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Residuals and Detachments

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Abstract. We give a compendium of algebraic calculation rules for the operations of residuation and detachment in semirings.

1 Introduction

Residuals [1,2] and detachments have many useful applications. This report serves as a compendium of laws for these operations, many of which are known from early residuation theory. However, there is also some new material relating residuals with tests and (pre)domain, in particular, a characterisation of locality of composition [7] without recourse to the domain operation.

2 Definitions and Proof Principles

Definition 2.1 1. A structure $(S, \leq, 0, \top, \cdot, 1)$ is called a *left (right) quantale* if $(S, \leq, 0, \top)$ is a complete lattice with least element 0 and greatest element $\top$ such that $(S, \cdot, 1)$ is a monoid and $\cdot$ preserves arbitrary suprema in its left (right) argument. The supremum of elements x and y is denoted by $x + y$. Any left (right) quantale satisfies $0 \cdot x = 0$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ ($x \cdot 0 = 0$ and $x \cdot (y + z) = x \cdot y + x \cdot z$). S is called a *quantale* [8] if it is both a left and right quantale. Quantales have been called *standard Kleene algebras* in [3].

2. A (left or right) quantale is called *Boolean* if its underlying lattice is a completely distributive Boolean algebra.

3. In a left quantale, the *left residual* and *right detachment* operations are defined as usual:

$$z \leq x/y \stackrel{\text{def}}{\Leftrightarrow} z \cdot y \leq x \ , \qquad\qquad x \lfloor y \stackrel{\text{def}}{=} \overline{\overline{x}/y} \ . \qquad \text{(GC)}$$

By these definitions, the function $\lambda x \,.\, x/y$ is the upper adjoint and the function $\lambda z \,.\, z \cdot y$ the lower adjoint of a Galois connection.

Symmetric definitions and laws apply to the *right residual* $\backslash$ and *left detachment* $\rfloor$ in a right quantale.

A useful tool for working with elements of a poset are the rules of *indirect inequality*:

$$\begin{array}{l} x \leq y \;\Leftrightarrow\; (\forall\, z : z \leq x \;\Rightarrow\; z \leq y) \;, \\ x \leq y \;\Leftrightarrow\; (\forall\, z : y \leq z \;\Rightarrow\; x \leq z) \;. \end{array}$$

Moreover, we have the rules of *indirect equality*:

$$x = y \quad\Leftrightarrow\quad (\forall\, z : z \leq x \;\Leftrightarrow\; z \leq y) \quad\Leftrightarrow\quad (\forall\, z : x \leq z \;\Leftrightarrow\; y \leq z) \;.$$

As special cases of this, we get

$$\begin{array}{l} x = \top \;\Leftrightarrow\; (\forall\, z : z \leq x \;\Leftrightarrow\; \mathrm{TRUE}) \;, \\ x = 0 \;\;\Leftrightarrow\; (\forall\, z : x \leq z \;\Leftrightarrow\; \mathrm{TRUE}) \;, \end{array}$$

A related principle is provided by the universal characterisations of infima and suprema:

$$\begin{array}{l} y \leq \sqcap X \;(\forall\, x \in X : y \leq x) \;, \\ \sqcup X \leq y \;(\forall\, x \in X : x \leq y) \;. \end{array} \qquad \text{(Inf/Sup)}$$

We will use all these rules tacitly in the remainder.

Definition 2.2 1. The *dual* $f^\natural$ of an endofunction f on a Boolean algebra is defined by

$$f^\natural(x) \stackrel{\mathrm{def}}{=} \overline{f(\overline{x})} \;.$$

2. Two functions f, g between Boolean algebras are called *conjugate* [6] if they satisfy

$$f(x) \leq \overline{y} \;\Leftrightarrow\; g(y) \leq \overline{x} \;. \qquad (*)$$

By straightforward Boolean algebra, the property that f and g are conjugate is equivalent to the Galois connection

$$f(x) \leq y \;\Leftrightarrow\; x \leq g^\natural(y) \;.$$

Lemma 2.3 *Assume that f, g are conjugate.*

1. $f\big(\overline{g(y)}\big) \le \overline{y}$.
2. $g\big(\overline{f(x)}\big) \le \overline{x}$.
3. *f and g preserve all suprema and hence are isotone and strict.*

Proof. 1. Set $x = \overline{g(y)}$ in $(*)$.
2. Set $y = \overline{f(x)}$ in $(*)$.
3. By the above remark, both f and g are lower adjoints in Galois connections.

