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Numerical solution of stochastic partial differential equations with correlated noise

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Abstract

The aim of this paper is to investigate the numerical solution of stochastic partial differential equations (SPDEs) for a wider class of stochastic equations with colored noise instead of the usual space-time white noise. By applying Galerkin method for spatial discretization we obtain the rate of path-wise convergence in the uniform topology. Numerical examples illustrate the theoretically predicted convergence rate.

Keywords: stochastic partial differential equations, colored noise, spectral Galerkin approximation, time discretization, order of convergence, uniform bounds.
MSC2010: 60H35, 60H15, 60H10, 65M12, 65M60

1 Introduction

Let $T > 0$, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and V is a Banach space. Suppose the space-time continuous stochastic process $X : [0,T] \times \Omega \to V$ is the unique solution of the the following stochastic partial differential equation (SPDE)

$$dX_t = [AX_t + F(X_t)]\,dt + dW_t, \qquad X_t(0) = X_t(1) = 0, \quad X_0 = 0, \tag{1}$$

for $t \in [0,T]$ and $x \in (0,1)$, where the operator A denotes an unbounded operator, for example the Laplacian. The noise is given by a Wiener process W_t, $t \in [0,T]$ defined later.

The main purpose of this article is to consider a spectral Galerkin approximation of (1) in L^∞, where the noise is colored. The main results are formulated in an abstract way so that in principle they should apply to other approximation methods like finite elements, but here we only verify the applicability for spectral Galerkin methods for simplicity of presentation. A key point is the uniform bound on the numerical data, which is for spectral methods usually straightforward to verify using energy-type a-priori estimates.

Of course the result should apply for higher dimensional domains, differential operators of higher order, or other boundary conditions like Dirichlet, but as an example we stick with this relatively simple situation here.

In [4] the Galerkin approximation was already considered for a stochastic Burgers equation with colored noise, but here we present this method in a more

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general setting, and not only for the Burgers equation. The main novelty, as in [4] or [3], is to bound the spatial and temporal discretization error in the uniform topology. The space of continuous or Hölder-continuous functions is a natural space for stochastic convolutions. For instance, if for space-time white noise the stochastic convolution is in L^2 in space, it is already continuous. In a recent publication [5] Cox & van Neerven established a time-discretization error in Hölder spaces, but the spatial error in UMD-spaces. We strongly believe, that working in fractional Sobolev-spaces $W^{\alpha,p}$ with $\alpha > 0$ small and $p \gg 1$ large should yield similar results than ours, but we present here a simple proof yielding uniform bounds in time only.

In [3, 12] the Galerkin approximation was considered for a simple case of SPDEs of the type of (1), either without time-discretization or in different spaces. Moreover, the Brownian motions in the Fourier expansion of the noise are independent. But in general the spatial covariance operator of the forcing does not necessarily commute with the linear operator A, thus we consider here the case where the Brownian motions are not independent.

Many authors have investigated the spectral Galerkin method for this kind of equation with space-time white noise. See for example [9, 10, 11, 12, 13, 14, 15]. There are also many articles about finite difference methods [1, 8, 9, 17, 18]. The existence and uniqueness of solutions of the stochastic equation was studied in [6, 7] for space-time white noise. In our proofs, as the nonlinearity allows for polynomial growth, we do not rely on the global existence of solutions, but assume that the numerical approximation remains uniformly bounded. In the limit of fine discretization, this will ensure global existence of the solutions and a global error bound for the numerical approximation.

Our aim here is to extend the results of [4] to the case of more general nonlinearities, with local Lipschitz conditions and polynomial growth. For spatial discretization of equation (1) we apply a spectral Galerkin approximation as already discussed in [3] and for the time discretization we follow the method proposed in [12].

It should be mentioned that the spatial discretization error is obtained by the results of [3, 4]. We will recall their main results in Section 1. In this article we focus on the time discretization. Not treated in [3] but already in [4], we consider here also the case of colored noise being not diagonal with respect to the eigenfunctions of the Laplacian. As the final result we obtain an error estimate for the full space-time discretization. The main result of [4] in combination with the results presented in this paper yields the convergence results in the uniform topology of continuous functions for the numerical approximations of a wider class of SPDEs with colored noise. The key assumption is a uniform bound on the numerical approximations, that allows for local Lipschitz-conditions only.

The paper is organized as follows. Section 2 gives the setting and the assumptions. In Section 3 we recall the results on the spatial discretization error, and in Section 4 estimates for the temporal error are derived. Finally, in the last section a simple numerical example is presented, in order to illustrate the results.

2 Setting and assumptions

Let V, W be $\mathbb{R}$-Banach spaces of continuous functions from $[0,1]$ to $\mathbb{R}$. Suppose that the unbounded and invertible linear operator A generates a strongly continuous semigroup $S_t : W \to V$. Especially, $S_{t+s} = S_t S_s$ and $S_0 = Id$.

Consider the following equation

$$dX_t = [AX_t + F(X_t)]\,dt + dW_t, \qquad X_t(0) = X_t(1) = 0, \quad X_0 = 0, \tag{2}$$

for $t \in [0,T]$ and $x \in (0,1)$.

Suppose there are bounded linear operators $P_N : V \to V$. The example we have in mind is the spectral Galerkin method given by the orthogonal projection $P_N v = \sum_{i=1}^{N} \int_0^1 e_i(s)v(s)ds \cdot e_i$, where $\{e_i\}_{i\in\mathbb{N}}$ are an orthonormal basis of eigenfunctions of A. But any other approximation method like finite elements should work in a similar way, if we can satisfy our assumptions for the projections.

Consider the following assumptions already made in [3].

Assumption 1 (Semigroup)**.** *Suppose for the semigroup, that $S : (0,T] \to L(W,V)$ is a continuous mapping satisfying for given constants $\alpha, \theta \in [0,1)$ and $\gamma \in (0,\infty)$*

$$\sup_{0<t\le T} \left(t^\alpha \|S_t\|_{L(W,V)}\right) < \infty, \quad \sup_{N\in\mathbb{N}} \sup_{0\le t\le T} \left(t^\alpha N^\gamma \|S_t - P_N S_t\|_{L(W,V)}\right) < \infty, \tag{3}$$

and

$$\|A^\theta S_t\|_{L(V,V)} \le C t^{-\theta} \quad \text{together with} \quad \|A^{-\theta}(S_t - I)\|_{L(V,V)} \le t^\theta. \tag{4}$$

The first assumption is crucial for the spatial discretization, while the second assumption (4) is mainly needed for the result on time-discretization, in order to bound differences of the semigroup. For example, for analytic semigroups generated by the Laplacian, this is usually straightforward to verify. See for example [16].

