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Biproportional Matrix Scaling and the Iterative Proportional Fitting procedure

Dedicato alla memoria di Bruno Simeone (1945–2010)

Friedrich Pukelsheim

Abstract Convergence of the Iterative Proportional Fitting procedure is analyzed. The input comprises a nonnegative weight matrix, and positive target marginals for rows and columns. The output sought is what is called the biproportional fit, a scaling of the input weight matrix by means of row and column divisors so as to equate row and column sums to target marginals. The procedure alternates between the fitting of rows, and the fitting of columns. We monitor progress with an L_1-error function measuring the distance between current row and column sums and target row and column marginals. The procedure converges to the biproportional fit if and only if the L_1-error tends zero. In case of non-convergence the procedure appears to oscillate between two accumulation points. The oscillation result is contingent on the "IPF conjecture" that row and column divisors are always convergent. The conjecture is established in the specific case when the even-step subsequence admits an accumulation point that is connected, but remains open in general.

Keywords Alternating scaling algorithm · Biproportional fitting · Entropy · Matrix scaling · RAS procedure

AMS 2010 subject classification: 62P25, 62H17

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1 Introduction

We present a novel, L_1-based analysis of the Iterative Proportional Fitting (IPF) procedure. The IPF procedure is an algorithm for scaling rows and columns of an input $k \times \ell$ weight matrix $A = ((a_{ij}))$ so that the output matrix $B = ((b_{ij}))$ achieves row sums equal to a prespecified vector of row marginals, $r = (r_1, \ldots, r_k)$, and column sums equal to a prespecified vector of column marginals, $c = (c_1, \ldots, c_\ell)$. All weights are assumed nonnegative, $a_{ij} \geq 0$, with at least one entry in each row and column of A being positive. All marginals are taken to be positive, $r_i > 0$ and $c_j > 0$.

The problem has a continuous variant, the biproportional fitting problem, and a discrete variant, the biproportional apportionment problem. In the continuous variant, the entries of the output matrix B are nonnegative real numbers, $b_{ij} \in [0, \infty)$. The output B is called a biproportional fit of the weight matrix A to the target marginals r and c. The IPF procedure iteratively calculates scaled matrices $A(t) = \big((a_{ij}(t))\big)$, where for odd steps row sums are matching, $a_{i+}(t+1) = r_i$ for all $i \leq k$, while for even steps column sums match, $a_{+j}(t+2) = c_j$ for all $j \leq \ell$. If a biproportional fit B exists, the sequence of scaled matrices $A(t)$, $t \geq 0$, converges to B.

In the discrete problem variant the entries of B are restricted to be nonnegative integers, $b_{ij} \in \{0, 1, 2, \ldots\}$. Then the output matrix B is called a biproportional apportionment, for the weight matrix A and the target marginals r and c. The procedure to solve the discrete problem is the Alternating Scaling (AS) algorithm. It produces matrices $A(t)$ with entries $a_{ij}(t)$ both, scaled and rounded. There are (rare) instances when a biproportional apportionment B exists while the AS algorithm stalls and fails to converge to it. An example is given by Gaffke and Pukelsheim (2008b, page 157).

Our research arose from the desire to better understand the interplay between the continuous IPF procedure, and the discrete AS algorithm. The present paper focuses on the continuous fitting problem. Yet our major tool, the L_1-error function

$$f\big(A(t)\big) = \sum_{i \leq k} \Big| a_{i+}(t) - r_i \Big| + \sum_{j \leq \ell} \Big| a_{+j}(t) - c_j \Big|,$$

is borrowed from Balinski and Demange's (1989a, 1989b) inquiry into the discrete apportionment problem. In the discrete case the error function is quite suggestive, simply counting how many units are wrongly allocated in step t. For the continuous problem the L_1-error is, at first glance, just one out of many ways to assess lack of fit. At second glance it is a most appropriate way, as this paper endeavors to show.

1.1 The literature on biproportional fitting

The continuous biproportional fitting problem is the senior member of the two problem families. It has created an enormous body of literature of which we review only the papers that influenced the present research. The term *IPF procedure* prevails in Statistics, see Fienberg and Meyer (2006), or Speed (2005). Some Statisticians speak of *matrix raking*, such as Fagan and Greenberg (1987). In Operations Research and Econometrics the label *RAS method* is popular, pointing to a (diagonal) matrix R of row multipliers, the weight matrix A, and a (diagonal) matrix S of column multipliers, as mentioned already by Bacharach (1965, 1970). Computer scientists prefer the term *matrix scaling*, as in Rote and Zachariasen (2007).

Deming and Stephan (1940) are first to popularize the IPF procedure though there are earlier papers on the subject, see Fienberg and Meyer (2006). Deming and Stephan (1940, page 440) recommend terminating iterations when *the table reproduces itself*. This closeness is what is measured by the L_1-error function $f\big(A(t)\big)$, see the remarks leading to our Lemma 1. While successfully advocating the merits of the algorithm, Deming and Stephan were somewhat led astray in its analysis, as communicated by Stephan (1942).

Brown (1959) proposes a convergence proof which Ireland and Kullback (1968) criticize to lack rigor. The latter authors establish convergence by relating the IPF procedure to the minimum entropy solution. Csiszár (1975, page 155) notes that their argument is flawed, and that the generalization to measure-spaces by Kullback (1968) suffers from a similar deficiency. Csiszár (1975) salvages the entropy approach, and Rüschendorf (1995) extends it to general measure-spaces. Rüschendorf and Thomsen (1993, 1997) rectify a technical detail that escaped Csiszár's (1975) attention.

The ultimate arguments of Ireland and Kullback (1968, eqs. (4·32) and (4·33)) substitute convergence of entropy by convergence in L_1, referring to a result of Kullback (1966). Also Bregman (1967) starts out with entropy, and then uses the L_1-error function. Here we dispose of the entropy detour, and use L_1 from start to finish. Ireland and Kullback (1968, page 184) prove that the entropy criterion decreases monotonically, as does the likelihood function of Bishop, Fienberg and Holland (1975, page 86), and the L_1-error function, see Bregman (1967, page 197). Marshall and Olkin (1968) and Macgill (1977) minimize a quadratic objective function. Pretzel (1980) uses a geometric matrix-mean and makes do with the arithmetic-geometric-mean inequality, as reviewed after our concluding Theorem 5. The computational complexity of the IPF procedure is investigated by Kalantari, Lari, Ricca and Simeone (2008).

The question when a biproportional fit exists is discussed by Brualdi, Parter and Schneider (1966), Schneider (1990), and Brown, Chase and Pittenger (1993). Many of them use network and graph theory by viewing the issue as a transportation problem, see the review of Pukelsheim, Ricca, Simeone, Scozzari and Serafini (2012).

Fienberg (1970) opens up a different route by embedding the IPF procedure into the geometry of the manifold of constant interaction in a $(k\ell-1)$-dimensional simplex of reference. The author works with the assumption that all input weights are positive, $a_{ij} > 0$. He points out (page 915) that the extension to problems involving zero weights is *quite complex*, which is attested to by much of the literature. Ireland and Kullback's (1968, page 182) plea of assuming positive weights in order *to simplify the argument* is a friendly understatement, unless it is meant to be the utter truth.

Yet another approach, staying as close to calculus as possible, is due to Bacharach (1965, 1970), and Sinkhorn (1964, 1966, 1967, 1972, 1974) and Sinkhorn and Knopp (1967). Much of the present paper is owed to the work of Bacharach and Sinkhorn.

Michael Owen Leslie Bacharach (b. 1936, d. 2002) was an Oxford econometrician. In 1965 he earned a PhD degree in Mathematics from Cambridge. His thesis was published as Bacharach (1965), and became Section 4 of Bacharach (1970). Richard Dennis Sinkhorn (b. 1934, d. 1995) received his Mathematics PhD in 1962 from the University of Wisconsin–Madison, with a thesis entitled *On Two Problems Concerning Doubly Stochastic Matrices.* Throughout his career he served as a Mathematics professor with the University of Houston. Though contemporaries, neither of the two ever quoted the other.

1.2 The literature on biproportional apportionment

The discrete biproportional apportionment problem is the junior problem family, first put forward by Balinski and Demange (1989a, 1989b), see also Balinski and Rachev (1997) and Simeone and Pukelsheim (2006). The operation of rounding scaled quantities to integers sounds most attractive for the statistical analysis of frequency tables, as noted by Wainer (1998) and Pukelsheim (1998). It disposes of any disclaimer that the adjusted figures are *rounded off, hence when summed may occasionally disagree a unit or so*, as warned in Table I of Deming and Stephan (1940, page 433). When calculating percentages, as in Table 3.6-4 of Bishop, Fienberg and Holland (1975, page 99), the method finishes off with 100 percent and does not stop short with 99 percent. Yet Balinski's motivation was not contingency table analysis in statistics, but proportional representation systems for parliamentary elections.

The task of allocating seats of a parliamentary body to political parties does not tolerate any disclaimer excusing residual rounding errors. Methods must account for each seat. This is achieved by biproportional methods. In 2003, the Swiss Canton of Zurich adopted a doubly proportional system, the biproportional divisor method with standard rounding, see Pukelsheim and Schuhmacher (2004, 2011), and Balinski and Pukelsheim (2006). The method may be attractive also for other countries as investigated by Pennisi (2006) for Italy, Zachariassen and Zachariasen (2006) for the Farœ Islands, Ramírez, Pukelsheim, Palomares and Martínez (2008) for Spain, and Oelbermann and Pukelsheim (2011) for the European Union.

When I had the privilege of advising the Zurich politicians on the amendment of the electoral law, I felt it inappropriate to present double proportionality as a method that minimizes entropy, or that is justified through differential geometry of smooth manifolds in high-dimensional simplexes. The procedure simply does what proportionality is about: Scale and round! Scaling within electoral districts (rows) achieves proportionality among the parties campaigning in that district. Scaling within parties (columns) secures district lists of any party to be handled proportionally. The final rounding step is inevitable, since deputies are counted in whole numbers and not measured in fractions.