Lemma 2.4 (Modularity; Dedekind) *For conjugate f and g,*

$$f(x) \sqcap y \le f(x \sqcap g(y)) \ .$$

Proof.

$$
\begin{array}{ll}
 & f(x) \sqcap y \\
= & \quad \{\!|\ \text{Boolean algebra}\ |\!\} \\
 & f((x \sqcap g(y)) \sqcup (x \sqcap \overline{g(y)})) \sqcap y \\
= & \quad \{\!|\ f \text{ preserves suprema}\ |\!\} \\
 & (f(x \sqcap g(y)) \sqcup f(x \sqcap \overline{g(y)})) \sqcap y \\
\le & \quad \{\!|\ \text{definition of } \sqcap \text{ and isotony of } f\ |\!\} \\
 & (f(x \sqcap g(y)) \sqcup f(\overline{g(y)})) \sqcap y \\
\le & \quad \{\!|\ \text{by Lemma 2.3.1}\ |\!\} \\
 & (f(x \sqcap g(y)) \sqcup \overline{y}) \sqcap y \\
= & \quad \{\!|\ \text{Boolean algebra}\ |\!\} \\
 & f(x \sqcap g(y)) \sqcap y \\
\le & \quad \{\!|\ \text{definition of } \sqcap\ |\!\} \\
 & f(x \sqcap g(y)) \ .
\end{array}
$$

As our final proof tool in Boolean algebras we mention

$$x \sqcap y \le z \ \Leftrightarrow\ x \le \overline{y} \sqcup z \ . \qquad \text{(Shunting)}$$

3 Laws for Residuals

Law 3.1 (Left-Conjunctivity) $(\sqcap X)/y = \sqcap (X/y)$.
Proof: Upper adjoints preserve all infima.

Law 3.2 $\top/y = \top$.
Proof: Set $X = \emptyset$ in the previous law.

Law 3.3 $u \leq v \Rightarrow u/y \leq v/y$.
Proof: Immediate from left-conjunctivity.

Law 3.4 (Right-Antidisjunctivity) $x/(\sqcup Y) = \sqcap (x/Y)$.

Proof:

$$
\begin{array}{ll}
 & z \leq x/(\sqcup Y) \\
\Leftrightarrow & \quad \{\!\![\text{ by (GC) }]\!\!\} \\
 & z \cdot \sqcup Y \leq x \\
\Leftrightarrow & \quad \{\!\![\text{ disjunctivity of } \cdot \ ]\!\!\} \\
 & \sqcup (z \cdot Y) \leq x \\
\Leftrightarrow & \quad \{\!\![\text{ lattice algebra }]\!\!\} \\
 & \forall\, y \in Y : z \cdot y \leq x \\
\Leftrightarrow & \quad \{\!\![\text{ by (GC) }]\!\!\} \\
 & \forall\, y \in Y : z \leq x/y \\
\Leftrightarrow & \quad \{\!\![\text{ lattice algebra }]\!\!\} \\
 & z \leq \sqcap x/Y \ .
\end{array}
$$

Law 3.5 $x/0 = \top$.
Proof: Set $Y = \emptyset$ in the previous law.

Law 3.6 $u \leq v \Rightarrow x/u \geq x/v$.
Proof: Immediate from right-antidisjunctivity.

Law 3.7 $1 \leq x/x$.
Proof: Immediate from (GC) and neutrality of 1.

Law 3.8 $(x/y) \cdot y \leq x$.
Proof: Set $z = x/y$ in (GC).

Law 3.9 $(x/y) \cdot y = x \Leftrightarrow \exists\, z : x = z \cdot y$.
Proof: The implication ($\Rightarrow$) is trivial. For ($\Leftarrow$) assume $z \cdot y = x$. Then $z \cdot y \leq x$ and hence by (GC) we get $z \leq x/y$. Since also $x \leq z \cdot y$ we obtain from this by isotony $x \leq (x/y) \cdot y$. The reverse inequality is given by Law 3.8.

Law 3.10 $(x/x) \cdot x = x$.
Proof: Use Law 3.9 and set $x = y$ and $z = 1$.

Law 3.11 $x/1 = x$.
Proof: Use Law 3.9 and set $y = 1$ and $z = x$.

Law 3.12 $(0/y) \cdot y = 0$.
Proof: For ($\leq$) set $x = 0$ in Law 3.8. ($\geq$) is trivial.

Law 3.13 $x/(y \cdot z) = (x/z)/y$.

Proof:
$$
\begin{array}{ll}
 & u \leq x/(y \cdot z) \\
\Leftrightarrow & \quad \{\!\![\text{ by (GC) }]\!\!\} \\
 & u \cdot y \cdot z \leq x \\
\Leftrightarrow & \quad \{\!\![\text{ by (GC) }]\!\!\} \\
 & u \cdot y \leq x/z \\
\Leftrightarrow & \quad \{\!\![\text{ by (GC) }]\!\!\} \\
 & u \leq (x/z)/y \ .
\end{array}
$$

Law 3.14 ("Euclid" for Residual) $x \cdot (y/z) \leq (x \cdot y)/z$.

Proof:
$$
\begin{array}{ll}
 & x \cdot (y/z) \leq (x \cdot y)/z \\
\Leftrightarrow & \quad \{\!\![\text{ by (GC) }]\!\!\} \\
 & x \cdot (y/z) \cdot z \leq x \cdot y \\
\Leftarrow & \quad \{\!\![\text{ by Law 3.8 and isotony }]\!\!\} \\
 & \text{TRUE} \ .
\end{array}
$$

Law 3.15 $x \leq (x \cdot y)/y$.
Proof: Immediate by (GC) and reflexivity of $\leq$.

