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A STABLE LIMIT LAW FOR RECURRENCE TIMES OF THE SIMPLE RANDOM WALK ON THE LATTICE $\mathbf{Z}^2$

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Abstract. We consider the random walk of a particle on the two-dimensional integer lattice starting at the origin and moving from each site (independently of the previous moves) with equal probabilities to any of the 4 nearest neighbours. When τ_i denotes the even number of steps between the $(i-1)$st and ith return to the origin, we shall prove that the geometric mean of $\tau_1, \ldots, \tau_n$ divided by n^π converges in distribution to some positive random variable having a logarithmic stable law. We also obtain a rate of this convergence and improve an asymptotic estimate of the tail probability of τ_1 due to Erdös and Taylor (1960).

Keywords: Simple random walk, square lattice, first return time, geometric mean, characteristic function, elliptic integral of first kind, asymptotic expansion, Esseen's inequality, mathematical constants

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1 Introduction and Main Results

We consider the simple symmetric random walk on the $d-$dimensional integer lattice $\mathbb{Z}^d$ starting in the origin $\mathbf{o}$ and moving in each step to one of the $2d$ nearest neighbours with probability $1/2d$. According to a famous result by G. Pólya (see [12] or the monographs [13],[14]) this random walk is recurrent only for $d = 1$ and $d = 2$ but for both cases the *recurrence time* τ_i (= ith *return time* to $\mathbf{o}$ = even number of steps between the $(i-1)$st and ith return to $\mathbf{o}$) has infinite mean. Our aim is to study the asymptotic behaviour of the i.i.d. sequence $\{\tau_i, i \geq 1\}$ for $d = 2$. Whereas for $d = 1$ the τ_i's belong to the domain of attraction of the stable random variable (r.v.) with characteristic exponent $\alpha = 1/2$ and skewness parameter $\beta = 1$, see [4] (p. 171), the situation is completely different for $d = 2$. The crucial difference is reasoned by the logarithmic decay of the tail probability $\mathbf{P}(\tau_1 \geq x)$ for $d = 2$ in contrast to the decaying rate $(\pi\, x)^{-1/2} + \mathcal{O}(x^{-3/2})$ for $d = 1$, see [2] (p. 22). By applying a Tauberian theorem for power series in [4] (p. 423), it follows for $d = 2$ that $\mathbf{P}(\tau_1 \geq x)\, \log x \to \pi$ as $x \to \infty$ and Erdös and Taylor [3] could show somewhat more precisely that $\mathbf{P}(\log \tau_1 \geq x) = \pi/x + \mathcal{O}(x^{-2})$ as $x \to \infty$, see also [14], Chapter 20. As an immediate consequence of the latter estimate one can deduce that $\log \tau_1$ belongs to the domain of attraction of the stable r.v. S with characteristic exponent $\alpha = 1$, skewness parameter $\beta = 1$ and scale parameter $\lambda = \pi^2/2$ having the characteristic function (c.f.)

$\mathbf{E}\exp\{itS\} = \exp\{\ell(t)\}$, where $\ell(t) := -\pi^2\,|t|/2 - i\,\pi\,t\,\log|t|$ for $t \in \mathbb{R}^1$. For a detailed discussion of stable distributions and their c.f's. we refer the reader to [2], [4], [13] and [15]. In Sect. 2 we represent the tail probabilities $\mathbb{P}(\tau_1 > 2n)$ as integral over the reciprocal of an elliptic integral of first kind $\mathsf{K}(e^{it/2})$, see (9) and (15). A thourough study of the asymptotic behaviour of the integrand as $t \to 0$ enables us to improve the tail estimate of $\log\tau_1$ given in [3] as follows:

Theorem 1. *For the first return time τ_1 of the simple random walk on $\mathbb{Z}^2$ we have, as $x \to \infty$,*

$$\mathbf{P}(\,\log\tau_1 > x\,) = \pi/x - c_0/x^2 + \mathcal{O}(x^{-3}) \quad \textit{with} \quad c_0 = \pi\,(\,\gamma + 3\,\log 2\,)\,, \tag{1}$$

where $\gamma := \lim_{n\to\infty}\big[\sum_{k=1}^n 1/k - \log n\big] \approx 0.57721$ denotes the Euler-Mascheroni constant, see [5].