That biproportional apportionment also won administrative support is a victory of the IPF procedure. Its discrete sibling, the AS algorithm, enables officials to calculate district divisors and party divisors. Once suitable divisors are publicized any voter can easily double-check the outcome. She or he only needs to take the vote count of the party of their choice in the district where they live, divide it by the respective district and party divisors, and round the result to the nearest seat number. A computer program for carrying out the apportionment is provided at `www.uni-augsburg.de/bazi`, see Pukelsheim (2004), Joas (2005), Maier (2009). The user may choose to run the AS algorithm, the Tie-and-Transfer (TT) algorithm of Balinski and Demange (1989b), or various hybrid combinations. The performance of these algorithms is studied by Maier, Zachariassen and Zachariasen (2010).

In the electoral application the entries a_{ij} in the weight matrix A signify vote counts, and the occurrence of zero counts is unavoidable. When a party j does not campaign in a district i it enters final evaluations with $a_{ij} = 0$. Zero weights must be properly dealt with, even if the labor entailed becomes quite complex. It is no longer appropriate *to simplify the argument* by assuming all weights to be positive.

1.3 Section overview

A brief overview of the paper is as follows. Section 2 investigates biproportional scalings of a given weight matrix A. The additional requirement, of matching prespecified row marginals r and column marginals c, is treated in Sections 3 and 4.

A scaling of a matrix A as in Section 2 reweighs rows and columns moderately enough to not totally annihilate any row nor column. If two scalings share the same row and column sums, then they coincide (Theorem 1). A scaling is called direct when the limits that are allowed by definition become superfluous. In such a case the input matrix A and the output matrix B decompose accordingly (Lemma 1), see Balinski and Demange (1989a), Gietl (2009). Theorem 2 puts forward five conditions to check for directness. Though not needed in the sequel, Theorems 3 and 4 further explore the structural properties of matrix scalings. The proof of Theorem 4 refers to the linear space spanned by cycle matrices, as in Gaffke and Pukelsheim (2008a, page 178).

Section 3 turns to the IPF procedure. The procedure uses biproportional scalings in order to fit a $k \times \ell$ weight matrix A to prespecified row marginals r and column marginals c. In each step t a scaled weight matrices $A(t)$ is produced that either has matching rows, or matching columns. The goodness-of-fit of $A(t)$ is measured by the L_1-error function mentioned in the beginning. Lemma 2 ascertains that the L_1-error is nonincreasing, and that every row subset I bounds it from below according to

$$f\big(A(t)\big) \geq r_I - c_{J_A(I)} + c_{J_A(I)'} - r_{I'},$$

where r_I and $c_{J_A(I)}$ are partial sums of row and column marginals, a prime indicating set complements. Lemma 3 recalls an intriguing set of interlacing inequalities among row and column divisors that is due to Bacharach (1970).

Section 4 aims to show that the L_1-error achieves the lower bound displayed above. For the case of nonconvergence Jiroušek and Vomlel (1995) and Vomlel (2004) report that the IPF procedure tends to cycle. Our study of the set of accumulation points is contingent on what we call the "IPF conjecture", namely that the incremental divisors of any row constitute a convergent sequence. Lemma 4 proves that the IPF conjecture holds true when the IPF sequence $A(t)$, $t \geq 0$, admits an accumulation point that is connected. Then the L_1-error achieves the lower bound from above (Lemma 5). Finally Theorem 5 assembles five necessary and sufficient conditions for the IPF sequence to converge to the biproportional fit sought. Two proofs are provided. The first proof relies on the IPF conjecture and is short and elegant. Without reference to the IPF conjecture the second proof becomes a bit more cumbersome.

1.4 Notation

A plus-sign is used as a subscript to indicate summation over the index that otherwise appears in its place, as in $r_+ = \sum_{i \leq k} r_i$, $c_+ = \sum_{j \leq \ell} c_j$, or $a_{++} = \sum_{i \leq k} \sum_{j \leq \ell} a_{ij}$. Partial sums are written with the range of summation in place of the index, $r_I = \sum_{i \in I} r_i$, $c_J = \sum_{j \in J} c_j$, or $a_{I \times J} = \sum_{i \in I} \sum_{j \in J} a_{ij}$. A prime signifies the complement of a set, $I' = \{1, \ldots, k\} \setminus I$. In a matrix A with nonnegative entries, $a_{ij} \geq 0$, the columns connected with a row subset I are assembled in the subset $J_A(I) = \{\, j \leq \ell \mid a_{ij} > 0 \text{ for some } i \in I \,\}$.

2 Biproportional scalings

Let $A = ((a_{ij}))$ be a given $k \times \ell$ *weight matrix*, that is, A is assumed to have nonnegative entries and no zero row nor zero column. We concentrate on true matrix problems, $k \geq 2$ and $\ell \geq 2$. Whether row $i \leq k$ or column $j \leq \ell$ does not vanish is conveniently read off from their component sums, $a_{i+} > 0$ and $a_{+j} > 0$. Another weight matrix B is said to *preserve the zeros of* A when all zeros of A are zeros also of B, $a_{ij} = 0 \Rightarrow b_{ij} = 0$. Two matrices A and B *have the same zeros* when $a_{ij} = 0 \Leftrightarrow b_{ij} = 0$.

Definition A $k \times \ell$ matrix $B = ((b_{ij}))$ is defined to be a *biproportional scaling of* A when for all rows $i \leq k$ and for all columns $j \leq \ell$ there exist sequences of positive *row divisors* $\rho_i(1), \rho_i(2), \ldots$ and of positive *column divisors* $\sigma_j(1), \sigma_j(2), \ldots$ satisfying

$$b_{ij} = \lim_{n \to \infty} \frac{a_{ij}}{\rho_i(n)\sigma_j(n)}, \quad b_{i+} > 0, \quad b_{+j} > 0.$$

A biproportional scaling B is termed *direct* when its associated divisor sequences can be chosen to be constant, that is, for all rows $i \leq k$ and for all columns $j \leq \ell$ there are positive divisors μ_i and ν_j such that $b_{ij} = a_{ij}/(\mu_i \nu_j)$. □

A cell (i, j) is said to be *fading* when $a_{ij} > 0 = b_{ij}$. Fading cells cannot arise with direct scalings. If they arise with general scalings, then the denominators $\rho_i(n)\sigma_j(n)$ diverge to infinity. Their speed of growth is damped by the requirement that row and column sums of B stay positive. Otherwise the divisors could always be exaggerated so as to annihilate a whole row, or a whole column. No precautions have to be taken at the other end of the range where the denominators get close to zero. They can do so only if the numerator vanishes, $a_{ij} = 0$, and then the denominators' speed of convergence to zero, or even lack of convergence, is irrelevant.

The side condition that biproportional scalings have positive row and column sums has consequences for their multitude.

Theorem 1 (Uniqueness) *If two biproportional scalings B and C of a weight matrix A share the same row and column sums, $b_{i+} = c_{i+}$ for all rows $i \leq k$ and $b_{+j} = c_{+j}$ for all columns $j \leq \ell$, then they coincide, $B = C$.*

Proof The proof is by contraposition. Assuming the two scalings to be distinct, $B \neq C$, their difference $B - C$ is nonzero, but has row and column sums vanishing. We construct a cycle of cells

$$(i_1, j_1),\ (i_2, j_1),\ (i_2, j_2),\ (i_3, j_2),\ \ldots,\ (i_{q-1}, j_{q-1}),\ (i_q, j_{q-1}),\ (i_q, j_q),\ (i_1, j_q) \quad \text{[CC]}$$

along which the entries in $B - C$ are alternately positive or negative. First we assemble a "long list" of cells (i_1, j_1), (i_2, j_1), (i_2, j_2), $\ldots$, (i_Q, j_Q), (i_1, j_Q), as follows. We start with a cell (i_1, j_1) where $b_{i_1 j_1} > c_{i_1 j_1}$. In column j_1 there is a cell (i_2, j_1) with $b_{i_2 j_1} < c_{i_2 j_1}$. Next we search in row i_2 a column j_2 where $b_{i_2 j_2} > c_{i_2 j_2}$. Then we look for a row i_3 such that $b_{i_3 j_2} > c_{i_3 j_2}$. The long list terminates when encountering a row i_Q already listed, that is, when for some $P < Q$ we find $i_Q = i_P$. The initial $P - 1$ cells are discarded, and the remaining "short list" is relabeled as in [CC].

A *cyclic ratio* in a matrix is a ratio having the entries along a given cell cycle alternately appear in the denominator and in the numerator. Since $a_{ij} = 0$ implies $b_{ij} = c_{ij} = 0$, the cell cycle [CC] touches only upon positive entries of the weight matrix A. Let $\rho_i(n)$ and $\sigma_j(n)$ denote the divisor sequences for B, and $\mu_i(n)$ and $\nu_j(n)$ for C. As biproportionality preserves cyclic ratios, the cyclic ratios in A, B, and C are seen to be equal,

$$\prod_{p\le q} \frac{a_{i_{p+1}j_p}}{a_{i_pj_p}} = \prod_{p\le q} \frac{\frac{a_{i_{p+1}j_p}}{\rho_{i_{p+1}}(n)\sigma_{j_p}(n)}}{\frac{a_{i_pj_p}}{\rho_{i_p}(n)\sigma_{j_p}(n)}} = \prod_{p\le q} \frac{b_{i_{p+1}j_p}}{b_{i_pj_p}} = \prod_{p\le q} \frac{\frac{a_{i_{p+1}j_p}}{\mu_{i_{p+1}}(n)\nu_{j_p}(n)}}{\frac{a_{i_pj_p}}{\mu_{i_p}(n)\nu_{j_p}(n)}} = \prod_{p\le q} \frac{c_{i_{p+1}j_p}}{c_{i_pj_p}},$$

where $i_{q+1} = i_1$. The first and third equation signs are obvious. The second equality involves a passage to the limit as n tends to ∞ and is justified since, by construction, the limiting denominator is positive, $b_{i_pj_p} > c_{i_pj_p} \ge 0$. As the left hand side is positive, the numerator must be positive, too, $b_{i_{p+1}j_p} > 0$. A similar argument establishes the last equality, with the roles of numerator and denominator interchanged.

