.y \perp \Sigma$ and therefore the K-action is polar with section Σ, and the theorem follows from Dadok's main result [D].

3. Characterisation of extrinsic symmetric s-orbits

Let K be a connected Lie group and $H \subset K$ a closed subgroup. The homogeneous space K/H is called *(strongly) isotropy irreducible* if H (the connected component of H containing the identity) acts irreducibly on the tangent space or equivalently on $\mathfrak{k}/\mathfrak{h}$ where $\mathfrak{k}$ and $\mathfrak{h}$ are the Lie algebras of K and H, respectively. We say that the universal cover $\tilde{M}$ of $M = K/H$ *decomposes into a product of isotropy irreducible spaces* if we have $\tilde{M} = \tilde{M}_1 \times ... \times \tilde{M}_r$ with isotropy irreducible $\tilde{M}_i = K_i/H_i$ such that $\tilde{K} = K_1 \times ... \times K_r$ is a connected covering group of K. Note that $\tilde{H} := H_1 \times ... \times H_r$ is connected since otherwise $\tilde{M} = \tilde{K}/\tilde{H}$ would have a nontrivial covering. Therefore, this condition is equivalent to the splitting of the Lie algebras into ideals

$$\mathfrak{k} = \mathfrak{k}_1 \oplus ... \oplus \mathfrak{k}_r, \quad \mathfrak{h} = \mathfrak{h}_1 \oplus ... \oplus \mathfrak{h}_r$$

with $\mathfrak{h}_i \subset \mathfrak{k}_i$, such that $\mathfrak{k}_i/\mathfrak{h}_i$ is an irreducible $\mathfrak{h}_i$-module. Examples are symmetric spaces K/H where (K, H) is a symmetric pair, and of course strongly isotropy irreducible spaces.

Lemma *Let $H \subset K$ be as above and assume in addition that $\mathfrak{k}$ carries a biinvariant metric and that K acts effectively on K/H. Then the following conditions are equivalent:*

(i) The universal cover of K/H decomposes into a product of isotropy irreducible spaces.

(ii) $[W, W^{\perp}] = 0$ for any $ad(\mathfrak{h})$-invariant subspace W of $\mathfrak{h}^{\perp}$ where $W^{\perp}$ denotes the orthogonal complement of W in $\mathfrak{h}^{\perp}$.

Proof. Assume (i). Let $\mathfrak{z}_i$ be the center of $\mathfrak{k}_i$ and $\mathfrak{m}_i$ the orthogonal complement of $\mathfrak{h}_i$ in $\mathfrak{k}_i$. The image of the orthogonal projection of $\mathfrak{z}_i$ to $\mathfrak{m}_i$ commutes with $\mathfrak{h}_i$. Therefore it is zero or $\mathfrak{m}_i$, by the irreducibility assumption. In the first case, $\mathfrak{z}_i \subset \mathfrak{h}_i \subset \mathfrak{h}$ and hence $\mathfrak{z}_i = 0$, since $\mathfrak{h}$ does not contain any nontrivial ideal of $\mathfrak{k}$, by the effectivity assumption. In the second case, $\mathfrak{h}_i$ acts trivially on $\mathfrak{m}_i$ which implies $\mathfrak{h}_i = 0$ (by effectivity) and $\mathfrak{z}_i = \mathfrak{m}_i$. Hence $\mathfrak{k}_i$ is either semisimple or a (one-dimensional) subspace of the center. Since the semisimple ideals are orthogonal to each other as well as to the center, we get

$$\mathfrak{h}^{\perp} = \mathfrak{z} + \sum_{j=1}^{s} \mathfrak{m}_j$$

where $\mathfrak{z}$ is the center of $\mathfrak{k}$ and $\mathfrak{k}_1, ..., \mathfrak{k}_s$ are precisely the semisimple ideals among the $\mathfrak{k}_i$. Since $\mathfrak{h}_j$ acts only on $\mathfrak{m}_j$, the $\mathfrak{h}$-modules $\mathfrak{m}_1, ..., \mathfrak{m}_s$ are not only irreducible but also inequivalent. Hence after a suitable renummeration, there exists some $t \leq s$ such that

$$W = W \cap \mathfrak{z} + \sum_{j=1}^{t} \mathfrak{m}_j, \quad W^{\perp} = W^{\perp} \cap \mathfrak{z} + \sum_{j=t+1}^{s} \mathfrak{m}_j$$

and (ii) follows.