Law 3.16 $(x \cdot y)/y = x \Leftrightarrow \exists\, z : x = z/y$.
Proof: ($\Rightarrow$) is trivial. For ($\Leftarrow$) assume $x = z/y$. Then

$$
\begin{array}{ll}
 & (x \cdot y)/y \\
= & \quad \{\!\![\text{ assumption }]\!\!\} \\
 & ((z/y) \cdot y)/y \\
\leq & \quad \{\!\![\text{ by Law 3.8 and left-isotony of } / \text{ }]\!\!\} \\
 & z/y \\
= & \quad \{\!\![\text{ assumption }]\!\!\} \\
 & x
\end{array}
$$

The reverse inequality is given by Law 3.15.

Law 3.17 $(\top \cdot y)/y = \top$.
Proof: For ($\geq$) set $x = \top$ in Law 3.15. ($\leq$) is trivial.

Law 3.18 $x \cdot y = ((x \cdot y)/y) \cdot y$.
Proof: By GC and standard Galois theory.

Law 3.19 $x/y = ((x/y) \cdot y)/y$.
Proof: By GC and standard Galois theory.

Law 3.20 $1/x \leq y/(x \cdot y)$.

Proof:

$$
\begin{array}{ll}
 & 1/x \\
\leq & \quad \{\!\![\text{ by Law 3.7 and left-isotony of } / \text{ }]\!\!\} \\
 & (y/y)/x \\
= & \quad \{\!\![\text{ by Law 3.13 }]\!\!\} \\
 & y/(x \cdot y) \ .
\end{array}
$$

Law 3.21 $(x/y) \cdot (y/z) \leq x/z$.

Proof:

$$
\begin{array}{ll}
 & (x/y) \cdot (y/z) \leq x/z \\
\Leftrightarrow & \quad \{\!\!\{ \text{ by (GC) } \}\!\!\} \\
 & (x/y) \cdot (y/z) \cdot z \leq x \\
\Leftarrow & \quad \{\!\!\{ \text{ by Law 3.8 and isotony of } \cdot \}\!\!\} \\
 & (x/y) \cdot y \leq x \\
\Leftrightarrow & \quad \{\!\!\{ \text{ by Law 3.8 } \}\!\!\} \\
 & \text{TRUE} .
\end{array}
$$

Law 3.22 x/x is a preorder.
Proof: Immediate from Law 3.7 and Law 3.21.

Law 3.23 $x/y \leq (x/z)/(y/z)$.

Proof:

$$
\begin{array}{ll}
 & u \leq (x/z)/(y/z) \\
\Leftrightarrow & \quad \{\!\!\{ \text{ by (GC) } \}\!\!\} \\
 & u \cdot (y/z) \leq x/z \\
\Leftrightarrow & \quad \{\!\!\{ \text{ by (GC) } \}\!\!\} \\
 & u \cdot (y/z) \cdot z \leq x \\
\Leftarrow & \quad \{\!\!\{ \text{ by Law 3.8 and isotony } \}\!\!\} \\
 & u \cdot y \leq x \\
\Leftrightarrow & \quad \{\!\!\{ \text{ by (GC) } \}\!\!\} \\
 & u \leq x/y .
\end{array}
$$

4 Interaction Between Residuals

Law 4.1 $(x\backslash y)/z = x\backslash(y/z)$.

Proof:

$$
\begin{array}{ll}
 & u \leq (x\backslash y)/z \\
\Leftrightarrow & \quad \{\!\!\{ \text{ by (GC) } \}\!\!\} \\
 & u \cdot z \leq x\backslash y
\end{array}
$$

$$\Leftrightarrow \quad \{\!\![\text{ by (GC) }]\!\!\}$$
$$x \cdot u \cdot z \le y$$
$$\Leftrightarrow \quad \{\!\![\text{ by (GC) }]\!\!\}$$
$$x \cdot u \le y/z$$
$$\Leftrightarrow \quad \{\!\![\text{ by (GC) }]\!\!\}$$
$$u \le x\backslash(y/z)\ .$$

5 Laws for Detachments

Law 5.1 (Exchange; Schröder) $x \cdot y \le z \Leftrightarrow \overline{z} \lfloor y \le \overline{x}$.

Proof:
$$x \cdot y \le z$$
$$\Leftrightarrow \quad \{\!\![\text{ by (GC) }]\!\!\}$$
$$x \le z/y$$
$$\Leftrightarrow \quad \{\!\![\text{ Boolean algebra }]\!\!\}$$
$$\overline{z/y} \le \overline{x}$$
$$\Leftrightarrow \quad \{\!\![\text{ Boolean algebra and definition } \lfloor\ ]\!\!\}$$
$$\overline{z} \lfloor y \le \overline{x}\ .$$

By this law the functions $\lambda x \,.\, x \cdot y$ and $\lambda z \,.\, z \lfloor y$ are conjugates:

Law 5.2 $x \cdot y \sqcap z = 0 \Leftrightarrow z \lfloor y \sqcap x = 0$.
Proof: Immediate from exchange by shunting (substitute $\overline{z}$ for z).

Law 5.3 (Dedekind)
$x \sqcap y \cdot z \le (x \lfloor z \sqcap y) \cdot z$ and $x \sqcap y \lfloor z \le (x \cdot z \sqcap y) \lfloor z$.
Proof: Set $f(z) \stackrel{\text{def}}{=} a \cdot z$ and $g(z) \stackrel{\text{def}}{=} a \lfloor z$ in Lemma 2.4.