However, the construction of the cycle [CC] precludes equality,

$$\prod_{p\le q} \frac{b_{i_{p+1}j_p}}{b_{i_pj_p}} < \prod_{p\le q} \frac{c_{i_{p+1}j_p}}{c_{i_pj_p}}.$$

Hence the assumption $B \ne C$ is untenable and uniqueness obtains, $B = C$. □

Directness of a biproportional scaling transpires to be closely related to the notion of connectedness. A nonzero matrix C is said to be *connected* when it is not disconnected. A nonzero matrix D is called *disconnected* when a suitable permutation of rows and a suitable permutation of columns give rise to a row subset I and a column subset J such that D acquires block structure,

$$D = \begin{array}{c c} & \begin{array}{cc} J & J' \end{array} \\ \begin{array}{c} I \\ I' \end{array} & \begin{pmatrix} D^{(1)} & 0 \\ 0 & D^{(2)} \end{pmatrix} \end{array},$$

where at least one of the subsets I or J is nonempty and proper, $\emptyset \subsetneq I \subsetneq \{1, \dots, k\}$ or $\emptyset \subsetneq J \subsetneq \{1, \dots, \ell\}$. In most applications both subsets are nonempty and proper.

For keeping track of the nonzero entries in a weight matrix A we associate with every row subset $I \subseteq \{1, \dots, k\}$ the *set of columns connected in A with I*,

$$J_A(I) = \left\{ j \le \ell \;\middle|\; a_{ij} > 0 \text{ for some } i \in I \right\}.$$

The complement $J_A(I)'$ embraces the columns j with entries $a_{ij} = 0$ for all $i \in I$. Hence the $I \times J_A(I)'$ submatrix of A vanishes and the sum of its entries is zero, $a_{I\times J_A(I)'} = 0$. The extreme settings provide simple examples. If we choose $I = \{1, \dots, k\}$ then we get $J_A(I) = \{1, \dots, \ell\}$, since no row nor column of A vanishes. If $I = \emptyset$ then $J_A(I) = \emptyset$.

When the input weight matrix A decomposes into several connected components, the calculation of any biproportional scaling B decomposes into several separate instances. Hence there is no loss of generality of assuming A to be connected. Yet its associated biproportional scalings may be connected, or disconnected. When B is disconnected its structure has repercussions on the structure of A, as follows.

Lemma 1 (Joint decomposition) *For every connected weight matrix A and for every disconnected biproportional scaling B of A there exists a nonempty and proper subset I of rows, $\emptyset \subsetneq I \subsetneq \{1,\dots,k\}$, such that A acquires blocktriangular structure and B acquires blockdiagonal structure,*

$$A = \begin{array}{c c} & \begin{array}{cc} J_A(I) & J_A(I)' \end{array} \\ \begin{array}{c} I \\ I' \end{array} & \begin{pmatrix} A^{(1)} & 0 \\ A^{(2,1)} & A^{(2)} \end{pmatrix} \end{array}, \qquad B = \begin{array}{c c} & \begin{array}{cc} J_B(I) & J_B(I)' \end{array} \\ \begin{array}{c} I \\ I' \end{array} & \begin{pmatrix} B^{(1)} & 0 \\ 0 & B^{(2)} \end{pmatrix} \end{array}$$

where the sets of columns connected with I in A or B are equal, $J_A(I) = J_B(I)$.

Proof Without loss of generality we may assume the largest accumulation point of the column divisors to be positive and finite, $\limsup_{n\to\infty} \sigma_{\max}(n) = M \in (0,\infty)$, where $\sigma_{\max}(n) = \max\{\sigma_1(n),\dots,\sigma_\ell(n)\}$. If need be, we would adjust the divisors according to $\widetilde{\rho}_i(n) = \rho_i(n)\sigma_{\max}(n)$ and $\widetilde{\sigma}_j(n) = \sigma_j(n)/\sigma_{\max}(n) \le 1$, and use $M = 1$.

Let an arrow $\to$ indicate a passage to the limit as n tends to infinity. The set I is defined to contain the rows with divisors not degenerating, in the sense of not diverging to infinity,

$$I = \left\{ i \le k \,\middle|\, \rho_i(n) \not\to \infty \right\}, \qquad I' = \left\{ i \le k \,\middle|\, \rho_i(n) \to \infty \right\}.$$

Likewise the columns connected in B with I will turn out to have their divisors not degenerating, but in the sense of not converging to zero,

$$J_B(I) = \left\{ j \le \ell \,\middle|\, \sigma_j(n) \not\to 0 \right\}, \qquad J_B(I)' = \left\{ j \le \ell \,\middle|\, \sigma_j(n) \to 0 \right\}.$$

With A connected and B disconnected there exists a cell (i,j) that is fading, $a_{ij} > 0 = b_{ij}$. This implies $\lim_{n\to\infty} \rho_i(n)\sigma_j(n) = \infty$ and, since column divisors stay bounded by assumption, $\lim_{n\to\infty} \rho_i(n) = \infty$. Hence I' is not empty, nor is $J_B(I')$—which in the end will turn out to coincide with $J_B(I)'$. By definition of $J_B(I')$, the $I' \times J_B(I')'$ block of B is zero.

The columns $j \in J_B(I')$ have their divisors converge to zero, $\lim_{t\to\infty} \sigma_j(n) = 0$. Indeed, there exists a row $i \in I'$ with $b_{ij} > 0$. In this cell we have $\lim_{n\to\infty} \rho_i(n)\sigma_j(n) = a_{ij}/b_{ij} < \infty$. Since the divisors of row $i \in I'$ diverge to infinity, column j has its divisors converge to zero.

By assumption the column divisors satisfy $\sigma_{\max}(n) > M/2$ infinitely often. Thus there exists a column j with divisors fulfilling $\sigma_j(n) > M/2$ again and again and not converging to zero. Hence $J_B(I')'$ is not empty, nor is I.

Now every column $j \in J_B(I')'$ has its divisors not converging to zero, $\limsup_{n\to\infty} \sigma_j(n) > 0$. Indeed, there is a row $i \in I$ with $b_{ij} > 0$. In this cell we have $\lim_{n\to\infty} \rho_i(n) \sigma_j(n) = a_{ij}/b_{ij} > 0$. Since column j admits a divisor subsequence bounded away from zero, the row divisors that go along cannot diverge to infinity.

The $I \times J_B(I')$ top right block of A has $a_{ij} = 0$. Indeed, the case $a_{ij} > 0$ would admit a row divisor subsequence bounded from above, in the presence of column divisors converging to zero. But $a_{ij} > 0$ and $\lim_{n\to\infty} \rho_i(n)\sigma_j(n) = 0$ lead to the contradiction $b_{ij} = \infty$. The top right block of B inherits the zeros of A. The structures of B and A entail $J_B(I')' = J_B(I) = J_A(I)$. □

Given a biproportional scaling B, there are various ways to check for directness.

Theorem 2 (Directness) *For every connected weight matrix A and for every biproportional scaling B of A the following five statements are equivalent:*

(1) *The biproportional scaling B is direct.*

(2) *The matrices A and B have the same zeros.*

(3) *There exists a weight matrix D sharing the same zeros with A and the same row and column sums with B.*

(4) *For every nonempty and proper subset I of rows, $\emptyset \subsetneqq I \subsetneqq \{1,\dots,k\}$, partial row and column sums of B fulfill $\sum_{i\in I} b_{i+} < \sum_{j\in J_A(I)} b_{+j}$.*

(5) *The matrix B is connected.*

Proof (1) $\Rightarrow$ (2). A direct scaling, $b_{ij} = a_{ij}/(\mu_i\nu_j)$, has the same zeros as has A.

(2) $\Rightarrow$ (3). The scaling B, sharing all zeros with A, is of the type asked for in (3).

(3) $\Rightarrow$ (4). For every row subset I we have $a_{I\times J_A(I)'} = 0$, and hence $d_{I\times J_A(I)'} = 0$. If I is nonempty and proper then $d_{I'\times J_A(I)} > 0$, as otherwise D is disconnected and so would be A. We get $\sum_{i\in I} b_{i+} = d_{I\times J_A(I)} < d_{I\times J_A(I)} + d_{I'\times J_A(I)} = \sum_{j\in J_A(I)} b_{+j}$.

(4) $\Rightarrow$ (5). The proof is by contraposition. If B is disconnected, then Lemma 1 provides a nonempty and proper row set I fulfilling $\sum_{i\in I} b_{i+} = b_{I\times J_A(I)} = \sum_{j\in J_A(I)} b_{+j}$.

(5) $\Rightarrow$ (1). Row divisors μ_i and column divisors ν_j for B are constructed in the course of a scanning process. The process is initialized by standardizing the given divisor sequences according to $\widetilde{\rho}_i(n) = \rho_i(n)/\rho_1(n)$ and $\widetilde{\sigma}_j(n) = \rho_1(n)\sigma_j(n)$, thus equipping the first row with constant divisor unity, $\widetilde{\rho}_1(n) = 1 = \mu_1$, $n \geq 1$. Then the process scans all columns j with $b_{1j} > 0$, and sets

$$0 < \nu_j = \frac{a_{1j}}{\mu_1 b_{1j}} = \frac{\lim_{n\to\infty} \widetilde{\rho}_1(n)\widetilde{\sigma}_j(n)}{\lim_{n\to\infty} \widetilde{\rho}_1(n)} = \lim_{n\to\infty} \widetilde{\sigma}_j(n), \qquad \text{whence } b_{1j} = \frac{a_{1j}}{\mu_1\nu_j}.$$

Next all unscanned rows i with $b_{ij} > 0$ for some scanned column j are scanned, setting

$$0 < \mu_i = \frac{a_{ij}}{b_{ij}\nu_j} = \frac{\lim_{n\to\infty} \widetilde{\rho}_i(n)\widetilde{\sigma}_j(n)}{\lim_{n\to\infty} \widetilde{\sigma}_j(n)} = \lim_{n\to\infty} \widetilde{\rho}_i(n), \qquad \text{whence } b_{ij} = \frac{a_{ij}}{\mu_i\nu_j}.$$

Thereafter the process turns to columns again, then rows. In this fashion it keeps enlarging the scanned sets of rows and columns, terminating after at most $k+\ell$ steps. The terminal scanned row set I and column set J enforce a block structure upon B,

$$B = \begin{array}{c c} & \begin{array}{cc} J & J' \end{array} \\ \begin{array}{c} I \\ I' \end{array} & \begin{pmatrix} B^{(1)} & 0 \\ 0 & B^{(2)} \end{pmatrix} \end{array}.$$

Connectedness of B lets the scanned sets be exhaustive, $I = \{1,\dots,k\}$ and $J = \{1,\dots,\ell\}$. All rows and all columns having constant divisors, the scaling is direct. □

Generally a biproportional scaling B may decompose into several connected components. Theorem 3 permutes these components in such a way that A acquires block-triangular structure, and that row and column divisors are grouped by the order with which they diverge to infinity, or converge to zero.