Assumption 2 (Nonlinearity)**.** *Let $F : V \to W$ be a continuous mapping, which satisfies*

$$\|F(u) - F(v)\|_W \le L\|u - v\|_V (1 + \|u\|_V^p + \|v\|_V^p), \tag{5}$$

for nonnegative integer p and every $u, v \in V$, where L is a given real number.

Let us remark that it is not a major restriction that we assumed the operator A to be invertible, as we can always consider for some constant c the operator $\tilde{A} = A + cI$ and the nonlinearity $\tilde{F} = F - cI$.

2.1 The Ornstein-Uhlenbeck process

Assumption 3 (Ornstein-Uhlenbeck)**.** *Let $O : [0,T] \times \Omega \to V$ be a stochastic process with continuous sample paths and*

$$\sup_{N\in\mathbb{N}} \sup_{0\le t\le T} N^\gamma \|O_t(\omega) - P_N(O_t(\omega))\|_V < \infty, \tag{6}$$

for every $\omega \in \Omega$, where $\gamma \in (0, \infty)$ is given in Assumption 1. Moreover,

$$\sup_{0 \leq t_1 \leq t_2 \leq T} \frac{\left\| O_{t_2}(\omega) - O_{t_1}(\omega) \right\|_V}{(t_2 - t_1)^\theta} < \infty, \tag{7}$$

for some $\theta \in (0, \frac{1}{2})$.

In order to give an example for this assumption, let $\beta^i : [0, T] \times \Omega \to \mathbb{R}, i \in \mathbb{N}^d$, be a family of Brownian motions that are not necessarily independent. They are correlated as given by

$$\mathbb{E}(\beta^k(t)\beta^l(t)) =< Qe_k, e_l > t, \quad k, l \in \mathbb{N}.$$

The basis functions e_k are given by the standard Dirichlet basis, where for every $k \in \mathbb{N}$

$$e_k : [0, 1] \to \mathbb{R}, \quad e_k(x) = \sqrt{2} \sin(k\pi x), \quad x \in [0, 1],$$

are smooth functions. Furthermore, the covariance operator Q is a symmetric non-negative operator, given by the convolution with a translation invariant positive definite kernel q, which is the correlation function of the noise. This means

$$< Qe_k, e_l >= \int_0^1 \int_0^1 e_k(x)e_l(y)q(x - y)dydx, \tag{8}$$

for $k, l \in \mathbb{N}^d$. Note that Q is diagonal with respect to the standard Fourier basis, but in general not with the Dirichlet basis. See for example [2] for a detailed discussion.

For the regularity assume that for some $\rho > 0$ we have

$$\sum_{i \in N} \sum_{j \in N} \|i\|_2^{\rho - 1} \|j\|_2^{\rho - 1} |\langle Qe_i, e_j \rangle| < \infty. \tag{9}$$

This is for a diagonal operator Q a condition on the trace of $\Delta^{\rho-1}Q$ being finite.

Using (9) together with Lemma 4 in [4], there exists a stochastic process $O : [0, T] \times \Omega \to V$, which is the Ornstein-Uhlenbeck process (or stochastic convolution) given by the semigrup generated by the Dirichlet Laplacian and the Wiener process $W(t) = \sum_{k \in \mathbb{N}} \beta_k(t)e_k$. Furthermore, Lemma 4 in [4] assures that O satisfies Assumption 3, for all $\theta \in (0, \min\{\frac{1}{2}, \frac{\rho}{2}\})$, with

$$\mathbb{P}\Big[\lim_{N \to \infty} \sup_{0 \leq t \leq T} \Big\| O_t - \sum_{i \in \{1, \ldots, N\}} \Big(-\lambda_i \int_0^t e^{-\lambda_i(t-s)} \beta_s^i ds + \beta_t^i \Big) e_i \Big\|_V = 0\Big] = 1. \tag{10}$$

Let us comment a little bit more on the Q-Wiener process. As Q is a symmetric Hilbert-Schmidt operator, there exists an orthonormal basis f_k given by eigenfunctions of Q with $\alpha_k^2 f_k = Qf_k$. Using standard theory of [6], there is a family of i.i.d. Brownian motions $\{B_k\}_{k \in \mathbb{N}}$ such that $W(t) = \sum_{k \in \mathbb{N}} \alpha_k B_k(t) f_k \in L^2([0, \pi])$. We can then define

$$\beta_k(t) = \langle W(t), e_k \rangle_{L^2} = \sum_{\ell \in \mathbb{N}} \alpha_\ell B_\ell(t) \langle f_\ell, e_k \rangle_{L^2}.$$

2.2 Bounds and solutions

Let us first assume boundedness of the spectral Galerkin approximation. This will assure the existence of mild solutions later on. We will discuss later how to relax this condition to boundedness of the numerical data alone.

Assumption 4. *Let $X^N : [0,T] \times \Omega \to V, \quad N \in \mathbb{N}$, be a sequence of stochastic processes with continuous sample paths such that*

$$\sup_{M\in\mathbb{N}} \sup_{0\le s\le T} \|X_s^M(\omega)\|_V < \infty \tag{11}$$

and

$$X_t^N(\omega) = \int_0^t P_N S_{t-s} F(X_s^N(\omega))ds + P_N(O_t(\omega)), \tag{12}$$

for every $t \in [0,T], \omega \in \Omega$ and every $N \in \mathbb{N}$.

From [3] we have the following theorem about existence of solutions.