The estimate (1) is the key to obtain a bound of the difference $\log\mathbf{E}\exp\{it\log\tau_1\} - \ell(t)$ of order $\mathcal{O}\big(t^2\,\log^2(1/t)\big)$ as $t \to 0$ which turns out small enough to provide the distributional limits

$$\frac{1}{n}\sum_{i=1}^n \log\tau_i - \pi\,\log n - \kappa \xrightarrow[n\to\infty]{\mathrm{d}} S \qquad \text{or equivalently} \qquad \frac{\sqrt[n]{\tau_1\cdots\tau_n}}{e^\kappa\,n^\pi} \xrightarrow[n\to\infty]{\mathrm{d}} e^S$$

with a constant κ specified in the below Theorem 2. Stable limit theorems for suitably scaled and centered sums of random variables with limiting r.v. S are comparatively rare in applications; for further quite different examples see e.g. [7] and [8].

Moreover, by using Esseen smoothness lemma, we find a uniform bound of the difference of the corresponding distribution functions in the above limit theorems which seems to be the best possible in comparison with a similar problem in the metric theory of continued fractions, see in [7].

Theorem 2. *Let $p(u)$ denote the probability density function of the r.v. S having the c.f. $\exp\{\ell(t)\}$ for $t \in \mathbb{R}^1$. Then*

$$\sup_{x\in\mathbb{R}^1}\Big|\,\mathbf{P}\Big(\frac{1}{n}\sum_{i=1}^n \log\tau_i - \pi\,\log n - \kappa \le x\Big) - \int_{-\infty}^x p(u)\,\mathrm{d}u\,\Big| = \mathcal{O}\Big(\frac{\log^2 n}{n}\Big) \quad \textit{as} \quad n \to \infty \tag{2}$$

or equivalently

$$\sup_{x\ge 0}\Big|\,\mathbf{P}\Big(\frac{\sqrt[n]{\tau_1\cdots\tau_n}}{e^\kappa\,n^\pi} \le x\Big) - \int_0^x p(\log u)\,\frac{\mathrm{d}u}{u}\,\Big| = \mathcal{O}\Big(\frac{\log^2 n}{n}\Big) \quad \textit{as} \quad n \to \infty \tag{3}$$

with the constant $\kappa := \pi\,(1-\gamma) + \log 2 + \lim_{n\to\infty}\big[\sum_{k=2}^n \mathbb{P}(\tau_1 = 2k)\,\log k - \pi\,\log\log n\big] \approx -6.6947$.

Remark 1. We may write $p(u) = \frac{2}{\pi^2}\,p^*\big(\frac{2}{\pi^2}\,u + \frac{2}{\pi}\,\log\frac{2}{\pi^2}\big)$, where $p^*(u)$ denotes standard stable density function (for $\alpha = \beta = \lambda = 1$) defined by its c.f. $\exp\{2\,\ell(t)/\pi^2\}$, more precisely, by applying the Fourier inversion formula we have $p^*(u) = (2\,\pi)^{-1}\int_{-\infty}^{\infty}\exp\{-iut + 2\,\ell(t)/\pi^2\}\,\mathrm{d}t$ for any $u \in \mathbb{R}^1$, see [2] and [15] for various series expansions of $p^*(u)$ and the decaying rates $p^*(u) = \mathcal{O}(u^{-2})$, $p^*(-u) = \mathcal{O}(\exp\{-e^u\})$ as $u \to \infty$. Further, we mention that there are numerical Fourier inversion procedures for a pointwise calculation of $p^*(u)$, see [11].

The rest of the paper is organized as follows: In Sect. 2 we put together the analytical tools needed to prove Theorem 1 and 2. In particular, we shall express the tail probability $\mathbf{P}(\tau_1 > 2n)$ by an integral formula containing the elliptic integral of the first kind $\mathsf{K}(e^{it})$ and obtain asymptotic estimates of the integrand by using classical expansions of $\mathsf{K}(e^{it})$ for $t \in (0, \pi/2]$ and $t \downarrow 0$, see [10]. The proofs of Theorem 1 and Theorem 2 are presented in Sect. 3 and 4, respectively. In Sect. 5 we discuss different expressions and the computation of the analytic constant κ which seems to be of interest in its own right. Due to some numerical uncertainties the above approximate value of κ should be considered as preliminary and might be subject to slight changes.