Theorem 3 (Connectedness structure) *For every weight matrix A and for every biproportional scaling B of A, the $K \geq 1$ connected components of B may be permuted in such a way that A acquires blocktriangular structure,*

$$A = \begin{array}{c} \\ I_1 \\ I_2 \\ \vdots \\ I_K \end{array} \begin{array}{c} \begin{array}{cccc} J_1 & J_2 & \cdots & J_K \end{array} \\ \begin{pmatrix} A^{(1)} & 0 & \cdots & 0 \\ A^{(2,1)} & A^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{(K,1)} & A^{(K,2)} & \cdots & A^{(K)} \end{pmatrix} \end{array}, \qquad B = \begin{array}{c} \\ I_1 \\ I_2 \\ \vdots \\ I_K \end{array} \begin{array}{c} \begin{array}{cccc} J_1 & J_2 & \cdots & J_K \end{array} \\ \begin{pmatrix} B^{(1)} & 0 & \cdots & 0 \\ 0 & B^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^{(K)} \end{pmatrix} \end{array}.$$

Moreover, there exist positive scalars $\mu_1, \ldots, \mu_k$ and $\nu_1, \ldots, \nu_\ell$ such that B results from using the divisors $\rho_i(n) = \mu_i n^{m-1}$ for rows $i \in I_m$ and $\sigma_j(n) = \nu_j / n^{m-1}$ for columns $j \in J_m$, for all $n \geq 1$ and for all connected components $m \leq K$. Within every component $m \leq K$, one of the row scalars μ_i, $i \in I_m$, may be standardized to be unity.

Proof In case A is disconnected, we may treat each of its connected components as a separate instance. In case B is connected, $K = 1$, the current assertions reduce to those of Theorem 2.

In case B is disconnected, we recursively apply Lemma 1. Indeed, if block $B^{(1)}$ of Lemma 1 is connected, then we copy it into the present display. If block $B^{(1)}$ of Lemma 1 is disconnected, then we first decompose block $A^{(1)}$ of Lemma 1 into its (that is, those of $A^{(1)}$) connected components, and thereafter operate on each of the evolving instances separately by again applying Lemma 1. This recursive process terminates after finitely many steps, and eventually leaves us with a structure as displayed above.

Moreover, the connected component $B^{(m)}$ is a biproportional scaling of the block $A^{(m)}$. Hence Theorem 2 secures the existence of constant divisors μ_i and ν_j, for all $i \in I_m$ and $j \in J_m$. As row divisors $\rho_i(n)$ diverge of order n^{m-1} and column divisors $\sigma_j(n)$ vanish with the same order, diagonal blocks remain unaffected and lower diagonal blocks are annihilated. □

We adjoin a result, not needed in the sequel, to elucidate the interplay of biproportional scalings and cyclic ratios. Two weight matrices A and B are said to be *biproportionally equivalent* when they are direct biproportional scalings of each other. The *support set* of a weight matrix A is constituted by the cells where the entry is positive, $\operatorname{supp}(A) = \{(i,j) \in \{1, \ldots, k\} \times \{1, \ldots, \ell\} \mid a_{ij} > 0\}$. If A and B are biproportionally equivalent, then they have the same support sets.

Let $S \subseteq \{1, \ldots, k\} \times \{1, \ldots, \ell\}$ denote a subset of cells. A *cell cycle on S* is defined to consist of a sequence of $2q$ cells, as in display [CC] in the proof of Theorem 1, involving $q \geq 2$ distinct rows $i_1, \ldots, i_q$ and distinct columns $j_1, \ldots, j_q$ that satisfy $(i_p, j_p) \in S$ and $(i_{p+1}, j_p) \in S$ for all $p \leq q$. We adopt the convention of always setting $i_{q+1} = i_1$. Two weight matrices A and B are said to be *cyclically equivalent* when they share a common support set, $\operatorname{supp}(A) = \operatorname{supp}(B) = S$ say, and all cell cycles on S fulfill $\prod_{p \leq q} a_{i_{p+1} j_p} / a_{i_p j_p} = \prod_{p \leq q} b_{i_{p+1} j_p} / b_{i_p j_p}$.

Theorem 4 (Equivalence) *Any two weight matrices A and B are biproportionally equivalent if and only if they are cyclically equivalent.*

Proof As in the proof of Theorems 1 the direct part is a one-liner,

$$\prod_{p\leq q}\frac{a_{i_{p+1}j_p}}{a_{i_pj_p}}=\prod_{p\leq q}\frac{\frac{a_{i_{p+1}j_p}}{\mu_{i_{p+1}}\nu_{j_p}}}{\frac{a_{i_pj_p}}{\mu_{i_p}\nu_{j_p}}}=\prod_{p\leq q}\frac{b_{i_{p+1}j_p}}{b_{i_pj_p}}.$$

For the converse part let S denote the support set common to A and B. We need to establish the existence of some positive numbers μ_i and ν_j such that $a_{ij}/b_{ij}=\mu_i\nu_j$. That is, we are looking for solutions $x_i=\log\mu_i$ and $y_j=\log\nu_j$ to the system of linear equations $x_i+y_j=\log(a_{ij}/b_{ij})$ for all $(i,j)\in S$.

Denoting by E_{ij} the $k\times\ell$ Euclidean unit matrix with entry unity in cell (i,j) and zeros elsewhere, we work in the linear space $V=\operatorname{span}\{E_{ij}\mid(i,j)\in S\}$ with inner product $\langle C,D\rangle=\operatorname{trace}C'D$. Consider the subspace

$$L=\Big\{\sum\nolimits_{(i,j)\in S}(x_i+y_j)E_{ij}\;\Big|\;x_1,\dots,x_k,y_1,\dots,y_\ell\in\mathbb{R}\Big\}.$$

It suffices to show that $C=\sum_{(i,j)\in S}\log(a_{ij}/b_{ij})E_{ij}$ lies in L. Equivalently, we verify that C is orthogonal to $L^\perp$. The orthogonal complement $L^\perp$ consists of all matrices D in V having vanishing row and column sums. Indeed, the inner products

$$\Big\langle\sum_{(i,j)\in S}(x_i+y_j)E_{ij},D\Big\rangle=\sum_{(i,j)\in S}(x_i+y_j)d_{ij}=\sum_{i\leq k}x_id_{i+}+\sum_{j\leq\ell}y_jd_{+j}$$

vanish for all x_i and y_j if and only if all $d_{i+}=0$ and $d_{+j}=0$.

For a cell cycle $(i_1,j_1),\dots,(i_q,j_q)$ the *cycle matrix* $D\big((i_1,j_1),\dots,(i_q,j_q)\big)=\sum_{p\leq q}\big(E_{i_pj_p}-E_{i_{p+1}j_p}\big)$ is defined to have entry 1 in cells (i_p,j_p) and entry -1 in cells (i_{p+1},j_p). The cycle matrices from cell cycles in S provide a spanning set for $L^\perp$,

$$L^\perp=\operatorname{span}\Big\{D\big((i_1,j_1),\dots,(i_q,j_q)\big)\;\Big|\;(i_1,j_1),\dots,(i_q,j_q)\text{ cell cycle in }S\Big\}.$$

Evidently the right hand subspace is included in $L^\perp$ since every cycle matrix has all row and column sums equal to zero. Conversely, every nonzero matrix $B\in L^\perp$ can be represented as a linear combination of cycle matrices, as follows. Since B has vanishing row and column sums, we may identify an initial cell cycle in $\operatorname{supp}(B)\subseteq S$ by proceeding just as in the proof of Theorem 1. The initial cell cycle induces a cycle matrix $D(0)$ with $\operatorname{supp}\big(D(0)\big)\subseteq\operatorname{supp}(B)$. We choose some cell (i,j) in the support of $D(0)$ and set $\lambda(0)=b_{ij}/d_{ij}(1)\neq 0$. Now $B(1)=B-\lambda(0)D(0)$ has a support set strictly smaller than that of B, $\operatorname{supp}\big(B(1)\big)\subsetneq\operatorname{supp}(B)$. In case $B(1)$ is a cycle matrix or zero, $B=\lambda(0)D(0)+B(1)$ lies in the right hand span. Otherwise, the reduction processes is applied to $B(1)$, and leads to $B=\lambda(0)D(0)+\lambda(1)D(1)+B(2)$. The reduction process may have to be repeated, but terminates after finitely many steps.

Hence it suffices to show that $C=\sum_{(i,j)\in S}\log(a_{ij}/b_{ij})E_{ij}$ is orthogonal to every cycle matrix $D=\sum_{p\leq q}\big(E_{i_pj_p}-E_{i_{p+1}j_p}\big)$. But this follows from cyclic equivalence,

$$\langle C,D\rangle=\sum_{p\leq q}\Big(c_{i_pj_p}-c_{i_{p+1}j_p}\Big)=\log\prod_{p\leq q}\frac{a_{i_pj_p}}{b_{i_pj_p}}\frac{b_{i_{p+1}j_p}}{a_{i_{p+1}j_p}}=\log 1=0.\qquad\square$$

3 The IPF procedure

Let A be a $k \times \ell$ weight matrix. Furthermore let $r = (r_1, \ldots, r_k)$ and $c = (c_1, \ldots, c_\ell)$ be vectors with positive entries, called *target row marginals* and *target column marginals*.

