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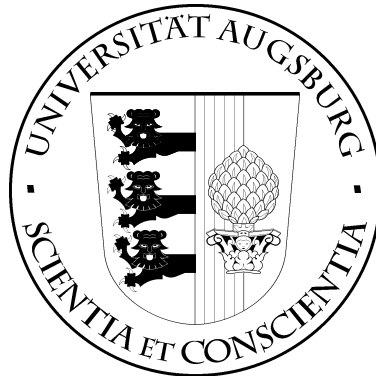
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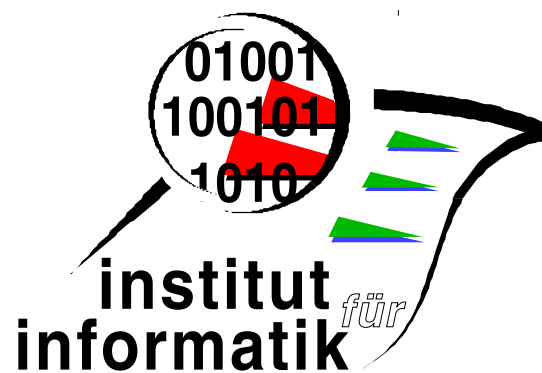


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Residuals and Detachments

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Abstract. We give a compendium of algebraic calculation rules for the operations of residuation and detachment in semirings.

1 Introduction

Residuals [1,2] and detachments have many useful applications. This report serves as a compendium of laws for these operations, many of which are known from early residuation theory. However, there is also some new material relating residuals with tests and (pre)domain, in particular, a characterisation of locality of composition [7] without recourse to the domain operation.

2 Definitions and Proof Principles

- Definition 2.1** 1. A structure $(S, \leq, 0, \top, \cdot, 1)$ is called a *left (right) quantale* if $(S, \leq, 0, \top)$ is a complete lattice with least element 0 and greatest element \top such that $(S, \cdot, 1)$ is a monoid and \cdot preserves arbitrary suprema in its left (right) argument. The supremum of elements x and y is denoted by $x + y$. Any left (right) quantale satisfies $0 \cdot x = 0$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ ($x \cdot 0 = 0$ and $x \cdot (y + z) = x \cdot y + x \cdot z$). S is called a *quantale* [8] if it is both a left and right quantale. Quantales have been called *standard Kleene algebras* in [3].
2. A (left or right) quantale is called *Boolean* if its underlying lattice is a completely distributive Boolean algebra.
 3. In a left quantale, the *left residual* and *right detachment* operations are defined as usual:

$$z \leq x/y \stackrel{\text{def}}{\Leftrightarrow} z \cdot y \leq x, \quad x \lfloor y \stackrel{\text{def}}{=} \overline{x/y}. \quad (\text{GC})$$

By these definitions, the function $\lambda x . x/y$ is the upper adjoint and the function $\lambda z . z \cdot y$ the lower adjoint of a Galois connection.

Symmetric definitions and laws apply to the *right residual* \backslash and *left detachment* \rfloor in a right quantale.

A useful tool for working with elements of a poset are the rules of *indirect inequality*:

$$\begin{aligned} x \leq y &\Leftrightarrow (\forall z : z \leq x \Rightarrow z \leq y) , \\ x \leq y &\Leftrightarrow (\forall z : y \leq z \Rightarrow x \leq z) . \end{aligned}$$

Moreover, we have the rules of *indirect equality*:

$$x = y \Leftrightarrow (\forall z : z \leq x \Leftrightarrow z \leq y) \Leftrightarrow (\forall z : x \leq z \Leftrightarrow y \leq z) .$$

As special cases of this, we get

$$\begin{aligned} x = \top &\Leftrightarrow (\forall z : z \leq x \Leftrightarrow \text{TRUE}) , \\ x = 0 &\Leftrightarrow (\forall z : x \leq z \Leftrightarrow \text{TRUE}) , \end{aligned}$$

A related principle is provided by the universal characterisations of infima and suprema:

$$\begin{aligned} y \leq \prod X &(\forall x \in X : y \leq x) , \\ \bigsqcup X \leq y &(\forall x \in X : x \leq y) . \end{aligned} \quad (\text{Inf/Sup})$$

We will use all these rules tacitly in the remainder.

Definition 2.2 1. The *dual* f^\natural of an endofunction f on a Boolean algebra is defined by

$$f^\natural(x) \stackrel{\text{def}}{=} \overline{f(\bar{x})} .$$

2. Two functions f, g between Boolean algebras are called *conjugate* [6] if they satisfy

$$f(x) \leq \bar{y} \Leftrightarrow g(y) \leq \bar{x} . \quad (*)$$

By straightforward Boolean algebra, the property that f and g are conjugate is equivalent to the Galois connection

$$f(x) \leq y \Leftrightarrow x \leq g^\natural(y) .$$

Lemma 2.3 *Assume that f, g are conjugate.*

1. $f(\overline{g(y)}) \leq \bar{y}$.
2. $g(\overline{f(x)}) \leq \bar{x}$.
3. f and g preserve all suprema and hence are isotone and strict.

