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ABSOLUTE REGULARITY AND BRILLINGER MIXING OF STATIONARY POINT PROCESSES

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Abstract. We study the following problem: How to verify Brillinger-mixing of stationary point processes in $\mathbb{R}^d$ by imposing conditions on a suitable mixing coefficient? For this, we define an absolute regularity (or β-mixing) coefficient for point processes and derive an explicit condition in terms of this coefficient which implies finite total variation of the kth-order reduced factorial cumulant measure of the point process for fixed $k \geq 2$. To prove this, we introduce higher-order covariance measures and use Statulevičius' representation formula for mixed cumulants in case of random (counting) measures. To illustrate our results, we consider some Brillinger-mixing point processes occurring in stochastic geometry.

Keywords: Palm distribution, (reduced) factorial cumulant measure, Brillinger-mixing, higher-order covariance measure, β-mixing coefficient, germ-grain model, dependently thinned point process

MSC 2010: Primary 60 G 55, 60 D 05; Secondary 60 G 60, 60 F 99

1 INTRODUCTION AND BASIC DEFINITIONS

Point processes (briefly PPs) are adequate models to describe randomly or irregularly scattered points in some Euclidean space $\mathbb{R}^d$ (often $d = 1, 2, 3$ in applications). Statistics of PPs is mostly based on a single observation of a point pattern in some large sampling window which is assumed to expand unboundedly in all directions, see Chapt. 4 in [17]. Provided the underlying PP model is homogeneous (i.e. stationary) the asymptotic behaviour of parameter estimators and other empirical characteristics can only be determined under ergodicity and (strong) mixing assumptions, respectively. We encounter a similar situation in statistical physics, where stationary PPs are used to describe limits of configurations of interacting particles given in a "large (expanding) container", see [12, 15].

Throughout, let $\Psi := \sum_{i \geq 1} \delta_{X_i} \sim P$ denote a simple stationary PP on $\mathbb{R}^d$ with distribution P defined on the σ-algebra $\mathcal{N}$ generated by sets of the form $\{\psi \in N : \psi(B) = n\}$ for any $n \in \mathbb{N} \cup \{0\}$ and $B \in \mathcal{B}_b^d$ (= bounded sets of the Borel-σ-algebra $\mathcal{B}^d$ in $\mathbb{R}^d$), where N denotes the family of locally finite counting measures ψ on $\mathcal{B}^d$ satisfying $\psi(\{x\}) \leq 1$ for all $x \in \mathbb{R}^d$. In other words, Ψ is a random counting measure with random atoms $\{X_i, i \geq 1\}$ of multiplicity one which nowhere accumulate. Shortly spoken, Ψ is a random element defined on some probability space $[\Omega, \mathcal{F}, \mathbf{P}]$ taking values in $[N, \mathcal{N}, P]$ with $P = \mathbf{P} \circ \Psi^{-1}$. Stationarity of $\Psi \sim P$ means that $T_x\Psi := \sum_{i \geq 1} \delta_{X_i - x} \sim P$

or, equivalently, that $P(\{T_x\psi : \psi \in Y\}) = P(Y)$ for any $Y \in \mathcal{N}$ and all $x \in \mathbb{R}^d$, where $T_x\psi(\cdot) = \psi((\cdot)+x)$. For an all-embracing and rigorous introduction to the theory of PPs the reader is referred to [2]. Further, we define the *reduced Palm distribution* $P_{\mathbf{o}}^!$ of $\Psi \sim P$ by

$$P_{\mathbf{o}}^!(Y) := \frac{1}{\lambda} \int_N \int f(x)\, \mathbf{1}_Y(T_x\psi - \delta_{\mathbf{o}})\, \psi(\mathrm{d}x)\, P(\mathrm{d}\psi) \quad \text{for any} \quad Y \in \mathcal{N}\,, \tag{1.1}$$

where the *intensity* $\lambda := \mathbf{E}\Psi(E_{\mathbf{o}})$ is assumed to be positive and finite and f can be any non-negative, Borel-measurable function satisfying $\int f(x)\,\mathrm{d}x = 1$. Here and below, $\int$ stands for integration over $\mathbb{R}^d$ and $E_{\mathbf{o}}$ denotes the half-open unit cube $[-1/2, 1/2)^d$ centered at the origin $\mathbf{o} = (0, \ldots, 0)$. Note that the left-hand side of (1.1) does not depend on the choice of f due to the stationarity of $\Psi \sim P$ and the shift-invariance of the Lebesgue measure ν_d on $\mathbb{R}^d$.

The stationary Poisson process $\Psi \sim \Pi_\lambda$ with intensity $\lambda > 0$ is the most important PP model which is defined by the following two properties:

1. $\mathbf{P}(\Psi(B) = n) = (n!)^{-1}\, (\lambda\, \nu_d(B))^n\, \exp\{-\lambda\, \nu_d(B)\}$ for $n \in \mathbb{N} \cup \{0\}$ and $B \in \mathcal{B}_b^d$ and

2. $\Psi(B_1), \ldots, \Psi(B_k)$ are mutually independent for any pairwise disjoint $B_1, \ldots, B_k \in \mathcal{B}_b^d$, $k \geq 2$.

We recall that a stationary Poisson process $\Psi \sim P = \Pi_\lambda$ is characterized by the identity $P_{\mathbf{o}}^! = P$ (Slivnyak's theorem), see Chapt. 13 in [2].

Next, we define the *absolute regularity* or *β-mixing coefficient* $\beta(\mathcal{F}_1, \mathcal{F}_2)$ to measure the dependence between two sub-σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$ by

$$\beta(\mathcal{F}_1, \mathcal{F}_2) := \frac{1}{2} \sup \sum_{i=1}^{I} \sum_{j=1}^{J} |\, \mathbf{P}(A_i \cap B_j) - \mathbf{P}(A_i)\, \mathbf{P}(B_j)\, |\,, \tag{1.2}$$

where the supremum is taken over all pairs of finite partitions $\{A_1, \ldots, A_I\}$ and $\{B_1, \ldots, B_J\}$ of Ω such that $A_i \in \mathcal{F}_1$ for each i and $B_j \in \mathcal{F}_2$ for each j. This measure of dependence has been introduced by Volkonskii and Rozanov [20] (to prove asymptotic normality of sums of weakly dependent random variables) and later studied and used by many others, see e.g. [5, 7, 16, 21].

Our first result illustrates that (1.2) is the appropriate mixing coefficient (which is not replaceable by the α-mixing coefficient, see [1, 16]) to estimate the distance between expectations w.r.t. $P_{\mathbf{o}}^!$ and expectations w.r.t. P. In particular, it yields effective bounds of the total variation distance between $P_{\mathbf{o}}^!$ and P on the σ-algebra $\mathcal{N}(G) = \mathcal{N} \cap N(G)$ with $N(G) = \{\psi \in N : \psi(G^c) = 0\}$ for sets $G \in \mathcal{B}^d$ being far away from the origin $\mathbf{o}$. For any $B \in \mathcal{B}^d$, put $\psi_B(\cdot) := \psi((\cdot) \cap B)$ and $\mathcal{F}_\Psi(B) := \{\Psi^{-1}Y : Y \in \mathcal{N}(B)\}$ denotes the sub-σ-algebra of $\mathcal{F}$ generated by the restriction Ψ_B of the PP Ψ on $B \in \mathcal{B}^d$.

Theorem 1. *Assume that the support F of the function f in (1.1) is bounded such that $F \cap (G \oplus F) = \emptyset$. Then, for any $\mathcal{N}$-measurable function $g|N \mapsto \mathbb{R}^1$ and $p, q \geq 1$ satisfying $p + q \leq p\,q$, the bound*

$$\Big| \int_N g(\psi_G)\big(P_{\mathbf{o}}^! - P\big)(\mathrm{d}\psi) \Big|$$

$$\leq \frac{2}{\lambda} \Big(\mathbf{E}\big(\sum_{i \geq 1} f(X_i)\big)^p \Big)^{\frac{1}{p}} \Big(\mathbf{E} \sup_{x \in F} \big| g((T_x\Psi)_G) \big|^q \Big)^{\frac{1}{q}} \big(\beta(\mathcal{F}_\Psi(F), \mathcal{F}_\Psi(G \oplus F)) \big)^{1 - \frac{1}{p} - \frac{1}{q}} \tag{1.3}$$

holds, which remains valid for $p = 1$ and $q = \infty$, if $g(\psi_G)$ is bounded $\mathbf{P}$-a.s. In particular, for any $\delta \geq 0$,

$$(1.4) \qquad \sup_{Y \in \mathcal{N}(G)} |\, P^!_{\mathbf{o}}(Y) - P(Y)\,| \leq \frac{1}{\lambda\,\nu_d(F)} \big(\beta(\mathcal{F}_\Psi(F), \mathcal{F}_\Psi(G \oplus F))\big)^{\frac{\delta}{1+\delta}} \big(\mathbf{E}\big(\Psi(F)\big)^{1+\delta}\big)^{\frac{1}{1+\delta}} .$$

2 FACTORIAL MOMENT AND CUMULANT MEASURES AND B_k-MIXING

Assume that $\mathbf{E}\Psi(E_{\mathbf{o}})^k < \infty$ for some fixed $k \in \mathbb{N}$. The *kth-order factorial moment measure* $\alpha^{(k)}$ (on $[\mathbb{R}^{dk}, \mathcal{B}^{dk}]$) of $\Psi = \sum_{i \geq 1} \delta_{X_i} \sim P$ is defined by

$$(2.5) \qquad \alpha^{(k)}\big(\mathop{\times}_{i=1}^{k} B_i\big) := \mathbf{E} \sum_{i_1,\ldots,i_k \geq 1}^{\neq} \mathbf{1}_{B_1}(X_{i_1}) \cdots \mathbf{1}_{B_k}(X_{i_k}) = \int_N \sum_{x_1,\ldots,x_k \in \mathrm{supp}(\psi)}^{\neq} \prod_{i=1}^{k} \mathbf{1}_{B_i}(x_i)\, P(\mathrm{d}\psi)$$

for any $B_1, \ldots, B_k \in \mathcal{B}^d_b$, where the sum $\sum^{\neq}$ runs over all k-tuples of pairwise distinct elements.

According to the general relationship between mixed moments and mixed cumulant, see [11] or [16], the *kth-order factorial cumulant measure* is a locally finite, signed measure (on $[\mathbb{R}^{dk}, \mathcal{B}^{dk}]$) given by

$$(2.6) \qquad \gamma^{(k)}\big(\mathop{\times}_{i=1}^{k} B_i\big) := \sum_{j=1}^{k} (-1)^{j-1} (j-1)! \sum_{K_1 \cup \cdots \cup K_j = K} \prod_{i=1}^{j} \alpha^{(\kappa_i)}(B_{k_{i,1}} \times \cdots \times B_{k_{i,\kappa_i}}) \,,$$

for any $B_1, \ldots, B_k \in \mathcal{B}^d_b$, where the inner sum is taken over all decompositions of $K := \{1, \ldots, k\}$ into j disjoint non-empty subsets $K_1, \ldots, K_j$ and $\kappa_i := \# K_i$ denotes the number of elements of $K_i := \{k_{i,1}, \ldots, k_{i,\kappa_i}\}$. Further, note that $P = \Pi_\lambda$ implies $\alpha^{(k)} = \lambda^k\, \nu_{dk}$ for $k \geq 1$ and vice versa, and this in turn is equivalent to $\gamma^{(1)} = \lambda\, \nu_d$ and $\gamma^{(k)} = 0$ for $k \geq 2$.

