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AMPLITUDE EQUATION FOR THE GENERALIZED SWIFT HOHENBERG EQUATION WITH NOISE

KONRAD KLEPEL, DIRK BLÖMKER, AND WAEL W. MOHAMMED

ABSTRACT. We derive an amplitude equation for a stochastic partial differential equation (SPDE) of Swift-Hohenberg type with a nonlinearity that is composed of a stable cubic and an unstable quadratic term, under the assumption that the noise acts only on the constant mode. Due to the natural separation of timescales, solutions are approximated well by the slow modes. Nevertheless, via the nonlinearity, the noise gets transmitted to those modes too, such that multiplicative noise appears in the amplitude equation.

1. INTRODUCTION

The Swift Hohenberg equation is a model equation used to study pattern formation in driven systems. It was originally derived in [SH77] as a qualitative description of the convective instability in the Rayleigh Bernard model. Originally, it takes the form

$$\partial_t u = ru - (1 + \nabla^2)^2 u - u^3 \ , \tag{1}$$

where $r \in \mathbb{R}$ is the bifurcation parameter. At $r = 0$ is the change of stability that corresponds to the convective instability. A variant is the so called generalized Swift Hohenberg model with quadratic and cubic nonlinearity:

$$\partial_t u = ru - (1 + \nabla^2)^2 u + \alpha u^2 - u^3, \tag{2}$$

where $\alpha > 0$ is an additional parameter, measuring the strength of the quadratic instability. Equation (2) is also derived, when a general nonlinearity is expanded via Taylor's formula. The dynamics of (2) was studied in [CH93], [HMBD95], [BK06] and recently [BD12] among others. In these articles the usual approach of amplitude equations is the derivation of a simplified model in the vicinity of the change of stability at $r = 0$. To be more precise, both (1) and (2) are very well approximated by

$$u(t, x) \approx \sqrt{|r|} \cdot A(|r|t) \cdot e^{ix} + \sqrt{|r|} \cdot \overline{A(|r|t)} \cdot e^{-ix}. \tag{3}$$

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where the complex amplitude $A(T)$ of the dominant mode e^{ix} is the solution of

$$\partial_T A = rA + 3(\tfrac{38}{27}\alpha^2 - 1)|A|^2 A, \tag{4}$$

which is accordingly named amplitude equation (AE, for short) of (2).
For the deterministic Swift-Hohenberg equation on an unbounded domain solutions are approximated via the Ginzburg-Landau PDE. For more results on the deterministic Swift-Hohenberg equation, see for instance [KMS92], [CE90], [MSZ00] and [Sch96].
It is the aim of this article to provide rigorous error estimates and to verify the existence of an amplitude equation for (2). We also add noise constant in space. This does not cover thermal noise, but only perturbations acting on the whole system. This assumption is only for simplicity of presentation. Completely analogous, we could treat all kind of spatial noise not acting on the dominant modes directly. If the additive noise acts on the dominant modes, then we need to change scaling and consider smaller noise. See for example [BH04] or [Blö07]
Thus we consider the following stochastic generalized Swift Hohenberg equation:

$$\partial_t u = \nu\varepsilon^2 u - (1+\Delta)^2 u + \alpha u^2 - u^3 + \varepsilon\sigma\partial_t\beta, \tag{SH}$$

where $\beta(t)$ is a real valued standard Brownian motion. For simplicity of presentation we consider (SH) with periodic boundary conditions on $[0, 2\pi]$ only. Here α, σ and ν are real-valued constants. The small parameter $\varepsilon > 0$ relates the distance from bifurcation to the noise strength. Of course different scalings are possible, but then in the amplitude equation, either the noise or the linear term disappears. We show that in our scaling, though the constant mode is non-dominant, the noise appears also in the AE through coupling by the nonlinear terms. Additional terms on the right-hand side are created, and the noise is multiplicative. To be more precise, both (SH) is well approximated by

$$u(t,x) \approx \varepsilon A(\varepsilon^2 t)\cdot e^{ix} + \varepsilon\overline{A(\varepsilon^2 t)}\cdot e^{-ix}. \tag{5}$$

where the complex-valued amplitude $A(T)$ solves the Itô differential equation

$$dA = (\nu A + 3(\tfrac{38}{27}\alpha^2 - 1)A|A|^2 + 3(\alpha^2 - \tfrac{1}{2})\sigma^2 A)dT + 2\alpha\sigma A d\tilde{\beta}\ . \tag{AE}$$

with $\tilde{\beta}(T) := \varepsilon\beta(\varepsilon^{-2}T)$ being a rescaled version of $\beta(t)$. It is an interesting observation, that due to the quadratic nonlinearity both cubic and linear unstable terms arise in the amplitude equation. This is significantly different to other quadratic nonlinearities like Burgers, for example, where these terms are all stabilizing. See [BHP07, BMNW11].
Our research was initiated originally by the observations of Axel Hutt and collaborators, who treated the case with $\alpha = 0$. By numerical simulations and argumentation based on formal application of center manifold theory they studied the standard Swift Hohenberg equation with noise constant in space [HLSG07, Hut08, HLSG08]. For a rigorous result in this direction see [BM12] on bounded domains and [MBK12] on unbounded domains.

The paper is organized as follows. Section 2 provides the setting of the problem, while section 3 states the main result. In section 4 we collect all proofs.

2. Setting

We consider mild solutions of (SH) with values in the space $C^0 = C^0_{per}([0, 2\pi])$, i.e. the space of 2π periodic continuous functions, defined by

Definition 1. *A stochastic process $u(t)$, $t \in [0, T_0]$ with continuous paths in C^0 is a mild solution of* (SH) *if the following variation of constants formula holds in C^0 for all $t \in [0, T_0]$:*

$$u(t) = e^{-t(1+\partial_x^2)^2}u(0) + \int_0^t e^{-(t-s)(1+\partial_x^2)^2}[\nu\varepsilon^2 u(s) + \alpha u^2(s) - u^3(s)]ds + \varepsilon \int_0^t e^{-(t-s)(1+\partial_x^2)^2}\sigma d\beta(s) \;, \tag{6}$$

where $e^{-t(1+\partial_x^2)^2}$ is the semigroup created by the operator $-(1+\partial_x^2)^2$ (cf. [Paz83]*).*

Using standard theory given in [DPZ92], it is straightforward to verify that such a mild solution exists. This is, for example, done via Banach's fixed-point theorem for unique local solutions and energy estimates for global solutions.

Remark 2. *The stochastic integral on the right-hand side of* (6) *can be simplified to*

$$Z(t) := \varepsilon\sigma \int_0^t e^{-(t-s)(1+\partial_x^2)^2} 1 d\beta(s) = \varepsilon\sigma \int_0^t e^{-(t-s)} d\beta(s), \tag{7}$$

which is a simple real-valued Ornstein-Uhlenbeck process.

Our approximation result states the error in terms of the distance to the bifurcation point ($r = \nu = 0$) using big $\mathcal{O}$ notation modified for random variables. This is defined by the following:

Definition 3. *Let X_ε with $\varepsilon > 0$ be a family of stochastic processes and $f(\varepsilon)$ be a function of ε. Then X_ε is of order $f(\varepsilon)$, which we abbreviate by*

$$X_\varepsilon = \mathcal{O}(f(\varepsilon)),$$

if and only if for every p-th moment of X_ε there is a constant C_p such that the following is valid for all $\varepsilon > 0$:

$$\mathbb{E}\left(|X_\varepsilon|^p\right) \le C_p |f(\varepsilon)|^p.$$

3. Main result

The main result is the following approximation theorem for the stochastic generalized Swift Hohenberg equation (SH).