Proof. 1. Set $x = \overline{g(y)}$ in (*).

2. Set $y = \overline{f(x)}$ in (*).

3. By the above remark, both f and g are lower adjoints in Galois connections.

Lemma 2.4 (Modularity; Dedekind) *For conjugate f and g ,*

$$f(x) \sqcap y \leq f(x \sqcap g(y)) .$$

Proof. $f(x) \sqcap y$

$$\begin{aligned}
&= \{ \text{Boolean algebra} \} \\
&\quad f((x \sqcap g(y)) \sqcup (x \sqcap \overline{g(y)})) \sqcap y \\
&= \{ f \text{ preserves suprema} \} \\
&\quad (f(x \sqcap g(y)) \sqcup f(x \sqcap \overline{g(y)})) \sqcap y \\
&\leq \{ \text{definition of } \sqcap \text{ and isotony of } f \} \\
&\quad (f(x \sqcap g(y)) \sqcup f(\overline{g(y)})) \sqcap y \\
&\leq \{ \text{by Lemma 2.3.1} \} \\
&\quad (f(x \sqcap g(y)) \sqcup \bar{y}) \sqcap y \\
&= \{ \text{Boolean algebra} \} \\
&\quad f(x \sqcap g(y)) \sqcap y \\
&\leq \{ \text{definition of } \sqcap \} \\
&\quad f(x \sqcap g(y)) .
\end{aligned}$$

As our final proof tool in Boolean algebras we mention

$$x \sqcap y \leq z \Leftrightarrow x \leq \bar{y} \sqcup z . \quad (\text{Shunting})$$

3 Laws for Residuals

Law 3.1 (Left-Conjunctivity) $(\sqcap X)/y = \sqcap (X/y)$.

Proof: Upper adjoints preserve all infima.

Law 3.2 $\top/y = \top$.

Proof: Set $X = \emptyset$ in the previous law.

Law 3.3 $u \leq v \Rightarrow u/y \leq v/y$.

Proof: Immediate from left-conjunctivity.

Law 3.4 (Right-Antidisjunctivity) $x/(\sqcup Y) = \sqcap (x/Y)$.

Proof:

$$\begin{aligned} z &\leq x/(\sqcup Y) \\ \Leftrightarrow &\quad \{ \text{by (GC)} \} \\ z \cdot \sqcup Y &\leq x \\ \Leftrightarrow &\quad \{ \text{disjunctivity of } \cdot \} \\ \sqcup (z \cdot Y) &\leq x \\ \Leftrightarrow &\quad \{ \text{lattice algebra} \} \\ \forall y \in Y : z \cdot y &\leq x \\ \Leftrightarrow &\quad \{ \text{by (GC)} \} \\ \forall y \in Y : z &\leq x/y \\ \Leftrightarrow &\quad \{ \text{lattice algebra} \} \\ z &\leq \sqcap x/Y . \end{aligned}$$

Law 3.5 $x/0 = \top$.

Proof: Set $Y = \emptyset$ in the previous law.

Law 3.6 $u \leq v \Rightarrow x/u \geq x/v$.

Proof: Immediate from right-antidisjunctivity.

Law 3.7 $1 \leq x/x$.

Proof: Immediate from (GC) and neutrality of 1.

Law 3.8 $(x/y) \cdot y \leq x$.

Proof: Set $z = x/y$ in (GC).

Law 3.9 $(x/y) \cdot y = x \Leftrightarrow \exists z : x = z \cdot y$.

Proof: The implication (\Rightarrow) is trivial. For (\Leftarrow) assume $z \cdot y = x$. Then $z \cdot y \leq x$ and hence by (GC) we get $z \leq x/y$. Since also $x \leq z \cdot y$ we obtain from this by isotony $x \leq (x/y) \cdot y$. The reverse inequality is given by Law 3.8.

Law 3.10 $(x/x) \cdot x = x$.

Proof: Use Law 3.9 and set $x = y$ and $z = 1$.

Law 3.11 $x/1 = x$.

Proof: Use Law 3.9 and set $y = 1$ and $z = x$.

Law 3.12 $(0/y) \cdot y = 0$.

Proof: For (\leq) set $x = 0$ in Law 3.8. (\geq) is trivial.

Law 3.13 $x/(y \cdot z) = (x/z)/y$.