By stationarity of $\Psi \sim P$, it follows that both $\alpha^{(k)}$ and $\gamma^{(k)}$ are invariant under diagonal shifts, i.e.

$$\alpha^{(k)}\big(\mathop{\times}_{i=1}^{k} B_i\big) = \alpha^{(k)}\big(\mathop{\times}_{i=1}^{k} (B_i + x)\big) \quad \text{and} \quad \gamma^{(k)}\big(\mathop{\times}_{i=1}^{k} B_i\big) = \gamma^{(k)}\big(\mathop{\times}_{i=1}^{k} (B_i + x)\big)$$

for any $B_1, \ldots, B_k \in \mathcal{B}^d_b$ and all $x \in \mathbb{R}^d$. This enables us to introduce the (uniquely determined) *reduced kth-order factorial moment (and cumulant) measure* $\alpha^{(k)}_{\mathrm{red}}$ (and $\gamma^{(k)}_{\mathrm{red}}$) by disintegration w.r.t. ν_d giving

$$\alpha^{(k)}\big(\mathop{\times}_{i=1}^{k} B_i\big) = \lambda \int_{B_1} \alpha^{(k)}_{\mathrm{red}}\big(\mathop{\times}_{i=2}^{k} (B_i - x)\big)\, \mathrm{d}x \quad \text{and} \quad \gamma^{(k)}\big(\mathop{\times}_{i=1}^{k} B_i\big) = \lambda \int_{B_1} \gamma^{(k)}_{\mathrm{red}}\big(\mathop{\times}_{i=2}^{k} (B_i - x)\big)\, \mathrm{d}x \,.$$

By standard measure-theoretic arguments and using the uniqueness of $\alpha^{(k)}_{\mathrm{red}}$ and $\gamma^{(k)}_{\mathrm{red}}$, it follows from (2.5) and (1.1) that $\alpha^{(k)}_{\mathrm{red}}$ coincides with the $(k-1)$st-order factorial moment measure w.r.t. $P^!_{\mathbf{o}}$ and $\gamma^{(k)}_{\mathrm{red}}$ can be expressed by $\gamma^{(k)}$ as follows:

$$(2.7) \quad \gamma^{(k)}_{\mathrm{red}}(B_2 \times \cdots \times B_k) = \frac{1}{\lambda\,\nu_d(F)} \int_{(\mathbb{R}^d)^k} \mathbf{1}_F(x)\, \mathbf{1}_{B_2}(x_2 - x) \cdots \mathbf{1}_{B_k}(x_k - x)\, \gamma^{(k)}(\mathrm{d}(x, x_2, \ldots, x_k))$$

for any $F \in \mathcal{B}_b^d$ with $\nu_d(F) > 0$. In view of Jordan's decomposition theorem, the signed measure $\gamma_{\rm red}^{(k)}$ (on $[\mathbb{R}^{d(k-1)}, \mathcal{B}^{d(k-1)}]$) can be expressed as the difference of measures $\gamma_{\rm red}^{(k)+}$ (positive part) and $\gamma_{\rm red}^{(k)-}$ (negative part) and the corresponding *total variation measure* $|\gamma_{\rm red}^{(k)}|$ is then the sum of its positive and negative part:

$$\gamma_{\rm red}^{(k)} = \gamma_{\rm red}^{(k)+} - \gamma_{\rm red}^{(k)-} \quad \text{and} \quad |\gamma_{\rm red}^{(k)}| = \gamma_{\rm red}^{(k)+} + \gamma_{\rm red}^{(k)-} .$$

In view of the corresponding Hahn decomposition, the locally finite measures $\gamma_{\rm red}^{(k)+}$ and $\gamma_{\rm red}^{(k)-}$ are concentrated on two disjoint Borel sets H_{k-1}^+ and H_{k-1}^- with $H_{k-1}^+ \cup H_{k-1}^- = (\mathbb{R}^d)^{k-1}$. The total variation $\|\gamma_{\rm red}^{(k)}\|_{\rm TV}$ of $\gamma_{\rm red}^{(k)}$ can then be expressed by

$$\|\gamma_{\rm red}^{(k)}\|_{\rm TV} = |\gamma_{\rm red}^{(k)}|((\mathbb{R}^d)^{k-1}) = \gamma_{\rm red}^{(k)+}(H_{k-1}^+) + \gamma_{\rm red}^{(k)-}(H_{k-1}^-) = \gamma_{\rm red}^{(k)}(H_{k-1}^+) - \gamma_{\rm red}^{(k)}(H_{k-1}^-) .$$

Definition. (see e.g. [6, 10]) *A simple stationary PP $\Psi \sim P$ satisfying $\mathbf{E}\Psi(E_{\mathbf{o}})^k < \infty$ for some integer $k \geq 2$ is said to be $\mathbf{B}_k$-mixing if $\|\gamma_{\rm red}^{(j)}\|_{\rm TV} < \infty$ for $j = 2, \ldots, k$. The PP $\Psi \sim P$ is called Brillinger mixing if it is $\mathbf{B}_k$-mixing for all $k \geq 2$.*

To formulate our main result we need assumptions on the decay of dependence between the restrictions Ψ_{F_a} and $\Psi_{F_{a+r}^c}$ of the PP Ψ for large r, where $F_a := [-a,a]^d$ and $F_a^c := \mathbb{R}^d \setminus [-a,a]^d$ for $a > 0$.

Theorem 2. *Let $\Psi \sim P$ be a simple stationary PP on $\mathbb{R}^d$. Assume that there exists a non-increasing β-mixing rate $\beta_\Psi|[1/2,\infty) \mapsto [0,1]$ such that*

$$\beta(\mathcal{F}_\Psi(F_a), \mathcal{F}_\Psi(F_{a+r}^c)) \leq \max\Big\{1, \frac{a}{r}\Big\}^{d-1} \beta_\Psi(r) \quad \textit{for} \quad a, r \geq 1/2 . \tag{2.8}$$

Then $\Psi \sim P$ is $\mathbf{B}_k$-mixing for some $k \geq 2$ if additionally

$$\mathbf{E}\Psi(E_{\mathbf{o}})^{k+\delta} < \infty \quad \textit{and} \quad \int_1^\infty r^{(k-1)d-1} \beta_\Psi(r)^{\delta/(k+\delta)} \, \mathrm{d}r < \infty \ \textit{for some} \ \delta > 0 . \tag{2.9}$$

In the particular cases $k = 2$ and $k = 3$ condition (2.8) is only needed for $r \geq a \geq 1/2$.

Corollary 1. *Assume that $\mathbf{E}\Psi(E_{\mathbf{o}})^k < \infty$ for all $k \in \mathbb{N}$. Further, let the β-mixing rate in (2.8) satisfy the bound $\beta_\Psi(r) \leq \mathrm{e}^{-g(r)}$ for $r \geq 1/2$, where the function $g|[1/2,\infty) \mapsto [0,\infty]$ is non-decreasing such that $g(r)/\log r \underset{r\to\infty}{\longrightarrow} \infty$. Then $\Psi \sim P$ is Brillinger-mixing.*

3 HIGHER-ORDER COVARIANCE MEASURES AND A COVARIANCE INEQUALITY

In this section we derive a representation of $\gamma^{(k)}$ in terms of *higher order covariance measures* $\overset{\frown}{\zeta}^{(j)}$. Such representations of higher-order mixed cumulants $\mathbf{Cum}_n(Y_{t_1}, \ldots, Y_{t_n})$, see e.g. [11], of (discrete-time) stochastic processes $\{Y_t, t \in \mathbb{N}\}$ in terms of higher-order covariances $\widehat{\mathbf{E}}\ Y_{t_1} Y_{t_2} \cdots Y_{t_k}$ have been introduced in the early 1960s by V. A. Statulevičius first to prove large deviations relations for sums of random variables connected in a Markov chain and later for other types of weakly dependent random sequences, see [16] for a survey of these results. In [3] the equivalence of the original with the following recursive definition of the kth-order covariance $\widehat{\mathbf{E}}\ Y_1 Y_2 \cdots Y_k$ has been shown: $\widehat{\mathbf{E}}\ Y_1 := \mathbf{E} Y_1$ and

$$\widehat{\mathbf{E}}\ Y_1 Y_2 \cdots Y_k := \mathbf{E} Y_1 Y_2 \cdots Y_k - \sum_{j=1}^{k-1} \widehat{\mathbf{E}}\ Y_1 Y_2 \cdots Y_j\ \mathbf{E} Y_{j+1} \cdots Y_k$$

for $k \geq 2$. By induction on $k \in \mathbb{N}$ it follows that $\widehat{\mathbf{E}}\ Y_1 Y_2 \cdots Y_k = \widehat{\mathbf{E}}\ Y_k \cdots Y_2 Y_1$.

In analogy to these higher-order covariances of random variables we introduce the *kth-order (factorial) covariance measure* $\overset{\frown}{\zeta}^{(k)}$ of $\Psi \sim P$ by recursion: $\overset{\frown}{\zeta}^{(1)}(B_1) := \alpha^{(1)}(B_1) = \mathbf{E}\Psi(B_1)$ and

$$\overset{\frown}{\zeta}^{(k)}(B_1 \times \cdots \times B_k) := \alpha^{(k)}(B_1 \times \cdots \times B_k) - \sum_{j=1}^{k-1} \overset{\frown}{\zeta}^{(j)}(B_1 \times \cdots \times B_j)\ \alpha^{(k-j)}(B_{j+1} \times \cdots \times B_k) \tag{3.10}$$

for any $B_1, \ldots, B_k \in \mathcal{B}_b^d$ and $k \geq 2$. Note that $\alpha^{(k)}$ as well as the signed measure $\gamma^{(k)}$ are completely symmetric in their arguments while this is not true for the signed measure $\overset{\frown}{\zeta}^{(k)}$, but the relation $\overset{\frown}{\zeta}^{(k)}(\times_{i=1}^{k} B_i) = \overset{\frown}{\zeta}^{(k)}(\times_{i=1}^{k} B_{k-i+1})$ holds. It is easily seen that $\overset{\frown}{\zeta}^{(1)} = \lambda\,\nu_d$ and $\overset{\frown}{\zeta}^{(k)} = 0$ for $k \geq 2$ yields a further characterization of $\Psi \sim \Pi_\lambda$. The total variation of the signed measures $\overset{\frown}{\zeta}^{(k)}$ in case of renewal processes on $\mathbb{R}^1$ has been studied in [9]. For such type of one-dimensional stationary PP we have $\beta_\Psi(r) \underset{r\to\infty}{\longrightarrow} 0$ if and only if the distribution of the typical inter-renewal time possesses a convolution power with an absolutely continuous part, see [13]. Rates of decay of $\beta_\Psi(r)$ have been obtained in [4].