Theorem 1. *Let Assumptions 1-4 be fulfilled. Then, there exists a unique stochastic process $X : [0,T] \times \Omega \to V$ with continuous sample paths, which fulfills*

$$X_t(\omega) = \int_0^t S_{t-s} F(X_s(\omega))ds + O_t(\omega), \tag{13}$$

for every $t \in [0,T]$ and every $\omega \in \Omega$. Moreover, there exists a $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $C : [0,\infty) \to \Omega$ such that

$$\sup_{0\le t\le T} \| X_t(\omega) - X_t^N(\omega) \|_V \le C(\omega) \cdot N^{-\gamma}, \tag{14}$$

holds for every $N \in \mathbb{N}$ and every $\omega \in \Omega$, where $\gamma \in (0,\infty)$ is given in Assumption 1.

3 Time discretization

For time discretization of the finite dimensional SDEs (12) we follow the method in [12], which was also used in [4]. Fix a small time-step $\Delta t > 0$ and define the discretized points via the mapping $Y_m^{N,M} : \Omega \to V$ for $m \in \{1, ..., M\}$ by

$$Y_{m+1}^{N,M}(\omega) = S_{\Delta t}\Big(Y_m^{N,M}(\omega) + \Delta t (P_N F)(Y_m^{N,M}(\omega))\Big) + P_N\Big(O_{(m+1)\Delta t}(\omega) - S_{\Delta t} O_{m\Delta t}(\omega)\Big). \tag{15}$$

Thus $Y_m^{N,M}$, $m \in \{1, ..., M\}$ should be the approximation of the spectral Galerkin approximation X^N (see (18) below) at times $m \cdot (\Delta t)$.

For simplicity of presentation, we assume in addition to (11) that our numerical data is uniformly bounded:

Assumption 5. *For the numerical scheme (15) we assume*

$$\sup_{0\le m\le M} \sup_{N,M\in\mathbb{N}} \|Y_m^{N,M}\|_V < \infty. \tag{16}$$

Therefore for all the examples that one wants to study, at first one should verify this. We will comment later on the effect of either (16) or (15) not being true.

Our aim is to obtain the discretization error in time

$$\|X^N_{m\Delta t}(\omega) - Y^{N,M}_m(\omega)\|_V, \tag{17}$$

where

$$X^N_{m\Delta t}(\omega) = \int_0^{m\Delta t} P_N S_{m\Delta t - s} F(X^N_s(\omega))ds + O^N_{m\Delta t}(\omega), \tag{18}$$

is the solution of the spatial discretization, which is evaluated at the grid points. It should be mentioned that for simplicity of notation, during this section $C(\omega,\alpha,\theta) > 0$ is a random constant which changes from line to line.

Lemma 2. *Suppose Assumptions 1-4 are true. Let $X^N : [0,T]\times\Omega \to V$ be the unique adapted stochastic process with continuous sample paths in (12) and $O^N : [0,T]\times\Omega \to V$ is the stochastic process defined in Assumption 4 in (10). Then we obtain*

$$\left\|(X^N_{t_2}(\omega) - O^N_{t_2}(\omega)) - (X^N_{t_1}(\omega) - O^N_{t_1}(\omega))\right\|_V \leq C(\omega)(t_2 - t_1)^{\theta},$$

for every $\omega \in \Omega$ and $\theta \in \min\{\frac{1}{2}, \frac{\rho}{2}, 1-\alpha\}$ and all $t_1, t_2 \in [0,T]$, with $t_1 < t_2$ where C is a finite random variable $C : \Omega \to [0,\infty)$.

Proof.

$$\begin{aligned}
&\left\|(X^N_{t_2}(\omega) - O^N_{t_2}(\omega)) - (X^N_{t_1}(\omega) - O^N_{t_1}(\omega))\right\|_V \\
&\leq \left\| \int_{t_1}^{t_2} P_N S_{t_2-s} F(X^N_s(\omega))ds + \int_0^{t_1} P_N(S_{t_2-s} - S_{t_1-s})F(X^N_s(\omega))ds \right\|_V \\
&\leq \int_{t_1}^{t_2} \|P_N S_{t_2-s}\|_{L(W,V)}.\|F(X^N_s(\omega)\|_W ds + \int_0^{t_1} \|P_N(S_{t_2-s} - S_{t_1-s})\|_{L(W,V)}\|F(X^N_s(\omega))\|_W ds \\
&\leq \sup_{0\leq s\leq T}\|F(X^N_s(\omega))\|_W \int_{t_1}^{t_2}(t_2-s)^{-\alpha}ds + \int_0^{t_1}\|P_N S_{t_1-s}(S_{t_2-t_1} - I)\|_{L(W,V)}\|F(X^N_s(\omega))\|_W ds \\
&\leq \Big(\int_{t_1}^{t_2}(t_2-s)^{-\alpha}ds + \int_0^{t_1}\|P_N S_{t_1-s}A^{\theta}\|_{L(W,V)}\|A^{-\theta}(S_{t_2-t_1} - I)\|_{L(V,V)}ds\Big)\sup_{0\leq s\leq T}\|F(X^N_s(\omega))\|_W \\
&\leq C(\omega)\Big(\int_{t_1}^{t_2}(t_2-s)^{-\alpha}ds + \int_0^{t_1}\|P_N S_{\frac{t_1-s}{2}}\|_{L(W,V)}\|S_{\frac{t_1-s}{2}}A^{\theta}\|_{L(V,V)}\|A^{-\theta}(S_{t_2-t_1} - I)\|_{L(V,V)}ds\Big) \\
&\leq C(\omega)\Big((t_2-t_1)^{1-\alpha} + \int_0^{t_1}(t_1-s)^{-\alpha-\theta}ds\cdot(t_2-t_1)^{\theta}\Big) \\
&\leq C(\omega)\Big((t_2-t_1)^{1-\alpha} + (t_2-t_1)^{\theta}\Big) \leq C(\omega)(t_2-t_1)^{\theta},
\end{aligned} \tag{19}$$

where we have used (4). □

Before we start to bound the first part of the error, we define

$$
\begin{aligned}
R(\omega) := & \sup_{N\in\mathbb{N}} \sup_{0\le s\le T} \|F(X_s^N(\omega))\|_W + \sup_{N\in\mathbb{N}} \sup_{0\le s\le T} \|X_s^N(\omega)\|_V \\
& + \sup_{0\le t_1,t_2\le T} \|O_{t_2}(\omega) - O_{t_1}(\omega)\|_V |t_2-t_1|^{-\theta} \\
& + \sup_{N\in\mathbb{N}} \sup_{0\le t_1,t_2\le T} \|X_{t_2}^N(\omega) - O_{t_2}^N(\omega) - (X_{t_1}^N(\omega) - O_{t_1}^N(\omega))\|_V |t_2-t_1|^{\theta}.
\end{aligned}
$$

From Assumption 4, (7) and Lemma 2, $R : \Omega \to \mathbb{R}$ is a finite random variable.