Theorem 4. *Let $T_0 > 0$ be a time of order 1, $\alpha \in \mathbb{R}$ with $\alpha^2 < \frac{27}{38}$ and $0 < \kappa < \frac{1}{17}$. Let u be a stochastic process with continuous paths in C^0 that is a mild solution of* (SH) *with $\|u(0)\|_\infty = \mathcal{O}(\varepsilon^{1-\kappa})$.*
Furthermore, let $A(T)$, $T \in [0, T_0]$ be a stochastic process with continuous paths in $\mathbb{C}$ that solves (AE) *with*

$$A(0) = \frac{1}{2\pi} \int_0^{2\pi} \varepsilon^{-1} u(0,x) e^{ix} dx,$$

Then for all $p \in \mathbb{N}$ there is a constant C_p such that the following holds:

$$\mathbb{P}\Big(\sup_{t\in[0,T_0]} \|u(t) - u_A(t) - \varepsilon Z_\varepsilon(\varepsilon^2 t) - e^{-t(1+\partial_x^2)^2} u_s(0)\|_\infty > \varepsilon^{2-19\kappa} \Big) \leq C_p \varepsilon^p, \tag{8}$$

with the approximation

$$u_A(t,x) = \varepsilon A(\varepsilon^2 t) e^{ix} - \varepsilon \bar{A}(\varepsilon^2 t) e^{-ix}$$

where Z_ε is the Ornstein-Uhlenbeck process defined by

$$Z_\varepsilon(T) := \varepsilon^{-1} \sigma \int_0^T e^{-\varepsilon^{-2}(T-s)} d\tilde{\beta}(s). \tag{9}$$

Here we easily see that $Z_\varepsilon(\varepsilon^2 t) = Z(t)$ with Z defined in (7).

Remark 5. *We see in* (AE) *surprising deterministic terms. The origin of these lie in nonlinear interaction of the noise together with averaging results (see Lemma 11). There is a stabilizing linear term from the cubic term, that was already observed in* [Hut08]. *The quadratic term leads to destabilizing terms both cubic and linear. But if α is not too large, increasing the noise strength σ may lead to a stabilization effect.*

Remark 6. *We assume in Theorem 4 that $\alpha^2 < \frac{27}{38}$. This means that the amplitude equation* (AE) *has a stable cubic nonlinearity. Nevertheless as long as the solution $A(T)$ to the AE stays small enough (for example $|A(T)| \leq \varepsilon^{-\kappa}$) our result still holds for $\alpha^2 \geq \frac{27}{38}$. The proof is basically the same except T_0 is exchanged for the stopping time $\tau_A = \inf\{t : |A(t)| \geq \varepsilon^{-\kappa}\} \wedge T_0$. For simplicity of presentation, we refrain from giving details here.*

Remark 7. *The interesting case $\alpha^2 = \frac{27}{38}$ was studied in the deterministic case. See for example* [BD12], *where an even more general case was treated. In this case* (AE) *loses its cubic nonlinearity. Thus we can change the scaling and consider larger solutions and, moreover, larger noise. Still a meaningful amplitude equation is obtained but now with a quintic nonlinearity.*

Using the methods presented in this paper it is straightforward but lengthy to derive the quintic amplitude equation also in the stochastic case. We refrain from giving details here.

4. Proof of the main result

We start by rescaling $u(t,x)$ to the slow time-scale by

$$v(T,x) := \varepsilon^{-1}u(\varepsilon^{-2}T,x) .$$

Its stochastic differential is given by

$$dv = (-\varepsilon^{-2}(1+\partial_x^2)^2 v + \nu v + \varepsilon^{-1}\alpha v^2 - v^3)dT + \varepsilon^{-1}\sigma d\tilde{\beta} .$$

The mild formulation is:

$$\begin{aligned} v(T) &= e^{-T\varepsilon^{-2}(1+\partial_x^2)^2}v(0) + Z_\varepsilon(T) \\ &\quad + \int_0^T e^{-(T-s)\varepsilon^{-2}(1+\partial_x^2)^2}[\nu v(s) + \varepsilon^{-1}\alpha v^2(s) - v^3(s)]ds . \end{aligned} \tag{10}$$

Here Z_ε is the fast Ornstein-Uhlenbeck process defined in (9). It is the solution of

$$dZ_\varepsilon = -\varepsilon^{-2}Z_\varepsilon dT + \sigma\varepsilon^{-1}d\tilde{\beta}, \quad Z_\varepsilon(0) = 0 . \tag{11}$$

Also we define the stopping time

$$\tau^* = \inf\{T > 0 : \|v(T)\|_\infty > \varepsilon^{-\kappa_0}\} \wedge T_0, \tag{12}$$

where κ_0 is any small real value with $\kappa_0 > \kappa$, which asserts that $\tau^* > 0$ almost surely. Later we fix $\kappa_0 = \frac{9}{8}\kappa$. Expanding $v(T,x)$ as a complex Fourier series yields

$$v(T,x) = \sum_{k=-\infty}^{\infty} v_k(T)e^{ikx} . \tag{13}$$

Define a splitting of the Fourier modes into the non-dominant modes

$$v_s(T,x) = \sum_{|k|\neq 1} v_k(T)e^{ikx} \tag{14}$$

and the dominant modes

$$v_c(T,x) = v(T,x) - v_s(T,x) = v_1(T)e^{ix} + c.c. . \tag{15}$$

Finally for technical reasons, we define

$$v_\infty(T,x) = \sum_{|k|\geq 3}[v_k(T) - e^{-T\varepsilon^{-2}(1-k^2)^2}v_k(0)] \cdot e^{ikx} \tag{16}$$

For $|k| \geq 1$ from the mild solution (10), each v_k is given by

$$\begin{aligned} v_k(T) &= e^{-\varepsilon^{-2}(1-k^2)^2T}v_k(0) \\ &\quad + \int_0^T e^{-\varepsilon^{-2}(1-k^2)^2(T-s)}\left[\nu v_k(s) + \varepsilon^{-1}\alpha\widehat{(v^2)}_k(s) - \widehat{(v^3)}_k(s)\right]ds, \end{aligned} \tag{17}$$

where the hat indicates the discrete Fourier transform and the lower index k denotes its k-th mode.

4.1. **Removing non-dominant modes.** we show first that the non-dominant modes ($|k| \neq 1$) can be approximated by the fast OU-process Z_ε. With a slight abuse of the $\mathcal{O}$-notation, our result states:

$$v_s(T) = e^{-T\varepsilon^{-2}(1+\partial_x^2)^2} v_s(0) + Z_\varepsilon(T) + \mathcal{O}(\varepsilon^{1-3\kappa_0}) \ .$$

Or, to be more precise:

Lemma 8. *Under the assumptions of Theorem 4, with stopping time τ^* defined by* (12) *and v_k as in* (13)*, the following statements are true:*

$$\sup_{T\in[0,\tau^*]} \|\sum_{|k|\geq 2} [v_k(T) - e^{-T\varepsilon^{-2}(1-k^2)^2} v_k(0)] \cdot e^{ikx}\|_\infty = \mathcal{O}(\varepsilon^{1-2\kappa_0}) \ , \tag{18}$$

$$\sup_{T\in[0,\tau^*]} \|v_0(T) - Z_\varepsilon(T) - e^{-T\varepsilon^{-2}} v_0(0)\| = \mathcal{O}(\varepsilon^{1-2\kappa_0}) \ . \tag{19}$$

Proof. Since $\|v\|_\infty \leq \varepsilon^{-\kappa_0}$, it follows that for any $k \in \mathbb{Z}$ and $n \in \mathbb{N}$

$$|(\widehat{v^n})_k| \leq \Big(\sum_{k\in\mathbb{Z}} |(\widehat{v^n})_k|^2\Big)^{1/2} = \|\widehat{v^n}\|_{L_2} = \|v^n\|_{L_2} \leq \sqrt{2\pi}\|v^n\|_\infty \leq \sqrt{2\pi}\varepsilon^{-n\kappa_0}. \tag{20}$$

In combination with the simple inequality (for $|k| \neq 1$)

$$\int_0^T e^{-\varepsilon^{-2}(1-k^2)^2(T-s)} ds \leq (1-k^2)^{-2}\varepsilon^2,$$

we can bound the right side of (17) by

$$\Big|v_k(T) - e^{-T\varepsilon^{-2}(1-k^2)^2} v_k(0)\Big| \leq \varepsilon^{1-2\kappa_0} \cdot (1-k^2)^{-2} \cdot (2 + |\nu| + |\alpha|). \tag{21}$$