For any stationary PP $\Psi \sim P$ the first-order measures $\alpha^{(1)}$, $\gamma^{(1)}$ and $\overset{\frown}{\zeta}^{(1)}$ coincide with $\lambda\,\nu_d$, and we have $\gamma^{(2)} = \overset{\frown}{\zeta}^{(2)}$. For $k = 3$ and any $B_1, B_2, B_3 \in \mathcal{B}_b^d$, the above definitions (2.6) and (3.10) give

$$\begin{aligned}
\gamma^{(3)}(B_1 \times B_2 \times B_3) &= \alpha^{(3)}(B_1 \times B_2 \times B_3) - \alpha^{(1)}(B_1)\,\alpha^{(2)}(B_2 \times B_3) - \alpha^{(1)}(B_2)\,\alpha^{(2)}(B_1 \times B_3) \\
&\quad - \alpha^{(1)}(B_3)\,\alpha^{(2)}(B_1 \times B_2) + 2\,\alpha^{(1)}(B_1)\,\alpha^{(1)}(B_2)\,\alpha^{(1)}(B_3)\,, \\
\overset{\frown}{\zeta}^{(3)}(B_1 \times B_2 \times B_3) &= \alpha^{(3)}(B_1 \times B_2 \times B_3) - \alpha^{(1)}(B_1)\,\alpha^{(2)}(B_2 \times B_3) \\
&\quad - \alpha^{(2)}(B_1 \times B_2)\,\alpha^{(1)}(B_3) + \alpha^{(1)}(B_1)\,\alpha^{(1)}(B_2)\alpha^{(1)}(B_3)\,,
\end{aligned}$$

$$\gamma^{(3)}(B_1 \times B_2 \times B_3) = \overset{\frown}{\zeta}^{(3)}(B_1 \times B_2 \times B_3) - \overset{\frown}{\zeta}^{(1)}(B_2)\ \overset{\frown}{\zeta}^{(2)}(B_1 \times B_3)\,. \tag{3.11}$$

For general $k \geq 2$, there are the following representations of $\overset{\frown}{\zeta}^{(k)}$ and $\gamma^{(k)}$, see [16], p. 13, for the case of random processes,

$$\overset{\frown}{\zeta}^{(k)}(B_1 \times \cdots \times B_k) = \sum_{j=1}^{k} (-1)^{j-1} \sum_{0=k_0<k_1<\cdots<k_j=k}\ \prod_{i=1}^{j} \alpha^{(k_i-k_{i-1})}(B_{k_{i-1}+1} \times \cdots \times B_{k_i}) \tag{3.12}$$

and

$$\gamma^{(k)}(B_1 \times \cdots \times B_k) = \sum_{j=1}^{k} (-1)^{j-1} \sum_{K_1 \cup \cdots \cup K_j = K} N_j(K_1, \ldots, K_j) \prod_{i=1}^{j} \overset{\frown}{\zeta}^{(\kappa_i)} (B_{k_{i,1}} \times \cdots \times B_{k_{i,\kappa_i}}) \tag{3.13}$$

for any $B_1, \ldots, B_k \in \mathcal{B}^d$, where the inner sum is taken over all decompositions of $K = \{1, \ldots, k\}$ into j disjoint non-empty subsets $K_1, \ldots, K_j$ and $K_i = \{k_{i,1}, \ldots, k_{i,\kappa_i}\}$ with $k_{i,1} < \cdots < k_{i,\kappa_i}$. We always assume that $k_{1,1} = 1$. The non-negative integers $N_j(K_1, \ldots, K_j)$ depend on all the sets $K_1, \ldots, K_j$ and are positive if and only if either $j = 1$ (since $N_1(K) = 1$) or for any $i = 2, \ldots, j$ there exists an $\ell \in \{1, \ldots, j\}$ such that $k_{\ell,1} < k_{i,1} < k_{\ell,\kappa_\ell}$, see p. 80 in [16], for a detailed description and calculation of these numbers.

After some rearrangement on the right-hand side of (3.12) we are led to the following representation of the signed measure $\overset{\frown}{\zeta}^{(k)}$:

(3.14)

$$\overset{\frown}{\zeta}^{(k)} (B_1 \times \cdots \times B_k) = \sum_{p=0}^{q-1} \sum_{r=q+1}^{k} \overset{\frown}{\zeta}^{(p)} (B_1 \times \cdots \times B_p)\, \Delta_q(B_{p+1} \times \cdots \times B_r)\, \overset{\frown}{\zeta}^{(k-r)} (B_{r+1} \times \cdots \times B_k)$$

with the convention that $\overset{\frown}{\zeta}^{(0)} (B_{k+1} \times B_k) = -1$ for $k = 0, 1, \ldots$ and

$$\Delta_q(B_{p+1} \times \cdots \times B_r) := \alpha^{(r-p)}(B_{p+1} \times \cdots \times B_r) - \alpha^{(q-p)}(B_{p+1} \times \cdots \times B_q)\, \alpha^{(r-q)}(B_{q+1} \times \cdots \times B_r) \tag{3.15}$$

for $0 \le p < q < r \le k$. Formula (3.14) can be proved by induction on $k \ge 2$ and $1 \le q \le k-1$ using the above recursive definition of $\overset{\frown}{\zeta}^{(k)}$. The details are left to the reader.

In order to obtain bounds of $\overset{\frown}{\zeta}^{(k)}$ we need estimates of the covariances (3.15). We may rewrite verbatim the proof of Lemma 1 in [21] to our point process setting leading to the subsequent bound of a general covariance-type expression in terms of the β-mixing coefficient (1.2), see also [7].

Lemma 1. *Let $\Psi_B, \Psi_{B'}$ be the restrictions of a simple stationary PP $\Psi \sim P$ to Borel subsets $B, B' \subset \mathbb{R}^d$. Furthermore, let $\widetilde{\Psi}_B$ and $\widetilde{\Psi}_{B'}$ be independent copies of Ψ_B and $\Psi_{B'}$, respectively. Then for any $\mathcal{N} \otimes \mathcal{N}$-measurable function $f | N \times N \mapsto \mathbb{R}^1$ and for any $\eta \ge 0$,*

$$\big| \mathbf{E} f(\Psi_B, \Psi_{B'}) - \mathbf{E} f(\widetilde{\Psi}_B, \widetilde{\Psi}_{B'}) \big| \le 2 \big(\beta(\mathcal{F}_\Psi(B), \mathcal{F}_\Psi(B'))\big)^{\frac{\eta}{1+\eta}} \times \max\Big\{ \big(\mathbf{E}|f(\Psi_B, \Psi_{B'})|^{1+\eta}\big)^{\frac{1}{1+\eta}}, \big(\mathbf{E}|f(\widetilde{\Psi}_B, \widetilde{\Psi}_{B'})|^{1+\eta}\big)^{\frac{1}{1+\eta}} \Big\} .$$

In combination with Lemma 1 we will use several times the following result.

Lemma 2. *Under the assumptions of Lemma 1 put $B = F_{1/2} \cup \bigcup_{j=2}^{q} (F_1 + z_j)$ and $B' = \bigcup_{j=q+1}^{k} (F_1 + z_j)$ for some $q = 1, \ldots, k-1$ and $z_2, \ldots, z_k \in \mathbb{Z}^d$. If the function $f | N \times N \mapsto \mathbb{R}^1$ admits the estimate $|\, f(\Psi_B, \Psi_{B'}) \,| \le \Psi(F_{1/2})\, \Psi(F_1 + z_2) \cdots \Psi(F_1 + z_k)$, then*

$$\max\Big\{ \big(\mathbf{E}|\, f(\Psi_B, \Psi_{B'})\,|^{1+\eta}\big)^{\frac{1}{1+\eta}}, \big(\mathbf{E}|\, f(\widetilde{\Psi}_B, \widetilde{\Psi}_{B'})\,|^{1+\eta}\big)^{\frac{1}{1+\eta}} \Big\} \le 2^{(k-1)d} \Big(\mathbf{E}\Psi(E_{\mathbf{o}})^{k(1+\eta)} \Big)^{\frac{1}{1+\eta}} \quad \textit{for any } \eta \ge 0.$$

Proof of Lemma 2. By Hölder's inequality and the fact that $\Psi(F_{1/2}\setminus E_{\mathbf{o}})=0$ $\mathbf{P}$-a.s., we obtain

$$\mathbf{E}|\,f(\Psi_B,\Psi_{B'})\,|^{1+\eta}\le\Big(\mathbf{E}\Psi(E_{\mathbf{o}})^{k(1+\eta)}\Big)^{1/k}\prod_{j=2}^{k}\Big(\mathbf{E}\Psi(F_1+z_j)^{k(1+\eta)}\Big)^{1/k}.$$

Together with $\mathbf{E}\Psi(F_1+z_j)^{k(1+\eta)}=\mathbf{E}\Psi(F_1)^{k(1+\eta)}\le 2^{d\,k(1+\eta)}\,\mathbf{E}\Psi(E_{\mathbf{o}})^{k(1+\eta)}$ for $j=2,\ldots,k$ it is easily seen that

$$\mathbf{E}|\,f(\Psi_B,\Psi_{B'})\,|^{1+\eta}\le 2^{(k-1)d(1+\eta)}\mathbf{E}\Psi(E_{\mathbf{o}})^{k(1+\eta)}.$$

The same upper bound can be shown for $\mathbf{E}|\,f(\widetilde{\Psi}_B,\widetilde{\Psi}_{B'})\,|^{1+\eta}$ which completes the proof of Lemma 2. □

4 THE SPECIAL CASES B_2- AND B_3-MIXING

For any $z=(z_1,\ldots,z_d)\in\mathbb{Z}^d$ put $E_z:=E_{\mathbf{o}}+z=\mathop{\times}\limits_{i=1}^{d}[-1/2+z_i,1/2+z_i)$ and $|z|:=\max\{|z_1|,\ldots,|z_d|\}$. For $k\in\{2,3\}$, Condition (2.8) is only needed for $r\ge a\ge 1/2$, which means that $\beta(\mathcal{F}(F_a),\mathcal{F}(F^c_{a+r}))\le\beta_\Psi(r)$ for $r\ge a\ge 1/2$. Since $\gamma^{(2)}_{\mathrm{red}}=\alpha^{(2)}_{\mathrm{red}}-\lambda\,\nu_d$ with $\alpha^{(2)}_{\mathrm{red}}(B)=\int_N\psi(B)\,P^!_{\mathbf{o}}(\mathrm{d}\psi)$ and $\lambda\,\nu_d(B)=\alpha^{(1)}(B)=\int_N\psi(B)\,P(\mathrm{d}\psi)$ for $B\in\mathcal{B}^d_b$, we may apply (1.3) with $F=E_{\mathbf{o}}$, $G=E_z$ for $|z|\ge 2$, $f(x)=\mathbf{1}_{E_{\mathbf{o}}}(x)$, $g(\psi_G)=\psi(E_z\cap H_2^+)-\psi(E_z\cap H_2^-)$ and $p=q=2+\delta$ and get the estimates

$$\begin{aligned}|\gamma^{(2)}_{\mathrm{red}}|(E_z)&=\gamma^{(2)}_{\mathrm{red}}(E_z\cap H_1^+)-\gamma^{(2)}_{\mathrm{red}}(E_z\cap H_1^-)\\&\le\frac{2}{\lambda}\left(\mathbf{E}\Psi(E_{\mathbf{o}})^{2+\delta}\,\mathbf{E}\Psi(E_z\oplus E_{\mathbf{o}})^{2+\delta}\right)^{\frac{1}{2+\delta}}\left(\beta(\mathcal{F}_\Psi(E_{\mathbf{o}}),\mathcal{F}_\Psi(E_z\oplus E_{\mathbf{o}}))\right)^{\frac{\delta}{2+\delta}}\\&\le\frac{2^{d+1}}{\lambda}\left(\mathbf{E}\Psi(E_{\mathbf{o}})^{2+\delta}\right)^{\frac{2}{2+\delta}}\left(\beta_\Psi(|z|-3/2)\right)^{\frac{\delta}{2+\delta}}\quad\text{for }\ |z|\ge 2\,.\end{aligned}$$

The last line is a consequence of (2.8) and $E_z\oplus E_{\mathbf{o}}\subset F^c_{|z|-1}\cup\partial F_{|z|-1}$, where $\Psi(\partial F_{|z|-1})=0$ $\mathbf{P}$-a.s. due to the stationarity of Ψ. From

$$\#\{z\in\mathbb{Z}^d:|z|=m\}=(2m+1)^d-(2m-1)^d\le 2\,d\,(2\,m+1)^{d-1}\quad\text{for }\ m\in\mathbb{N}\tag{4.16}$$

and (2.9) for $k=2$ we obtain immediately that $|\gamma^{(2)}_{\mathrm{red}}|(\mathbb{R}^d)<\infty$. This result has already been proved by slightly different arguments in [7].