Now assume (ii) and let $\mathfrak{h}^{\perp} = \mathfrak{m}_1 + ... + \mathfrak{m}_r$ be an orthogonal decomposition into $ad(\mathfrak{h})$-irreducible subspaces. Let $\mathfrak{k}_i := \mathfrak{m}_i + [\mathfrak{m}_i, \mathfrak{m}_i]$. Then from $[\mathfrak{m}_i, \mathfrak{m}_j] = 0$ we get $\langle \mathfrak{k}_i, \mathfrak{m}_j \rangle = 0$ and $[\mathfrak{k}_i, \mathfrak{m}_j] = 0$ for all $i \neq j$. Hence $\mathfrak{k}_i \subset \mathfrak{h} + \mathfrak{m}_i$ and $\mathfrak{k}_i$ is an ideal of $\mathfrak{k}$. Furthermore, $\mathfrak{k} = \mathfrak{k}_1 + ... + \mathfrak{k}_r$ since for any $X \in \mathfrak{k}$ perpendicular to $\mathfrak{k}_1 + ... + \mathfrak{k}_r$ we have

$X \in \mathfrak{h}$ and $\langle [X, \mathfrak{m}_i], \mathfrak{m}_i\rangle = \langle X, [\mathfrak{m}_i, \mathfrak{m}_i]\rangle = 0$, i.e. $[X, \mathfrak{h}^\perp] = 0$ and thus $X = 0$ by the effectivity assumption. Moreover, $\mathfrak{k}_i \perp \mathfrak{k}_j$ for $i \neq j$. Let $\mathfrak{h}_i := \mathfrak{h} \cap \mathfrak{k}_i$. Then $\mathfrak{h} = \mathfrak{h}_1 + ... + \mathfrak{h}_r$ since any $X \in \mathfrak{h}$ can be written as $X = \sum_{i=1}^r X_i$ with $X_i \in \mathfrak{k}_i$, and each X_i in turn as $X_i' + X_i''$ with $X_i' \in \mathfrak{h}$ and $X_i'' \in \mathfrak{m}_i$. But $X_i'' = 0$ since $\sum_{i=1}^r X_i'' = 0$. Furthermore, $\mathfrak{h}$ and thus $\mathfrak{h}_i$ act irreducibly on $\mathfrak{m}_i \cong \mathfrak{k}_i/\mathfrak{h}_i$. This finishes the proof. □

In the following theorem, we characterize the extrinsic symmetric orbits of an s-representation as those which split locally into a product of isotropy irreducible ones. Since it is no extra work, we include other known characterizations.

Theorem 2. *Let G/K be an irreducible symmetric space of compact type with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and let $M = K.x = Ad(K)x$ for some $x \in \mathfrak{p}$. Then the following conditions are equivalent:*

(i) (K, K_x) is a symmetric pair,

(ii) the universal cover of $M = K/K_x$ decomposes into a product of isotropy irreducible spaces,

(iii) $ad(x)^3 = -\lambda^2 ad(x)$ for some $\lambda > 0$,

(iv) M is extrinsic symmetric, and the transvections of M belong to K.

Proof. "(i) $\Rightarrow$ (ii)" is obvious.

"(ii) $\Rightarrow$ (iii)": For arbitrary $x \in \mathfrak{p}$, the endomorphism $ad(x)$ of $\mathfrak{g}$ is skew-symmetric with respect to an $Ad(G)$-invariant metric $\langle\ ,\ \rangle$. Let $E_\lambda \subset \mathfrak{g} \otimes \mathbb{C}$ be the eigenspace corresponding to an eigenvalue λi with $\lambda \in \mathbb{R}$ and put $\mathfrak{g}_\lambda = (E_\lambda + E_{-\lambda}) \cap \mathfrak{g}$. By the Jacobi identity,

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu} + \mathfrak{g}_{\lambda-\mu}$$

for any two eigenvalues λi, μi. Since the involution of $\mathfrak{g}$ corresponding to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ maps x onto $-x$, it preserves $\mathfrak{g}_\lambda$, and so we obtain $\mathfrak{g}_\lambda = \mathfrak{k}_\lambda + \mathfrak{p}_\lambda$ where $\mathfrak{k}_\lambda$ and $\mathfrak{p}_\lambda$ are the intersections of $\mathfrak{g}_\lambda$ with $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Thus we get orthogonal decompositions