Definition A $k \times \ell$ nonnegative matrix $B = ((b_{ij}))$ is said to *match the target marginals r and c* when its row sums are equal to r and its column sums are equal to c, that is, $b_{i+} = r_i$ for all $i \leq k$ and $b_{+j} = c_j$ for all $j \leq \ell$. A weight matrix B is called a *biproportional fit of the weight matrix A to the target marginals r and c* when B is a biproportional scaling of A and B matches the target marginals r and c. □

The Iterative Proportional Fitting (IPF) procedure calculates a biproportional fit, if existing, of the weight matrix A to the target marginals r and c. It operates alternately on the rows and columns of A, in that odd steps scale rows to match target row marginals, and even steps scale columns to match target column marginals.

The notation $B(A, r, c)$ for a biproportional fit would exhibit the input more visibly, but is dismissed as too cumbersome. If a biproportional fit exists then it is unique, since by Theorem 1 there is at most one. An existing fit B necessitates equal marginal totals, $r_+ = b_{++} = c_+$. It would seem tempting to demand equality of marginal totals right from the beginning. We do not do so, though, since the IPF procedure may well be run without target marginals sharing the same total, and since different marginal subtotals evolve naturally when IPF subsequences converge to limit matrices that decompose into several connected components.

We initialize the IPF procedure by scaling the given weight matrix A into a matrix $A(0)$ which has column sums equal to target column marginals. The initialization routine uses column divisors $\beta_j(0) = a_{+j}/c_j$ and sets $a_{ij}(0) = a_{ij}/\beta_j(0)$, for all columns $j \leq \ell$ and rows $i \leq k$. This fits columns, $a_{+j}(0) = c_j$, and the sum of the initialized weights becomes $a_{++}(0) = c_+$. Thereafter the procedure advances in pairs of an odd step $t+1$ and an even step $t+2$, for $t = 0, 2, \ldots$:

- Odd steps $t+1$ fit row sums to target row marginals by calculating row divisors $\alpha_i(t+1)$ from the preceding even step t, and scaled weights $a_{ij}(t+1)$:

$$\alpha_i(t+1) = \frac{a_{i+}(t)}{r_i}, \qquad \text{[IPF1]}$$

$$a_{ij}(t+1) = \frac{a_{ij}(t)}{\alpha_i(t+1)}, \qquad \text{[IPF2]}$$

  for all rows $i \leq k$ and for all columns $j \leq \ell$.

- Even steps $t+2$ fit column sums to target column marginals by calculating column divisors $\beta_j(t+2)$ from the preceding odd step $t+1$, and scaled weights $a_{ij}(t+2)$:

$$\beta_j(t+2) = \frac{a_{+j}(t+1)}{c_j}, \qquad \text{[IPF3]}$$

$$a_{ij}(t+2) = \frac{a_{ij}(t+1)}{\beta_j(t+2)}, \qquad \text{[IPF4]}$$

  for all columns $j \leq \ell$ and for all rows $i \leq k$.

Definitions [IPF1] and [IPF3] are reminiscent of likelihood ratios, of fitted distributions relative to target distributions. All divisors stay positive since no row nor column of A is allowed to vanish. When weighted by their corresponding marginal distributions, row and column divisors have means that are ratios of marginal totals,

$$\sum_{i\le k}\alpha_i(t+1)\,\frac{r_i}{r_+}=\frac{a_{++}(t)}{r_+}=\frac{c_+}{r_+},\qquad \sum_{j\le\ell}\beta_j(t+2)\,\frac{c_j}{c_+}=\frac{a_{++}(t+1)}{c_+}=\frac{r_+}{c_+}.$$

It follows that the divisor products, $\alpha_i(t+1)\beta_j(t+2)$, have mean unity relative to the marginal product distribution,

$$\sum_{i\le k}\sum_{j\le\ell}\alpha_i(t+1)\beta_j(t+2)\,\frac{r_i}{r_+}\frac{c_j}{c_+}=\frac{c_+}{r_+}\frac{r_+}{c_+}=1.$$

The mean is unity also relative to the probability distribution $(1/c_+)A(t+2)$. Indeed, with $a_{ij}(t)=\alpha_i(t+1)\beta_j(t+2)a_{ij}(t+2)$ from [IPF2] and [IPF4] we get

$$\frac{1}{c_+}\sum_{i\le k}\sum_{j\le\ell}\alpha_i(t+1)\beta_j(t+2)\,a_{ij}(t+2)=\frac{a_{++}(t)}{c_+}=1.$$

The *incremental row divisors* $\alpha_i(1),\alpha_i(3),\ldots$ from [IPF1] give rise to *cumulative row divisors* ρ_i, and the *incremental column divisors* $\beta_j(2),\beta_j(4),\ldots$ from [IPF3] generate *cumulative column divisors* σ_j, defined for steps $t=0,2,\ldots$ through

$$\alpha_i(1)\,\alpha_i(3)\,\cdots\,\alpha_i(t+1)=\rho_i(t+1)=\rho_i(t+2),$$
$$\beta_j(0)\,\beta_j(2)\,\beta_j(4)\,\cdots\,\beta_j(t+2)=\sigma_j(t+2)=\sigma_j(t+3).$$

Adjoining $\rho_i(0)=1$ and $\sigma_j(0)=\sigma_j(1)=\beta_j(0)$, cumulative divisors are defined for all steps $t\ge 0$. The scaled weights take the form $a_{ij}(t)=a_{ij}/\big(\rho_i(t)\sigma_j(t)\big)$. The *scaled weight matrices* $A(t)=\big(\!\big(a_{ij}(t)\big)\!\big)$, $t\ge 0$, constitute the *IPF sequence*, for the fitting of the weight matrix A to the target marginals r and c.

An L_1-*error function* f is employed to assess the goodness-of-fit of a scaled weight matrix $A(t)$. It checks whether any row is *underfitted*, $a_{i+}(t)<r_i$, or *overfitted*, $a_{i+}(t)>r_i$, as well as whether any column is under- or overfitted, and then totals the absolute deviations between current sums and target marginals,

$$f\big(A(t)\big)=\sum_{i\le k}\Big|a_{i+}(t)-r_i\Big|+\sum_{j\le\ell}\Big|a_{+j}(t)-c_j\Big|.$$

Odd steps t have row sums matching their target marginals, whence the L_1-error $f\big(A(t)\big)$ is equal to the (second) column-error sum. For even steps t, the (first) row-error sum is decisive.

The L_1-error admits another interpretation, as the L_1-distance between a scaled weight matrix and its successor. To see this for an even step t, we substitute $r_i=a_{i+}(t)/\alpha_i(t+1)$ from [IPF1] and $a_{ij}(t)/\alpha_i(t+1)=a_{ij}(t+1)$ from [IPF2] to obtain

$$f\big(A(t)\big)=\sum_{i\le k}\Big|1-\frac{1}{\alpha_i(t+1)}\Big|a_{i+}(t)=\sum_{i\le k}\sum_{j\le\ell}\Big|a_{ij}(t)-a_{ij}(t+1)\Big|.$$

Definitions [IPF3] and [IPF4] confirm the result for odd steps t. Deming and Stephan (1940, page 440) recommend that the IPF procedure *is continued until the table reproduces itself.* This is exactly what is captured by the error function f: The table reproduces itself, $A(t)=A(t+1)$, if and only if the L_1-error is zero, $f\big(A(t)\big)=0$.

Lemma 2 shows that the L_1-error is nonincreasing, and decreasing when a mass transport between under- and overfitted rows (or columns) becomes feasible. It admits a transparent lower bound, which Lemma 5 proves to be attained in the limit.

Lemma 2 (Monotonicity) *Let $A(t)$, $t \geq 0$, be the IPF sequence for the fitting of the weight matrix A to the target marginals r and c. Then we have for all steps $t \geq 0$:*

(i) *The L_1-error function is nonincreasing, $f\big(A(t)\big) \geq f\big(A(t+1)\big)$.*

(ii) *If $A(t)$ is connected and features an underfitted row as well as an overfitted row, then the L_1-error decreases at the latest after another $k-1$ steps, $f\big(A(t)\big) > f\big(A(t+k-1)\big)$.*

(iii) *Any row subset $I \subseteq \{1, \ldots, k\}$ bounds the L_1-errors from below via*

$$f\big(A(t)\big) \geq r_I - c_{J_A(I)} + c_{J_A(I)'} - r_{I'}.$$

Equality holds if and only if (a) the set I contains all currently underfitted rows and no currently overfitted rows, $U(t) \subseteq I \subseteq O(t)'$ with $U(t) = \{i \leq k \mid a_{i+}(t) < r_i\}$ and $O(t) = \{i \leq k \mid a_{i+}(t) > r_i\}$, and (b) the weight matrix A is blockdiagonal according to

$$A = \begin{array}{c} \\ I \\ I' \end{array} \!\! \begin{array}{c} \begin{array}{cc} J_A(I) & J_A(I)' \end{array} \\ \left(\begin{array}{cc} A^{(1)} & 0 \\ 0 & A^{(2)} \end{array} \right) \end{array}.$$

Proof (i) Let step $t \geq 0$ be even, whence $A(t)$ has fitted columns and its L_1-error originates from rows. From [IPF1] and $a_{i+}(t) = \alpha_i(t+1) r_i$ we get

$$f\big(A(t)\big) = \sum_{i \leq k} \Big| a_{i+}(t) - r_i \Big| = \sum_{i \leq k} \Big| \alpha_i(t+1) - 1 \Big| r_i.$$

Inserting $r_i = \sum_{j \leq \ell} a_{ij}(t+1)$ we apply the triangle inequality within each column j,

$$\sum_{j \leq \ell} \sum_{i \leq k} \Big| 1 - \alpha_i(t+1) \Big| a_{ij}(t+1) \geq \sum_{j \leq \ell} \Big| \sum_{i \leq k} \Big(1 - \alpha_i(t+1) \Big) a_{ij}(t+1) \Big|. \qquad \text{[TI]}$$