Next we derive a bound of $|\gamma^{(3)}_{\mathrm{red}}|(\mathbb{R}^d\times\mathbb{R}^d)=\gamma^{(3)}_{\mathrm{red}}(H_2^+)-\gamma^{(3)}_{\mathrm{red}}(H_2^-)$. Using (2.7) for $k=3$ and $F=E_{\mathbf{o}}$, and (3.11) we find for any $y,z\in\mathbb{Z}^d$,

$$\begin{aligned}\lambda\,\gamma^{(3)}_{\mathrm{red}}((E_y\times E_z)\cap H_2^+)&=\int\!\!\int\!\!\int\mathbf{1}_{E_{\mathbf{o}}}(x)\mathbf{1}_{(E_y\times E_z)\cap H_2^+}(x_2-x,x_3-x)\,\gamma^{(3)}(\mathrm{d}(x,x_2,x_3))\\&=\int\!\!\int\!\!\int\mathbf{1}_{E_{\mathbf{o}}}(x)\mathbf{1}_{(E_y\times E_z)\cap H_2^+}(x_2-x,x_3-x)\ \overset{\frown}{\zeta}{}^{(3)}\ (\mathrm{d}(x,x_2,x_3))\\&\quad-\lambda\int\!\!\int\!\!\int\mathbf{1}_{E_{\mathbf{o}}}(x)\,\mathbf{1}_{(E_y\times E_z)\cap H_2^+}(x_2-x,x_3-x)\,\mathrm{d}x_2\ \overset{\frown}{\zeta}{}^{(2)}\ (\mathrm{d}(x,x_3))\\&=:I_1-I_2\,.\end{aligned}\tag{4.17}$$

The first term I_1 can be rewritten as

$$I_1 = \int\int\int \mathbf{1}_{E_{\mathbf{o}}}(x)\mathbf{1}_{(E_y\times E_z)\cap H^+}(x_2-x,x_3-x)\,\alpha^{(3)}(\mathrm{d}(x,x_2,x_3)) - \lambda\,\alpha^{(2)}((E_y\times E_z)\cap H_2^+)$$
$$- \lambda \int\int\int \mathbf{1}_{E_{\mathbf{o}}}(x)\,\mathbf{1}_{(E_y\times E_z)\cap H_2^+}(x_2-x,x_3-x)\,\alpha^{(2)}(\mathrm{d}(x,x_2))\,\mathrm{d}x_3$$
$$+ \lambda\,(\alpha^{(1)}\times\alpha^{(1)})((E_y\times E_z)\cap H_2^+)$$

(4.18)

$$= \mathbf{E}\sum_{i,j,k\geq 1}^{\neq} \mathbf{1}_{E_{\mathbf{o}}}(X_i)\mathbf{1}_{(E_y\times E_z)\cap H_2^+}(X_j-X_i,X_k-X_i) - \lambda\,\alpha^{(2)}((E_y\times E_z)\cap H_2^+)$$
$$- \lambda\int \mathbf{E}\sum_{i,j\geq 1}^{\neq} \mathbf{1}_{E_{\mathbf{o}}}(X_i)\,\mathbf{1}_{(E_y\times E_z)\cap H_2^+}(X_j-X_i,x_3-X_i)\,\mathrm{d}x_3 + \lambda\,(\alpha^{(1)}\times\alpha^{(1)})((E_y\times E_z)\cap H_2^+)\,,$$

and the second term I_2 becomes

$$I_2 = \lambda\int\int \mathbf{1}_{E_{\mathbf{o}}}(x)\,\mathbf{1}_{(E_y\times E_z)\cap H_2^+}(x_2-x,x_3-x)\,\mathrm{d}x_2\,\alpha^{(2)}(\mathrm{d}(x,x_3)) - \lambda\,(\alpha^{(1)}\times\alpha^{(1)})((E_y\times E_z)\cap H_2^+)$$
$$= \lambda\int \mathbf{E}\sum_{i,k\geq 1}^{\neq} \mathbf{1}_{E_{\mathbf{o}}}(X_i)\,\mathbf{1}_{(E_y\times E_z)\cap H_2^+}(x_2-X_i,X_k-X_i)\,\mathrm{d}x_2 - \lambda\,(\alpha^{(1)}\times\alpha^{(1)})((E_y\times E_z)\cap H_2^+)\,.$$

We have now to distinguish different cases according to the norms of y and z. The set $S_2 := \{(y,z)\in\mathbb{Z}^d\times\mathbb{Z}^d : |y|\leq|z|\}$ decomposes into three disjoint sets $S_2^{(1)} := \{(y,z)\in S_2 : |y|\leq 1, |z|\leq|y|+2\}$,

$$S_2^{(2)} := \{(y,z)\in S_2 : |y|\geq 2, |z|\leq 2\,|y|\}\,,\quad\text{and}\quad S_2^{(3)} := \{(y,z)\in S_2 : |z|\geq\max\{2\,|y|+1,|y|+3\}\}\,.$$

Since $S_2^{(1)}$ is finite with cardinality $\#S_2^{(1)} = 5^d + (3^d-1)(7^d-1)$, we need only a uniform bound of (4.17). Replacing $\gamma^{(3)}$ in (4.17) by $\alpha^{(3)} + 2\,\alpha^{(1)}\times\alpha^{(1)}\times\alpha^{(1)}$ and the fact that $X_i\in E_{\mathbf{o}}$ and $(X_j-X_i,X_k-X_i)\in(E_y\times E_z)\cap H_2^+$ imply $X_j\in E_{\mathbf{o}}\oplus E_y\subset F_1+y$ and $X_k\in E_{\mathbf{o}}\oplus E_z\subset F_1+z$ yield the estimate

$$I_1 - I_2 \leq \alpha^{(3)}(E_{\mathbf{o}}\times(F_1+y)\times(F_1+z)) + 2\,\lambda^3\int\int\int \mathbf{1}_{E_{\mathbf{o}}}(x)\,\mathbf{1}_{E_y}(x_2-x)\,\mathbf{1}_{E_z}(x_3-x)\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x.$$

By applying Hölder's inequality and the stationarity of Ψ (like in the proof of Lemma 2) we obtain that

$$I_1 - I_2 \leq 2^{2d}\,\mathbf{E}\Psi(E_{\mathbf{o}})^3 + 2\,\lambda^3 =: C_1 < \infty\,.$$

For any pair $(y,z)\in S_2^{(2)}$ we get the relations

$$I_1 = \mathbf{E}f(\Psi_{E_{\mathbf{o}}},\Psi_{(F_1+y)\cup(F_1+z)}) - \mathbf{E}f(\widetilde{\Psi}_{E_{\mathbf{o}}},\widetilde{\Psi}_{(F_1+y)\cup(F_1+z)})$$
$$- \lambda\int_{F_1+z}\Big[\mathbf{E}g_{x_3}(\Psi_{E_{\mathbf{o}}},\Psi_{F_1+y}) - \mathbf{E}g_{x_3}(\widetilde{\Psi}_{E_{\mathbf{o}}},\widetilde{\Psi}_{F_1+y})\Big]\,\mathrm{d}x_3$$

and

$$I_2 = \lambda\int_{F_1+y}\Big[\mathbf{E}h_{x_2}(\Psi_{E_{\mathbf{o}}},\Psi_{F_1+z}) - \mathbf{E}h_{x_2}(\widetilde{\Psi}_{E_{\mathbf{o}}},\widetilde{\Psi}_{F_1+z})\Big]\,\mathrm{d}x_2\,,$$

where $\widetilde{\Psi}_B$ and $\widetilde{\Psi}_{B'}$ are defined as in Lemma 1 with $B = E_{\mathbf{o}}$ and $B' \in \{F_1 + y, F_1 + z, (F_1 + y) \cup (F_1 + z)\}$, respectively, and

$$\begin{aligned} f(\Psi_{E_{\mathbf{o}}}, \Psi_{(F_1+y)\cup(F_1+z)}) &:= \sum_{i\geq 1} \sum_{j,k\geq 1}^{\neq} \mathbf{1}_{E_{\mathbf{o}}}(X_i)\, \mathbf{1}_{(E_y\times E_z)\cap H_2^+}(X_j - X_i, X_k - X_i) \\ &\leq \Psi(E_{\mathbf{o}})\, \Psi(F_1 + y)\, \Psi(F_1 + z)\,, \\ g_{x_3}(\Psi_{E_{\mathbf{o}}}, \Psi_{F_1+y}) &:= \sum_{i,j\geq 1}^{\neq} \mathbf{1}_{E_{\mathbf{o}}}(X_i)\, \mathbf{1}_{(E_y\times E_z)\cap H_2^+}(X_j - X_i, x_3 - X_i) \leq \Psi(E_{\mathbf{o}})\Psi(F_1 + y)\,, \end{aligned}$$

and

$$h_{x_2}(\Psi_{E_{\mathbf{o}}}, \Psi_{F_1+z}) := \sum_{i,k\geq 1}^{\neq} \mathbf{1}_{E_{\mathbf{o}}}(X_i)\, \mathbf{1}_{(E_y\times E_z)\cap H_2^+}(x_2 - X_i, X_k - X_i) \leq \Psi(E_{\mathbf{o}})\Psi(F_1 + z)\,.$$

Since $\Psi(\partial F_1) = 0$ $\mathbf{P}$-a.s., the foregoing formulas with f, g_{x_3} and h_{x_2} remain unchanged when F_1 is replaced by the open square $F_1^{int} = (-1, 1)^d$. In view of $E_{\mathbf{o}} \subset F_{1/2}$ and $(F_1^{int}+y)\cup(F_1^{int}+z) \subset F_{|y|-1}^c$, we may apply Lemma 1 and obtain together with Lemma 2 and (2.8) the following estimates:

$$|I_1| \leq 2^{2d+1}\, \beta_\Psi(|y| - 3/2)^{\frac{\eta}{1+\eta}} \left(\mathbf{E}\Psi(E_{\mathbf{o}})^{3+3\eta}\right)^{\frac{1}{1+\eta}} + \lambda\, \nu_d(F_1)\, 2^{d+1}\, \beta_\Psi(|y| - 3/2)^{\frac{\eta}{1+\eta}} \left(\mathbf{E}\Psi(E_{\mathbf{o}})^{2+2\eta}\right)^{\frac{1}{1+\eta}}$$

and

(4.19)
$$|I_2| \leq \lambda\, \nu_d(F_1)\, 2^{d+1}\, \beta_\Psi(|z| - 3/2)^{\frac{\eta}{1+\eta}} \left(\mathbf{E}\Psi(E_{\mathbf{o}})^{2+2\eta}\right)^{\frac{1}{1+\eta}}\,.$$

For $\eta = \delta/3$ the expressions on the right-hand sides are finite so that

$$\lambda\, \gamma_{\text{red}}^{(3)+}(E_y \times E_z) \leq |I_1| + |I_2| \leq C_2\, \beta_\Psi(|y| - 3/2)^{\frac{\delta}{3+\delta}} \quad \text{for some constant} \quad C_2 > 0\,.$$