Proof:

$$\begin{aligned} u &\leq x/(y \cdot z) \\ \Leftrightarrow &\quad \{ \text{by (GC)} \} \\ &u \cdot y \cdot z \leq x \\ \Leftrightarrow &\quad \{ \text{by (GC)} \} \\ &u \cdot y \leq x/z \\ \Leftrightarrow &\quad \{ \text{by (GC)} \} \\ &u \leq (x/z)/y . \end{aligned}$$

Law 3.14 (“Euclid” for Residual) $x \cdot (y/z) \leq (x \cdot y)/z$.

Proof:

$$\begin{aligned} &x \cdot (y/z) \leq (x \cdot y)/z \\ \Leftrightarrow &\quad \{ \text{by (GC)} \} \\ &x \cdot (y/z) \cdot z \leq x \cdot y \\ \Leftarrow &\quad \{ \text{by Law 3.8 and isotony} \} \\ &\text{TRUE} . \end{aligned}$$

Law 3.15 $x \leq (x \cdot y)/y$.

Proof: Immediate by (GC) and reflexivity of \leq .

Law 3.16 $(x \cdot y)/y = x \Leftrightarrow \exists z : x = z/y$.

Proof: (\Rightarrow) is trivial. For (\Leftarrow) assume $x = z/y$. Then

$$\begin{aligned} & (x \cdot y)/y \\ = & \{ \text{assumption} \} \\ & ((z/y) \cdot y)/y \\ \leq & \{ \text{by Law 3.8 and left-isotony of } / \} \\ & z/y \\ = & \{ \text{assumption} \} \\ & x \end{aligned}$$

The reverse inequality is given by Law 3.15.

Law 3.17 $(\top \cdot y)/y = \top$.

Proof: For (\geq) set $x = \top$ in Law 3.15. (\leq) is trivial.

Law 3.18 $x \cdot y = ((x \cdot y)/y) \cdot y$.

Proof: By GC and standard Galois theory.

Law 3.19 $x/y = ((x/y) \cdot y)/y$.

Proof: By GC and standard Galois theory.

Law 3.20 $1/x \leq y/(x \cdot y)$.

Proof:

$$\begin{aligned} & 1/x \\ \leq & \{ \text{by Law 3.7 and left-isotony of } / \} \\ & (y/y)/x \\ = & \{ \text{by Law 3.13} \} \\ & y/(x \cdot y) . \end{aligned}$$

Law 3.21 $(x/y) \cdot (y/z) \leq x/z$.

Proof: $(x/y) \cdot (y/z) \leq x/z$
 \Leftrightarrow { by (GC) }
 $(x/y) \cdot (y/z) \cdot z \leq x$
 \Leftarrow { by Law 3.8 and isotony of \cdot }
 $(x/y) \cdot y \leq x$
 \Leftrightarrow { by Law 3.8 }
TRUE .

Law 3.22 x/x is a preorder.

Proof: Immediate from Law 3.7 and Law 3.21.

Law 3.23 $x/y \leq (x/z)/(y/z)$.

Proof: $u \leq (x/z)/(y/z)$
 \Leftrightarrow { by (GC) }
 $u \cdot (y/z) \leq x/z$
 \Leftrightarrow { by (GC) }
 $u \cdot (y/z) \cdot z \leq x$
 \Leftarrow { by Law 3.8 and isotony }
 $u \cdot y \leq x$
 \Leftrightarrow { by (GC) }
 $u \leq x/y$.

4 Interaction Between Residuals

Law 4.1 $(x \setminus y)/z = x \setminus (y/z)$.

Proof: $u \leq (x \setminus y)/z$
 \Leftrightarrow { by (GC) }
 $u \cdot z \leq x \setminus y$

$$\begin{aligned}
&\Leftrightarrow \{ \text{by (GC)} \} \\
&\quad x \cdot u \cdot z \leq y \\
&\Leftrightarrow \{ \text{by (GC)} \} \\
&\quad x \cdot u \leq y/z \\
&\Leftrightarrow \{ \text{by (GC)} \} \\
&\quad u \leq x \setminus (y/z) .
\end{aligned}$$

5 Laws for Detachments

Law 5.1 (Exchange; Schröder) $x \cdot y \leq z \Leftrightarrow \bar{z} \lfloor y \leq \bar{x}$.

Proof:

$$\begin{aligned}
&x \cdot y \leq z \\
&\Leftrightarrow \{ \text{by (GC)} \} \\
&\quad x \leq z/y \\
&\Leftrightarrow \{ \text{Boolean algebra} \} \\
&\quad \overline{z/y} \leq \bar{x} \\
&\Leftrightarrow \{ \text{Boolean algebra and definition } \lfloor \} \\
&\quad \bar{z} \lfloor y \leq \bar{x} .
\end{aligned}$$

By this law the functions $\lambda x . x \cdot y$ and $\lambda z . z \lfloor y$ are conjugates:

Law 5.2 $x \cdot y \sqcap z = 0 \Leftrightarrow z \lfloor y \sqcap x = 0$.