Therefore with $\sum_{|k|\geq 2}(1-k^2)^{-2} \leq \sum_{k=1}^\infty k^{-2} = \frac{\pi^2}{6}$ we obtain (using $\kappa_0 < 1$ for the cubic term)

$$\sum_{|k|\geq 2} \Big|v_k(T) - e^{-T\varepsilon^{-2}(1-k^2)^2} v_k(0)\Big| \leq \varepsilon^{1-2\kappa_0} \cdot \frac{\pi^2}{3}(2 + |\nu| + |\alpha|),$$

which proves (18). Projecting the mild solution (6), the constant mode v_0 has the form

$$\begin{aligned} v_0(T) =& e^{-\varepsilon^{-2}T} v_0(0) + Z_\varepsilon(T) \\ &+ \int_0^T e^{-\varepsilon^{-2}(T-s)} (\nu v_k(s) + \varepsilon^{-1}\alpha(\widehat{v^2})_0(s) - (\widehat{v^3})_0(s)) ds \ . \end{aligned} \tag{22}$$

Thus with similar arguments as before, for all $T < \tau^*$ the left side of (19) is bounded by

$$\Big|v_0(T) - Z_\varepsilon(T) - e^{-\varepsilon^{-2}T} v_0(0)\Big| \leq \varepsilon^{1-2\kappa_0}(2 + |\nu| + |\alpha|).$$

□

4.2. **Rewriting the first Fourier-Mode.** The next step is to show that the dominant mode $v_1(T)$ is well approximated by $A(T)$. For simplicity of presentation let us define the following functions:

$$a(T) := v_1(T), \qquad \Phi(T) := \varepsilon^{-1}\left(v_2(T) - e^{-9T\varepsilon^{-2}}v_2(0)\right),$$
$$\Psi(T) := \varepsilon^{-1}\left(v_0(T) - Z_\varepsilon(T) - e^{-T\varepsilon^{-2}}v_0(0)\right).$$

Lemma 9. *Under the assumptions of Lemma 8, the stochastic differential of $a(T)$ is given by*

$$da = (\nu a + 3(\tfrac{38}{27}\alpha^2 - 1)a|a|^2 + 6(\alpha^2 - \tfrac{1}{2})aZ_\varepsilon^2)dT + 2\alpha\sigma a d\tilde{\beta} + dR, \tag{23}$$

where $R(t)$ is a stochastic processes with $\sup_{t\in[0,\tau^*]}|R(t)| = \mathcal{O}(\varepsilon^{1-8\kappa_0})$.

Proof. In Lemma 8 in (20) and (21) we established:

$$\sup_{T\in[0,\tau^*]}|v_1(T)| \leq \varepsilon^{-\kappa_0} \tag{24}$$

$$\sup_{T\in[0,\tau^*]}\big(\sup_{|k|\geq 2}|v_k(T) - e^{-\varepsilon^{-2}(1-k^2)^2}v_k(0)|\big) = \mathcal{O}(\varepsilon^{1-2\kappa_0}). \tag{25}$$

This readily implies

$$\sup_{T\in[0,\tau^*]}|a(T)| = \mathcal{O}(\varepsilon^{-2\kappa_0}), \quad \sup_{T\in[0,\tau^*]}|\Phi(T)| = \mathcal{O}(\varepsilon^{-2\kappa_0}), \quad \sup_{T\in[0,\tau^*]}|\Psi(T)| = \mathcal{O}(\varepsilon^{-2\kappa_0}).$$

The infinite-dimensional part is bounded by

$$\sup_{T\in[0,\tau^*]}\|v_\infty(T)\|_\infty = \mathcal{O}(\varepsilon^{1-2\kappa_0}). \tag{26}$$

The OU-process can be bounded by

$$\sup_{T\in[0,\tau^*]}|Z_\varepsilon(T)| = \mathcal{O}(\varepsilon^{-\gamma}) \tag{27}$$

for all positive $\gamma \in \mathbb{R}$. For a proof of this well-known result see for example [BM12] p. 9 (Lemma 14).

Now we can directly calculate the stochastic differentials $da, d\Phi$ and $d\Psi$ by writing v as

$$v = ae^{ix} + \varepsilon\Phi e^{i2x} + \bar{a}e^{-ix} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + Z_\varepsilon + v_\infty + e^{-T\varepsilon^{-2}(1+\varepsilon^2\partial_x^2)^2}v_s(0)$$

and multiplying it with itself to bound $(\widehat{v^2})_k$ and $(\widehat{v^3})_k$ for $k \in \{0, 1, 2\}$. Note that we can bound the Fourier transform by the L^∞ norm. We have

$$\begin{aligned} v^2 &= 2(ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi}e^{-i2x} + \varepsilon\Psi + v_\infty) \\ &\quad + (ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)^2 + r_1 \\ v^3 &= (ae^{ix} + \bar{a}e^{-ix} + Z_\varepsilon)^3 + r_2 \end{aligned} \tag{28}$$

with

$$
\begin{aligned}
r_1 &= (\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi} e^{-i2x} + \varepsilon\Psi + v_\infty)^2 + (e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0))^2 \\
&\quad + 2(a e^{ix} + \varepsilon\Phi e^{i2x} + \bar{a} e^{-ix} + \varepsilon\bar{\Phi} e^{-i2x} + \varepsilon\Psi + Z_\varepsilon + v_\infty) e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0) \\
r_2 &= (\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi} e^{-i2x} + \varepsilon\Psi + v_\infty)^3 + (e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0))^3 \\
&\quad + 3(a e^{ix} + \bar{a} e^{-ix} + Z_\varepsilon)(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi} e^{-i2x} + \varepsilon\Psi + v_\infty)^2 \\
&\quad + 3(a e^{ix} + \bar{a} e^{-ix} + Z_\varepsilon)^2(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi} e^{-i2x} + \varepsilon\Psi + v_\infty) \\
&\quad + 3(a e^{ix} + \bar{a} e^{-ix} + Z_\varepsilon)(e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0))^2 + 3(a e^{ix} + \bar{a} e^{-ix} + Z_\varepsilon)^2(e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0)) \\
&\quad + 3(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi} e^{-i2x} + \varepsilon\Psi + v_\infty)(e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0))^2 \\
&\quad + 3(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi} e^{-i2x} + \varepsilon\Psi + v_\infty)^2(e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0)) \\
&\quad + 6(a e^{ix} + \bar{a} e^{-ix} + Z_\varepsilon)(\varepsilon\Phi e^{i2x} + \varepsilon\bar{\Phi} e^{-i2x} + \varepsilon\Psi + v_\infty)(e^{-\varepsilon^{-2}T\mathcal{L}} v_s(0)).
\end{aligned}
$$

Because of

$$
\sup_{T\in[0,\tau^*]} \|\varepsilon\Phi(T) e^{i2x} + \varepsilon\bar{\Phi}(T) e^{-i2x} + \varepsilon\Psi(T) + v_\infty(T)\|_\infty = \mathcal{O}(\varepsilon^{1-2\kappa_0}),
$$

$$
\sup_{T\in[0,\tau^*]} \|a(T) e^{ix} + \bar{a}(T) e^{-ix} + Z_\varepsilon(T)\|_\infty = \mathcal{O}(\varepsilon^{-2\kappa_0}),
$$

which follows from (24), (25), (26) and (27), together with

$$
\begin{aligned}
\|\int_0^T e^{-\varepsilon^{-2}s\mathcal{L}} v_s(0) ds\|_\infty &\le \varepsilon^2 \sum_{|k|\neq 1} (1-k^2)^{-2} |(\widehat{v_s(0)})_k| \\
&\le \varepsilon^2 \sqrt{2\pi} \sum_{|k|\neq 1} (1-k^2)^{-2} \|v_s(0)\|_\infty = \mathcal{O}(\varepsilon^{2-\kappa_0})
\end{aligned}
$$

we can bound the integral in time of r_1 and r_2 by

$$
\sup_{T\in[0,\tau^*]} \|\int_0^T r_1 dt\|_\infty = \mathcal{O}(\varepsilon^{2-6\kappa_0})
$$

$$
\sup_{T\in[0,\tau^*]} \|\int_0^T r_2 dt\|_\infty = \mathcal{O}(\varepsilon^{1-6\kappa_0}).
$$