3.1 Theorem

The first main result of this section is stated below.

Theorem 3. *Let Assumptions 1-5 be fulfilled. There exists a finite random variable $C : \Omega \to [0,\infty)$ such that for all $m \in \{0,1,...,M\}$ and every $M,N \in \mathbb{N}$*

$$
\|X_{m\Delta t}^N(\omega) - Y_m^{N,M}(\omega)\|_V \le C(\omega)(\Delta t)^\theta,
$$

where $X^N : [0,T]\times\Omega \to V$ is the unique adapted stochastic process with continuous sample paths, defined in Assumption 4, and $Y_m^{N,M} : \Omega \to V$, for $m \in \{0,1,...,M\}$, and $N,M \in \mathbb{N}$, is given in (15).

Proof. For the proof it is sufficient to prove the result for sufficiently small $|t_2 - t_1|$. From (18) we have

$$
\begin{aligned}
X_{m\Delta t}^N(\omega) &= \int_0^{m\Delta t} P_N S_{m\Delta t - s} F(X_s^N(\omega)) ds + O_{m\Delta t}^N(\omega) \\
&= \sum_{k=0}^{m-1} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - s} F(X_s^N(\omega)) ds + O_{m\Delta t}^N(\omega),
\end{aligned}
\tag{20}
$$

for every $m \in \{0,1,...,M\}$, and every $M \in \mathbb{N}$.

as an intermediate discretization, we consider the mapping $Y_m^N : \Omega \to V,\ m = 1,2,...,M$ by

$$
Y_m^N(\omega) = \sum_{k=0}^{m-1} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - k\Delta t} F(X_{k\Delta t}^N(\omega)) ds + O_{m\Delta t}^N(\omega). \tag{21}
$$

Our aim is to bound $\|X_{m\Delta t}^N(\omega) - Y_m^{N,M}(\omega)\|_V$.

We split this into two Lemmas. First for an error between Y_m^N and the spectral Galerkin method.

Lemma 4. *Under the assumptions of Theorem 3 there exists a finite random variable $C : \Omega \to [0,\infty)$ such that for all $m \in \{0,1,...,M\}$ and every $M,N \in \mathbb{N}$*

$$
\|X_{m\Delta t}^N(\omega) - Y_m^N(\omega)\|_V \le C(\omega)(\Delta t)^\theta. \tag{22}
$$

Secondly for the difference between Y_m^N and the full discretization in time

Lemma 5. *Under the assumptions of Theorem 3 there exists a finite random variable $C : \Omega \to [0,\infty)$ such that for all $m \in \{0,1,...,M\}$ and every $M,N \in \mathbb{N}$*

$$
\|Y_m^N(\omega) - Y_m^{N,M}(\omega)\|_V \le C(\omega)(\Delta t)^\theta. \tag{23}
$$

3.2 Proof of Lemma 4

For estimating the first error term stated in (22) we have

$$\begin{aligned} X^N_{m\Delta t}(\omega) - Y^N_m(\omega) = &\sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - s} F(X^N_s(\omega)) ds \\ &- \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - k\Delta t} F(X^N_{k\Delta t}(\omega)) ds \\ &+ \int_{(m-1)\Delta t}^{m\Delta t} P_N S_{m\Delta t - s} F(X^N_s(\omega)) ds \\ &- \int_{(m-1)\Delta t}^{m\Delta t} P_N S_{\Delta t} F(X^N_{k\Delta t}(\omega)) ds. \end{aligned} \tag{24}$$

At first we obtain the bound for the last two integrals in (24). For the first one, we get

$$\begin{aligned} \Big\| \int_{(m-1)\Delta t}^{m\Delta t} P_N S_{m\Delta t - s} F(X^N_s(\omega)) ds \Big\|_V &\leq \int_{(m-1)\Delta t}^{m\Delta t} \|P_N S_{m\Delta t - s}\|_{L(W,V)} \cdot \|F(X^N_s(\omega))\|_W ds \\ &\leq \int_{(m-1)\Delta t}^{m\Delta t} (m\Delta t - s)^{-\alpha} ds \sup_{0\leq s\leq t} \|F(X^N_s(\omega))\|_W \\ &\leq CR(\omega)(\Delta t)^{1-\alpha}. \end{aligned}$$

For the second term we obtain similarly

$$\begin{aligned} \Big\| \int_{(m-1)\Delta t}^{m\Delta t} P_N S_{\Delta t} F(X^N_{k\Delta t}(\omega)) ds \Big\|_V &\leq \sup_{0\leq s\leq t} \|F(X^N_s(\omega))\|_W \int_{(m-1)\Delta t}^{m\Delta t} (\Delta t)^{-\alpha} ds \\ &\leq R(\omega)(\Delta t)^{1-\alpha}. \end{aligned}$$

Therefore we have

$$\begin{aligned} \|X^N_{m\Delta t}(\omega) - Y^N_m(\omega)\|_V \leq &\Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - s} F(X^N_s(\omega)) ds \Big\|_V \\ &+ \Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - k\Delta t} F(X^N_{k\Delta t}(\omega)) ds \Big\|_V \\ &+ R(\omega)(\Delta t)^{1-\alpha}. \end{aligned} \tag{25}$$