Proof: Immediate from exchange by shunting (substitute \bar{z} for z).

Law 5.3 (Dedekind)

$$x \sqcap y \cdot z \leq (x \lfloor z \sqcap y) \cdot z \quad \text{and} \quad x \sqcap y \lfloor z \leq (x \cdot z \sqcap y) \lfloor z .$$

Proof: Set $f(z) \stackrel{\text{def}}{=} a \cdot z$ and $g(z) \stackrel{\text{def}}{=} a \lfloor z$ in Lemma 2.4.

Law 5.4 (Left-Disjunctivity) $(\sqcup X) \lfloor y = \sqcup (X \lfloor y)$.

Proof: Conjugates preserve suprema (Lemma 2.3.3).

Law 5.5 $0 \lfloor y = 0$.

Proof: Set $X = \emptyset$ in the previous law.

Law 5.6 $u \leq v \Rightarrow u \sqcup y \leq v \sqcup y$.

Proof: Immediate from left-disjunctivity.

Law 5.7 (Right-Disjunctivity) $x \sqcup (\sqcup Y) = \sqcup (x \sqcup Y)$.

Proof:

$$\begin{aligned} & x \sqcup (\sqcup Y) \\ &= \quad \{ \text{definition} \} \\ & \quad \overline{\bar{x} / \sqcup Y} \\ &= \quad \{ \text{Law 3.4} \} \\ & \quad \overline{\sqcap \bar{x} / Y} \\ &= \quad \{ \text{de Morgan} \} \\ & \quad \sqcup \overline{\bar{x} / Y} \\ &= \quad \{ \text{definition} \} \\ & \quad \sqcup x \sqcup Y . \end{aligned}$$

Law 5.8 $x \sqcup 0 = 0$.

Proof: Set $Y = \emptyset$ in the previous law.

Law 5.9 $u \leq v \Rightarrow x \sqcup u \leq x \sqcup v$.

Proof: Immediate from right-disjunctivity.

Law 5.10 $x \sqcup 1 = x$.

Proof:

$$\begin{aligned} & x \sqcup 1 \leq u \\ &= \quad \{ \text{exchange and neutrality of 1} \} \\ & \quad \bar{u} \leq \bar{x} \\ &= \quad \{ \text{shunting} \} \\ & \quad x \leq u . \end{aligned}$$

Law 5.11 $\top \sqcup \top = \top$.

Proof: Immediate from the previous law, $1 \leq \top$ and isotony.

Law 5.12 $x[(y \cdot z)] = (x[z])[y]$.

$$\begin{aligned}
 \text{Proof: } & x[(y \cdot z)] \\
 &= \frac{\{\text{definition}\}}{\overline{\overline{x/(y \cdot z)}}} \\
 &= \frac{\{\text{by Law 3.13}\}}{(\overline{\overline{x/z}})/y} \\
 &= \frac{\{\text{involution}\}}{\overline{\overline{(\overline{\overline{x/z}})/y}}} \\
 &= \frac{\{\text{definitions}\}}{(x[z])[y]} .
 \end{aligned}$$

6 Interaction Between Detachments

Law 6.1 $(x]y)[z] = x](y[z])$.

$$\begin{aligned}
 \text{Proof: } & (x]y)[z] \\
 &= \frac{\{\text{definitions}\}}{\overline{\overline{x \setminus \overline{y}/z}}} \\
 &= \frac{\{\text{involution}\}}{(\overline{\overline{x \setminus \overline{y}}})/z} \\
 &= \frac{\{\text{by Law 4.1}\}}{x \setminus (\overline{\overline{\overline{y}/z}})} \\
 &= \frac{\{\text{involution}\}}{\overline{\overline{x \setminus \overline{y}/z}}} \\
 &= \frac{\{\text{definitions}\}}{x](y[z])} .
 \end{aligned}$$

7 Residuals and Detachment in Particular Quantales

For a completely distributive Boolean algebra (M, \leq) , the structure $B(M) \stackrel{\text{def}}{=} (M, \leq, 0, \top, \sqcap, \sqcup)$ is a Boolean quantale.