From [IPF2] and $\sum_{i \leq k} \alpha_i(t+1) a_{ij}(t+1) = a_{+j}(t) = c_j$ we conclude

$$\sum_{j \leq \ell} \Big| a_{+j}(t+1) - c_j \Big| = f\big(A(t+1)\big).$$

With definitions [IPF3] and [IPF4] the argument carries over to odd steps $t \geq 1$. Thus monotonicity is established.

$$\begin{matrix} - \\ + \end{matrix}\begin{pmatrix} a_{11}(t) & \cdots \\ a_{21}(t) & \cdots \end{pmatrix} \qquad \begin{matrix} - \\ = \\ + \end{matrix}\begin{pmatrix} a_{11}(t) & 0 & \cdots \\ a_{21}(t) & a_{22}(t) & \cdots \\ 0 & a_{32}(t) & \cdots \end{pmatrix} \qquad \begin{matrix} - \\ = \\ = \\ + \end{matrix}\begin{pmatrix} a_{11}(t) & 0 & 0 & \cdots \\ a_{21}(t) & a_{22}(t) & 0 & \cdots \\ 0 & a_{32}(t) & a_{33}(t) & \cdots \\ 0 & 0 & a_{43}(t) & \cdots \end{pmatrix}$$

$$k = 2 \qquad\qquad k = 3 \qquad\qquad k = 4$$

Exhibit 1: *Mass transport.* A connected weight matrix $A(t)$ is shown, with $k = 2, 3, 4$ rows, together with a longest path linking an underfitted row $(-)$ via fitted rows $(=)$ to an overfitted row $(+)$. Zeros indicate void cells. All other entries are assumed positive, $a_{ij}(t) > 0$.

(ii) To keep the notation simple let row 1 be underfitted, and row k be overfitted. By [IPF1] under- and overfittedness are reflected by how the divisors relate to unity,

$$a_{1+}(t) < r_1 \iff \alpha_1(t+1) < 1, \qquad a_{k+}(t) > r_k \iff \alpha_k(t+1) > 1.$$

The proof is by induction on the number of rows. In case $k = 2$, connectedness of $A(t)$ gives rise to some column, $j = 1$ say, where the weights $a_{11}(t+1)$ and $a_{21}(t+1)$ are positive. This injects two nonzero terms of opposite sign into the jth sum in display [TI]. Hence the triangle inequality turns strict, $f\big(A(t)\big) > f\big(A(t+1)\big)$.

The case $k = 3$ reduces to the previous case whenever there is a single column connecting the under- and overfitted rows. Otherwise a connecting path involves two columns, $j = 1, 2$ say, and visits an intermediate row that is fitted, see Exhibit 1. Column 1, intersecting only such rows that are fitted or underfitted, acquires a divisor larger than unity, $\beta_1(t+2) > 1$. In contrast column 2, meeting only fitted or overfitted rows, gets a divisor smaller than unity, $\beta_2(t+2) < 1$. By interchanging the roles of rows and columns we recover the previous case, and get $f\big(A(t+1)\big) > f\big(A(t+2)\big)$.

The case $k = 4$ indicates how to complete the induction. Unless it reduces to the previous case, we need to contemplate a path involving three columns, $j = 1, 2, 3$ say, see Exhibit 1. With such a path step $t+2$ has column 1 overfitted, column 2 fitted, and column 3 underfitted. This is the previous case, with the roles of rows and columns interchanged. Continuing as in the previous case step $t+3$ has rows 2 and 3 connect in column 2, with $\alpha_2(t+3) < 1$ and $\alpha_3(t+3) > 1$. This yields $f\big(A(t+2)\big) > f\big(A(t+3)\big)$.

(iii) Because of monotonicity we may assume step t to be even. Let I be any row subset. As $\sum_{i\in I}\big(a_{i+}(t) - r_i\big) + \sum_{i\in I'}\big(a_{i+}(t) - r_i\big) = \sum_{i\le k}\big(a_{i+}(t) - r_i\big) = c_+ - r_+$, the complement I' satisfies $\sum_{i\in I'}\big(a_{i+}(t) - r_i\big) = c_+ - r_+ + \sum_{i\in I}\big(r_i - a_{i+}(t)\big)$. We get

$$f\big(A(t)\big) \ge \sum_{i\in I}\Big(r_i - a_{i+}(t)\Big) + \sum_{i\in I'}\Big(a_{i+}(t) - r_i\Big) = c_+ - r_+ + 2\sum_{i\in I}\Big(r_i - a_{i+}(t)\Big).$$

Equality holds if and only if condition (a) applies. The sum $\sum_{i\in I} a_{i+}(t)$ is broken up into four terms,

$$\begin{aligned}\sum_{i\in I}\Big(r_i - a_{i+}(t)\Big) &= r_I - \left(\sum_{i\in I}\sum_{j\in J_A(I)} + \sum_{i\in I}\sum_{j\in J_A(I)'} + \sum_{i\in I'}\sum_{j\in J_A(I)} - \sum_{i\in I'}\sum_{j\in J_A(I)}\right) a_{ij}(t) \\ &\ge r_I - c_{J_A(I)} - 0 + 0.\end{aligned}$$

The last line uses $a_{I\times J_A(I)'}(t) = 0$ that is inherited from $a_{I\times J_A(I)'} = 0$. It also employs the estimate $a_{I'\times J_A(I)}(t) \ge 0$, where equality holds if and only if condition (b) applies. This proves $f\big(A(t)\big) \ge c_+ - r_+ + 2\,\big(r_I - c_{J_A(I)}\big) = r_I - c_{J_A(I)} + c_{J_A(I)'} - r_{I'}$. $\square$

Bacharach (1970, page 50) establishes an intriguing succession of interlacing inequalities between incremental row divisors and incremental column divisors. As the IPF procedure advances, the smallest of the incremental divisors is nondecreasing, the largest, nonincreasing. The smallest incremental row and column divisors are denoted by $\alpha_{\min}(t+1)$ and $\beta_{\min}(t+2)$, and the largest by $\alpha_{\max}(t+1)$ and $\beta_{\max}(t+2)$.

Lemma 3 (Bacharach inequalities) *Let step $t = 0, 2, 4, \ldots$ be even.*

(i) *The following interlacing inequalities hold true:*

$$\alpha_{\min}(t+1) \overset{(1)}{\leq} \frac{1}{\beta_{\max}(t+2)} \overset{(2)}{\leq} \alpha_{\min}(t+3) \leq \frac{c_+}{r_+} \leq \alpha_{\max}(t+3) \overset{(3)}{\leq} \frac{1}{\beta_{\min}(t+2)} \overset{(4)}{\leq} \alpha_{\max}(t+1).$$

(ii) *If A is connected and the smallest or largest row divisors stay constant over $k-1$ subsequent row adjustments, $\alpha_{\min}(t+1) = \alpha_{\min}\big(t+2k-1\big)$ or $\alpha_{\max}(t+1) = \alpha_{\max}\big(t+2k-1\big)$, then all row divisors are identical, $\alpha_i(t+1) = c_+/r_+$ for all $i \leq k$.*

Proof (i) We start from $a_{+j}(t) = a_{+j}(t+2) = c_j$ and $a_{i+}(t+1) = a_{i+}(t+3) = r_i$, for all columns $j \leq \ell$ and all rows $i \leq k$. This yields

$$1 = \frac{a_{+j}(t+2)}{c_j} = \frac{1}{c_j} \sum_{p \leq k} \frac{a_{pj}(t)}{\alpha_p(t+1)\beta_j(t+2)} \begin{cases} \leq \dfrac{1}{\alpha_{\min}(t+1)\beta_j(t+2)}, & (1j) \\ \geq \dfrac{1}{\alpha_{\max}(t+1)\beta_j(t+2)}; & (4j) \end{cases}$$

$$1 = \frac{a_{i+}(t+3)}{r_i} = \frac{1}{r_i} \sum_{q \leq \ell} \frac{a_{iq}(t+1)}{\beta_q(t+2)\alpha_i(t+3)} \begin{cases} \leq \dfrac{1}{\beta_{\min}(t+2)\alpha_i(t+3)}, & (3i) \\ \geq \dfrac{1}{\beta_{\max}(t+2)\alpha_i(t+3)}. & (2i) \end{cases}$$

Maxima and minima over $j \leq \ell$ in $(1j)$ and $(4j)$, and over $i \leq k$ in $(3i)$ and $(2i)$ yield

$$\begin{aligned} \alpha_{\min}(t+1)\beta_{\max}(t+2) &\overset{(1)}{\leq} 1 \overset{(4)}{\leq} \alpha_{\max}(t+1)\beta_{\min}(t+2), \\ \beta_{\min}(t+2)\alpha_{\max}(t+3) &\overset{(3)}{\leq} 1 \overset{(2)}{\leq} \beta_{\max}(t+2)\alpha_{\min}(t+3). \end{aligned}$$

This establishes inequalities (1)–(4) of the assertion. The middle inequalities follow from definition [IPF1], $\alpha_{\min}(t+3) \leq \sum_{i \leq k} \alpha_i(t+3) r_i / r_+ = c_+/r_+ \leq \alpha_{\max}(t+3)$.

(ii) The proof is by contraposition. Assuming $\alpha_{\min}(t+1) < \alpha_{\max}(t+1)$, we show that the concurrence with $\alpha_{\min}(t+1) = \alpha_{\min}(t+2k-1)$ forces A to be disconnected. The decomposition of A is afforded by the row subsets $I(z)$ where the row divisor is minimum, and the column subsets $J(z)$ where the column divisor is maximum,

$$\begin{aligned} I(z) &= \{\, i \leq k \mid \alpha_i(z) = \alpha_{\min}(z) \,\} \qquad \text{for } z \text{ odd}, \\ J(z) &= \{\, j \leq \ell \mid \beta_j(z) = \beta_{\max}(z) \,\} \qquad \text{for } z \text{ even}. \end{aligned}$$

These sets are nonempty. Due to the first assumption, $\alpha_{\min}(t+1) < \alpha_{\max}(t+1)$, the row subset $I(t+1)$ is proper, $\{1, \ldots, k\} \neq I(t+1)$.