In case of $(y, z) \in S_2^{(3)}$ we swap the second and third term in (4.18), and may rewrite I_1 as follows:

$$I_1 = \mathbf{E}f(\Psi_{E_{\mathbf{o}}\cup(F_1+y)}, \Psi_{F_1+z}) - \mathbf{E}f(\widetilde{\Psi}_{E_{\mathbf{o}}\cup(F_1+y)}, \widetilde{\Psi}_{F_1+z}) - \lambda \Big[\mathbf{E}g(\Psi_{F_1+y}, \Psi_{F_1+z}) - \mathbf{E}g(\widetilde{\Psi}_{F_1+y}, \widetilde{\Psi}_{F_1+z})\Big]\,,$$

where

$$f(\Psi_{E_{\mathbf{o}}\cup(F_1+y)}, \Psi_{F_1+z}) = \sum_{i,j\geq 1}^{\neq} \sum_{k\geq 1} \mathbf{1}_{E_{\mathbf{o}}}(X_i)\mathbf{1}_{(E_y\times E_z)\cap H_2^+}(X_j - X_i, X_k - X_i)$$

and

$$g(\Psi_{F_1+y}, \Psi_{F_1+z}) = \sum_{j,k\geq 1} \mathbf{1}_{(E_y\times E_z)\cap H_2^+}(X_j, X_k)\,.$$

In the same manner as above, the Lemmas 1 and 2 combined with (2.8) yield the estimate

$$|I_1| \leq 2^{2d+1}\, \beta_\Psi(|z| - |y| - 2)^{\frac{\eta}{1+\eta}} \left(\mathbf{E}\Psi(E_{\mathbf{o}})^{3+3\eta}\right)^{\frac{1}{1+\eta}} + \lambda\, 2^{d+1}\, \beta_\Psi(|z| - |y| - 2)^{\frac{\eta}{1+\eta}} \left(\mathbf{E}\Psi(E_{\mathbf{o}})^{2+2\eta}\right)^{\frac{1}{1+\eta}}\,.$$

The bound of I_2 is the same as in (4.19) and therefore, by setting $\eta = \delta/3$, we arrive at

$$\lambda\, \gamma_{\text{red}}^{(3)+}(E_y \times E_z) \leq |I_1| + |I_2| \leq C_3\, \beta_\Psi(|z| - |y| - 2)^{\frac{\delta}{3+\delta}} \quad \text{for some constant} \quad C_3 > 0\,.$$

Using the symmetry of the signed measure $\gamma^{(3)}_{\mathrm{red}}$ we can summarize three cases for the position of $(y,z) \in S_2$ and obtain that

$$\begin{aligned}\lambda\,\gamma^{(3)+}_{\mathrm{red}}(H_2^+) &= \sum_{y,z\in\mathbb{Z}^d} \lambda\,\gamma^{(3)+}_{\mathrm{red}}((E_y\times E_z)\cap H_2^+) \le 2 \sum_{(y,z)\in S_2} \lambda\,\gamma^{(3)+}_{\mathrm{red}}((E_y\times E_z)\cap H_2^+)\\ &\le 2\Big[C_1\,\#S_2^{(1)} + C_2 \sum_{(y,z)\in S_2^{(2)}} \beta_\Psi(|y|-3/2)^{\frac{\delta}{3+\delta}} + C_3 \sum_{(y,z)\in S_2^{(3)}} \beta_\Psi(|z|-|y|-2)^{\frac{\delta}{3+\delta}}\Big]\,.\end{aligned}$$

By means of (4.16) some simple rearrangements show that

$$\sum_{(y,z)\in S_2^{(2)}} \beta_\Psi(|y|-3/2)^{\frac{\delta}{3+\delta}} \le \sum_{m=2}^{\infty} 2d(2m+1)^{d-1}(2m+2)d(4m+1)^{d-1}\beta_\Psi(m-3/2)^{\frac{\delta}{3+\delta}}$$

and

$$\begin{aligned}\sum_{(y,z)\in S_2^{(3)}} \beta_\Psi(|y|-3/2)^{\frac{\delta}{3+\delta}} &\le \sum_{n=3}^{\infty}\beta_\Psi(n-2)^{\frac{\delta}{3+\delta}} + (3^d-1)\sum_{n=4}^{\infty}\beta_\Psi(n-3)^{\frac{\delta}{3+\delta}}\\ &\quad + \sum_{m=2}^{\infty} 2d(2m+1)^{d-1} \sum_{n=2m+1}^{\infty} 2d(2n+1)^{d-1}\beta_\Psi(n-m-2)^{\frac{\delta}{3+\delta}}\,.\end{aligned}$$

By condition (2.9) for $k=3$ it is not difficult to see that $\gamma^{(3)+}_{\mathrm{red}}(H_2^+) \le C_4 \sum_{n\ge1} n^{2d-1}\beta_\Psi(n)^{\frac{\delta}{3+\delta}} < \infty$ for some constant $C_4>0$ depending on d, λ, δ and $\mathbf{E}\Psi(E_{\mathbf{o}})^{3+\delta}$. In the same way we can prove that $\gamma^{(3)-}_{\mathrm{red}}(H_2^-) < \infty$ and thus $|\gamma^{(3)}_{\mathrm{red}}|(\mathbb{R}^d\times\mathbb{R}^d) < \infty$ completing the proof of Theorem 2 for $k=2,3$. □

5 PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1.

Let $f|\mathbb{R}^d \mapsto [0,\infty]$ be Borel-measurable with bounded support F and $\int f(x)\,\mathrm{d}x = 1$. Since $F\cap(G\oplus F)=\emptyset$, we have $\mathbf{o}\notin G$ implying $(T_x\psi-\delta_{\mathbf{o}})_G = (T_x\psi)_G$ for all $\psi\in N$. By applying the Campbell-Mecke formula, see Chapt. 13 in [2], to the stationary PP $\Psi\sim P$ we get the equality

$$\int_N\int_F f(x)\,g((T_x\psi-\delta_{\mathbf{o}})_G)\,\psi(\mathrm{d}x)\,P(\mathrm{d}\psi) = \lambda\int_F f(x)\,\mathrm{d}x \int_N g(\psi_G)\,P^!_{\mathbf{o}}(\mathrm{d}\psi)\,,$$

which combined with the simple Campbell formula

$$\mathbf{E}\sum_{i\ge1} f(X_i) = \int_N\int_F f(x)\,\psi(\mathrm{d}x)\,P(\mathrm{d}\psi) = \lambda\int_F f(x)\,\mathrm{d}x = \lambda$$

yields the relation

$$\begin{aligned}\lambda\int_N g(\psi_G)\,\big(P^!_{\mathbf{o}}-P\big)(\mathrm{d}\psi) &= \int_N\int_F f(x)g((T_x\psi-\delta_{\mathbf{o}})_G)\,\psi(\mathrm{d}x)\,P(\mathrm{d}\psi) - \lambda\int_N\int_F f(x)\,g((T_x\psi)_G)\,\mathrm{d}x\,P(\mathrm{d}\psi)\\ &= \mathbf{E}h(\Psi_F,\Psi_{G\oplus F}) - \mathbf{E}h(\widetilde{\Psi}_F,\widetilde{\Psi}_{G\oplus F})\,,\end{aligned}$$

where the $\mathcal{N}\otimes\mathcal{N}$-measurable function $h|N(F)\times N(G\oplus F)\mapsto \mathbb{R}^1$ is defined by

$$h(\Psi_F,\Psi_{G\oplus F}) := \sum_{i\geq 1} f(X_i)\,\mathbf{1}_F(X_i)\,g((T_{X_i}\Psi)_G)\,.$$

The independence of the restricted PPs $\widetilde{\Psi}_F$ and $\widetilde{\Psi}_{G\oplus F}$, Fubini's theorem, and the stationarity of $\Psi\sim P$ allow to write

$$\begin{aligned}\mathbf{E}h(\widetilde{\Psi}_F,\widetilde{\Psi}_{G\oplus F}) &= \int\limits_{N(F)}\int\limits_{N(G\oplus F)}\int\limits_{F} f(x)\,g((T_x\psi)_G)\,\varphi(\mathrm{d}x)\,P(\mathrm{d}\psi)\,P(\mathrm{d}\varphi)\\ &= \int\limits_{N}\int\limits_{F} f(x)\,\varphi(\mathrm{d}x)\,P(\mathrm{d}\varphi)\int\limits_{N} g(\psi_G)\,P(\mathrm{d}\psi) = \lambda\int\limits_{N} g(\psi_G)\,P(\mathrm{d}\psi).\end{aligned}$$

A straightforward application of Lemma 1 yields the estimate

$$\begin{aligned}\lambda\Big|\int\limits_{N} g(\psi_G)\,\big(P^!_{\mathbf{o}}-P\big)(\mathrm{d}\psi)\Big| &\leq 2\,\max\Big\{\big(\mathbf{E}|h(\Psi_F,\Psi_{G\oplus F})|^{1+\eta}\big)^{\frac{1}{1+\eta}},\big(\mathbf{E}|h(\widetilde{\Psi}_F,\widetilde{\Psi}_{G\oplus F})|^{1+\eta}\big)^{\frac{1}{1+\eta}}\Big\}\\ &\quad\times\ \big(\beta(\mathcal{F}_\Psi(F),\mathcal{F}_\Psi(G\oplus F))\big)^{1-\frac{1}{1+\eta}}\quad\text{for any}\quad \eta\geq 0\,.\end{aligned}$$

Further, for any $p,q\in[1,\infty]$ satisfying $\frac{1}{p}+\frac{1}{q}=\frac{1}{1+\eta}$, we employ Hölder's inequality to show that

$$\begin{aligned}\Big(\mathbf{E}\big|h(\Psi_F,\Psi_{G\oplus F})\big|^{1+\eta}\Big)^{\frac{1}{1+\eta}} &\leq \Big(\mathbf{E}\Big[\big(\sum_{i\geq 1} f(X_i)\big)^{1+\eta}\sup_{x\in F}\big|g((T_x\Psi)_G)\big|^{1+\eta}\Big]\Big)^{\frac{1}{1+\eta}}\\ &\leq \Big(\mathbf{E}\big(\sum_{i\geq 1} f(X_i)\big)^{p}\Big)^{\frac{1}{p}}\Big(\mathbf{E}\sup_{x\in F}\big|g((T_x\Psi)_G)\big|^{q}\Big)^{\frac{1}{q}}\,.\end{aligned}$$

Likewise, we get the same upper bound for $\big(\mathbf{E}|h(\widetilde{\Psi}_F,\widetilde{\Psi}_{G\oplus F})|^{1+\eta}\big)^{\frac{1}{1+\eta}}$. This provides immediately the desired estimate (1.3). To prove (1.4) we consider the Hahn decomposition $N^+(G)\cup N^-(G) = N(G)$ of the signed measure $P^!_{\mathbf{o}}((\cdot)\cap N(G))-P((\cdot)\cap N(G))$. Inserting $g(\psi)=\mathbf{1}_{N^+(G)}(\psi)-\mathbf{1}_{N^-(G)}(\psi)$ on both sides of the inequality (1.3) we can take $p=1+\delta$ and $q=\infty$ (since $|g((T_x\psi)_G)|\leq 1$ for $x\in F$) and $f(x)=\mathbf{1}_F(x)/\nu_d(F)$ on the right-hand side, whereas the left-hand side equals $2\,\sup_{Y\in\mathcal{N}(G)}|\,P^!_{\mathbf{o}}(Y)-P(Y)\,|$. Hence, (1.4) is shown and the proof of Theorem 1 is finished. □

Proof of Theorem 2.