Law 7.1 In $B(M)$ one has $x/y = y \rightarrow x$.

$$\begin{aligned}
\text{Proof: } \quad & z \leq x/y \\
& \Leftrightarrow \quad \{ \text{by (GC)} \} \\
& \quad z \sqcap y \leq x \\
& \Leftrightarrow \quad \{ \text{shunting} \} \\
& \quad z \leq \bar{y} \sqcup x .
\end{aligned}$$

Law 7.2 In $B(M)$ one has $x \lfloor y = x \sqcap y$.

$$\begin{aligned}
\text{Proof: } \quad & x \lfloor y \\
& = \quad \{ \text{definition} \} \\
& \quad \overline{\bar{x}/y} \\
& = \quad \{ \text{previous law} \} \\
& \quad \overline{\bar{y} \sqcup \bar{x}} \\
& = \quad \{ \text{de Morgan} \} \\
& \quad x \sqcap y .
\end{aligned}$$

Dually, $(M, \geq, \top, 0, \sqcup, 0)$ is again a Boolean quantale with analogous laws.

8 Interaction with Subidentities

In this section we deal with subidentities $p \leq 1$ in a Boolean quantale and their relative complements $\neg p \stackrel{\text{def}}{=} \bar{p} \sqcap 1$. As auxiliary properties we note the complement rules

$$\overline{p \cdot \top} = \neg p \cdot \top , \quad \overline{\top \cdot p} = \top \cdot \neg p , \quad (\text{CR})$$

and the restriction law

$$p \cdot x = x \sqcap p \cdot \top \quad (\text{RE})$$

(see e.g. [4]). In the remainder we assume $p, q \leq 1$.

Law 8.1 $x/p = x + \top \cdot \neg p$.

Proof:

$$\begin{aligned} z \leq x/p & \\ \Leftrightarrow \{ \text{by (GC)} \} & \\ z \cdot p \leq x & \\ \Leftrightarrow \{ \text{by (RE)} \} & \\ z \sqcap \top \cdot p \leq x & \\ \Leftrightarrow \{ \text{shunting} \} & \\ z \leq \overline{\top \cdot p} + x & \\ \Leftrightarrow \{ \text{by (CR)} \} & \\ z \leq \top \cdot \neg p + x . & \end{aligned}$$

Law 8.2 $0/p = \top \cdot \neg p$.

Proof: Set $x = 0$ in the previous law.

Law 8.3 $x/p = x + 0/p$.

Proof: Immediate from the previous two laws.

Law 8.4 $x \cdot p = x \sqcap 0/\neg p$.

Proof:

$$\begin{aligned} x \cdot p & \\ = \{ \text{by (RE)} \} & \\ x \sqcap \top \cdot p & \\ = \{ \text{by Law 8.2} \} & \\ x \sqcap 0/\neg p . & \end{aligned}$$

Law 8.5 $x \lfloor p = x \cdot p$.

Proof:

$$\begin{aligned} x \lfloor p & \\ = \{ \text{definition} \} & \\ \overline{\overline{x}/p} & \\ = \{ \text{by Law 8.1} \} & \end{aligned}$$

$$\begin{aligned}
& \overline{x + \top \cdot \neg p} \\
= & \quad \{ \text{de Morgan} \} \\
& x \sqcap \overline{\top \cdot \neg p} \\
= & \quad \{ \text{by (CR)} \} \\
& x \sqcap \top \cdot p \\
= & \quad \{ \text{by (RE)} \} \\
& x \cdot p .
\end{aligned}$$

9 Interaction with Predomain and Precodomain

Definition 9.1 The *predomain* operation in a Boolean left quantale is defined by the following Galois connection [7,5]:

$$\top a \leq p \Leftrightarrow a \leq p \cdot \top .$$

It is called *domain* operation if additionally it satisfies the axiom of *left locality of composition* [7]

$$\top(a \cdot b) = \top(a \cdot \top b) .$$

The *(pre)codomain* operation in a Boolean right quantale is defined symmetrically.

This is well defined, since one can show that \cdot preserves arbitrary infima of subidentities [4].

Law 9.2 $\top(p \cdot a) = p \cdot \top a$.

$$\begin{aligned}
\text{Proof:} \quad & p \cdot \top a \leq q \\
\Leftrightarrow & \quad \{ \text{by (GC)} \} \\
& \top a \leq p \backslash q \\
\Leftrightarrow & \quad \{ \text{definition of domain} \} \\
& a \leq (p \backslash q) \cdot \top \\
\Leftrightarrow & \quad \{ \text{dual of Law 8.1} \} \\
& a \leq (q + \neg p \cdot \top) \cdot \top
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \{ \text{distributivity and idempotence of } \top \} \\
&\quad a \leq q \cdot \top + \neg p \cdot \top \\
&\Leftrightarrow \{ \text{dual of Law 8.1} \} \\
&\quad a \leq p \setminus (q \cdot \top) \\
&\Leftrightarrow \{ \text{by (GC)} \} \\
&\quad p \cdot a \leq q \cdot \top \\
&\Leftrightarrow \{ \text{definition of domain} \} \\
&\quad \ulcorner (p \cdot a) \leq q \text{ .}
\end{aligned}$$