The second assumption turns inequalities (3) and (4) into equations, $\alpha_{\min}(t+1) = 1/\beta_{\max}(t+2) = \alpha_{\min}(t+3) = \cdots = \alpha_{\min}(t+2k-3) = 1/\beta_{\max}(t+2k-2) = \alpha_{\min}(t+2k-1)$. We work our way in sets of three,

$$\alpha_{\min}(t+z+1) = \frac{1}{\beta_{\max}(t+z+2)} = \alpha_{\min}(t+z+3), \qquad \text{with } z = 0, 2, \ldots, 2k-4.$$

The first set has $z = 0$. For rows $i \in I(t+3)$ we get $1/\beta_{\max}(t+2) = \alpha_{\min}(t+3) = \alpha_i(t+3)$. Therefore equality holds in $(2i)$, and all $q \notin J(t+2)$ have $a_{iq}(t+1) = 0$ and hence $a_{iq} = 0$. For columns $j \in J(t+2)$ equality obtains in $(1j)$, whence all $p \notin I(t+1)$ fulfill $a_{pj}(t) = 0$ and hence $a_{pj} = 0$. Any row $i \in I(t+3) \setminus I(t+1)$ would vanish, having $a_{ij} = 0$ for $j \in J(t+2)$ as well as for $j \notin J(t+2)$. Since vanishing rows in A are not allowed, we get $I(t+1) \supseteq I(t+3)$. The argument carries forward to build a chain of $k-1$ inclusions,

$$\{1, \ldots, k\} \neq I(t+1) \supseteq I(t+3) \supseteq \cdots \supseteq I(t+2k-3) \supseteq I(t+2k-1) \neq \emptyset.$$

At most $k-2$ inclusions can be strict. Somewhere between $z = 0$ and $z = 2k-4$ equality obtains, $I(t+z+1) = I(t+z+3)$. This forces A to be disconnected,

$$A = \begin{array}{r} \\ I(t+z+1) \\ I(t+z+3)' \end{array} \!\!\begin{array}{c} \begin{array}{cc} J(t+z+2) & J(t+z+2)' \end{array} \\ \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix} \end{array}.$$

Hence the first assumption is untenable. Instead we must have $\alpha_{\min}(t+1) = \alpha_{\max}(t+1)$, meaning that all incremental row divisors $\alpha_i(t+1)$ are identical. As then they cannot but equal their mean, we finally get $\alpha_{\min}(t+1) = c_+/r_+$ for all $i \leq k$. □

Lemma 3(ii) offers a sufficient condition for the IPF sequence to finally stabilize. If the weight matrix A is connected and and if after some even step t the smallest or largest divisors stay put long enough, then the fitting has come to an end and the remainder of the IPF sequence oscillates between $A(t)$ and $A(t+1)$. The condition proves instrumental when applied to an accumulation point B of the IPF sequence.

4 Accumulation points

For the fitting of a weight matrix A to the target marginals r and c, the induced IPF sequence $A(t)$, $t \geq 0$, stays in the compact set $[0,h]^{k\times\ell}$ of the matrix space $\mathbb{R}^{k\times\ell}$, where $h = \max\{r_+, c_+\}$. All row sums stay uniformly bounded away from zero. Indeed, for t even we have $a_{i+}(t) = \alpha_i(t+1)r_i \geq \alpha_{\min}(1)r_i > 0$, by Lemma 3(i), and $a_{i+}(t+1) = r_i > 0$. The same is true of column sums. Hence every accumulation point $B = \lim_{n\to\infty} A(t_n)$ has row sums and column sums nonvanishing, $b_{i+} > 0$ and $b_{+j} > 0$. Thus B is a biproportional scaling of A, and the structural results of Section 2 apply. Furthermore B is a legitimate input matrix on which to run the IPF procedure. Without loss of generality we concentrate on *even-step accumulation points* which, by definition, emerge as the limit along even steps t_n. Hence column sums of B match column marginals, $b_{+j} = c_j$ for all $j \leq \ell$.

For the fitting of an even-step accumulation point $B = \lim_{n\to\infty} A(t_n)$ to the target marginals r and c let $B(z)$, $z \ge 0$, designate the induced IPF sequence. The initialization is $B(0) = B$. The incremental row divisors are denoted by $\gamma_i(z+1)$, the column divisors by $\delta_j(z+2)$. Let an arrow $\to$ indicate a passage to the limit as n tends to infinity. Induction on $z = 0, 2, \dots$ proves convergence of the incremental divisors:

$$\alpha_i(t_n+z+1) = \frac{a_{i+}(t_n+z)}{r_i} \to \frac{b_{i+}(z)}{r_i} = \gamma_i(z+1),$$

$$a_{ij}(t_n+z+1) = \frac{a_{ij}(t_n+z)}{\alpha_i(t_n+z+1)} \to \frac{b_{ij}(z)}{\gamma_i(z+1)} = b_{ij}(z+1),$$

$$\beta_j(t_n+z+2) = \frac{a_{+j}(t_n+z+1)}{c_j} \to \frac{b_{+j}(z+1)}{c_j} = \delta_j(z+2), \qquad \text{[CID]}$$

$$a_{ij}(t_n+z+2) = \frac{a_{ij}(t_n+z+1)}{\beta_j(t_n+z+2)} \to \frac{b_{ij}(z+1)}{\delta_j(z+2)} = b_{ij}(z+2).$$

On the grounds of empirical calculations and simulation studies we claim that the incremental divisors converge not only along certain subsequences, but unconditionally and in any case. Pukelsheim and Simeone (2009) and Pukelsheim (2012) contain some worked examples. Yet the claim is a conjecture rather than an established result.

IPF conjecture *Every sequence of incremental divisors is convergent.*

The IPF conjecture has rather strong consequences, as testified by Lemmas 4 and 5. Lemma 4 shows that an even-step accumulation point B actually is an even-step limit, and that its IPF sequence oscillates between B and $B(1)$.

Lemma 4 (Even-step convergence) *Let $A(t)$, $t \ge 0$, be the IPF sequence for the fitting of the weight matrix A to the target marginals r and c. Then we have:*

(i) *If there is a connected accumulation point, then the IPF conjecture holds true.*

(ii) *If the IPF conjecture holds true, then the entire even-step IPF subsequence is convergent,* $\lim_{t=0,2,4,\dots} A(t) = B$ *say. The IPF sequence induced by B oscillates, $B(z) = B(z+2)$ and $B(z+1) = B(z+3)$ for all even z. The accumulation point $B(1)$ is the limit of the entire odd-step IPF subsequence,* $B(1) = \lim_{t=1,3,5,\dots} A(t)$.

Proof (i) Let $B = \lim_{n\to\infty} A(t_n)$ be an even-step accumulation point that is connected. Invoking [CID] and Lemma 3(i) we see that, for all steps $z = 0, 2, 4, \dots$,

$$\gamma_{\min}(z+1) = \lim_{n\to\infty} \alpha_{\min}(t_n+z+1) = \lim_{t=1,3,5,\dots} \alpha_{\min}(t),$$

$$\gamma_{\max}(z+1) = \lim_{n\to\infty} \alpha_{\max}(t_n+z+1) = \lim_{t=1,3,5,\dots} \alpha_{\max}(t).$$

That is, the smallest and largest incremental row divisors for B stay constant forever. Due to connectedness Lemma 3(ii) lets the incremental divisors of B share the common value, $\gamma_i(1) = c_+/r_+$ for all rows $i \le k$, giving

$$\frac{c_+}{r_+} = \lim_{t=1,3,5,\dots} \alpha_{\min}(t) \le \liminf_{t=1,3,5,\dots} \alpha_i(t) \le \limsup_{t=1,3,5,\dots} \alpha_i(t) \le \lim_{t=1,3,5,\dots} \alpha_{\max}(t) = \frac{c_+}{r_+}.$$

Hence the incremental row divisors of A converge unconditionally, $\lim_{t=1,3,5,\dots} \alpha_i(t) = c_+/r_+$ for all $i \le k$.

(ii) The IPF hypothesis provides the existence of the limits $\lim_{t=1,3,5,\dots} \alpha_i(t) = \gamma_i$ say, for all rows $i \le k$. Let B and D be two even-step accumulation points. Their column sums are the same, $b_{+j} = c_j = d_{+j}$. By [CID] so are the row sums, $b_{i+} = \gamma_i r_i = d_{i+}$. Theorem 1 yields $B = D$. Admitting just a unique accumulation point, the entire even-step IPF subsequence is convergent, $\lim_{t=0,2,4,\dots} A(t) = B$.

Let B decompose into the $I_m \times J_m$ connected components $B^{(m)}$, $m \le K$. Within each component, [CID] implies that the row divisors stay constant forever. Then Lemma 3(ii) says that they are equal, $\gamma_i = c_{J_m}/r_{I_m}$ for all rows $i \in I_m$ and all blocks $m \le K$. Hence the matrix $B(1)$ has connected components $\big(B(1)\big)^{(m)} = (r_{I_m}/c_{J_m})B^{(m)}$. It follows that the induced IPF sequence oscillates, $B(z) = B(z+2)$ and $B(z+1) = B(z+3)$ for all even z. Moreover $B(1)$ is the entire odd-step limit of the original IPF sequence $A(t)$. □

The last paragraph of the proof shows that the incremental divisors of the even-step limit B depend on its $I_m \times J_m$ connected components $B^{(m)}$ only through the partial sums of their row and column marginals,

$$\gamma_i(1) = \frac{c_{J_m}}{r_{I_m}}, \qquad \delta_j(2) = \frac{r_{I_m}}{c_{J_m}},$$

for all rows $i \in I_m$ and columns $j \in J_m$, and all blocks $m \le K$.

Lemma 5 (L_1-Error limit) *Suppose the IPF conjecture holds true. For the fitting of the weight matrix A to the target marginals r and c the induced IPF sequence $A(t)$, $t \ge 0$, has limiting L_1-error given by*

$$\lim_{t\to\infty} f\big(A(t)\big) = \max_{I\subseteq\{1,\dots,k\}} \Big(r_I - c_{J_A(I)} + c_{J_A(I)'} - r_{I'} \Big).$$

Proof Let $B = \lim_{t=0,2,4,\dots} A(t)$ denote the even-step accumulation point of the IPF sequence $A(t)$. By Lemma 2(i) its L_1-error is minimum, $\lim_{t\to\infty} f\big(A(t)\big) = \lim_{t=0,2,4,\dots} f\big(A(t)\big) = f(B)$. The L_1-error of B originates from its rows, $f(B) = \sum_{i\le k} \big|b_{i+} - r_i\big|$.