We have to show that $|\gamma^{(k)}_{\mathrm{red}}|(\mathbb{R}^d)^{k-1} = \gamma^{(k)}_{\mathrm{red}}(H^+_{k-1})-\gamma^{(k)}_{\mathrm{red}}(H^-_{k-1})<\infty$ for some fixed $k\geq 4$, where H^+_{k-1}, H^-_{k-1} denotes the Hahn decomposition of the signed measure $\gamma^{(k)}_{\mathrm{red}}$. Due to the complete symmetry of $\gamma^{(k)}_{\mathrm{red}}$ we have

$$\begin{aligned}\gamma^{(k)}_{\mathrm{red}}(H^+_{k-1}) &= \sum_{z_2,\ldots,z_k\in\mathbb{Z}^d}\gamma^{(k)}_{\mathrm{red}}((E_{z_2}\times\cdots\times E_{z_k})\cap H^+_{k-1})\\ &\leq (k-1)!\sum_{(z_2,\ldots,z_k)\in S_{k-1}}\gamma^{(k)+}_{\mathrm{red}}((E_{z_2}\times\cdots\times E_{z_k})\cap H^+_{k-1}),\end{aligned}$$

where $S_{k-1}:=\{(z_2,\ldots,z_k)\in(\mathbb{Z}^d)^{k-1}: 0\leq|z_2|\leq\cdots\leq|z_k|\}$. Let us fix $(z_2,\ldots,z_k)\in S_{k-1}$ and put $E^+_k:=(E_{z_2}\times\cdots\times E_{z_k})\cap H^+_{k-1}$ for notational ease. Our next aim is to derive an upper bound for $\gamma_{\mathrm{red}}^{(k)}(E_k^+)$. Using (3.13) we can express $\gamma_{\mathrm{red}}^{(k)}(E_k^+)$ in terms of higher order-covariance measures $\overset{\frown}{\zeta}^{(j)}$:

$$\text{(5.20)} \qquad \lambda\gamma_{\mathrm{red}}^{(k)}(E_k^+) = \int\cdots\int \mathbf{1}_{E_{\mathbf{o}}}(x_1)\,\mathbf{1}_{E_k^+}((x_2-x_1,\ldots,x_k-x_1))\,\gamma^{(k)}(\mathrm{d}(x_1,\ldots,x_k))$$
$$= \sum_{j=1}^{k}(-1)^{j-1}\sum_{K_1\cup\cdots\cup K_j=K} N_j(K_1,\ldots,K_j)I_j(K_1,\ldots,K_j),$$

where

$$I_j(K_1,\ldots,K_j) := \int\cdots\int \mathbf{1}_{E_{\mathbf{o}}}(x_1)\,\mathbf{1}_{E_k^+}((x_2-x_1,\ldots,x_k-x_1))\prod_{i=1}^{j}\overset{\frown}{\zeta}^{(\kappa_i)}(\mathrm{d}(x_{k_{i,1}},\ldots,x_{k_{i,\kappa_i}})).$$

Since $x_1\in E_{\mathbf{o}}$ and $(x_2-x_1,\ldots,x_k-x_1)\in E_k^+$, it follows that $x_i\in E_{\mathbf{o}}\oplus E_{z_i}\subset F_1+z_i$ for $i=2,\ldots,k$ and together with (5.20) we arrive at

$$\lambda\,\gamma_{\mathrm{red}}^{(k)}(E_k^+) \le |\gamma^{(k)}|(E_{\mathbf{o}}\times(F_1+z_2)\times\cdots\times(F_1+z_k)).$$

Obviously, $\alpha^{(j)}((F_1+z_{k_1})\times\cdots\times(F_1+z_{k_j}))\le \mathbf{E}\Psi(F_1+z_{k_1})\cdots\Psi(F_1+z_{k_j})$ and using Hölder's inequality and the stationarity of $\Psi\sim P$ we get that

$$\text{(5.21)} \qquad \alpha^{(j)}((F_1+z_{k_1})\times\cdots\times(F_1+z_{k_j}))\le 2^{jd}\mathbf{E}\Psi(E_{\mathbf{o}})^j \le \big(2^{kd}\mathbf{E}\Psi(E_{\mathbf{o}})^k\big)^{j/k}.$$

Inserting the latter estimate into (2.6) gives

$$|\gamma^{(k)}|(E_{\mathbf{o}}\times(F_1+z_2)\times\cdots\times(F_1+z_k))\le k!\,2^{kd}\,\mathbf{E}\Psi(E_{\mathbf{o}})^k.$$

Thus, each summand of the sum $\sum_{(z_2,\ldots,z_k)\in S_{k-1}}\gamma_{\mathrm{red}}^{(k)}(E_k^+)$ is finite and, consequently, it suffices to show that

$$\sum_{(z_2,\ldots,z_k)\in S_{k-1}:|z_k|\ge 2k-1}|I_j(K_1,\ldots,K_j)|<\infty$$

for any decomposition of $K=\{1,\ldots,k\}$ into $j\in\{1,\ldots,k-1\}$ disjoint non-empty subsets $K_1,\ldots,K_j$ such that $N_j(K_1,\ldots,K_j)>0$.

Let $z_1=\mathbf{o}$ and $m(z_2,\ldots,z_k):=\max\{|z_j|-|z_{j-1}|,j=2,\ldots,k\}$ be the largest gap in the sequence $0=|z_1|\le|z_2|\le\cdots\le|z_k|$. If $|z_k|\ge 2k-1$, then the maximal gap $m(z_2,\ldots,z_k)$ is at least 3. Let $q\in\{1,\ldots,k-1\}$ be such that $|z_{q+1}|-|z_q|=m(z_2,\ldots,z_k)$, i.e. the largest gap occurs between $|z_q|$ and $|z_{q+1}|$. We start with the case $j=1$.

Making use of the formula (3.14) with (3.15) we may express $I_1(K)$ as

$$I_1(K)=\sum_{p=0}^{q-1}\sum_{r=q+1}^{k}\int_{(\mathbb{R}^d)^p}\int_{(\mathbb{R}^d)^{r-p}}\int_{(\mathbb{R}^d)^{k-r}}\mathbf{1}_{E_{\mathbf{o}}}(x_1)\,\mathbf{1}_{E_k^+}((x_2-x_1,\ldots,x_k-x_1))$$
$$\times\overset{\frown}{\zeta}^{(p)}(\mathrm{d}(x_1,\ldots,x_p))\,\Delta_q(\mathrm{d}(x_{p+1},\ldots,x_r))\,\overset{\frown}{\zeta}^{(k-r)}(\mathrm{d}(x_{r+1},\ldots,x_k))$$
$$=\sum_{p=0}^{q-1}\sum_{r=q+1}^{k}\int_{(\mathbb{R}^d)^p}\int_{(\mathbb{R}^d)^{k-r}}\Big[\mathbf{E}f(\Psi_{B_p},\Psi_{B_r'};x_1,\ldots,x_p,x_{r+1},\ldots,x_k)$$
$$-\mathbf{E}f(\widetilde{\Psi}_{B_p},\widetilde{\Psi}_{B_r'};x_1,\ldots,x_p,x_{r+1},\ldots,x_k)\Big]\,\overset{\frown}{\zeta}^{(k-r)}(\mathrm{d}(x_{r+1},\ldots,x_k))\,\overset{\frown}{\zeta}^{(p)}(\mathrm{d}(x_1,\ldots,x_p)),$$

where $B_p = \bigcup_{\ell=p+1}^{q}(F_1+z_\ell)$, $B'_r = \bigcup_{\ell=q+1}^{r}(F_1+z_\ell)$, $\widetilde{\Psi}_{B_p}$ and $\widetilde{\Psi}_{B'_r}$ are copies of Ψ_{B_p} and $\Psi_{B'_r}$, respectively, and are assumed to be independent, and

$$f(\Psi_{B_p},\Psi_{B'_r};x_1,\ldots,x_p,x_{r+1},\ldots,x_k) := \sum_{i_{p+1},\ldots,i_q\geq 1}^{\neq}\ \sum_{i_{q+1},\ldots,i_r\geq 1}^{\neq} \mathbf{1}_{E_{\mathbf{o}}}(x_1)\,\mathbf{1}_{E_k^+}(x_2-x_1,\ldots,x_p-x_1,$$

$$X_{i_{p+1}}-x_1,\ldots,X_{i_q}-x_1,X_{i_{q+1}}-x_1,\ldots,X_{i_r}-x_1,x_{r+1}-x_1,\ldots,x_k-x_1) \leq \prod_{\ell=p+1}^{r}\Psi(F_1+z_\ell).$$

The latter inequality holds $\mathbf{P}$-a.s. if F_1 is replaced by $F_1^{int}=(-1,1)^d$. Thus, we can apply Lemma 1 for $B_1 = B_p \subset F_{|z_q|+1}$ and $B_2 = B'_r \subset F^c_{|z_{q+1}|-1}$, and together with the assumption (2.8) and Lemma 2 (with obvious modifications for $p\geq 1$), we obtain the inequality

$$|\mathbf{E}f(\Psi_{B_p},\Psi_{B'_r};x_1,\ldots,x_p,x_{r+1},\ldots,x_k) - \mathbf{E}f(\widetilde{\Psi}_{B_p},\widetilde{\Psi}_{B'_r};x_1,\ldots,x_p,x_{r+1},\ldots,x_k)|$$

$$\leq 2^{(r-p)d+1}\Big(\mathbf{E}\Psi(E_{\mathbf{o}})^{(r-p)(1+\eta)}\Big)^{\frac{1}{1+\eta}} \max\Big\{1,\frac{|z_q|+1}{|z_{q+1}|-|z_q|-2}\Big\}^{d-1}\beta_\psi(|z_{q+1}|-|z_q|-2)^{\frac{\eta}{1+\eta}}$$

$$\times\,\mathbf{1}_{E_{\mathbf{o}}}(x_1)\prod_{j=2}^{p}\mathbf{1}_{E_{z_j}}(x_j-x_1)\prod_{j=r+1}^{k}\mathbf{1}_{E_{z_j}}(x_j-x_1)$$

for any $\eta\geq 0$, where the right-hand side (with $0\leq p<r\leq k$) is finite for $\eta=\delta/k$. From (3.12) and (5.21) we get that the total variation measures $|\overset{\frown}{\zeta}^{(p)}|(\cdot)$ and $|\overset{\frown}{\zeta}^{(k-r)}|(\cdot)$ for $0<p<r<k$ satisfy the estimates

$$|\overset{\frown}{\zeta}^{(p)}|(\mathop{\times}_{j=1}^{p}(F_1+z_j)) \leq 2^{(d+1)p-1}\mathbf{E}\Psi(E_{\mathbf{o}})^p \quad\text{and}\quad |\overset{\frown}{\zeta}^{(k-r)}|(\mathop{\times}_{j=r+1}^{k}(F_1+z_j)) \leq 2^{(d+1)(k-r)-1}\mathbf{E}\Psi(E_{\mathbf{o}})^{k-r}.$$

Combining the previous estimates with $\eta=\delta/k$ and applying again Hölder's inequality we find that

$$|I_1(K)| \leq 2^{(k+1)d}\Big(\mathbf{E}\Psi(E_{\mathbf{o}})^{k+\delta}\Big)^{\frac{k}{k+\delta}} \max\Big\{1,\frac{|z_q|+1}{|z_{q+1}|-|z_q|-2}\Big\}^{d-1}\beta_\Psi(|z_{q+1}|-|z_q|-2)^{\frac{\delta}{k+\delta}}$$

for any $(z_2,\ldots,z_k)\in S_{k-1}$ satisfying $|z_k|\geq 2k-1$ and $m(z_2,\ldots,z_k)=|z_{q+1}|-|z_q|$ (≥ 3). The number of such $(k-1)$-tuples $(z_2,\ldots,z_k)$ is at most

$$(2\,|z_q|+1)^{d(q-1)}\Big((2\,|z_k|+1)^d-(2\,|z_k|-1)^d\Big)^{k-q} \leq 2\,d\,(2\,|z_k|+1)^{d(k-2)+d-1}$$

for $2\leq q\leq k-1$, where the latter bound is justified by $|z_q|<|z_{q+1}|\leq|z_k|$ and (4.16).