Law 9.3 $\ulcorner (x \lfloor y) \leq \ulcorner x$.

$$\begin{aligned}
\textit{Proof:} \quad &\ulcorner (x \lfloor y) \leq q \\
&\Leftrightarrow \{ \text{definition of domain} \} \\
&\quad x \lfloor y \leq q \cdot \top \\
&\Leftrightarrow \{ \text{exchange} \} \\
&\quad \overline{q \cdot \top} \cdot y \leq \bar{x} \\
&\Leftrightarrow \{ \text{by (CR)} \} \\
&\quad \neg q \cdot \top \cdot y \leq \bar{x} \\
&\Leftarrow \{ \text{by } \top \cdot y \leq \top \text{ and isotony} \} \\
&\quad \neg q \cdot \top \leq \bar{x} \\
&\Leftrightarrow \{ \text{by shunting and (CR)} \} \\
&\quad x \leq q \cdot \top \\
&\Leftrightarrow \{ \text{definition of domain} \} \\
&\quad \ulcorner x \leq q \text{ .}
\end{aligned}$$

Law 9.4 $\ulcorner x \leq p \Leftrightarrow x = p \cdot x$.

$$\begin{aligned}
\textit{Proof:} \quad &\ulcorner x \leq p \\
&\Leftrightarrow \{ \text{definition of domain} \} \\
&\quad x \leq p \cdot \top \\
&\Leftrightarrow \{ \text{lattice algebra} \}
\end{aligned}$$

$$\begin{aligned}
& x = x \sqcap p \cdot \top \\
\Leftrightarrow & \quad \{ \text{by (RE)} \} \\
& x = p \cdot x .
\end{aligned}$$

Law 9.5 $\lceil x \leq p \Leftrightarrow x \leq p \cdot x$.

Proof: By $p \leq 1$ the inclusion $p \cdot x \leq x$ of the previous law is trivial.

Law 9.6 $\lceil x \leq p \Leftrightarrow \neg p \cdot x = 0$.

$$\begin{aligned}
\textit{Proof:} \quad & \lceil x \leq p \\
\Leftrightarrow & \quad \{ \text{previous law} \} \\
& x \leq p \cdot x \\
\Leftrightarrow & \quad \{ \text{shunting} \} \\
& x \sqcap \overline{p \cdot x} = 0 \\
\Leftrightarrow & \quad \{ \text{by (CR)} \} \\
& x \sqcap (\neg p \cdot x + \bar{x}) = 0 \\
\Leftrightarrow & \quad \{ \neg p \cdot x \leq x \text{ and Boolean algebra} \} \\
& \neg p \cdot x = 0 .
\end{aligned}$$

Law 9.7 $\lceil x = x \lfloor x \sqcap 1$.

$$\begin{aligned}
\textit{Proof:} \quad & x \lfloor x \sqcap 1 \leq p \\
\Leftrightarrow & \quad \{ \text{shunting} \} \\
& x \lfloor x \leq \bar{1} \sqcup p \\
\Leftrightarrow & \quad \{ \text{definition of } \neg p \text{ and Boolean algebra} \} \\
& x \lfloor x \leq \overline{\neg p} \\
\Leftrightarrow & \quad \{ \text{exchange} \} \\
& \neg p \cdot x \leq \bar{x} \\
\Leftrightarrow & \quad \{ \text{Boolean algebra} \} \\
& \neg p \cdot x \sqcap x = 0 \\
\Leftrightarrow & \quad \{ \text{by } p \leq 1 \text{ and lattice algebra} \} \\
& \neg p \cdot x = 0
\end{aligned}$$

$$\Leftrightarrow \{ \text{by Law 9.6} \}$$

$$\lceil x \leq p .$$

The next law provides a computationally more pleasing expression for the domain, since the variable x is not repeated on the right hand side.

Law 9.8 $\lceil x = \top \lfloor x \sqcap 1.$

Proof:

$$\lceil x \leq p$$

$$\Leftrightarrow \{ \text{by Law 9.6} \}$$

$$\neg p \cdot x = 0$$

$$\Leftrightarrow \{ \text{by (GC)} \}$$

$$\neg p \leq 0/x$$

$$\Leftrightarrow \{ \text{Boolean algebra} \}$$

$$\overline{0/x} \leq \overline{\neg p}$$

$$\Leftrightarrow \{ \text{definition and Boolean algebra} \}$$

$$\top \lfloor x \leq p + \bar{1}$$

$$\Leftrightarrow \{ \text{shunting} \}$$

$$\top \lfloor x \sqcap 1 \leq p .$$

Law 9.9 $\neg \lceil x = 0/x \sqcap 1.$

Proof:

$$\neg \lceil x$$

$$= \{ \text{definition} \}$$

$$\overline{\lceil x} \sqcap 1$$

$$= \{ \text{previous law} \}$$

$$\overline{\top \lfloor x \sqcap 1} \sqcap 1$$

$$= \{ \text{de Morgan} \}$$

$$(\overline{\top \lfloor x + \bar{1}}) \sqcap 1$$

$$= \{ \text{Boolean algebra} \}$$

$$\overline{\top \lfloor x} \sqcap 1$$

$$= \{ \text{definition and Boolean algebra} \}$$

$$0/x \sqcap 1 .$$

Finally, using the dual of Law 8.4, we can give a different form of the overwrite operation $x | y \stackrel{\text{def}}{=} x + \neg \ulcorner x \cdot y$.