Lemma 2(iii) calls for some row subset I such that

$$f(B) = r_I - c_{J_A(I)} + c_{J_A(I)'} - r_{I'}. \qquad [\dagger]$$

In case all rows of B are underfitted or fitted, $b_{i+} \le r_i$ for all $i \le k$, the error is $f(B) = r_+ - c_+$ and [†] holds with $I = \{1,\dots,k\}$. In case all rows of B are fitted or overfitted, $b_{i+} \ge r_i$ for all $i \le k$, the error is $f(B) = c_+ - r_+$ and $I = \emptyset$ verifies [†].

The case left has some rows of B underfitted, others, overfitted. We show that [†] holds for the set of underfitted or fitted rows, $I = \big\{ i \le k \;\big|\; b_{i+} \le r_i \big\}$. The set I is the union of the row subsets I_m of the connected components $B^{(m)}$ of B where $\gamma_i(1) = c_{J_m}/r_{I_m} \le 1$. The L_1-error turns into

$$f(B) = r_I - c_{J_B(I)} + c_{J_B(I)'} - r_{I'}.$$

Thus [†] follows as soon as the sets of columns connected with I in B and in A are seen to be the same, $J_B(I) = J_A(I)$.

The direct inclusion $J_B(I) \subseteq J_A(I)$ holds since always $b_{ij} > 0$ comes with $a_{ij} > 0$. For the converse inclusion we note that rows $i \in I$ satisfy $\gamma_i(1) \le 1$, while rows $i \in I'$ fulfill $\gamma_i(1) > 1$. Therefore columns $j \in J_B(I)$ satisfy $\delta_j(2) \ge 1$, while columns $j \in J_B(I)'$ fulfill $\delta_j(2) < 1$. Denoting, by abuse of notation, the $I \times J_B(I)$ block by $B^{(1)}$ and the $I' \times J_B(I)'$ block by $B^{(2)}$ the state of affairs is depicted as follows:

$$B = \begin{array}{c} \\ I \\ I' \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} J_B(I) & J_B(I)' \end{array} \\ \begin{pmatrix} B^{(1)} & 0 \\ 0 & B^{(2)} \end{pmatrix} \\ \begin{array}{cc} \delta_j(2) \ge 1 & \delta_j(2) < 1 \end{array} \end{array} \begin{array}{c} \\ \gamma_i(1) \le 1 \\ \gamma_i(1) > 1 \\ \\ \end{array} .$$

Consider a row $i \in I$ and a column $j \in J_B(I)'$, and choose some positive ϵ such that $\delta_j(2) < 1 - \epsilon$. Since $\lim_{t=1,3,5,\dots} \alpha_i(t) = \gamma_i(1) \le 1$ and $\lim_{t=0,2,4,\dots} \beta_j(t) = \delta_j(2) < 1$ there exists an even step t_0 such that all even steps $t \ge t_0$ satisfy $\alpha_i(t+1) < 1 + \epsilon$ and $\beta_j(t+2) < 1 - \epsilon$. This bounds the denominator of $a_{ij}(t) = a_{ij} / \big(\rho_i(t)\sigma_j(t)\big)$,

$$\rho_i(t)\sigma_j(t) < \rho_i(t_0)\sigma_j(t_0) \left(1 - \epsilon^2\right)^{(t-t_0)/2}.$$

In fact the denominator converges to zero, $\lim_{t\to\infty} \rho_i(t)\sigma_j(t) = 0$. This necessitates $a_{ij} = 0$, thereby entailing the converse inclusion, $J_A(I) \subseteq J_B(I)$. □

We are now in a position to annunciate the necessary and sufficient conditions for the convergence of the IPF procedure. The arrangement of the five statements mimics the conditions of Theorem 2 securing directness. We present two proofs, the first assuming that the IPF conjecture holds true, the second making do without it.

Theorem 5 (Convergence) *For the fitting of a weight matrix A to the target marginals r and c the following five statements are equivalent:*

(1) *The IPF sequence $A(t)$, $t \ge 0$, is convergent.*

(2) *The biproportional fit of A to r and c exists.*

(3) *There exists a weight matrix D preserving the zeros of A and matching the target marginals r and c.*

(4) *Marginal totals are equal, $r_+ = c_+$ and, for every row subset $I \subseteq \{1, \dots, k\}$, marginal partial row and column sums fulfill $r_I \le c_{J_A(I)}$.*

(5) *The L_1-errors of the IPF sequence $A(t)$, $t \ge 0$, tend to zero, $\lim_{t\to\infty} f\big(A(t)\big) = 0$.*

First proof, assuming the IPF conjecture to hold true

(1) ⇒ (2). If the IPF sequence converges then its limit B is a biproportional scaling of A. It inherits matching row sums along odd steps and matching column sums along even steps. By Theorem 1, B is the unique biproportional fit.

(2) ⇒ (3). The biproportional fit clearly qualifies for a matrix D asked for in (3).

(3) ⇒ (4). As in the proof of Theorem 2, the definition of $J_A(I)$ yields $d_{I\times J_A(I)'} = 0 \le d_{I'\times J_A(I)}$, and $r_I = d_{I\times J_A(I)} + d_{I\times J_A(I)'} \le d_{I\times J_A(I)} + d_{I'\times J_A(I)} = c_{J_A(I)}$.

(4) ⇒ (5). Equal marginal totals entail $r_I - c_{J_A(I)} + c_{J_A(I)'} - r_{I'} = 2\big(r_I - c_{J_A(I)}\big)$. Assuming the IPF conjecture to hold true Lemma 5 applies and leads to (5), $0 \le \lim_{t\to\infty} f\big(A(t)\big) = 2 \max_{I\subseteq\{1,\dots,k\}} \big(r_I - c_{J_A(I)}\big) \le 0$.

(5) $\Rightarrow$ (1). Let B be an accumulation point along a subsequence $A(t_n)$, $n \geq 1$. From (5) we get $f(B) = \lim_{n\to\infty} f\big(A(t_n)\big) = \lim_{t\to\infty} f\big(A(t)\big) = 0$. With row and column sums fitted, B is a biproportional fit. By Theorem 1 there is but one. Hence the IPF sequence $A(t)$, $t \geq 0$, has B for its unique accumulation point, and converges. □

Second proof, not referring to the IPF conjecture

(3) $\Leftrightarrow$ (4). The Feasibility Theorem of Gale (1957, page 1075) states that (3) a feasible mass distribution exists if and only if (4) the flow inequalities hold true.

(3) $\Rightarrow$ (5). The implication that the existence of a feasible mass distribution D forces the limiting L_1-error to vanish is essentially Csiszár's (1975, page 154) Theorem 3.2 and Pretzel's (1980, page 380) Theorem 1. The arguments of Pretzel (1980) may be condensed as follows. With a feasible distribution D as in (3), let $g\big(A(t)\big)$ be the geometric mean of the entries $a_{ij}(t)$, with exponents d_{ij}/d_{++},

$$g\big(A(t)\big) = \prod_{i\leq k}\prod_{j\leq \ell} a_{ij}(t)^{\frac{d_{ij}}{d_{++}}}.$$

A base zero has exponent zero, $a_{ij}(t) = 0 \Rightarrow a_{ij} = 0 \Rightarrow d_{ij} = 0$, and contributes the factor $0^0 = 1$. Thus all means stay positive, $g\big(A(t)\big) > 0$. For the passage from an even step t to the next even step $t+2$ definitions [IPF2] and [IPF4] yield $a_{ij}(t)$ $= \alpha_i(t+1)\beta_j(t+2)a_{ij}(t+2)$. From $d_{i+} = r_i$, $d_{+j} = c_j$, and $d_{++} = r_+ = c_+$, we get

$$g\big(A(t)\big) = \left(\prod_{i\leq k}\alpha_i(t+1)^{\frac{r_i}{r_+}}\right)\left(\prod_{j\leq \ell}\beta_j(t+2)^{\frac{c_j}{c_+}}\right) g\big(A(t+2)\big) \leq g\big(A(t+2)\big).$$

The estimate twice employs the geometric-arithmetic-mean inequality,

$$\prod_{i\leq k}\alpha_i(t+1)^{\frac{r_i}{r_+}} \leq \sum_{i\leq k}\alpha_i(t+1)\frac{r_i}{r_+} = 1, \qquad \prod_{j\leq \ell}\beta_j(t+2)^{\frac{c_j}{c_+}} \leq \sum_{j\leq \ell}\beta_j(t+2)\frac{c_j}{c_+} = 1.$$

Therefore the even-step matrix-mean sequence is positive, isotonic, and bounded, $0 < g\big(A(t)\big) \leq g\big(A(t+2)\big) \leq d_{++}$, and converges to a nonzero and finite value. In the limit the geometric-arithmetic-mean inequalities thus hold with equality. Now the divisors $\alpha_i(t+1)$ converge to a common value, as do $\beta_j(t+2)$. Since in Section 3 we have seen that their mean is unity, the common value must be unity, too. We obtain $\lim_{t=1,3,\ldots}\alpha_i(t) = 1$ for all rows $i \leq k$, and $\lim_{t=0,2,\ldots}\beta_j(t) = 1$ for all columns $j \leq \ell$. Hence row sums converge to row marginals and column sums to column marginals, and the limiting L_1-error is zero. □

In the language of the present paper Pretzel's arguments may be paraphrased by saying that if a feasible distribution D exists, then the IPF conjecture holds true.

If all entries in the weight matrix A are positive and the target marginals share the same total, $r_+ = c_+ = h$ say, then the matrix D with entries $d_{ij} = r_i c_j/h > 0$ verifies statement (3) of Theorems 5 and 2. Hence the IPF sequence converges to the biproportional fit and the fit is direct, if A is positive and $r_+ = c_+$.

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