Now we insert the OU-process. Define

$$Z^N_{s,k\Delta t}(\omega) = O^N_s(\omega) - O^N_{k\Delta t}(\omega)$$

Thus for every $m \in \{0, 1, ..., M\}$ we have,

$$
\begin{aligned}
&\Big\| X^N_{m\Delta t}(\omega) - Y^N_m(\omega) \Big\|_V \\
&\leq \Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - s} \Big(F(X^N_s(\omega)) - F\big(X^N_{k\Delta t}(\omega) + Z^N_{s,k\Delta t}(\omega)\big) \Big) ds \Big\|_V \\
&+ \Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - s} \Big(F(X^N_{k\Delta t}(\omega) + Z^N_{s,k\Delta t}(\omega)) - F(X^N_{k\Delta t}(\omega)) \Big) ds \Big\|_V \\
&+ \Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} \Big(P_N S_{m\Delta t - s} - P_N S_{m\Delta t - k\Delta t} \Big) F(X^N_{k\Delta t}(\omega)) ds \Big\|_V \\
&+ R(\omega)(\Delta t)^{1-\alpha}.
\end{aligned} \tag{26}
$$

For the first term in (26) by using (5) and Lemma 2 we conclude

$$
\begin{aligned}
&\Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - s} \Big(F(X^N_s(\omega)) - F\big(X^N_{k\Delta t}(\omega) + Z^N_{s,k\Delta t}(\omega)\big) \Big) ds \Big\|_V \\
&\leq L \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} (m\Delta t - s)^{-\alpha} \Big\| X^N_s(\omega) - (X^N_{k\Delta t}(\omega) + Z^N_{s,k\Delta t}(\omega)) \Big\|_V \\
&\qquad\qquad \cdot \Big(1 + \|X^N_s(\omega)\|^p_V + \|X^N_{k\Delta t}(\omega) + Z^N_{s,k\Delta t}(\omega)\|^p_V \Big) ds \\
&\leq C(\omega) \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} (m\Delta t - s)^{-\alpha} (s - k\Delta t)^{\theta} \Big(1 + 2R^p(\omega) + (s - k\Delta t)^{p\theta} \Big) ds \\
&\leq C(\omega)(\Delta t)^{\theta}.
\end{aligned} \tag{27}
$$

For the second term in (26) by (5) we derive

$$
\begin{aligned}
&\Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - s} \Big(F\big(X^N_{k\Delta t}(\omega) + Z^N_{s,k\Delta t}(\omega)\big) - F(X^N_{k\Delta t}(\omega)) \Big) ds \Big\|_V \\
&\leq L \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} \Big\| P_N S_{m\Delta t - s} \Big\|_{L(W,V)} \Big\| X^N_{k\Delta t}(\omega) + Z^N_{s,k\Delta t}(\omega) - X^N_{k\Delta t}(\omega) \Big\|_V \\
&\qquad\qquad \cdot \Big(1 + \Big\| X^N_{k\Delta t}(\omega) + Z^N_{s,k\Delta t}(\omega) \Big\|^p_V + \|X^N_{k\Delta t}(\omega)\|^p_V \Big) \\
&\leq C(\omega) \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} (m\Delta t - s)^{-\alpha} (s - k\Delta t)^{\theta} \Big(1 + 2R^p(\omega) + (s - k\Delta t)^{p\theta} \Big) \\
&\leq C(\omega)(\Delta t)^{\theta}.
\end{aligned} \tag{28}
$$

Finally, for the third term in (26) we drive

$$
\begin{aligned}
&\Big\| \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} \left(P_N S_{m\Delta t-s} - P_N S_{m\Delta t-k\Delta t}\right) F(X_{k\Delta t}^N(\omega)) ds \Big\|_V \\
&\leq \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} \Big\| P_N S_{m\Delta t-k\Delta t}(S_{k\Delta t-s} - I) F(X_{k\Delta t}^N(\omega)) \Big\|_V ds \\
&\leq \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} \|P_N S_{\frac{m\Delta t-k\Delta t}{2}}\|_{L(W,V)} \cdot \|A^\theta S_{\frac{m\Delta t-k\Delta t}{2}}\|_{L(V,V)} \\
&\qquad\qquad \cdot \|A^{-\theta}(S_{k\Delta t-s} - I)\|_{L(V,V)} \cdot \|F(X_{k\Delta t}^N(\omega))\|_W ds \\
&\leq C(\omega) \sum_{k=0}^{m-2} \int_{k\Delta t}^{(k+1)\Delta t} (m\Delta t - k\Delta t)^{-\alpha-\theta} (k\Delta t - s)^\theta \|F(X_{k\Delta t}^N(\omega))\|_W ds \\
&\leq C(\omega)(\Delta t)^\theta,
\end{aligned}
\tag{29}
$$

where we used (4) from the assumption on the semigroup.

Hence, from (27), (28) and (29) we get

$$
\|X_{m\Delta t}^N(\omega) - Y_m^N(\omega)\|_V \leq C(\omega)(\Delta t)^\theta. \tag{30}
$$

3.3 Proof of Lemma 5

Now, for the second error term in (23) because $Y_m^{N,M} : \Omega \to V$ satisfies

$$
Y_m^{N,M}(\omega) = \sum_{k=0}^{m-1} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t-k\Delta t} F(Y_k^{N,M}(\omega)) ds + P_N O_{m\Delta t}(\omega). \tag{31}
$$

and by using (16), we can estimate

$$
\begin{aligned}
&\|Y_m^N - Y_m^{N,M}\|_V = \Big\| \sum_{k=0}^{m-1} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t-k\Delta t}(F(X_{k\Delta t}^N(\omega)) - F(Y_k^{N,M}(\omega))) ds \Big\|_V \\
&\leq L \sum_{k=0}^{m-1} \int_{k\Delta t}^{(k+1)\Delta t} (m\Delta t - k\Delta t)^{-\alpha} \Big\| X_{k\Delta t}^N(\omega) - Y_k^{N,M}(\omega) \Big\|_V \Big(1 + \Big\| X_{k\Delta t}^N(\omega) \Big\|_V^p + \Big\| Y_k^{N,M}(\omega) \Big\|_V^p \Big) ds \\
&\leq C(\omega) \sum_{k=0}^{m-1} \Delta t (m\Delta t - k\Delta t)^{-\alpha} \Big\| X_{k\Delta t}^N(\omega) - Y_k^{N,M}(\omega) \Big\|_V.
\end{aligned}
\tag{32}
$$

From (30) with (32), we get

$$
\begin{aligned}
\|X_{m\Delta t}^N(\omega) - Y_m^{N,M}(\omega)\|_V &\leq C\Big(R(\omega), \theta, T, L\Big)(\Delta t)^\theta \\
&\quad + C(\omega) \sum_{k=0}^{m-1} \|X_{k\Delta t}^N(\omega) - Y_k^{N,M}(\omega)\|_V.
\end{aligned}
\tag{33}
$$