Law 9.10 $x | y = (x + y) \sqcap \ulcorner x \setminus x$.

$$\begin{aligned}
\textit{Proof:} \quad & x | y \\
&= \quad \{ \text{definition} \} \\
&\quad x + \neg \ulcorner x \cdot y \\
&= \quad \{ \text{by Law 8.4} \} \\
&\quad x + (y \sqcap \ulcorner x \setminus 0) \\
&= \quad \{ \text{distributivity} \} \\
&\quad (x + y) \sqcap (x + \ulcorner x \setminus 0) \\
&= \quad \{ \text{by Law 8.3} \} \\
&\quad (x + y) \sqcap \ulcorner x \setminus x .
\end{aligned}$$

10 About Locality of Composition

The aim of this section is to give a characterisation of locality of composition without using the domain operation.

We first show

Lemma 10.1 *A Boolean left quantale satisfies left-locality of composition iff for all x*

$$\top \lfloor x = \top \lceil x .$$

Proof. (\Rightarrow)

$$\begin{aligned}
& \top \lfloor x \leq y \\
\Leftrightarrow & \quad \{ \text{exchange} \} \\
& \bar{y} \cdot x \leq 0 \\
\Leftrightarrow & \quad \{ \text{strictness of predomain} \} \\
& \ulcorner (\bar{y} \cdot x) \leq 0 \\
\Leftrightarrow & \quad \{ \text{left-locality of composition} \} \\
& \ulcorner (\bar{y} \cdot \ulcorner x) \leq 0 \\
\Leftrightarrow & \quad \{ \text{strictness of predomain} \}
\end{aligned}$$

$$\begin{aligned}
& \bar{y} \cdot \ulcorner x \leq 0 \\
\Leftrightarrow & \quad \{ \text{exchange} \} \\
& \top \lfloor \ulcorner x \leq y .
\end{aligned}$$

(\Leftarrow) By the defining Galois connection for predomain, the lattice of subidentities is isomorphic to the lattice of ideals $\{p \cdot \top \mid p \leq 1\}$. So to show $\ulcorner(a \cdot b) = \ulcorner(a \cdot \ulcorner b)$ it suffices to show $\top \cdot \ulcorner(a \cdot b) = \top \cdot \ulcorner(a \cdot \ulcorner b)$. By Law 8.5 this is equivalent to $\top \lfloor \ulcorner(a \cdot b) = \top \lfloor \ulcorner(a \cdot \ulcorner b)$.

$$\begin{aligned}
& \top \lfloor \ulcorner(a \cdot \ulcorner b) \\
= & \quad \{ \text{assumption} \} \\
& \top \lfloor (a \cdot \ulcorner b) \\
= & \quad \{ \text{by Law 5.12} \} \\
& (\top \lfloor \ulcorner b) \lfloor a \\
= & \quad \{ \text{assumption} \} \\
& (\top \lfloor b) \lfloor a \\
= & \quad \{ \text{by Law 5.12} \} \\
& \top \lfloor (a \cdot b) \\
= & \quad \{ \text{assumption} \} \\
& \top \lfloor \ulcorner(a \cdot b) .
\end{aligned}$$

Corollary 10.2 *A Boolean left quantale has left locality composition iff for all x*

$$0/x = 0/\ulcorner x .$$

Next we observe that, even without left locality of composition, we have

Law 10.3 $\top \lfloor x \leq \top \cdot \ulcorner x$.

$$\begin{aligned}
\textit{Proof:} \quad & \top \lfloor x \leq \top \cdot \ulcorner x \\
\Leftrightarrow & \quad \{ \text{exchange} \} \\
& \overline{\top \cdot \ulcorner x} \cdot x \leq 0 \\
\Leftrightarrow & \quad \{ \text{by (CR)} \} \\
& \top \cdot \neg \ulcorner x \cdot x \leq 0 \\
\Leftrightarrow & \quad \{ \text{domain law} \}
\end{aligned}$$

$$\begin{aligned}
& \top \cdot 0 \leq 0 \\
\Leftrightarrow & \quad \{ \text{strictness} \} \\
& \text{TRUE} .
\end{aligned}$$