In what follows, $c_1, c_2, \ldots$ denote positive constants (not depending on n) and θ stands for a complex number (eventually depending on a variable indicated in parentheses) with $|\theta| \le 1$ which may differ from one expression to another.

2 Preliminary Results

To avoid ambiguities we define once more the simple random walk on $\mathbb{Z}^2$ as sequence of partial sums $\{S_n := X_1 + \cdots + X_n : n \ge 1\}$ of i.i.d. random vectors X_k taking each of the values $(1,0), (0,-1), (-1,0), (0,1)$ with probability $1/4$ for $k \ge 1$ und set $S_0 := \mathbf{o}$. The first return time $\tau_1$to $\mathbf{o}$ taking only even positive integers is determined by the probabilities $Q_{2n} := \mathbf{P}(\tau_1 = 2n) = \mathbf{P}(E_{2n})$, where the events $E_n := \{S_1 \ne \mathbf{o}, \ldots, S_{n-1} \ne \mathbf{o}, S_n = \mathbf{o}\}$ satisfy the decomposition $\{S_{2n} = \mathbf{o}\} = \bigcup_{k=1}^{n}\big(E_{2k} \cap \{S_{2n} - S_{2k} = \mathbf{o}\}\big)$ for $n = 1, 2, \ldots$. By our independence assumptions on the X_k's we obtain the recurrence relations

$$P_{2n} = Q_{2n} + \sum_{k=1}^{n-1} Q_{2k}\,P_{2n-2k} \quad \text{for} \quad n = 1, 2, \ldots \tag{4}$$

connecting the probabilities Q_{2n} and $P_{2n} = \mathbf{P}(S_{2n} = \mathbf{o})$ for $n \ge 1$ and $P_0 = 1, Q_0 = 0$, where P_{2n} is equal to the number all possible paths from $\mathbf{o}$ to $\mathbf{o}$ in $2n$ steps divided by 4^{2n}, that is, for $n = 1, 2, \ldots$,

$$P_{2n} = \frac{1}{4^{2n}}\sum_{k=0}^{n}\frac{(2n)!}{(k!)^2\,((n-k)!)^2} = \frac{1}{4^{2n}}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \frac{1}{4^{2n}}\binom{2n}{n}^2. \tag{5}$$

An immediate consequence of (4) and (5) is the following closed-term expression for Q_{2n}:

$$Q_{2n} \;=\; \frac{1}{4^{2n}}\sum_{k=1}^{n}(-1)^{k-1}\sum_{\substack{n_1+\ldots+n_k=n\\ n_1,\ldots,n_k\ge 1}}\binom{2\,n_1}{n_1}^2\cdots\binom{2\,n_k}{n_k}^2. \tag{6}$$

Multiplying both sides of (4) by z^{2n} and summing up over $n \geq 1$ provide the equality $g(z) - 1 = g(z)\ \mathbf{E}z^{\tau_1}$ linking the probability generating functions (p.g.f.'s) $g(z) = 1 + \sum_{n\geq 1} P_{2n}\, z^{2n}$ and $\mathbf{E}z^{\tau_1} = \sum_{n\geq 1} Q_{2n}\, z^{2n}$ defined for $z \in \mathbb{C}^1 \setminus \{1\}$ satisfying $|z| \leq 1$. Furthermore, in view of (5) the p.g.f. $g(z)$ can be expressed by the *elliptic integral of first kind* $\mathsf{K}(z)$ defined by

$$\mathsf{K}(z) := \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1 - z^2\, \sin^2(\varphi)}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{z^{2n}}{4^{2n}} \quad \text{for} \quad |z| \leq 1, z \neq 1 \tag{7}$$

and this implies an explicit analytical expression for the p.g.f. of τ_1

$$\mathbf{E}z^{\tau_1} = 1 - \frac{1}{g(z)} = 1 - \frac{\pi}{2\,\mathsf{K}(z)} = 1 - \frac{1}{\sum_{n=0}^{\infty} \binom{2n}{n}^2 \left(\frac{z}{4}\right)^{2n}} \quad \text{for} \quad |z| \leq 1, z \neq 1\,. \tag{8}$$