Law 5.4 (Left-Disjunctivity) $(\bigsqcup X) \lfloor y = \bigsqcup (X \lfloor y)$.
Proof: Conjugates preserve suprema (Lemma 2.3.3).

Law 5.5 $0 \lfloor y = 0$.
Proof: Set $X = \emptyset$ in the previous law.

Law 5.6 $u \leq v \ \Rightarrow\ u\lfloor y \leq v\lfloor y$.
Proof: Immediate from left-disjunctivity.

Law 5.7 (Right-Disjunctivity) $x\lfloor(\sqcup Y) = \sqcup(x\lfloor Y)$.

Proof:

$$
\begin{array}{cl}
 & x\lfloor(\sqcup Y) \\
= & \quad \{\!\![\ \text{definition}\ ]\!\!\} \\
 & \overline{\overline{x}/\sqcup Y} \\
= & \quad \{\!\![\ \text{Law 3.4}\ ]\!\!\} \\
 & \overline{\sqcap\, \overline{x}/Y} \\
= & \quad \{\!\![\ \text{de Morgan}\ ]\!\!\} \\
 & \sqcup\, \overline{\overline{x}/Y} \\
= & \quad \{\!\![\ \text{definition}\ ]\!\!\} \\
 & \sqcup\, x\lfloor Y\ .
\end{array}
$$

Law 5.8 $x\lfloor 0 = 0$.
Proof: Set $Y = \emptyset$ in the previous law.

Law 5.9 $u \leq v \ \Rightarrow\ x\lfloor u \leq x\lfloor v$.
Proof: Immediate from right-disjunctivity.

Law 5.10 $x\lfloor 1 = x$.

Proof:

$$
\begin{array}{cl}
 & x\lfloor 1 \leq u \\
= & \quad \{\!\![\ \text{exchange and neutrality of } 1\ ]\!\!\} \\
 & \overline{u} \leq \overline{x} \\
= & \quad \{\!\![\ \text{shunting}\ ]\!\!\} \\
 & x \leq u\ .
\end{array}
$$

Law 5.11 $\top\lfloor\top = \top$.
Proof: Immediate from the previous law, $1 \leq \top$ and isotony.

Law 5.12 $x\lfloor(y\cdot z) = (x\lfloor z)\lfloor y$.

Proof:

$$
\begin{aligned}
& x\lfloor(y\cdot z) \\
= \quad & \{\!|\ \text{definition}\ |\!\} \\
& \overline{\overline{x}/(y\cdot z)} \\
= \quad & \{\!|\ \text{by Law 3.13}\ |\!\} \\
& \overline{(\overline{x}/z)/y} \\
= \quad & \{\!|\ \text{involution}\ |\!\} \\
& \overline{\overline{\overline{(\overline{x}/z)}}/y} \\
= \quad & \{\!|\ \text{definitions}\ |\!\} \\
& (x\lfloor z)\lfloor y\ .
\end{aligned}
$$

6 Interaction Between Detachments

Law 6.1 $(x\rfloor y)\lfloor z = x\rfloor(y\lfloor z)$.

Proof:

$$
\begin{aligned}
& (x\rfloor y)\lfloor z \\
= \quad & \{\!|\ \text{definitions}\ |\!\} \\
& \overline{\overline{\overline{x\backslash\overline{y}}}/z} \\
= \quad & \{\!|\ \text{involution}\ |\!\} \\
& \overline{(x\backslash\overline{y})/z} \\
= \quad & \{\!|\ \text{by Law 4.1}\ |\!\} \\
& \overline{x\backslash(\overline{y}/z)} \\
= \quad & \{\!|\ \text{involution}\ |\!\} \\
& \overline{x\backslash\overline{\overline{\overline{y}/z}}} \\
= \quad & \{\!|\ \text{definitions}\ |\!\} \\
& x\rfloor(y\lfloor z)\ .
\end{aligned}
$$

7 Residuals and Detachment in Particular Quantales

For a completely distributive Boolean algebra $(M,\leq)$, the structure $B(M) \overset{\text{def}}{=} (M,\leq,0,\top,\sqcap,\top)$ is a Boolean quantale.

Law 7.1 In $B(M)$ one has $x/y = y \rightarrow x$.

Proof:

$$
\begin{array}{ll}
 & z \leq x/y \\
\Leftrightarrow & \quad \{\!\![\text{ by (GC) }]\!\!\} \\
 & z \sqcap y \leq x \\
\Leftrightarrow & \quad \{\!\![\text{ shunting }]\!\!\} \\
 & z \leq \overline{y} \sqcup x \ .
\end{array}
$$

Law 7.2 In $B(M)$ one has $x \lfloor y = x \sqcap y$.

Proof:

$$
\begin{array}{ll}
 & x \lfloor y \\
= & \quad \{\!\![\text{ definition }]\!\!\} \\
 & \overline{\overline{x}/y} \\
= & \quad \{\!\![\text{ previous law }]\!\!\} \\
 & \overline{\overline{y} \sqcup \overline{x}} \\
= & \quad \{\!\![\text{ de Morgan }]\!\!\} \\
 & x \sqcap y \ .
\end{array}
$$

Dually, $(M, \geq, \top, 0, \sqcup, 0)$ is again a Boolean quantale with analogous laws.