Analogously we can bound integrals of any power of $\|r_i\|_\infty$. Inserting (28) into the mild solution formulas (17) respectively (22) gives

$$
da = (\nu a + 2\alpha\bar{a}\Phi + 2\alpha a\Psi - 3a|a|^2 - 3aZ_\varepsilon^2 + \varepsilon^{-1} 2\alpha a Z_\varepsilon + R_1) dT \tag{29}
$$

$$
d\Phi = (-9\varepsilon^{-2}\Phi + \varepsilon^{-2}\alpha a^2 + R_2) dT \tag{30}
$$

$$
d\Psi = (-\varepsilon^{-2}\Psi + \varepsilon^{-2}\alpha |a|^2 + \varepsilon^{-2}\alpha Z_\varepsilon^2 + R_3) dT \tag{31}
$$

where

$$R_1(t) = \varepsilon^{-1}\alpha(\widehat{r_1})_1 - (\widehat{r_2})_1,$$
$$R_2(t) = \nu\Phi + 2\varepsilon^{-1}\alpha Z_\varepsilon\Phi - 3\varepsilon^{-1}a^2 Z_\varepsilon + 2\varepsilon^{-2}\alpha v_3\bar{a} + \varepsilon^{-2}\alpha(\widehat{r_1})_2 - \varepsilon^{-1}(\widehat{r_2})_2$$

and

$$R_3(t) = \nu\Psi + \varepsilon^{-1}\alpha\Psi Z_\varepsilon - \varepsilon^{-1}Z_\varepsilon^3 + 6\varepsilon^{-1}|a|^2 Z_\varepsilon + \varepsilon^{-2}\alpha(\widehat{r_1})_0 - \varepsilon^{-1}(\widehat{r_2})_0$$

are stochastic processes with

$$\sup_{T\in[0,\tau^*]}\int_0^T |R_1|ds = \mathcal{O}(\varepsilon^{1-6\kappa_0}), \qquad \sup_{T\in[0,\tau^*]}\int_0^T |R_2| + |R_3|ds = \mathcal{O}(\varepsilon^{-1-6\kappa_0}).$$

In order to eliminate Φ and Ψ on the right side of (29) we apply the Itô formula to $\bar{a}\Phi$, $a\Psi$ and aZ_ε. Note that there is no Itô correction at this point.

$$d(\bar{a}\Phi) = (d\bar{a})\Phi + \bar{a}(d\Phi) = (\bar{a}(-9\varepsilon^{-2}\Phi + \varepsilon^{-2}\alpha a^2) + R_4)dT$$
$$d(a\Psi) = (da)\Psi + a(d\Psi) = (a(-\varepsilon^{-2}\Psi + 2\varepsilon^{-2}\alpha|a|^2 + \varepsilon^{-2}\alpha Z_\varepsilon^2) + R_5)dT$$
$$d(aZ_\varepsilon) = (da)Z_\varepsilon + a(dZ_\varepsilon) = (\varepsilon^{-1}2\alpha a Z_\varepsilon^2 - \varepsilon^{-2}aZ_\varepsilon + R_6)dT + a\varepsilon^{-1}\sigma d\tilde{\beta}$$

where

$$R_4(t) = \bar{a}R_2 + \Phi(\nu\bar{a} + 2\alpha a\bar{\Phi} + 2\alpha\bar{a}\bar{\Psi} - 3\bar{a}|a|^2 - 3\bar{a}Z_\varepsilon^2 + \varepsilon^{-1}2\alpha\bar{a}Z_\varepsilon + \bar{R}_1),$$
$$R_5(t) = aR_3 + \Psi(\nu a + 2\alpha\bar{a}\Phi + 2\alpha a\Psi - 3a|a|^2 - 3aZ_\varepsilon^2 + \varepsilon^{-1}2\alpha aZ_\varepsilon + R_1)$$

and

$$R_6(t) = Z_\varepsilon(\nu a + 2\alpha\bar{a}\Phi + 2\alpha a\Psi - 3a|a|^2 - 3aZ_\varepsilon^2 + R_1)$$

are stochastic processes with

$$\sup_{t\in[0,\tau^*]}\int_0^T |R_4| + |R_5|ds = \mathcal{O}(\varepsilon^{-1-8\kappa_0}), \qquad \sup_{t\in[0,\tau^*]}\int_0^T |R_6|ds = \mathcal{O}(\varepsilon^{-8\kappa_0}).$$

Therefore we have

$$\bar{a}\Phi dT = (\frac{1}{9}\alpha a|a|^2 + \varepsilon^2 R_4)dT - d(\varepsilon^2\bar{a}\Phi) \tag{32}$$

$$a\Psi dT = (2\alpha a|a|^2 + \alpha a Z_\varepsilon^2 + \varepsilon^2 R_5)dT - d(\varepsilon^2 a\Psi) \tag{33}$$

$$\varepsilon^{-1}aZ_\varepsilon dT = (2\alpha a Z_\varepsilon^2 + \varepsilon R_6)dT + \sigma a d\tilde{\beta}(T) - d(\varepsilon a Z_\varepsilon) \tag{34}$$

and by substituting (32) – (34) into (29) we get the desired result for da with

$$dR = 2\alpha\varepsilon^2((R_4 dT + R_5 dT - d(\bar{a}\Phi) - d(a\Psi)) + 2\alpha\varepsilon(R_6 dT - d(aZ)).$$

□

4.3. **Averaging with error bounds.** Next we have to get the equation for da to match the amplitude equation (AE). For this we need to remove $aZ_\varepsilon^2 dT$. This is done in this section. First we need the following technical Lemma

Lemma 10. *Let $X(t,\omega) \in \mathbb{C}$ be a stochastic process with*

$$X(t) = \int_0^t f(s)ds + \int_0^t g(s)d\tilde{\beta},$$

where $\sup_{t\in[0,T_0]} |f(t)| = \mathcal{O}(\varepsilon^\gamma)$ and $\sup_{t\in[0,T_0]} |g(t)| = \mathcal{O}(\varepsilon^\gamma)$ with $\gamma \in \mathbb{R}$. Then $X(t)$ has the same bound as $f(t)$ and $g(t)$:

$$\sup_{t\in[0,T_0]} |X(t)| = \mathcal{O}(\varepsilon^\gamma) \tag{35}$$

Let us remark that the same result is true, if we replace T_0 by the stopping time τ^*.

Proof. The proof is straightforward using Burkholder-Davis-Gundy, Hölder, and Young's inequality. □

Now we can substitute the aZ^2 term in (23). This is done by using the averaging property of Z_ε described in the next Lemma.

Lemma 11. *Let $X(t) \in \mathbb{C}$ be a stochastic process with $dX = f(T)dT + g(T)d\tilde{\beta}$, where $\sup_{T\in[0,T_0]} |f(T)| = \mathcal{O}(\varepsilon^{-\gamma})$ and $\sup_{T\in[0,T_0]} |g(T)| = \mathcal{O}(\varepsilon^{-\gamma})$ with $\gamma > 0$. Then with Z_ε as defined by (11) the following holds:*

$$\sup_{T\in[0,T_0]} \Big| \int_0^T X(s)Z_\varepsilon(s)^2 ds - \int_0^T \frac{1}{2}\sigma^2 X(s)ds \Big| = \mathcal{O}(\varepsilon^{1-\kappa_0-\gamma}). \tag{36}$$

Again the same result is true, if we replace T_0 by the stopping time τ^*.