Thus

$$
\begin{aligned}
\|X_{m\Delta t}^N(\omega) - Y_m^{N,M}(\omega)\|_V &\leq C\Big(R(\omega), \theta, T, L\Big)(\Delta t)^\theta \\
&\qquad + C(\omega) \sum_{k=0}^{m-1} \|X_{k\Delta t}^N(\omega) - Y_k^{N,M}(\omega)\|_V.
\end{aligned}
\tag{34}
$$

By the discrete Gronwall Lemma we finally conclude

$$\|X^N_{m\Delta t}(\omega) - Y^{N,M}_m(\omega)\|_V \leq C\Big(R(\omega), \theta, T, L\Big)(\Delta t)^\theta.$$

□

3.4 Main result – Full Discretization

Combining Theorem 3 for the time discretization and Theorem 1 for the spatial discretization, yields the following result on the full discretization

Theorem 6. *Let Assumptions 1-5 be true. Let* $X : [0,T] \times \Omega \to V$ *be the solution of the SPDE (13) and* $Y^{N,M}_m : \Omega \to V$, $m \in \{0, 1, ..., M\}, M, N \in \mathbb{N}$ *the numerical solution given by (15).*

Then there exists a finite random variable $C : \Omega \to [0, \infty)$ *such that*

$$\|X_{m\Delta t}(\omega) - Y^{N,M}_m(\omega)\|_V \leq C(\omega)\left(N^{-\gamma} + (\Delta t)^\theta\right) \tag{35}$$

for all $m \in \{0, 1, ..., M\}$ *and every* $M, N \in \mathbb{N}$.

For simplicity of presentation we supposed in our main result that both the full discretization (16) and the Galerkin approximation (15) to be uniformly bounded.

Following the proofs, it is easy to verify that it is sufficient to assume only one of those assumptions. If for instance, (16) for the full discretization fails to be true, we can bound every occurrence of $\|Y^{N,M}_m\|^p_V$ by the bounded $\|X^N_m\|_V$ and the error $e(t) = \sup_{m,N,M} \|Y^{N,M}_m - X^N_m\|_V$. If we now assume that $e(t) \leq 1$, then we can proceed as in the proofs of Theorem 6. As we verify in our results that for the error $e(t) \leq C(\omega)(\Delta t)^\theta$, we can conclude finally, that as long as $C(\omega)(\Delta t)^\theta \leq 1$ our first guess on e was true, and the result holds. The case when (15) fails, is verified in exactly the same way. We finally conclude:

Theorem 7. *Let Assumptions 1-5 be true, but only one bound of either (16) or (11) is true. Let* $X : [0,T] \times \Omega \to V$ *be the solution of the SPDE (13) and* $Y^{N,M}_m : \Omega \to V$, $m \in \{0, 1, ..., M\}, M, N \in \mathbb{N}$ *the numerical solution given by (15).*

Then there exists a finite random variable $C : \Omega \to [0, \infty)$ *such that the error estimate (35) holds provided* $0 < \Delta t < C(\omega)^{-\theta}$.

4 Numerical results

In this section we consider the numerical solution of stochastic equation by the method given in (15). Let $V = W = C([0, \pi])$ be the $\mathbb{R}$-Banach space of continuous functions from $[0, \pi]$ to $\mathbb{R}$ equipped with the norm $\|v\|_V = \|v\|_W := \sup_{x\in[0,\pi]} |v(x)|$ for every $v \in V = W$, where $|.|$ is the absolute value of a real number. Moreover, consider as orthonormal L^2-basis the smooth eigenfunctions

$$e_k : [0, \pi] \to \mathbb{R}, \quad e_k(x) = \sqrt{2/\pi}\sin(kx), \qquad \text{for every } x \in (0, \pi).$$

Denote the Laplacian with Dirichlet boundary conditions on $[0, \pi]$ by A, such that $Ae_k = -k^2 e_k$. Moreover, define the operators P_N as the L^2-orthogonal projections onto the span of the first N eigenfunctions e_k.

We define the mapping $S : [0, T] \to L(V)$ by

$$(S_t)v(x) = \sum_{i\in\mathbb{N}} e^{-\lambda_i t} \int e_i(s)v(s)ds \cdot e_i(x), \tag{36}$$

where $\lambda_i = -i^2$. It is well known that A generates the analytic semigroup $(S_t)_{t\geq 0}$ on V. See [16]. From Lemma 4.1 in [3] and Lemma 1 in [4] we recall that (3) is satisfied for $\gamma \in (0, \frac{3}{2})$ and $\alpha = 0$. Moreover, from [16] we know that (4) is satisfied for any $\theta \in (0, 1)$.

Consider for the nonlinearity the Nemytskii operator $F : V \to V$ given by $(F(v))(x) = f(v(x))$ for every $x \in [0, \pi]$ and every $v \in V$, where $f : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is given by

$$f(x, y) = 5\frac{1 - y}{1 + y^2}. \tag{37}$$

This generates a bounded and globally Lipschitz nonlinearity. Thus Assumption 2 is true.

Assume that the OU-process $O : [0, T] \times \Omega \to V$ is as defined in the example in Section 2.1. The covariance operator Q is given as a convolution operator

$$< Qe_k, e_l >= \int_0^\pi \int_0^\pi e_k(x)e_l(y)q(x - y)dydx. \tag{38}$$

We obtain our numerical result with two kernels

$$q_1(x - y) = \frac{1}{h} \max\{0, 1 - \frac{1}{h^2}|x - y|\} \tag{39}$$

and

$$q_2(x - y) = \max\{0, 1 - \frac{1}{h}|x - y|\}. \tag{40}$$

In Figures 1, 2 we plotted the Covariance Matrix for $h = 0.1$ and $h = 0.01$ with kernel (39) and kernel (40).

By some numerical calculations we can show that the condition on Q from (9) is satisfied for any $\rho \in (0, \frac{1}{2})$, as we can calculate explicitly the Fourier-coefficients of $(x, y) \mapsto q(x - y)$ and check for summability.