8 Interaction with Subidentities

In this section we deal with subidentities $p \leq 1$ in a Boolean quantale and their relative complements $\neg p \stackrel{\text{def}}{=} \overline{p} \sqcap 1$. As auxiliary properties we note the complement rules

$$\overline{p \cdot \top} = \neg p \cdot \top \ , \qquad\qquad \overline{\top \cdot p} = \top \cdot \neg p \ , \tag{CR}$$

and the restriction law

$$p \cdot x = x \sqcap p \cdot \top \tag{RE}$$

(see e.g. [4]). In the remainder we assume $p, q \leq 1$.

Law 8.1 $x/p = x + \top \cdot \neg p$.

Proof:

$$
\begin{aligned}
& z \le x/p \\
\Leftrightarrow \quad & \{\!\![\text{ by (GC) }]\!\!\} \\
& z \cdot p \le x \\
\Leftrightarrow \quad & \{\!\![\text{ by (RE) }]\!\!\} \\
& z \sqcap \top \cdot p \le x \\
\Leftrightarrow \quad & \{\!\![\text{ shunting }]\!\!\} \\
& z \le \overline{\top \cdot p} + x \\
\Leftrightarrow \quad & \{\!\![\text{ by (CR) }]\!\!\} \\
& z \le \top \cdot \neg p + x \ .
\end{aligned}
$$

Law 8.2 $0/p = \top \cdot \neg p$.
Proof: Set $x = 0$ in the previous law.

Law 8.3 $x/p = x + 0/p$.
Proof: Immediate from the previous two laws.

Law 8.4 $x \cdot p = x \sqcap 0/\neg p$.

Proof:

$$
\begin{aligned}
& x \cdot p \\
= \quad & \{\!\![\text{ by (RE) }]\!\!\} \\
& x \sqcap \top \cdot p \\
= \quad & \{\!\![\text{ by Law 8.2 }]\!\!\} \\
& x \sqcap 0/\neg p \ .
\end{aligned}
$$

Law 8.5 $x \lfloor p = x \cdot p$.

Proof:

$$
\begin{aligned}
& x \lfloor p \\
= \quad & \{\!\![\text{ definition }]\!\!\} \\
& \overline{\overline{x}/p} \\
= \quad & \{\!\![\text{ by Law 8.1 }]\!\!\}
\end{aligned}
$$

$$
\begin{array}{ll}
 & \overline{\overline{x} + \top \cdot \neg p} \\
= & \quad \{\![\text{ de Morgan }]\!\} \\
 & x \sqcap \overline{\top \cdot \neg p} \\
= & \quad \{\![\text{ by (CR) }]\!\} \\
 & x \sqcap \top \cdot p \\
= & \quad \{\![\text{ by (RE) }]\!\} \\
 & x \cdot p \ .
\end{array}
$$

9 Interaction with Predomain and Precodomain

Definition 9.1 The *predomain* operation in a Boolean left quantale is defined by the following Galois connection [7,5]:

$$\ulcorner a \leq p \Leftrightarrow a \leq p \cdot \top \ .$$

It is called *domain* operation if additionally it satisfies the axiom of *left locality of composition* [7]

$$\ulcorner(a \cdot b) = \ulcorner(a \cdot \ulcorner b) \ .$$

The *(pre)codomain* operation in a Boolean right quantale is defined symmetrically.

This is well defined, since one can show that $\cdot$ preserves arbitrary infima of subidentities [4].

Law 9.2 $\ulcorner(p \cdot a) = p \cdot \ulcorner a$.

Proof:

$$
\begin{array}{ll}
 & p \cdot \ulcorner a \leq q \\
\Leftrightarrow & \quad \{\![\text{ by (GC) }]\!\} \\
 & \ulcorner a \leq p \backslash q \\
\Leftrightarrow & \quad \{\![\text{ definition of domain }]\!\} \\
 & a \leq (p \backslash q) \cdot \top \\
\Leftrightarrow & \quad \{\![\text{ dual of Law 8.1 }]\!\} \\
 & a \leq (q + \neg p \cdot \top) \cdot \top
\end{array}
$$

$\Leftrightarrow$ $\{\!\![$ distributivity and idempotence of $\top$ $]\!\!\}$

$a \le q \cdot \top + \neg p \cdot \top$

$\Leftrightarrow$ $\{\!\![$ dual of Law 8.1 $]\!\!\}$

$a \le p \backslash (q \cdot \top)$

$\Leftrightarrow$ $\{\!\![$ by (GC) $]\!\!\}$

$p \cdot a \le q \cdot \top$

$\Leftrightarrow$ $\{\!\![$ definition of domain $]\!\!\}$

$\ulcorner(p \cdot a) \le q$.

Law 9.3 $\ulcorner(x \lfloor y) \le \ulcorner x$.