Proof. By using Itô's formula we get

$$d(XZ_\varepsilon^2) = (dX)Z_\varepsilon^2 + X(dZ_\varepsilon^2) + (dX)(dZ_\varepsilon^2)$$

and

$$d(Z_\varepsilon^2) = 2(dZ_\varepsilon)Z_\varepsilon + (dZ_\varepsilon)^2 = 2Z_\varepsilon(-\varepsilon^{-2}Z_\varepsilon dT + \varepsilon^{-1}\sigma d\tilde{\beta}) + \varepsilon^{-2}\sigma^2 dT.$$

This gives

$$d(XZ_\varepsilon^2) = fZ_\varepsilon dT + gZ_\varepsilon d\tilde{\beta} - \varepsilon^{-2}2XZ_\varepsilon^2 dT + \varepsilon^{-1}2\sigma XZ_\varepsilon d\tilde{\beta} + \varepsilon^{-2}\sigma^2 XdT + \varepsilon^{-1}\sigma g dT.$$

We already know from the proof of Lemma 9 that $\sup_{T\in[0,T_0]} |Z_\varepsilon(T)| = \mathcal{O}(\varepsilon^{-\kappa_0})$ and it follows from Lemma 10 that $\sup_{T\in[0,T_0]}|X(T)| = \mathcal{O}(\varepsilon^{-\gamma})$. Therefore $d(XZ_\varepsilon^2)$ can be written as

$$d(XZ_\varepsilon^2) = -\varepsilon^{-2}2XZ_\varepsilon^2 dT + \varepsilon^{-2}\sigma^2 XdT + R_7 dT + R_8 d\tilde{\beta},$$

where $R_7(T)$ and $R_8(T)$ are stochastic processes with

$$\sup_{[0,\tau^*]}|R_7| = \mathcal{O}(\varepsilon^{-1-\kappa_0-\gamma}), \qquad \sup_{[0,\tau^*]}|R_8| = \mathcal{O}(\varepsilon^{-1-\kappa_0-\gamma}).$$

By multiplying with ε^2 and integrating from 0 to T we get

$$\int_0^T \tfrac{1}{2}\sigma^2 X ds - \int_0^T XZ_\varepsilon^2 ds = \tfrac{1}{2}\varepsilon^2 XZ_\varepsilon^2\Big|_0^T - \varepsilon^2 \int_0^T R_7 ds - \varepsilon^2 \int_0^T R_8 d\tilde{\beta}$$

and the application of Hölder and Burkholder-Davis-Gundy yields the desired result. □

4.4. **SDE Lemma.** With Lemma 11 we have closed the gap between the SDEs (AE) and (23) down to some error on the right side which is of order $\varepsilon^{1-8\kappa_0}$. But to be able to compare the first Fourier mode a and the solution of the amplitude equation A we need the following Lemma.

Lemma 12. *Let $X_1(t), X_2(t) \in \mathbb{C}$ be stochastic processes given by*

$$\begin{aligned} X_1(t) &= X_1(0) + \int_0^t f(X_1) ds + \int_0^t g(X_1) d\beta \\ X_2(t) &= X_1(0) + \int_0^t f(X_2) ds + \int_0^t g(X_2) d\beta + R(t) \end{aligned} \tag{37}$$

with $sup_{t\in[0,\tau_0]}|R(t)| = \mathcal{O}(\varepsilon^\gamma)$, where $\gamma \in \mathbb{R}$ and $\tau_0 \le T_0$ is a stopping time. Let there be a constant $C > 0$ and a process $\hat{R}(t)$ with $\sup_{t\in[0,\tau_0]}|\hat{R}(t)| = \mathcal{O}(\varepsilon^\gamma)$ such that the functions f and g satisfy the following conditions:

$$\operatorname{Re}\{(f(X_1) - f(X_2))\overline{\varphi}\} \le C(|\varphi|^2 + |\hat{R}(t)|^2) \tag{38}$$

$$\forall x, y \in \mathbb{C}: \quad |g(x) - g(y)|^2 \le C|x-y|^2, \tag{39}$$

where $\varphi := X_1 - (X_2 - R)$. Then the difference between X_1 and X_2 can be bounded by

$$\sup_{t\in[0,\tau_0]} |X_1(t) - X_2(t)| = \mathcal{O}(\varepsilon^\gamma). \tag{40}$$

Note that condition (38) can be established by a bound of the type

$$\operatorname{Re}\{(f(x) - f(y))(x - y - z)\} \le C|x - y - z|^2 + p(y, z)$$

with polynomial p provided we have additional bounds on the process X_2.

Proof. Because of the unknown derivative of R it is much easier to split $X_1 - X_2$ into

$$X_1 - X_2 = \varphi - R \tag{41}$$

and bound $|\varphi|$ rather than the actual term.

Due to the stopping time the process φ is not easily bounded directly. Thus we extend all processes to $[0,T_0]$ and define

$$\tilde{R}(t) := \begin{cases} R(t) & \text{for } t \leq \tau_0 \\ R(\tau_0) & \text{for } t > \tau_0 \end{cases}$$

and modify X_1 and X_2:

$$\begin{aligned} \tilde{X}_1(t) &:= X_1(0) + \int_0^{\tau_0\wedge t} f(\tilde{X}_1)ds + \int_0^t g(\tilde{X}_1)d\beta \\ \tilde{X}_2(t) &:= X_1(0) + \int_0^{\tau_0\wedge t} f(\tilde{X}_2)ds + \int_0^t g(\tilde{X}_2)d\beta + \tilde{R}(t). \end{aligned}$$

With this we can define a suitable replacement for φ:

$$\begin{aligned} \varphi_{\tau_0}(t) &:= \tilde{X}_1(t) - (\tilde{X}_2(t) - \tilde{R}(t)) \\ &= \int_0^{\tau_0\wedge t} (f(X_1) - f(X_2))ds + \int_0^{\tau_0\wedge t} (g(\tilde{X}_1) - g(\tilde{X}_2))d\beta. \end{aligned}$$

Note that $\sup_{t\in[0,T_0]} |\tilde{R}(t)| = \mathcal{O}(\varepsilon^\gamma)$ and for any stopping time $\tau \leq \tau_0$ we have $\varphi_{\tau_0}(\tau) = \varphi(\tau)$, $\tilde{X}_1(\tau) = X_1(\tau)$ and $\tilde{X}_2(\tau) = X_2(\tau)$. This means

$$\sup_{t\in[0,\tau_0]} |\varphi(t)| = \sup_{t\in[0,\tau_0]} |\varphi_{\tau_0}(t)|.$$

Now in order to bound the moments of $\sup_{t\in[0,\tau_0]} |\varphi_{\tau_0}|$ we first need a bound on the moments of $|\varphi_{\tau_0}|$. We start by taking the differential of $|\varphi_{\tau_0}|^{2p}$ for $p \in \mathbb{N}$:

$$\begin{aligned} d|\varphi_{\tau_0}|^{2p} &= d(\overline{\varphi_{\tau_0}}\varphi_{\tau_0})^p = p(\overline{\varphi_{\tau_0}}\varphi_{\tau_0})^{p-1} d(\overline{\varphi_{\tau_0}}\varphi_{\tau_0}) \\ &= p|\varphi_{\tau_0}|^{2p-2} \left((d\overline{\varphi_{\tau_0}})\varphi_{\tau_0} + \overline{\varphi_{\tau_0}}(d\varphi_{\tau_0}) + (d\overline{\varphi_{\tau_0}})(d\varphi_{\tau_0}) \right). \end{aligned}$$

The derivative of φ_{τ_0} is given by

$$d\varphi_{\tau_0} = \chi_{[0,\tau_0\wedge t]}(f(X_1) - f(X_2))dt + (g(\tilde{X}_1) - g(\tilde{X}_2))d\beta\ .$$

Therefore

$$\begin{aligned} d|\varphi_{\tau_0}|^{2p} = p|\varphi_{\tau_0}|^{2p-2} [&\chi_{[0,\tau_0\wedge t]} 2\,\mathrm{Re}\left\{\overline{\varphi_{\tau_0}}(f(X_1) - f(X_2))\right\} dt \\ &+ 2\,\mathrm{Re}\left\{\overline{\varphi_{\tau_0}}(g(\tilde{X}_1) - g(\tilde{X}_2))\right\} d\beta + |g(\tilde{X}_1) - g(\tilde{X}_2)|^2 dt]. \end{aligned}$$