Corollary 10.4 *A Boolean left quantale has left locality of composition iff for all x*

$$\top \cdot \lceil x \leq \top \lfloor x .$$

Dually, we have

Corollary 10.5 *A Boolean left quantale has left locality of composition iff for all x*

$$0/x \leq \top \cdot \neg \lceil x .$$

With the expressions for predomain and its negation we get

Corollary 10.6 *The following statements are equivalent:*

1. *A Boolean left quantale satisfies left-locality of composition.*
2. $\forall x : \top \cdot (\top \lfloor x \sqcap 1) \leq \top \lfloor x .$
3. $\forall x : 0/x \leq \top \cdot (0/x \sqcap 1) .$

This admits a simple proof that Euclid's law

$$x \cdot (y \lfloor z) \leq (x \cdot y) \lfloor z$$

implies left-locality of composition:

$$\begin{aligned}
& \top \cdot (\top \lfloor x \sqcap 1) \leq \top \lfloor x \\
\Leftarrow & \quad \{ \text{definition of } \sqcap \text{ and isotony} \} \\
& \top \cdot (\top \lfloor x) \leq \top \lfloor x \\
\Leftarrow & \quad \{ \text{idempotence of } \top \} \\
& \top \cdot (\top \lfloor x) \leq (\top \cdot \top) \lfloor x \\
\Leftarrow & \quad \{ \text{Euclid} \} \\
& \text{TRUE} .
\end{aligned}$$

11 Totality and Local Composition

Motivated by the previous section we define

$$x \in \text{LLC} \stackrel{\text{def}}{\Leftrightarrow} 0/x \leq \top \cdot \neg x .$$

So LLC is the set of elements x that satisfy $\top \sqcup x = \top \sqcup x$. We have chosen the above formulation, since it is handy for the proofs to come.

Moreover, we introduce the set of *left-total* elements by

$$x \in \text{LT} \stackrel{\text{def}}{\Leftrightarrow} \forall y : y \cdot x = 0 \Rightarrow y = 0 .$$

This is a relaxation of the property of overall indivisibility of 0.

Law 11.1 $x \in \text{LT} \Leftrightarrow 0/x = 0$.

Proof:

$$\begin{aligned} & \forall y : y \cdot x = 0 \Rightarrow y = 0 \\ \Leftrightarrow & \quad \{ \text{leastness of } 0 \} \\ & \forall y : y \cdot x \leq 0 \Rightarrow y \leq 0 \\ \Leftrightarrow & \quad \{ \text{by (GC)} \} \\ & \forall y : y \leq 0/x \Rightarrow y \leq 0 \\ \Leftrightarrow & \quad \{ \text{indirect inequality} \} \\ & 0/x \leq 0 \\ \Leftrightarrow & \quad \{ \text{leastness of } 0 \} \\ & 0/x = 0 . \end{aligned}$$

By this law, $0/x$ is a good measure of the “left-definedness” of x : the smaller $0/x$, the more left-defined is x . This fits well with the law $0/x \sqcap 1 = \neg x$ which shows that $\neg x$ is a corresponding measure of left-definedness of x at the level of tests.

Law 11.2 $\text{LT} \subseteq \text{LLC}$.

Proof: Immediate from the definition of LLC and Law 11.1.

Law 11.3 $0 \in \text{LT} \Leftrightarrow 0 = \top$.

Proof: Immediate from Law 11.1 and Law 3.5.

Law 11.4 $0 \in \text{LLC}$.

Proof: Immediate from the definition of LLC, Law 3.5, $\top 0 = 0$ and Boolean algebra.

Law 11.5 $x \in \text{LT} \Rightarrow \top x = 1$.

Proof:

$$\begin{aligned} & \neg \top x \\ = & \quad \{ \text{by Law 9.9} \} \\ & 0/x \sqcap 1 \\ = & \quad \{ \text{by Law 11.1} \} \\ & 0 \sqcap 1 \\ = & \quad \{ \text{lattice algebra} \} \\ & 0 . \end{aligned}$$

Let now K be the set of elements of the underlying quantale. We call the quantale *left-total* iff $K = \text{LT} \cup \{0\}$.

Law 11.6 $K = \text{LT} \cup \{0\} \Rightarrow K = \text{LLC}$.

Proof: Immediate by Law 11.2 and Law 11.4.

In other words, every total algebra satisfies left-locality of composition.

Law 11.7 $K = \text{LT} \cup \{0\} \wedge p \in \{0, 1\}$.

Proof: Follows from Law 11.5 and $\top 0 = 0$.

In other words, a total algebra can only have a trivial subidentity structure.

For these last two laws there are also proofs without the use of residuals or detachment; however, they are a lot more cumbersome.

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