Proof: $\ulcorner(x \lfloor y) \le q$

$\Leftrightarrow$ $\{\!\![$ definition of domain $]\!\!\}$

$x \lfloor y \le q \cdot \top$

$\Leftrightarrow$ $\{\!\![$ exchange $]\!\!\}$

$\overline{q \cdot \top} \cdot y \le \overline{x}$

$\Leftrightarrow$ $\{\!\![$ by (CR) $]\!\!\}$

$\neg q \cdot \top \cdot y \le \overline{x}$

$\Leftarrow$ $\{\!\![$ by $\top \cdot y \le \top$ and isotony $]\!\!\}$

$\neg q \cdot \top \le \overline{x}$

$\Leftrightarrow$ $\{\!\![$ by shunting and (CR) $]\!\!\}$

$x \le q \cdot \top$

$\Leftrightarrow$ $\{\!\![$ definition of domain $]\!\!\}$

$\ulcorner x \le q$.

Law 9.4 $\ulcorner x \le p \Leftrightarrow x = p \cdot x$.

Proof: $\ulcorner x \le p$

$\Leftrightarrow$ $\{\!\![$ definition of domain $]\!\!\}$

$x \le p \cdot \top$

$\Leftrightarrow$ $\{\!\![$ lattice algebra $]\!\!\}$

$$x = x \sqcap p \cdot \top$$

$$\Leftrightarrow \quad \{\![\text{ by (RE }]\!\}$$

$$x = p \cdot x \ .$$

Law 9.5 $\ulcorner x \leq p \Leftrightarrow x \leq p \cdot x$.
Proof: By $p \leq 1$ the inclusion $p \cdot x \leq x$ of the previous law is trivial.

Law 9.6 $\ulcorner x \leq p \Leftrightarrow \neg p \cdot x = 0$.

Proof:

$$\ulcorner x \leq p$$

$$\Leftrightarrow \quad \{\![\text{ previous law }]\!\}$$

$$x \leq p \cdot x$$

$$\Leftrightarrow \quad \{\![\text{ shunting }]\!\}$$

$$x \sqcap \overline{p \cdot x} = 0$$

$$\Leftrightarrow \quad \{\![\text{ by (CR) }]\!\}$$

$$x \sqcap (\neg p \cdot x + \overline{x}) = 0$$

$$\Leftrightarrow \quad \{\![\ \neg p \cdot x \leq x \text{ and Boolean algebra }]\!\}$$

$$\neg p \cdot x = 0 \ .$$

Law 9.7 $\ulcorner x = x \lfloor x \sqcap 1$.

Proof:

$$x \lfloor x \sqcap 1 \leq p$$

$$\Leftrightarrow \quad \{\![\text{ shunting }]\!\}$$

$$x \lfloor x \leq \overline{1} \sqcup p$$

$$\Leftrightarrow \quad \{\![\text{ definition of } \neg p \text{ and Boolean algebra }]\!\}$$

$$x \lfloor x \leq \overline{\neg p}$$

$$\Leftrightarrow \quad \{\![\text{ exchange }]\!\}$$

$$\neg p \cdot x \leq \overline{x}$$

$$\Leftrightarrow \quad \{\![\text{ Boolean algebra }]\!\}$$

$$\neg p \cdot x \sqcap x = 0$$

$$\Leftrightarrow \quad \{\![\text{ by } p \leq 1 \text{ and lattice algebra }]\!\}$$

$$\neg p \cdot x = 0$$

$$\begin{array}{ll} \Leftrightarrow & \quad \{\![\text{ by Law 9.6 }]\!\} \\ & \ulcorner x \leq p \ . \end{array}$$

The next law provides a calculationally more pleasing expression for the domain, since the variable x is not repeated on the right hand side.

Law 9.8 $\ulcorner x = \top\lfloor x \sqcap 1$.

Proof:

$$\begin{array}{ll} & \ulcorner x \leq p \\ \Leftrightarrow & \quad \{\![\text{ by Law 9.6 }]\!\} \\ & \neg p \cdot x = 0 \\ \Leftrightarrow & \quad \{\![\text{ by (GC) }]\!\} \\ & \neg p \leq 0/x \\ \Leftrightarrow & \quad \{\![\text{ Boolean algebra }]\!\} \\ & \overline{0/x} \leq \overline{\neg p} \\ \Leftrightarrow & \quad \{\![\text{ definition and Boolean algebra }]\!\} \\ & \top\lfloor x \leq p + \overline{1} \\ \Leftrightarrow & \quad \{\![\text{ shunting }]\!\} \\ & \top\lfloor x \sqcap 1 \leq p \ . \end{array}$$

Law 9.9 $\neg\ulcorner x = 0/x \sqcap 1$.

Proof:

$$\begin{array}{ll} & \neg\ulcorner x \\ = & \quad \{\![\text{ definition }]\!\} \\ & \overline{\ulcorner x} \sqcap 1 \\ = & \quad \{\![\text{ previous law }]\!\} \\ & \overline{\top\lfloor x \sqcap 1} \sqcap 1 \\ = & \quad \{\![\text{ de Morgan }]\!\} \\ & (\overline{\top\lfloor x} + \overline{1}) \sqcap 1 \\ = & \quad \{\![\text{ Boolean algebra }]\!\} \\ & \overline{\top\lfloor x} \sqcap 1 \\ = & \quad \{\![\text{ definition and Boolean algebra }]\!\} \\ & 0/x \sqcap 1 \ . \end{array}$$

Finally, using the dual of Law 8.4, we can give a different form of the overwrite operation $x \,|\, y \stackrel{\text{def}}{=} x + \neg\ulcorner x \cdot y$.