For simplicity fix the smooth deterministic initial condition

$$\xi(x) = \frac{\sin x}{\sqrt{2}} + \frac{3\sqrt{2}}{5}\sin(3x), \qquad \text{for all } x \in [0, \pi].$$

The stochastic equation (1) now reads as

$$dX_t = \Big[\frac{\partial^2}{\partial x^2}X_t + 5\frac{1 - X(t)}{1 + X(t)^2}\Big]dt + dW_t, \quad X_0(x) = \frac{\sin x}{\sqrt{2}} + \frac{3\sqrt{2}}{5}\sin(3x), \tag{41}$$

with Dirichlet boundary conditions $X_t(0) = X_t(\pi) = 0$ for $t \in [0, 1]$.

The finite dimensional SDE (12) reduces to

$$dX_t^N = \Big[\frac{\partial^2}{\partial x^2}X_t^N + 5P_N\frac{1 - X_t^N}{1 + (X_t^N)^2}\Big]dt + dP_N W_t, \;\; X_0^N(x) = \frac{\sin x}{\sqrt{2}} + \frac{3\sqrt{2}}{5}\sin(3x), \tag{42}$$

with $X_t^N(0) = X_t^N(\pi) = 0$ for $t \in [0, 1]$ and $x \in [0, \pi]$, and all $N \in \mathbb{N}$.

Now in our simple example we can verify rigorously that the numerical data is uniformly bounded. For this recall that $\|F(X_s^N(\omega))\|_V \le C$ for some uniform constant $C > 0$. Therefore we derive

$$\begin{aligned}\|X_t^N(\omega)\|_V =& \Big\| \int_0^t P_N S_{t-s} F(X_s^N(\omega)) ds + P_N(O_t(\omega)) \Big\|_V \\ \le& \int_0^t \|P_N S_{t-s}\|_{L(V,V)} \|F(X_s^N(\omega))\|_V ds + \|P_N(O_t(\omega))\|_V \\ \le& Ct + C,\end{aligned} \tag{43}$$

and thus we conclude

$$\sup_{N\in\mathbb{N}} \sup_{0\le s\le T} \|X_s^N(\omega)\|_V < \infty. \tag{44}$$

In a similar way

$$\begin{aligned}\|Y_m^{N,M}(\omega)\|_V =& \Big\| \sum_{k=0}^{m-1} \int_{k\Delta t}^{(k+1)\Delta t} P_N S_{m\Delta t - k\Delta t} F(Y_k^{N,M}(\omega)) ds + P_N(O_{m\Delta t}(\omega)) \Big\|_V \\ \le& \sum_{k=0}^{m-1} \Delta t \|P_N S_{m\Delta t-k\Delta t}\|_{L(V,V)} \big\|F(Y_k^{N,M}(\omega))\big\|_V ds + \big\|P_N(O_{m\Delta t}(\omega))\big\|_V \\ \le& C \sum_{k=0}^{m-1} \Delta t + C.\end{aligned} \tag{45}$$

Again we conclude

$$\sup_{N,M\in\mathbb{N}} \sup_{0\le m\le M} \|Y_m^{N,M}(\omega)\|_V < \infty. \tag{46}$$

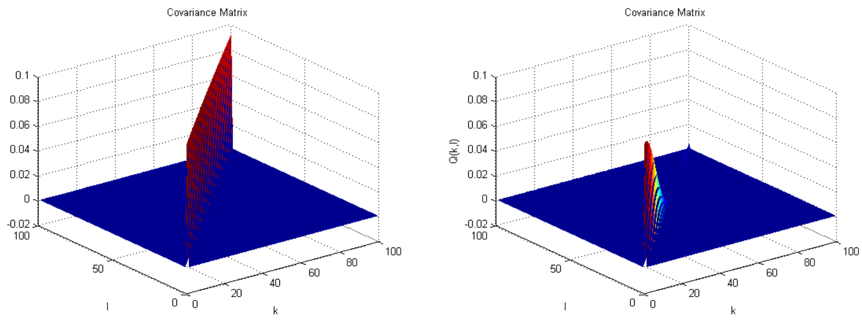
Theorem 6 yields the existence of a unique solution $X : [0,\pi] \times \Omega \to C^0([0,\pi])$ of the SPDE (41) such that

$$\sup_{0\le x\le\pi} |X_{m\Delta t}(\omega, x) - Y_m^{N,M}(\omega, x)| \le C(\omega) \left(N^{-\gamma} + (\Delta t)^\theta\right) \tag{47}$$

for $m = 1, ..., M,\quad M = \frac{1}{\Delta t}$, such that $\gamma \in (0, \frac{1}{2})$, $\theta \in (0, \frac{1}{4})$.
Let us now explain briefly how we implement our numerical results. The main part is generating the Brownian motions $X = (X_1, X_2, \cdots, X_N)$ that are correlated such that $X \sim N(0, \Sigma)$, which $Cov(X_i, X_j) = \Sigma_{ij}$. For this assume C is a $n \times m$ Matrix and let $Z = (Z_1, \cdots, Z_N)^T$, with $Z_i \sim N(0, 1)$, for $i = 1, \cdots, N$. Then obviously $C^T Z \sim N(0, C^T C)$. Therefore our aim clearly reduces to finding C such that $C^T C = \Sigma$, which can for instance be achieved by Cholesky.