Therefore, first fixing the largest gap $m(z_2,\ldots,z_k)=m$ and having in mind that $|z_\ell|\leq(\ell-1)\,m$ for $\ell=2,\ldots,k$, and then summing up over all $m\geq 3$ yields that

$$\sum_{(z_2,\ldots,z_k)\in S_{k-1}:|z_k|\geq 2k-1}|I_1(K)| \leq 2^{(k+1)d}\Big(\mathbf{E}\Psi(E_{\mathbf{o}})^{k+\delta}\Big)^{\frac{k}{k+\delta}}\sum_{m=3}^{\infty}2d\,\big(2(k-1)m+1\big)^{(k-1)d-1}$$

$$\times\max\Big\{1,\frac{(k-2)m+1}{m-2}\Big\}^{d-1}\beta_\Psi(m-2)^{\frac{\delta}{k+\delta}} \leq C_5(k,d,\delta)\sum_{m=1}^{\infty}m^{(k-1)d-1}\beta_\Psi(m)^{\frac{\delta}{k+\delta}},$$

where, by (2.9), the series in the last line converges and the constant $C_5(k,d,\delta)$ depends only on $d \geq 1$, $k \geq 2$ and $\mathbf{E}\Psi(E_{\mathbf{o}})^{k+\delta} < \infty$.

Next we regard the terms $I_j(K_1,\ldots,K_j)$ for $j \geq 2$ with decompositions $K_1,\ldots,K_j$ of $K = \{1,\ldots,k\}$ satisfying $N_j(K_1,\ldots,K_j) > 0$. These terms are multiple integrals over some subset of $E_{\mathbf{o}} \times (E_{z_2} \oplus E_{\mathbf{o}}) \times \cdots \times (E_{z_k} \oplus E_{\mathbf{o}})$ w.r.t. products of higher-order covariance measures (3.12). Let $q \in \{1,\ldots,k-1\}$ be the (largest) index such that $|z_{q+1}| - |z_q| = m$ is the maximal gap in the sequence $0 = |z_1| \leq |z_2| \leq \cdots \leq |z_k|$. Then there exists an (ordered) index set $K_\ell = \{k_{\ell,1},\ldots,k_{\ell,\kappa_\ell}\}$ such that $|z_{k_{\ell,r+1}}| - |z_{k_{\ell,r}}| \geq m$ for at least one $r \in \{1,\ldots,\kappa_\ell - 1\}$. This is obvious if q and $q+1$ belong to the same index set. Otherwise, we distinguish two cases. First, $q+1 \in K_\ell$ with $\kappa_\ell \geq 2$ and $k_{\ell,1} < q+1$ so that $|z_{q+1}| - |z_{k_{\ell,i}}| \geq m$, where $k_{\ell,i}$ is the largest index in K_ℓ less than $q+1$. Second, $q+1$ coincides with the smallest index $k_{p,1}$ in K_p for some $p \in \{2,\ldots,j\}$. Due to the positivity of $N_j(K_1,\ldots,K_j)$, see p. 80 in [16], there exists an index set K_ℓ with $\kappa_\ell \geq 2$ such that $k_{\ell,1} < q+1 < k_{\ell,\kappa_\ell}$ implying that $|z_{k_{\ell,i+1}}| - |z_{k_{\ell,i}}| \geq m$, where $k_{\ell,i}$ ($k_{\ell,i+1}$) is the largest (smallest) index in K_ℓ less (greater) than $q+1$.

In this way we have found a covariance measure $\overset{\frown(\kappa_\ell)}{\zeta}$ occurring in $I_j(K_1,\ldots,K_j)$ to which the same arguments as to $\overset{\frown(k)}{\zeta}$ in $I_1(K)$ can be applied. Hence, taking into account that

$$|\overset{\frown(j)}{\zeta}|((F_1 + z_{k_1}) \times \cdots \times (F_1 + z_{k_j})) \leq 2^{j-1} 2^{jd} \mathbf{E}\Psi(E_{\mathbf{o}})^j,$$

for any $\{k_1,\ldots,k_j\} \subset \{2,\ldots,q\}$, we obtain the estimate

$$|I_j(K_1,\ldots,K_j)| \leq C_6(k,d) \left(\mathbf{E}\Psi(E_{\mathbf{o}})^{k+\delta}\right)^{\frac{k}{k+\delta}} \beta_\Psi(m-2)^{\frac{\delta}{k+\delta}}.$$

Finally, repeating the above counting procedure and using (2.9) lead to

$$\sum_{(z_2,\ldots,z_k) \in S_{k-1} : |z_k| \geq 2k-1} |I_j(K_1,\ldots,K_j)| \leq C_7(k,d,\delta) \sum_{m=1}^{\infty} m^{(k-1)d-1} \beta_\Psi(m)^{\frac{\delta}{k+\delta}} < \infty,$$

where the constant $C_7(k,d,\delta)$ depends only on $d \geq 1$, $k \geq 2$ and $\mathbf{E}\Psi(E_{\mathbf{o}})^{k+\delta} < \infty$.

In the same way we can show that $-\gamma_{\mathrm{red}}^{(k)}(H_{k-1}^{-}) < \infty$ which terminates the proof. □

6 SOME EXAMPLES FROM STOCHASTIC GEOMETRY

Example 1. *m-dependent stationary PP* $\Psi \sim P$, i.e. $\mathcal{F}_\Psi(F_a)$ and $\mathcal{F}_\Psi(F_{a+m}^c)$ are independent for some fixed $m > 0$ and any $a > 0$, is B_k-mixing if $\mathbf{E}\Psi(E_{\mathbf{o}})^k < \infty$. Special cases of m-dependent PPs are Poisson cluster processes and dependently thinned Poisson processes with bounded cluster diameter and thinning procedures of bounded reach, respectively, see Example 4 below. Note that in Theorem 2 we can take $\beta_\Psi(m) = 0$ and $\delta = 0$.

Example 2. *Voronoi-tessellation* $V(\Psi) = \bigcup_{i\geq 1} \partial C_i(\Psi)$ generated by a simple stationary PP $\Psi = \sum_{i\geq 1} \delta_{X_i}$ in $\mathbb{R}^d$, where $\partial C_i(\Psi)$ denotes the boundary of the cell $C_i(\Psi)$ formed by all points in $\mathbb{R}^d$

which are closest to the atom X_i, i.e. $C_i(\Psi) = \{x \in \mathbb{R}^d : \|x - X_i\| < \|x - X_j\|, j \neq i\}$, see [18]. Let $\mathcal{F}_{V(\Psi)}(F)$ denote the σ-algebra generated by the random closed set $V(\Psi) \cap F$, see [5] for details. In case the X_i's are atoms of a Poisson process $\Psi \sim \Pi_\lambda$ the following bound could be shown in [5]:

$$\beta\big(\mathcal{F}_{V(\Psi)}(F_a), \mathcal{F}_{V(\Psi)}(F^c_{a+r})\big) \leq \begin{cases} c_3\, \left(\frac{r}{a}\right)^{d-1} \exp\{-\lambda\, c_1\, a^{d-1}\, r\} & \text{if } r \geq c_0\, a\,, \\ c_3\, \left(\frac{a}{r}\right)^{d-1} \exp\{-\lambda\, c_2\, r^d\} & \text{if } r \leq c_0\, a\,, \end{cases} \quad \text{for } a, r \geq 1/2,$$

giving $\beta_\Psi(r) = c_5\, r^{d-1} \exp\{-\lambda\, c_4\, r\}$ according to (2.8) with constants $c_0, c_1, \dots, c_5 > 0$ depending only on the dimension $d \geq 1$. Hence, the stationary PP of the cell vertices and other PPs associated with the cells $C_i(\Psi)$ (e.g. circumcentres of the $(d-1)$-facets or Cox processes supported by $V(\Psi)$) are Brillinger-mixing. Furthermore, the exponential decay of $\beta_\Psi(r)$ holds also for Poisson cluster processes with typical cluster diameter D_0 satisfying $\mathbf{E} \exp\{h\, D_0\} < \infty$ for some $h > 0$, see [5].

Example 3. *Germ-grain models* $\Xi = \bigcup_{i \geq 1} (X_i + \Xi_i)$ defined by a stationary PP $\Psi = \sum_{i \geq 1} \delta_{X_i}$ in $\mathbb{R}^d$ with intensity $\lambda > 0$ and a sequence $\{\Xi_i, i \geq 1\}$ (independent of Ψ) of independent copies of a compact set $\Xi_0 \subset \mathbb{R}^d$, called *typical grain*. In [8] the subsequent bound of the β-mixing coefficient between two σ-algebras generated by the random closed set Ξ on F_a and $\mathbb{R}^d \setminus F_{a+r}$, respectively, could be derived:

(6.22)

$$\begin{aligned} \beta\big(\mathcal{F}_\Xi(F_a), \mathcal{F}_\Xi(F^c_{a+r})\big) &\leq \beta\big(\mathcal{F}_\Psi(F_{a+r/4}), \mathcal{F}_\Psi(F^c_{a+3\,r/4})\big) \\ &\quad + \lambda\, d\, 2^{d+1} \left[\Big(1 + \frac{4\,a}{r}\Big)^{d-1} + \Big(3 + \frac{4\,a}{r}\Big)^{d-1} \right] \mathbf{E}\|\Xi_0\|^d \mathbf{1}(\|\Xi_0\| \geq r/4) \end{aligned}$$

for $a, r \geq 1/2$, where $\|\Xi_0\| := \sup\{\|x - y\| : x, y \in \Xi_0\}$ denotes the diameter of the typical grain Ξ_0. Taking into account condition (2.8) with β-mixing rate $\beta_\Psi(r)$, it is easily seen from (6.22) that

$$(6.23) \quad \beta\big(\mathcal{F}_\Xi(F_a), \mathcal{F}_\Xi(F^c_{a+r})\big) \leq \Big(\max\big\{1, \frac{4\,a}{r}\big\} \Big)^{d-1} \Big(\beta_\Psi(r/2) + \lambda\, d\, 8^d\, \mathbf{E}\|\Xi_0\|^d \mathbf{1}(\|\Xi_0\| \geq r/4) \Big)$$

for $a, r \geq 1/2$.

Note that (6.23) provides the β-mixing rate of a *cluster PP* $\Psi_{c\ell} := \sum_{i \geq 1} \sum_{j=1}^{N_i} \delta_{X_i + Y_j^{(i)}}$ if $\Xi_0 = \{Y_1, \dots, Y_N\}$ consists of ($\mathbf{P}$-a.s.) finitely many random points with typical cluster diameter $D_0 = \|\Xi_0\|$. Further, Cox processes Ψ_{co} are frequently used PP models, see e.g. [2] for a general definition, in particular so-called *interrupted Poisson processes* supported by a random set Ξ or its boundary $\partial\Xi$, see [7, 18]. For example, the atoms of a Poisson process $\Phi = \sum_{i \geq 1} \delta_{P_i} \sim \Pi_\mu$ being independent of the germ-grain model Ξ are only counted when they lie in Ξ, i.e. $\Psi_{co} = \sum_{i \geq 1} \mathbf{1}_\Xi(P_i)\, \delta_{P_i}$. Due to (6.23) and the properties of Φ, it is clear that the β-mixing rate $\beta_{\Psi_{co}}(r)$ satisfies (2.9) if $\beta_\Psi(r)$ does and

$$(6.24) \qquad \int_1^\infty r^{(k-1)d-1} \big(\mathbf{E}\|\Xi_0\|^d \mathbf{1}(\|\Xi_0\| \geq r/4) \big)^{\delta/(k+\delta)}\, \mathrm{d}r \leq \frac{4^{(k-1)d}}{(k-1)d}\, \mathbf{E}\|\Xi_0\|^{k\,d\,(1+\delta)/\delta} < \infty$$

for some $\delta > 0$. Hence, since $\mathbf{E}\Psi_{co}(E_o)^{k+\delta} < \infty$ obviously holds, both assumptions (2.9) and (6.24) imply that the stationary Cox PP Ψ_{co} turns out B_k-mixing.