Law 9.10 $x \,|\, y = (x + y) \sqcap \ulcorner x \backslash x$.

Proof:

$$\begin{array}{ll} & x \,|\, y \\ = & \quad \{\!\!\{\ \text{definition}\ \}\!\!\} \\ & x + \neg\ulcorner x \cdot y \\ = & \quad \{\!\!\{\ \text{by Law 8.4}\ \}\!\!\} \\ & x + (y \sqcap \ulcorner x \backslash 0) \\ = & \quad \{\!\!\{\ \text{distributivity}\ \}\!\!\} \\ & (x + y) \sqcap (x + \ulcorner x \backslash 0) \\ = & \quad \{\!\!\{\ \text{by Law 8.3}\ \}\!\!\} \\ & (x + y) \sqcap \ulcorner x \backslash x\ . \end{array}$$

10 About Locality of Composition

The aim of this section is to give a characterisation of locality of composition without using the domain operation.

We first show

Lemma 10.1 *A Boolean left quantale satisfies left-locality of composition iff for all x*

$$\top \lfloor x = \top \lfloor \ulcorner x\ .$$

Proof. ($\Rightarrow$)

$$\begin{array}{ll} & \top \lfloor x \leq y \\ \Leftrightarrow & \quad \{\!\!\{\ \text{exchange}\ \}\!\!\} \\ & \overline{y} \cdot x \leq 0 \\ \Leftrightarrow & \quad \{\!\!\{\ \text{strictness of predomain}\ \}\!\!\} \\ & \ulcorner(\overline{y} \cdot x) \leq 0 \\ \Leftrightarrow & \quad \{\!\!\{\ \text{left-locality of composition}\ \}\!\!\} \\ & \ulcorner(\overline{y} \cdot \ulcorner x) \leq 0 \\ \Leftrightarrow & \quad \{\!\!\{\ \text{strictness of predomain}\ \}\!\!\} \end{array}$$

$$
\begin{array}{ll}
 & \overline{y} \cdot \ulcorner x \leq 0 \\
\Leftrightarrow & \quad \{\!\![\text{ exchange }]\!\!\} \\
 & \top \lfloor \ulcorner x \leq y \ .
\end{array}
$$

($\Leftarrow$) By the defining Galois connection for predomain, the lattice of subidentities is isomorphic to the lattice of ideals $\{p \cdot \top \mid p \leq 1\}$. So to show $\ulcorner(a \cdot b) = \ulcorner(a \cdot \urcorner b)$ it suffices to show $\top \cdot \ulcorner(a \cdot b) = \top \cdot \ulcorner(a \cdot \urcorner b)$. By Law 8.5 this is equivalent to $\top \lfloor \ulcorner(a \cdot b) = \top \lfloor \ulcorner(a \cdot \urcorner b)$.

$$
\begin{array}{ll}
 & \top \lfloor \ulcorner(a \cdot \urcorner b) \\
= & \quad \{\!\![\text{ assumption }]\!\!\} \\
 & \top \lfloor (a \cdot \urcorner b) \\
= & \quad \{\!\![\text{ by Law 5.12 }]\!\!\} \\
 & (\top \lfloor \urcorner b) \lfloor a \\
= & \quad \{\!\![\text{ assumption }]\!\!\} \\
 & (\top \lfloor b) \lfloor a \\
= & \quad \{\!\![\text{ by Law 5.12 }]\!\!\} \\
 & \top \lfloor (a \cdot b) \\
= & \quad \{\!\![\text{ assumption }]\!\!\} \\
 & \top \lfloor \ulcorner(a \cdot b) \ .
\end{array}
$$

Corollary 10.2 *A Boolean left quantale has left locality composition iff for all* x

$$0/x = 0/\ulcorner x \ .$$

Next we observe that, even without left locality of composition, we have

Law 10.3 $\top \lfloor x \leq \top \cdot \ulcorner x$.

Proof:

$$
\begin{array}{ll}
 & \top \lfloor x \leq \top \cdot \ulcorner x \\
\Leftrightarrow & \quad \{\!\![\text{ exchange }]\!\!\} \\
 & \overline{\top \cdot \ulcorner x} \cdot x \leq 0 \\
\Leftrightarrow & \quad \{\!\![\text{ by (CR) }]\!\!\} \\
 & \top \cdot \neg \ulcorner x \cdot x \leq 0 \\
\Leftrightarrow & \quad \{\!\![\text{ domain law }]\!\!\}
\end{array}
$$

$$
\begin{array}{ll}
 & \top \cdot 0 \leq 0 \\
\Leftrightarrow & \quad \{\![\text{ strictness }]\!\} \\
 & \text{TRUE} .
\end{array}
$$