Next we integrate and split the right side into three parts:

$$
\begin{aligned}
|\varphi_{\tau_0}(t)|^{2p} = &\int_0^{\tau_0\wedge t} p|\varphi_{\tau_0}|^{2p-2} 2\,\mathrm{Re}\left\{\overline{\varphi_{\tau_0}}(f(X_1)-f(X_2))\right\} ds \\
&+ \int_0^t p|\varphi_{\tau_0}|^{2p-2} 2\,\mathrm{Re}\left\{\overline{\varphi_{\tau_0}}(g(\tilde X_1)-g(\tilde X_2))\right\} d\beta \\
&+ \int_0^t p|\varphi_{\tau_0}|^{2p-2} |g(\tilde X_1)-g(\tilde X_2)|^2 ds \\
:= &\, I_1 + I_2 + I_3
\end{aligned}
$$

For the first part we can exchange φ and φ_{τ_0} freely because the integral goes only up to the stopping time τ_0. Doing this and using (38) we get

$$
\begin{aligned}
I_1 &= \int_0^{\tau_0\wedge t} p|\varphi|^{2p-2} 2\,\mathrm{Re}\left\{\overline{\varphi}(f(X_1)-f(X_2))\right\} ds \\
&\le \int_0^{\tau_0\wedge t} p|\varphi|^{2p-2} 2C(|\varphi|^2+|\hat R|^2) ds \\
&\le \int_0^{\tau_0\wedge t} C_p(|\varphi_{\tau_0}|^{2p}+|\hat R|^{2p}) ds \le C_p(\int_0^t (|\varphi_{\tau_0}|^{2p} ds + \int_0^{\tau_0} |\hat R|^{2p}) ds,
\end{aligned}
$$

where C_p is a constant depending on p and we used Young's inequality in the last step. The third part can be bounded from above by using (39) and a simple application of the triangle inequality:

$$
\begin{aligned}
I_3 &\le \int_0^t p|\varphi_{\tau_0}|^{2p-2} |\tilde X_1-\tilde X_2|^2 ds \\
&\le \int_0^t p|\varphi_{\tau_0}|^{2p-2} (|\varphi_{\tau_0}|^2+|\tilde R|^2) ds \le \int_0^t C_p(|\varphi_{\tau_0}|^{2p}+|\tilde R|^{2p}) ds
\end{aligned}
$$

Again we used Young's inequality in the last step. Now since stochastic integration preserves the local martingale property, taking the expectation value of $|\varphi_{\tau_0}|^{2p}$ yields, for all $t \le T_0$,

$$
\begin{aligned}
\mathbb{E}(|\varphi_{\tau_0}(t)|^{2p}) &= \mathbb{E}(I_1) + \mathbb{E}(I_2) \\
&\le C_p\mathbb{E}\Big(\int_0^t |\varphi_{\tau_0}|^{2p} + |\tilde R|^{2p} ds + \int_0^{\tau_0} |\hat R|^{2p} ds \Big) \\
&\le \int_0^t C_p \mathbb{E}(|\varphi_{\tau_0}|^{2p}) ds + C_p T_0 R_{\mathrm{sup}}^{2p},
\end{aligned}
$$

where $R_{sup}^{2p} := \mathbb{E}(\sup_{t\in[0,\tau_0]}|\hat{R}(t)|^{2p} + \sup_{t\in[0,T_0]}|\tilde{R}(t)|^{2p})$. We apply Gronwall's Lemma to get

$$\begin{aligned}\mathbb{E}(|\varphi_{\tau_0}(t)|^{2p}) &\leq C_pT_0R_{sup}^{2p} + \int_0^t C_p^2T_0R_{sup}^{2p}e^{(T_0-s)C_p}ds \\ &\leq C_pT_0R_{sup}^{2p} + C_p^2T_0^2R_{sup}^{2p}e^{T_0C_p}. \qquad (42)\end{aligned}$$

With this we can now bound the moments of $\sup_{t\in[0,\tau_0]}|\varphi_{\tau_0}(t)|$. We start with $\mathbb{E}(\sup_{t\in[0,\tau_0]} I_3(t))$:

$$\begin{aligned}\mathbb{E}(\sup_{t\in[0,\tau_0]} I_3(t)) &= \mathbb{E}\sup_{t\in[0,\tau_0]}(\int_0^t 2\,\mathrm{Re}\left\{\overline{\varphi_{\tau_0}}(g(\tilde{X}_1)-g(\tilde{X}_2))\right\}d\beta) \\ &\leq \mathbb{E}\big(\int_0^{\tau_0} C_p^2|\varphi_{\tau_0}|^{4p-2}|g(\tilde{X}_1)-g(\tilde{X}_2)|^2ds\big)^{1/2} \\ &\leq \big(\mathbb{E}\int_0^{T_0} C_p^2|\varphi_{\tau_0}|^{4p-2}(|\varphi_{\tau_0}|^2+|\tilde{R}|^2)ds\big)^{1/2} \\ &\leq C_p\big(\mathbb{E}\int_0^{T_0}|\varphi_{\tau_0}|^{4p}+|\tilde{R}|^{4p}ds\big)^{1/2},\end{aligned}$$

where we used the Burkholder Davis Gundy theorem in the second step, the Hölder inequality in the third and Young's inequality in the last step.
The whole term is now easily bounded by

$$\begin{aligned}\mathbb{E}(\sup_{t\in[0,\tau_0]}|\varphi(t)|)^{2p} &= \mathbb{E}(\sup_{t\in[0,\tau_0]}(I_1+I_2+I_3)) \\ &\leq C_p\mathbb{E}(\int_0^{T_0}(|\varphi_{\tau_0}|^{2p}+|\tilde{R}|^{2p})ds + \int_0^{\tau_0}|\hat{R}|^{2p}ds) \\ &\qquad + C_p\big(\mathbb{E}\int_0^{T_0}|\varphi_{\tau_0}|^{4p}+|\tilde{R}|^{4p}ds\big)^{1/2} \\ &\leq C_p(\int_0^{T_0}\mathbb{E}|\varphi_{\tau_0}|^{2p}ds) + C_p(\int_0^{T_0}\mathbb{E}|\varphi_{\tau_0}|^{4p}ds)^{1/2} \\ &\qquad + C_p(T_0+T_0^{1/2})R_{\text{sup}}^{2p}.\end{aligned}$$

Using (42) we get

$$\begin{aligned}\mathbb{E}(\sup_{t\in[0,\tau_0]}|\varphi_{\tau_0}(t)|)^{2p} &\leq C_pT_0(C_pT_0R_{\text{sup}}^{2p} + C_p^2T_0^2R_{\text{sup}}^{2p}e^{T_0C_p}) \\ &\quad + C_pT_0^{1/2}(C_2pT_0R_{\text{sup}}^{4p} + C_2p^2T_0^2R_{\text{sup}}^{4p}e^{T_0C_2p}) + C_pT_0^{3/2}R_{\text{sup}}^{2p}.\end{aligned}$$

Finally any moment can be bounded by even moments through Hölder interpolation, which proves that $\sup_{t\in[0,\tau_0]}|\varphi(t)| = \sup_{t\in[0,\tau_0]}|\varphi_{\tau_0}(t)| = \mathcal{O}(\varepsilon^\gamma)$. By assumption we also have that $\sup_{t\in[0,\tau_0]}|R(t)| = \mathcal{O}(\varepsilon^\gamma)$, so the result follows from (41). □

From what we have proven it is easily shown that the theorem holds at least until the time τ^*, but we still need to show that τ^* is large enough. For this we prove bounds on moments of A which are a direct application of Lemma 12.