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda>0} \mathfrak{k}_\lambda, \quad \mathfrak{p} = \mathfrak{p}_0 + \sum_{\lambda>0} \mathfrak{p}_\lambda,$$

which are $Ad(K_x)$-invariant since x is fixed and $\mathfrak{k}$ and $\mathfrak{p}$ are invariant under $Ad(K_x)$. We have

$$\mathfrak{k}_0 = \mathfrak{k}_x, \quad \mathfrak{k}_0^\perp = \sum_{\lambda>0} \mathfrak{k}_\lambda.$$

Moreover, $ad(x)$ is an isomorphism between $\mathfrak{k}_\lambda$ and $\mathfrak{p}_\lambda$ for $\lambda > 0$, and

$$\mathfrak{p}_0 = \nu_x M, \quad \mathfrak{p}_0^\perp = \tau_x M$$

since $\xi \in \nu_x M$ is equivalent to $0 = \langle Ax, \xi\rangle = \langle A, [x, \xi]\rangle$ for all $A \in \mathfrak{k}$ and thus to $ad(x)\xi = 0$.

Now assume that $x \in \mathfrak{p}$ satisfies (ii). We claim

$$[\mathfrak{k}_\lambda, \mathfrak{k}_\mu] = 0, \quad [\mathfrak{p}_\lambda, \mathfrak{p}_\mu] = 0, \quad [\mathfrak{k}_\lambda, \mathfrak{p}_\mu] = 0$$

for different $\lambda, \mu > 0$.

In fact, the first equation follows from the lemma above. Moreover, since $ad(x)^2$ preserves $\mathfrak{k}_\lambda$ and $\mathfrak{k}_\mu$, we get

$$0 = ad(x)^2[\mathfrak{k}_\lambda, \mathfrak{k}_\mu] = [ad(x)\mathfrak{k}_\lambda, ad(x)\mathfrak{k}_\mu] = [\mathfrak{p}_\lambda, \mathfrak{p}_\mu].$$

Finally, since

$$ad(x)[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset [\mathfrak{p}_\lambda, \mathfrak{p}_\mu] + [\mathfrak{k}_\lambda, \mathfrak{k}_\mu] = 0,$$

we have $[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{p}_0$ which shows $[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] = 0$ since $\lambda = \pm\mu$ is excluded.
Thus

$$\mathfrak{p}^\lambda := \mathfrak{p}_\lambda + [\mathfrak{k}_\lambda, \mathfrak{p}_\lambda] + [\mathfrak{k}_\lambda, [\mathfrak{k}_\lambda, \mathfrak{p}_\lambda]] + ...$$

is a $\mathfrak{k}$-invariant subspace of $\mathfrak{p}$ which is perpendicular to any $\mathfrak{p}_\mu$ for $\mu \neq \pm\lambda$. Since the symmetric space G/K is irreducible by assumption, we have $\mathfrak{p}^\lambda = \mathfrak{p}$ and so there are no eigenvalues for $ad(x)$ on $\mathfrak{g} \otimes \mathbb{C}$ other than $\pm\lambda i$ and 0. This proves (iii).

"(iii) $\Rightarrow$ (iv)": Normalizing x suitably, we may assume $\lambda = 1$, i.e. $ad(x)^3 = -ad(x)$. Thus we have

$$Ad(\exp\, tx) = \sum \frac{t^k}{k!} ad(x)^k = I + ad(x)^2 + \sin(t)ad(x) - \cos(t)ad(x)^2,$$

hence $\sigma_x := Ad(\exp\, \pi x) = I + 2ad(x)^2$. So $\sigma_x = I$ on $\nu_x M$ (recall that $ad(x)|\nu_x M = 0$, see above), and for any $Ax \in T_x Kx$ (where $A \in \mathfrak{k}$),

$$\sigma_x Ax = -(I + 2ad(x)^2)ad(x)A = ad(x)A = -Ax.$$

Thus σ_x preserves $\mathfrak{p}$ and $\mathfrak{k}$ and is the reflection at the subspace $\nu_x = x + \nu_x$ in $\mathfrak{p}$. In particular, $\exp\, \pi x$ normalizes K, and hence σ_x preserves $M = Kx$. So the subgroup $\hat{K}$ of G generated by K and $\exp\, \pi x$ is K or an extension of K with index 2, and $Ad(\hat{K})$ preserves $\mathfrak{p}$ and M and contains the reflections at all normal spaces of M. Hence M is extrinsic symmetric and the transvections (being the compositions of any two reflections) lie in K.

"(iv) $\Rightarrow$ (i)" is obvious. □

Remarks 1. If we assume that the symmetric space G/K has rank ≥ 2, then we may replace *(iv)* in the above theorem by

(iv') M is extrinsic symmetric.