Corollary 10.4 *A Boolean left quantale has left locality of composition iff for all x*

$$\top \cdot \ulcorner x \leq \top \lfloor x \ .$$

Dually, we have

Corollary 10.5 *A Boolean left quantale has left locality of composition iff for all x*

$$0/x \leq \top \cdot \neg \ulcorner x \ .$$

With the expressions for predomain and its negation we get

Corollary 10.6 *The following statements are equivalent:*

1. *A Boolean left quantale satisfies left-locality of composition.*
2. $\forall\, x : \top \cdot (\top \lfloor x \sqcap 1) \leq \top \lfloor x$ *.*
3. $\forall\, x : 0/x \leq \top \cdot (0/x \sqcap 1)$ *.*

This admits a simple proof that Euclid's law

$$x \cdot (y \lfloor z) \leq (x \cdot y) \lfloor z$$

implies left-locality of composition:

$$
\begin{array}{ll}
 & \top \cdot (\top \lfloor x \sqcap 1) \leq \top \lfloor x \\
\Leftarrow & \quad \{\![\text{ definition of } \sqcap \text{ and isotony }]\!\} \\
 & \top \cdot (\top \lfloor x) \leq \top \lfloor x \\
\Leftrightarrow & \quad \{\![\text{ idempotence of } \top \ ]\!\} \\
 & \top \cdot (\top \lfloor x) \leq (\top \cdot \top) \lfloor x \\
\Leftrightarrow & \quad \{\![\text{ Euclid }]\!\} \\
 & \text{TRUE} .
\end{array}
$$

11 Totality and Local Composition

Motivated by the previous section we define

$$x \in \text{LLC} \overset{\text{def}}{\Leftrightarrow} 0/x \leq \top \cdot \neg\ulcorner x\ .$$

So LLC is the set of elements x that satisfy $\top\lfloor\, x = \top\lfloor\ulcorner x$. We have chosen the above formulation, since it its handy for the proofs to come.

Moreover, we introduce the set of *left-total* elements by

$$x \in \text{LT} \overset{\text{def}}{\Leftrightarrow} \forall\, y : y \cdot x = 0 \;\Rightarrow\; y = 0\ .$$

This is a relaxation of the property of overall indivisibility of 0.

Law 11.1 $x \in \text{LT} \;\Leftrightarrow\; 0/x = 0$.

Proof:

$$\begin{array}{ll}
 & \forall\, y : y \cdot x = 0 \;\Rightarrow\; y = 0 \\
\Leftrightarrow & \quad \{\!\![\ \text{leastness of } 0\ ]\!\!\} \\
 & \forall\, y : y \cdot x \leq 0 \;\Rightarrow\; y \leq 0 \\
\Leftrightarrow & \quad \{\!\![\ \text{by (GC)}\ ]\!\!\} \\
 & \forall\, y : y \leq 0/x \;\Rightarrow\; y \leq 0 \\
\Leftrightarrow & \quad \{\!\![\ \text{indirect inequality}\ ]\!\!\} \\
 & 0/x \leq 0 \\
\Leftrightarrow & \quad \{\!\![\ \text{leastness of } 0\ ]\!\!\} \\
 & 0/x = 0\ .
\end{array}$$

By this law, $0/x$ is a good measure of the "left-definedness" of x: the smaller $0/x$, the more left-defined is x. This fits well with the law $0/x \sqcap 1 = \neg\ulcorner x$ which shows that $\neg\ulcorner x$ is a corresponding measure of left-definedness of x at the level of tests.

Law 11.2 LT $\subseteq$ LLC.
Proof: Immediate from the definition of LLC and Law 11.1.

Law 11.3 $0 \in \text{LT} \;\Leftrightarrow\; 0 = \top$.
Proof: Immediate from Law 11.1 and Law 3.5.

Law 11.4 $0 \in \mathrm{LLC}$.
Proof: Immediate from the definition of LLC, Law 3.5, $\ulcorner 0 = 0$ and Boolean algebra.

Law 11.5 $x \in \mathrm{LT} \Rightarrow \ulcorner x = 1$.

Proof:

$$\begin{array}{ll} & \neg\ulcorner x \\ = & \quad \{\![\ \text{by Law 9.9}\ ]\!\} \\ & 0/x \sqcap 1 \\ = & \quad \{\![\ \text{by Law 11.1}\ ]\!\} \\ & 0 \sqcap 1 \\ = & \quad \{\![\ \text{lattice algebra}\ ]\!\} \\ & 0\ . \end{array}$$

Let now K be the set of elements of the underlying quantale. We call the quantale *left-total* iff $K = \mathrm{LT} \cup \{0\}$.

Law 11.6 $K = \mathrm{LT} \cup \{0\} \Rightarrow K = \mathrm{LLC}$.
Proof: Immediate by Law 11.2 and Law 11.4.

In other words, every total algebra satisfies left-locality of composition.

Law 11.7 $K = \mathrm{LT} \cup \{0\} \wedge p \in \{0, 1\}$.
Proof: Follows from Law 11.5 and $\ulcorner 0 = 0$.

In other words, a total algebra can only have a trivial subidentity structure.

For these last two laws there are also proofs without the use of residuals or detachment; however, they are a lot more cumbersome.

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