Corollary 13. *Let $A(t)$ be the solution to the amplitude equation* (AE) *then the following holds:*

$$\sup_{t\in[0,T_0]} |A(t)| = \mathcal{O}(\varepsilon^{-\kappa}). \tag{43}$$

Proof. We define f, g and R by

$$\begin{aligned} R(t) &:= -A(0) \\ f(A) &:= \nu A + 3(\tfrac{38}{27}\alpha^2 - 1)A|A|^2 + 3(\alpha^2 - \tfrac{1}{2})\sigma^2 A \\ g(A) &:= 2\sigma\alpha A. \end{aligned} \tag{44}$$

With this we can write A and zero as in (37):

$$\begin{aligned} A(t) &= A(0) + \int_0^t f(A)dt + \int_0^t g(A)d\beta \\ 0 &= A(0) + \int_0^t f(0)dt + \int_0^t g(0)d\beta + R. \end{aligned}$$

Since $f(0) = g(0)$ we obtain $\sup_{t\in[0,T_0]}|R(t)| = \sup_{t\in[0,T_0]}|A(0)| = \mathcal{O}(\varepsilon^{-\kappa})$, and we derive the desired result directly from Lemma 12, provided we can prove the conditions (38) and (39). Because g is linear (39) is readily verified:

$$|g(x) - g(y)|^2 = |2\sigma(x-y)|^2 \le 4\sigma^2|x-y|^2. \tag{45}$$

This leaves (38). For better readability we write f as

$$f(X) = C_1X - C_2|X|^2X$$

with positive constants C_1 and C_2. For the linear part of f we are in the same position as for g, there is no dependency on X_1 or X_2:

$$\operatorname{Re}\{(\overline{X_1} - (\overline{X_2} - \overline{R}))(C_1X_1 - X_2)\} \le 3C_1(|X_1 - (X_2 - R)|^2 + |R|^2). \tag{46}$$

For the cubic term, to keep this proof simple, we note that it is sufficient to bound it here just for the special case $X_1 = A$ and $X_2 = 0$.

$$\begin{aligned} \operatorname{Re}\{(\overline{A} - (0 - \overline{R}))(-C_2|A|^2A - 0)\} &= -C_2|A|^4 + \operatorname{Re}\{\overline{R}A\} \\ &\le 2(|A - (0-R)|^2 + |R|^2) \end{aligned}$$

□

4.5. **Removing the error.** Combining the Lemmas of the previous sections, we are now able to prove Theorem 4.

Proof of theorem 4. By Lemma 8 $u(t)$ can be approximated by $a = v_1$ and Z_ε until the time τ^*:

$$\sup_{t\in[0,\tau^*]} \|u(t) - \varepsilon a(\varepsilon^2 t)e^{ix} - \varepsilon\overline{a}(\varepsilon^2 t)e^{-ix} - \varepsilon Z_\varepsilon - e^{T\varepsilon^{-2}(1+\partial_x^2)^2}v_s(0)\|_\infty = \mathcal{O}(\varepsilon^{2-8\kappa_0}).$$

Now we bound the difference between a and A until time τ^*. The initial condition $A(0)$ is exactly the coefficient of the first Fourier mode of $v(0,x)$). This means $A(0) = a(0)$, thus by Lemma 9 and Lemma 10 we know that a is given by

$$a(t) = A(0) + \int_0^t (\nu a + 3(\tfrac{38}{27}\alpha^2 - 1)a|a|^2 + 6(\alpha^2 - \tfrac{1}{2})aZ_\varepsilon^2)ds + \int_0^t 2\sigma a d\tilde{\beta} + R_9,$$

where $\sup_{[0,\tau^*]} |R_9| = \mathcal{O}(\varepsilon^{1-8\kappa_0})$. Next we split the aZ_ε^2 term into

$$aZ_\varepsilon^2 = (a - R_9)Z_\varepsilon^2 + R_9 Z_\varepsilon^2.$$

The second part is bounded by $\sup_{[0,\tau^*]} |R_9 Z_\varepsilon^2| = \mathcal{O}(\varepsilon^{1-10\kappa_0})$ and the first part can be exchanged by using Lemma 11. Set $\kappa_0 = \frac{9}{8}\kappa$. Because

$$\sup_{[0,\tau^*]} |\nu a + 3(\frac{38}{27}\alpha^2 - 1)a|a|^2 + 6(\alpha^2 - \tfrac{1}{2})aZ_\varepsilon^2| = \mathcal{O}(\varepsilon^{-6\kappa_0}) \tag{47}$$

$$\sup_{[0,\tau^*]} |2\sigma a| = \mathcal{O}(\varepsilon^{-6\kappa_0}) \tag{48}$$

and $10\kappa_0 = \frac{45}{4}\kappa \leq 12\kappa$ we get

$$a(t) = A(0) + \int_0^t (\nu a + 3(\tfrac{38}{27}\alpha^2 - 1)a|a|^2 + 3(\alpha^2 - \tfrac{1}{2})\sigma^2 a)ds + \int_0^t 2\sigma a d\tilde{\beta} + R_{10},$$

where $\sup_{t\in[0,\tau^*]} |R_{10}(t)| = \mathcal{O}(\varepsilon^{1-12\kappa})$.

With f and g defined as in (44) we show that there exists a process $\hat{R}$ with

$$\sup_{t\in[0,\tau^*]} |\hat{R}(t)| = \mathcal{O}(\varepsilon^{1-18\kappa}) \tag{49}$$

such that the conditions (38) and (39) are fulfilled and we can apply Lemma 12. Since $\sup_{t\in[0,\tau^*]} |R_{10}| = \mathcal{O}(\varepsilon^{1-9\kappa})$ the condition on g and the linear term of f are already covered by (45) respectively (46). Because of this we only need show that there is a positive constant C and a process $\hat{R}$ conforming to (49) such that

$$\rho := \operatorname{Re}\left\{-C_2(\overline{A} - (\overline{a} - \overline{R_{10}}))(|A|^2 A - |a|^2 a)\right\} \leq C(|A - (a - R_{10})|^2 + |\hat{R}|^2),$$

where $C_2 = -3(\frac{38}{27}\alpha^2 - 1)$ is a positive constant. We do this by splitting ρ into two parts:

$$\begin{aligned}\rho &= \mathrm{Re}\left\{-C_2\left(\overline{A} - (\overline{a} - \overline{R_{10}})\right)\left(|A|^2A - |a|^2a\right)\right\}\\ &= \mathrm{Re}\left\{-C_2\left(\overline{A} - (\overline{a} - \overline{R_{10}})\right)\left(|A|^2A - |a-R_{10}|^2(a-R_{10})\right)\right\}\\ &\quad + \mathrm{Re}\left\{-C_2\left(\overline{A} - (\overline{a} - \overline{R_{10}})\right)\left(|a-R_{10}|^2(a-R_{10}) - |a|^2a\right)\right\}\\ &=: \rho_1 + \rho_2.\end{aligned}$$

The first term is negative because for any two complex numbers z, w we have

$$\begin{aligned}&2\,\mathrm{Re}\{(\overline{z} - \overline{w})(|z|^2z - |w|^2w\}\\ &\qquad = 2|z-w|^2(|z|^2+|w|^2) + 2\,\mathrm{Re}\{(z-w)^2\overline{zw}\}\\ &\qquad \geq 2|z-w|^2(|z|^2+|w|^2) - |z-w|^2(|z|^2+|w|^2)\\ &\qquad \geq |z-w|^2(|z|^2+|w|^2) \geq 0.\end{aligned}$$

This means ρ_1 can be bounded from above by 0. The second term can be bounded by

$$\begin{aligned}|\rho_2| &\leq C_2|\overline{A} - (\overline{a} - \overline{R_{10}})|(3|a|^2|R_{10}| + 3|a||R_{10}|^2 + |R_{10}|^3)\\ &\leq C_2(|\overline{A} - (\overline{a} - \overline{R_{10}})|^2 + (3|a|^2|R_{10}| + 3|a||R_{10}|^2 + |R_{10}|^3)^2)\end{aligned}$$

and since $\sup_{t\in[0,\tau^*]}|a(t)| = \mathcal{O}(\varepsilon^{-3\kappa})$ we obtain (as $\kappa < \frac{1}{17}$)

$$\sup_{t\in[0,\tau^*]}\left(3|a|^2|R_{10}| + 3|a||R_{10}|^2 + |R_{10}|^3\right) = \mathcal{O}(\varepsilon^{1-18\kappa}).$$

Therefore Lemma 12 yields the following bound on $|A-a|$:

$$\sup_{t\in[0,\tau^*]}|A(t) - a(t)| = \mathcal{O}(\varepsilon^{1-18\kappa}).$$

Combining this with Corollary 13 we obtain

$$\sup_{t\in[0,\tau^*]}|a(t)| \leq \sup_{t\in[0,\tau^*]}|A(t)-a(t)| + \sup_{t\in[0,\tau^*]}|A(t)| = \mathcal{O}(\varepsilon^{-\kappa}). \tag{50}$$