In fact, (up to connected components) the isotropy group K of an irreducible symmetric space of rank ≥ 2 is the maximal subgroup of $O(\mathfrak{p})$ preserving one of its (nontrivial) orbits. This is a consequence for example of Simons' holonomy theorem [S]: In fact, any larger group $K' \supset K$ together with the same curvature tensor (the curvature tensor of G/K at the point eK) would still give an irreducible holonomy system which must be symmetric unless the group acts transitively on the unit sphere, but this is impossible if K' and K have a common orbit. Hence K' also preserves the curvature tensor and thus the connected components of K' and K agree. In particular, K contains the transvections of the orbit M. However, if G/K is a rank-one symmetric space, then (iv) and (iv)' are no longer equivalent since all nontrivial K-orbits are extrinsic symmetric (round spheres) but K does not contain all transvections of the sphere unless G/K has constant curvature.

2. It is easy to get the classification of all extrinsic symmetric spaces from condition (iii) (cf. [KN]). If $\alpha_1, ..., \alpha_r$ are the simple roots of the symmetric space with respect to a Weyl chamber C and if $\delta = n_1\alpha_1 + ... + n_r\alpha_r$ is the highest root, then $x \in \bar{C}$ obviously satisfies (iii) with $\lambda = 1$ if and only if

$$\alpha_j(x) = 1, \; n_j = 1$$

for exactly one $j \in \{1, ..., r\}$ while $\alpha_i(x) = 0$ for all $i \neq j$. Thus $K.x$ is extrinsic symmetric for $x \in \bar{C}$ if and only if x lies on a (one-dimensional) edge of $\bar{C}$ for which the opposite face corresponds to a root α_j with $n_j = 1$. These roots can be read off for example from the table in [H], p. 476.

3. Theorem 2 says in particular that strongly isotropy irreducible orbits of K in $\mathfrak{p}$ are extrinsic symmetric. However, the converse is not true: If G/K is the real Grassmannian of p-planes in $\mathbb{R}^{p+q}$, there are isotropy reducible extrinsic symmetric K-orbits which are covered by $S^{p-1} \times S^{q-1}$. But these spaces do not split as Riemannian manifolds as follows from the next theorem:

Theorem 3. *Let $M \subset V = \mathbb{R}^N$ be an extrinsic symmetric space which splits intrinsically as a Riemannian product $M = M_1 \times M_2$. Then the splitting is extrinsic, i.e. M_i lies in some subspace $V_i \subset V$ $(i = 1, 2)$ with $V_1 \perp V_2$ such that*

$$M = \{x_1 + x_2; \; x_1 \in M_1, \, x_2 \in M_2\}.$$

Proof. Since M has parallel second fundamental form α, the same holds for any totally geodesic submanifold, so the factors $M_1 = M_1 \times \{x_2\}$ and $M_2 = \{x_1\} \times M_2$ (for some fixed $x = (x_1, x_2) \in M$) are also extrinsically symmetric. Let $\sigma, \sigma_1, \sigma_2 : V \to V$ denote the extrinsic symmetries of M, M_1, M_2 at the point x. Then σ_* fixes $\nu_x M$ and reflects $\tau_x M = \tau_x M_1 + \tau_x M_2$ while σ_{i*} fixes $\nu_x M_i = \nu_x M + \tau_x M_j$ (for $j \neq i \in \{1, 2\}$) and reflects $\tau_x M_i$. Thus $\sigma\sigma_i = \sigma_j$ which shows that M_i is fixed by σ_j. Hence $M_i \subset \nu_x M_j$. Moreover, for all $v_i \in \tau_x M_i$ we have

$$\alpha(v_1, v_2) = \alpha(\sigma_{1*}v_1, \sigma_{1*}v_2) = \alpha(-v_1, v_2),$$

hence $\alpha(v_1, v_2) = 0$. Then by the Gauss equations, $\alpha(v_1, v_1) \perp \alpha(v_2, v_2)$ since the mixed curvature $K(v_1, v_2)$ is zero in the Riemannian product $M = M_1 \times M_2$. Since α is parallel, the normal bundle $\nu = \nu M$ splits into two parallel orthogonal subbundles ν_1 and ν_2 such that $\alpha|M_i$ takes values in ν_i. Due to Erbacher's theorem (cf. [E]), the codimension of $M_i \subset \nu_x M_j = \tau_x M_i + \nu_x M$ can be reduced by $\dim(\nu_j)$, hence $M_i \subset V_i := \tau_x M_i + \nu_i(x)$. We have $V_2 = V_1^{\perp}$ and the result now follows easily. □

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