From the view point of statistics of germ-grain models, see [14], the family of PPs Ψ_u of *exposed tangent points* associated with the germ-grain model Ξ in direction (of a unit vector) u contain a lot of information on Ξ_0 and Ψ. Assuming additionally that Ξ_0 is convex and $\mathbf{o} \in \Xi_0$ the PP Ψ_u is defined by

$$\Psi_u := \sum_{i\geq 1} \delta_{\ell(u,\Xi_i)+X_i} \prod_{j:j\neq i} \big(1 - \mathbf{1}_{\Xi_j+X_j}(\ell(u,\Xi_i)+X_i)\big)\,,$$

where $\ell(u,\Xi_i)$ denotes the lexicographically smallest tangent point of the convex grain Ξ_i in direction u. This means that the atoms of Ψ_u are those tangent points of the shifted grains $\Xi_i + X_i$ being not covered by any other shifted grain $\Xi_j + X_j$, $j \neq i$, see Figure 1. Note that the PP Ψ_u turns out to be stationary (but not isotropic even if Ψ and Ξ_0 do so).

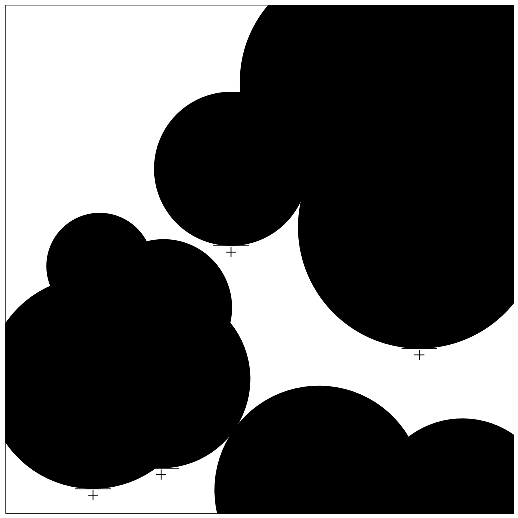

Figure 1: Exposed tangents points in a Boolean model with discs

The very definition of Ψ_u reveals that the β-mixing coefficient on the l.h.s. of (6.23) can be replaced by $\beta\big(\mathcal{F}_{\Psi_u}(F_a), \mathcal{F}_{\Psi_u}(F^c_{a+r})\big)$. Together with the obvious fact that the moments of $\Psi_u(E_{\mathbf{o}})$ do not exceed the moments of $\Psi(E_{\mathbf{o}})$ we arrive at the conclusion that Ψ_u is B_k-mixing for any u if Ψ fulfills (2.9) and $\mathbf{E}\|\Xi_0\|^{k\,d\,(1+\delta)/\delta}$ exists for some $\delta > 0$.

The best studied and most frequently used germ-grain model is the so-called *Boolean model* Ξ, where the germs form a Poisson process $\Psi \sim \Pi_\lambda$. The random union set Ξ is $\mathbf{P}$-a.s. closed if $\mathbf{E}\|\Xi_0\|^d < \infty$, see e.g. [14, 18] for more on this basic model of stochastic geometry. Since in this special case $\beta_\Psi(r) = 0$ for $r > 0$ and all moments of $\Psi(E_{\mathbf{o}})$ exist, the number $\delta > 0$ in (2.9) can be taken arbitrarily large which relaxes the moment assumption on $\|\Xi_0\|$ to $\mathbf{E}\|\Xi_0\|^{k\,d+\varepsilon} < \infty$ for an arbitrarily small $\varepsilon > 0$ in order to insure $\mathbf{B}_k$-mixing of Ψ_{co} and Ψ_u. It is noteworthy that for Boolean models the intensity λ_u of $\Psi_u \sim P_u$ can be simply expressed by $\lambda_u = \lambda\,\exp\{-\lambda\,\mathbf{E}\nu_d(\Xi_0)\}$ and the Lebesgue density $\varrho^{(k)}_u$ of the kth-order factorial moment measure (2.5) (with P_u instead of P) exists for any $k \geq 2$ and takes the form

$$\varrho^{(k)}_u(x_1,\dots,x_k) = \lambda^k \prod_{p=1}^k \mathbf{E}\Big(\prod_{\substack{q=1\\q\neq p}}^k \big(1-\mathbf{1}_{\Xi_0(u)}(x_q-x_p)\big)\Big)\,\exp\Big\{\lambda\int\Big[\mathbf{E}\prod_{r=1}^k\big(1-\mathbf{1}_{\Xi_0(u)}(x-x_r)\big)-1\Big]\,\mathrm{d}x\Big\},$$

where $\Xi_0(u) := -\Xi_0 + \ell(u, \Xi_0)$. This formula allows to check the B_k-mixing property in a direct way showing that indeed $\mathbf{E}\|\Xi_0\|^{k\,d} < \infty$ is sufficient. Furthermore, $\varrho_u^{(k)}(\cdot)$ is uniformly bounded by λ^k for $k \geq 2$, which is significant for so-called *sub-Poisson processes*.

Example 4. *$\pi(x)$-thinning of point processes*: Let $\{\pi(x), x \in \mathbb{R}^d\}$ be a stationary random field on $[\Omega, \mathcal{F}, \mathbf{P}]$ taking values in $[0,1]$ and being independent of the stationary PP $\Psi = \sum_{i\geq 1} \delta_{X_i}$ in $\mathbb{R}^d$, see [18]. Define the 0–1-valued random mark field $\{M(x), x \in \mathbb{R}^d\}$ with finite-dimensional distributions $\mathbf{P}(M(x_1) = 1, \ldots, M(x_k) = 1) = \mathbf{E}[\pi(x_1) \cdots \pi(x_k)]$ for any $x_1, \ldots, x_k \in \mathbb{R}^d$ and $k \in \mathbb{N}$. In this way we obtain a so-called $\pi(x)$-thinned stationary PP $\Psi_\pi = \sum_{i\geq 1} \delta_{X_i} M(X_i)$. This thinning procedure means that, for a given realization of the probabilities $\pi(x) = p(x)$, $x \in \mathbb{R}^d$, the atom X_i survives with probability $p(X_i)$ independently of the survival of the other atoms X_j, $j \neq i$. As special cases we mention $\pi(x) = \mathbf{1}(\xi \in B)$ or $\pi(x) = (\xi(x) - a)\mathbf{1}(a \leq \xi(x) \leq b)/(b - a)$ for some stationary random field $\{\xi(x), x \in \mathbb{R}^d\}$ and certain fixed $B \in \mathcal{B}^1$ and $a, b \in \mathbb{R}^1$. As particular case of geostatistical marking of PPs we deduce from Lemma 5.1 in [7] (with σ-algebra $\mathcal{F}_\pi(F)$ generated by $\{\pi(x), x \in F\}$) that

$$\beta\big(\mathcal{F}_{\Psi_\pi}(F_a), \mathcal{F}_{\Psi_\pi}(F^c_{a+r})\big) \leq \beta\big(\mathcal{F}_\Psi(F_a), \mathcal{F}_\Psi(F^c_{a+r})\big) + \beta\big(\mathcal{F}_\pi(F_a), \mathcal{F}_\pi(F^c_{a+r})\big)$$

for $a, r \geq 1/2$, which gives $\beta_{\Psi_\pi}(r) \leq \beta_\Psi(r) + \beta_\pi(r)$ for the corresponding β-mixing rates. This enables us to check $\mathbf{B}_k$-mixing of Ψ_π. On the other hand, this property of Ψ_π holds for any $\mathbf{B}_k$-mixing PP Ψ if additionally $\int_{(\mathbb{R}^d)^j} |\,\mathbf{Cum}_j(\pi(\mathbf{o}), \pi(x_2), \ldots, \pi(x_j))\,|\, \mathrm{d}(x_2, \ldots, x_j) < \infty$ for $j = 2, \ldots, k$.

Example 5. *Generalized Stoyan soft-core process I and II*: As in Example 3 let $\Psi = \sum_{i\geq 1} \delta_{X_i}$ be a simple stationary PP in $\mathbb{R}^d$ independently marked by a sequence of random vectors $\{(\Xi_i, U_i), i \geq 1\}$ with independent components, where the first ones are independent copies of a compact set $\Xi_0 \subset \mathbb{R}^d$ containing $\mathbf{o}$ and the second ones are independently uniformly distributed in $(0,1)$. Then we are in a position to define two types of *dependently thinned* PP generalizing two thinning procedures suggested in [19]:

$$\Psi_{th,1} := \sum_{i\geq 1} \delta_{X_i} \prod_{j\neq i} \big(1 - \mathbf{1}_{\Xi_i + X_i}(X_j)\big) \quad \text{and} \quad \Psi_{th,2} := \sum_{i\geq 1} \delta_{X_i} \prod_{j\neq i: X_j \in \Xi_i + X_i} \mathbf{1}_{[U_i,1)}(U_j)$$

To be precise, in the first model an atom X_i of Ψ survives if and only if no other atom X_j (of Ψ) lies in $\Xi_i + X_i$, whereas in the second model X_i will survive iff either no other atom X_j lies in $\Xi_i + X_i$ or all atoms $X_j \in \Xi_i + X_i, j \neq i$, have marks U_j greater than or equal to U_i. In [19], $\Psi_{th,1}$ and $\Psi_{th,2}$ were introduced and studied in the special case of a random ball $\Xi_0 = b(\mathbf{o}, R_0)$ centred at the origin with the aim to generalize Matérn's hard-core process I and II for which $\mathbf{P}(R_0 = \text{const} > 0) = 1$, see e.g. [18]. Note that both of Stoyan's soft-core PPs inherit the isotropy of Ψ, whereas a non-circular set Ξ_0 can generate a high degree of anisotropy in $\Psi_{th,i}$, $i = 1, 2$, even if $\Psi \sim \Pi_\lambda$.

Finally, it is easily checked that the β-mixing coefficients $\beta\big(\mathcal{F}_{\Psi_{th,i}}(F_a), \mathcal{F}_{\Psi_{th,i}}(F^c_{a+r})\big)$, $i = 1, 2$, have the same bound as $\beta\big(\mathcal{F}_\Xi(F_a), \mathcal{F}_\Xi(F^c_{a+r})\big)$ in (6.23) with all consequences mentioned above. In case of $\Psi \sim \Pi_\lambda$ this implies that each of the soft-core Poisson processes $\Psi_{th,1}$ and $\Psi_{th,2}$ (with intensities $\lambda_1 = \lambda \exp\{-\lambda\, \mathbf{E}\nu_d(\Xi_0)\}$ and $\lambda_2 = \mathbf{E}\big[(1 - \exp\{-\lambda\, \nu_d(\Xi_0)\})/\nu_d(\Xi_0)\big]$, respectively) turns out Brillinger-mixing whenever $\mathbf{E}\|\Xi_0\|^n < \infty$ for any $n \in \mathbb{N}$, and they prove to be m-dependent (as defined in Example 1) if $\mathbf{P}(\|\Xi_0\| \leq \text{const}) = 1$.

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