Next we show that the probability $\mathbb{P}(\tau^* < T_0)$ is small. Define the following subset of the probability space Ω:

$$M := \{\omega \in \Omega : \tau^*(\omega) < T_0\}.$$

If $\omega \in M$ then it follows from the definition of τ^* that $\|v(\tau^*(\omega))\|_\infty = \varepsilon^{-\kappa_0}$. Therefore the moments of $\|v(\tau^*)\|_\infty$ can be written as follows

$$\mathbb{E}\|v(\tau^*)\|_\infty^p = \int_{M^c}\|v(\tau^*)\|_\infty^p d\mathbb{P} + \int_M (\varepsilon^{-\kappa_0})^p d\mathbb{P} \geq \mathbb{P}(M)\varepsilon^{-p\kappa_0},$$

where $M^c := \Omega \setminus M$ is the complement set of M. From (50), (27),(19) and (18) we have

$$\begin{aligned}\mathbb{E}\|v(\tau^*)\|_\infty^p &\le C_p \mathbb{E} \sup_{t\in[0,\tau^*]} (|a(t)|^p + |Z_\varepsilon(t)|^p + |v_0(t) - Z_\varepsilon(t) - e^{-\varepsilon^{-2}T} v_0(0)|^p) \\ &\quad + C_p \mathbb{E} \sup_{t\in[0,\tau^*]} \|\sum_{k\ge 2} v_k - e^{-\varepsilon^{-2}T(1-k^2)^2} v_k(0))e^{ikx}\|_\infty^p \\ &\quad + C_p \mathbb{E} \sup_{t\in[0,\tau^*]} \|e^{-\varepsilon^{-2}T\mathcal{L}} \sum_{k\ne 1} (v_k(0))e^{ikx}\|_\infty^p \\ &\le C_p \varepsilon^{-p\kappa}\end{aligned}$$

with a constant C_p depending on p, where we used that there is a constant C such that for all $u \in C^0$,

$$\|e^{-\varepsilon^{-2}T\mathcal{L}} u\|_\infty \le C\|u\|_\infty.$$

This is a direct consequence of Lemma 4.5 in [MBK12] which follows the ideas of Collet and Eckmann in [CE90]. Therefore the probability of M is bounded by

$$\mathbb{P}(M) \le C_p \varepsilon^{p(\kappa_0-\kappa)}.$$

Define

$$\xi := \sup_{t\in[0,T_0]} \|u(t) - \varepsilon A(\varepsilon^2 t)e^{ix} - \varepsilon \bar{A}(\varepsilon^2 t)e^{-ix} + \varepsilon Z_\varepsilon(\varepsilon^2 t) - e^{-t(1+\partial_x^2)^2} u_s(0)\|_\infty$$

The last step is now to bound the probability of $\sup_{t\in[0,T_0]}\|\xi\|_\infty$ being too large (i.e. $\mathbb{P}(\sup_{t\in[0,T_0]}\|\xi\|_\infty > \varepsilon^{2-19\kappa})$). We can split this into

$$\begin{aligned}\mathbb{P}(\sup_{t\in[0,T_0]} \|\xi\| > \varepsilon^{2-19\kappa})) &= \mathbb{P}(M \cap \{\sup_{t\in[0,T_0]} \|\xi\| > \varepsilon^{2-19\kappa})\}) \\ &\quad + \mathbb{P}(M^c \cap \{\sup_{t\in[0,T_0]} \|\xi\| > \varepsilon^{2-19\kappa})\}) \\ &=: P_1 + P_2.\end{aligned}$$

P_1 is easily bounded by

$$\mathbb{P}(M \cap \{\sup_{t\in[0,T_0]} \|\xi\| > \varepsilon^{2-19\kappa})\}) \le \mathbb{P}(M) \le C_p \varepsilon^{p(\kappa_0-\kappa)},$$

so the only thing left to do is to bound P_2. We get

$$P_2 = \mathbb{P}(M^c \cap \{\sup_{t\in[0,T_0]} \|\xi\| > \varepsilon^{2-19\kappa}\}) \le \mathbb{P}(\sup_{t\in[0,T_0]} \|\xi\| > \varepsilon^{2-19\kappa}).$$

Using the Chebychev inequality gives

$$P_2 \le C_q \frac{1}{\varepsilon^{q(2-19)\kappa}} \mathbb{E}(\sup_{t\in[0,T_0]} \|\xi\|^q) \le C_q \varepsilon^{q\kappa},$$

where q is any positive number and C_q is a constant depending on q. By choosing $q = p/\kappa$ we get the desired result. $\square$

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References

[Blö07] D. Blömker. Amplitude equations for stochastic partial differential equations. Interdisciplinary mathematical sciences-Vol. 3, World Scientific (2007).

[BH04] D. Blömker and M. Hairer. *Multiscale expansion of invariant measures for SPDEs.* Comm. Math. Phys., 251(3):515 – 555, (2004).

[BHP07] D. Blömker, M. Hairer, and G.A. Pavliotis. *Multiscale analysis for stochastic partial differential equations with quadratic nonlinearities.* Nonlinearity, 20:1–25 (2007).

[BM12] D. Blömker and W. W. Mohammed. *Amplitude equations for SPDEs with cubic nonlinearities.* To appear in Stochastics. An International Journal of Probability and Stochastic Processes.

[BMNW11] D. Blömker, W.W. Mohammed, C. Nolde, and F. Wöhrl. *Numerical Study of Amplitude Equations for SPDEs with Degenerate Forcing.* To appear in International Journal of Computer Mathematics, 2012.

[BD12] John Burke and Jonathan H. P. Dawes *Localized States in an Extended SwiftHohenberg Equation.* SIAM J. Appl. Dyn. Syst.,**11**(1):261–284 (2012)

[BK06] John Burke and Edgar Knobloch, *Localized states in the generalized swift-hohenberg equation*, Phys. Rev. E **73** (2006), 056211.

[CE90] P. Collet and J.-P. Eckmann. The time dependent amplitude equation for the Swift-Hohenberg problem. Comm. Math. Physics. 132:139–153 (1990)

[CH93] M. C. Cross and P. C. Hohenberg. Pattern formation outside of equilibrium, Rev. Mod. Phys. 65:581–1112 (1993).

[DPZ92] Giuseppe Da Prato and Jerzy Zabczyk, *Stochastic equations in infinite dimensions*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1992. MR 1207136 (95g:60073)

[HLSG07] Axel Hutt, Andre Longtin, and Lutz Schimansky-Geier, *Additive global noise delays turing bifurcations*, Phys. Rev. Lett. **98** (2007), 230601.

[HLSG08] Axel Hutt, Andre Longtin, and Lutz Schimansky-Geier, *Additive noise-induced turing transitions in spatial systems with application to neural fields and the swift-hohenberg equation*, Physica D: Nonlinear Phenomena **237** (2008), no. 6, 755 – 773.

[HMBD95] M. F. Hilali, S. Métens, P. Borckmans, and G. Dewel, *Pattern selection in the generalized swift-hohenberg model*, Phys. Rev. E **51** (1995), 2046–2052.

[Hut08] A. Hutt, *Additive noise may change the stability of nonlinear systems*, EPL (Europhysics Letters) **84** (2008), no. 3, 34003.

[KMS92] P. Kirrmann, G. Schneider, and A. Mielke. The validity of modulation equations for extended systems with cubic nonlinearities. Proc. R. Soc. Edinb., Sect. A. 122A:459–490 (1992).

[MSZ00] A. Mielke, G. Schneider, and A. Ziegra. Comparison of inertial manifolds and application to modulated systems. Math. Nachr. 214:53–69 (2000).

[MBK12] W. W. Mohammed, D. Blömker, and K. Klepel. *Modulation equation for stochastic Swift-Hohenberg equation.* Preprint, (2012).

[Paz83] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983. MR 710486 (85g:47061)

[Sch96] G. Schneider. The validity of generalized Ginzburg-Landau equations. Math. Methods Appl. Sci. 19(9): 717–736 (1996).

[SH77] J. Swift and P. C. Hohenberg, *Hydrodynamic fluctuations at the convective instability*, Phys. Rev. A **15** (1977), 319–328.

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