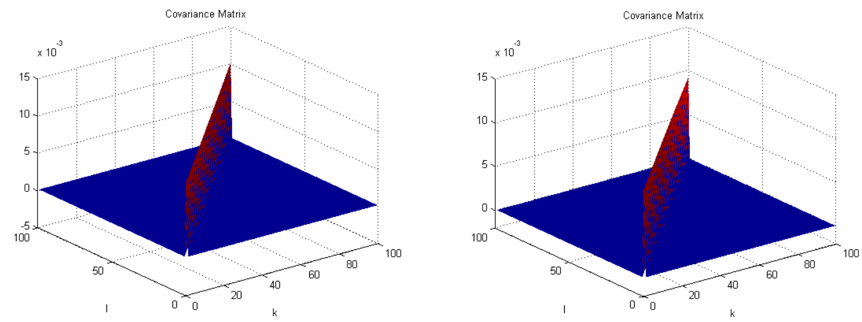
By using $\Delta t = \frac{T}{N^2}$, the solutions $X_t^N(\omega, x)$ of the finite dimensional SODEs (42) converge uniformly in $t \in [0, 1]$ and $x \in [0, \pi]$ to the solution $X_t(\omega, x)$ of the stochastic evolution equation (41) with the rate $\frac{1}{2}$, as N goes to infinity for all $\omega \in \Omega$. In Figure 3 the path-wise approximation error

$$\sup_{0\le x\le\pi} \sup_{0\le m\le M} |X_{m\Delta t}(\omega, x) - Y_m^{N,M}(\omega, x)| \tag{48}$$



(a) (b)

Figure 1: Covariance Matrix $< Qe_k, e_l >_{k,l}$ for $k, l \in \{1, 2, \cdots, 100\}$, for $h = 0.1$ by (a) kernel (39) and (b) kernel (40)



(a) (b)

Figure 2: Covariance Matrix $< Qe_k, e_l >_{k,l}$ for $k, l \in \{1, 2, \cdots, 100\}$, for $h = 0.01$ (a) kernel (39) and (b) kernel (40)

is plotted against N, for $N \in \{16, 32, \cdots, 256\}$. As a replacement for the true unknown solution, we use a numerical approximation for N sufficiently large.

Figure 3 confirms that, as we expected from Theorem 6, the order of convergence is $\frac{1}{2}$. Obviously, these are only two examples, but all out of a few hundred calculated examples behave similarly. Even their mean seem to behave with the same order of the error. Nevertheless, we did not calculate sufficiently many realizations to estimate the mean satisfactory, nor did we proof in the general setting, that the mean converges.

Finally, as an example in Figures 4, $X_t(\omega)$, are plotted for $t \in [0, T]$ for $T \in \{\frac{3}{200}, 0.2, 1\}$, for $h = 0.1$, with convolution operator (38) with kernel (39) and (40).

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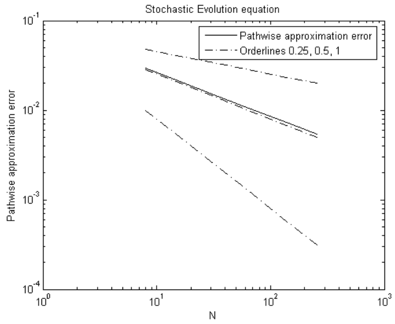
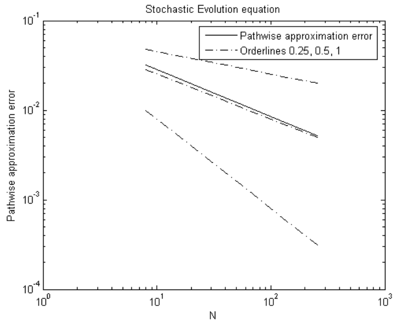
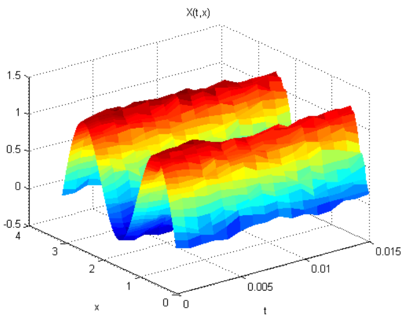


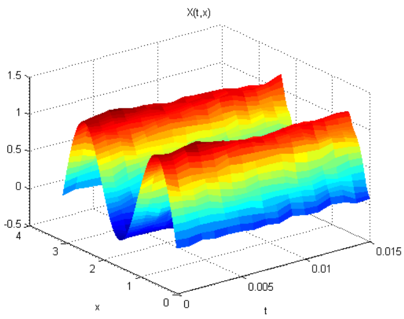
(a)

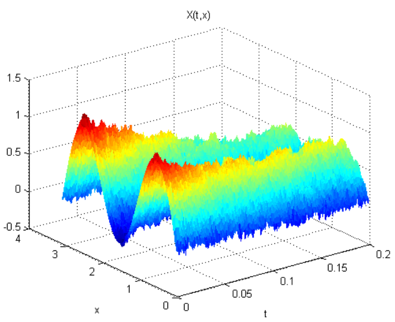


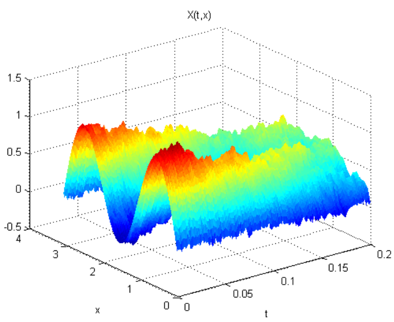
(b)

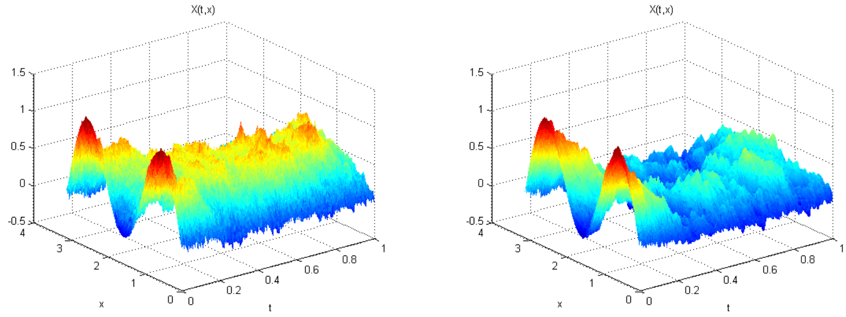
Figure 3: Pathwise approximation error (48) against N for $N \in \{16, 32, ..., 256\}$ with convolution operator with kernel (39) for (a) $h = 0.1$ and (b) $h = 0.01$, for one random $\omega \in \Omega$.











(a) (b)

Figure 4: $X_t(\omega, x)$, $x \in [0, \pi]$, $t \in (0, T)$ for $T \in \{3/200, 0.2, 1\}$, given by (41) for $h = 0.1$ with the covariance operator by (a) kernel (39) and (b) kernel (40), for one random $\omega \in \Omega$.

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