Lemma 1. *The tail probabilities of the first return time τ_1 can be expressed by*

$$\mathbf{P}(\tau_1 > 2\,n) = \frac{1}{2} \int_0^{\pi} \mathrm{Re}\left[\frac{e^{-itn}}{(1 - e^{it})\,\mathsf{K}(e^{it/2})}\right] \mathrm{d}t \quad \text{for} \quad n = 0, 1, 2, \ldots\,. \tag{9}$$

Proof of Lemma 1: The p.g.f. $\mathbf{E}z^{\tau_1} = \sum_{k\geq 1} Q_{2k}\, z^{2k}$ can be rewritten in the following way

$$1 - \mathbf{E}z^{\tau_1} = \sum_{k\geq 1} Q_{2k}\,(1 - z^{2k}) = (1 - z^2) \sum_{k\geq 1} Q_{2k} \sum_{n=0}^{k-1} z^{2n} = (1 - z^2) \sum_{n=0}^{\infty} z^{2n} \sum_{k\geq n+1} Q_{2k}$$

for any $z \in \mathbb{C}^1$ with $|z| \leq 1$. For $z = e^{it}$ the p.g.f. $\mathbf{E}z^{\tau_1}$ is transferred to the c.f. $h(t) = \mathbf{E}e^{it\,\tau_1}$ of the lattice r.v. τ_1 having span 2 so that $h(t+\pi) = h(t)$ for $t \in \mathbb{R}^1$. Together with $\sum_{k\geq n+1} Q_{2k} = \mathbf{P}(\tau_1 > 2\,n)$ we obtain the identity

$$\frac{e^{-2nti}}{1 - e^{2ti}}\,\big(1 - h(t)\big) = \sum_{m=0}^{\infty} e^{2(m-n)ti}\,\mathbf{P}(\tau_1 > 2\,m) \quad \text{for} \quad t \notin \pi\,\mathbb{Z}^1\,, \tag{10}$$

where the Fourier series on the r.h.s. converges uniformly in any interval $[a, b]$ with $0 < a < b < \pi$. The latter is seen from the identity

$$(e^{2ti} - 1) \sum_{m\geq N} e^{2mti}\mathbf{P}(\tau_1 > 2\,m) = it \int_{2N}^{\infty} e^{txi}\,\mathbf{P}(\tau_1 > x)\mathrm{d}x = t \int_{2N}^{\infty} \big(i\cos(tx) - \sin(tx)\big)\mathbf{P}(\tau_1 > x)\mathrm{d}x\,,$$

which, after applying the below Lemma 3 to the real and imaginary part on the r.h.s. combined with $|e^{2ti} - 1| = 2\,|\sin(t)|$, implies that

$$\sup_{a\leq t\leq b} \Big| \sum_{m\geq N} e^{2mti}\,\mathbf{P}(\tau_1 > 2\,m) \Big| \leq \frac{2\,\mathbf{P}(\tau_1 > 2\,N)}{\min\{\sin(a), \sin(b)\}} \underset{N\to\infty}{\longrightarrow} 0\,.$$

This uniform convergence allows the term-wise integration over $t \in [a, b]$ on the r.h.s. of (10). According to (8) we replace $1 - h(t)$ by $\pi/2\,\mathsf{K}(e^{it})$ on the l.h.s. of (10) and integrate the real parts on both sides over $t \in [\delta/2, \pi/2]$ for fixed $\delta \in (0,\pi)$. In this way we are led to

$$\int_{\delta/2}^{\pi/2} \mathrm{Re}\left[\frac{e^{-2nit}}{1-e^{2it}}\,\frac{\pi}{2\,\mathsf{K}(e^{it})}\right] \mathrm{d}t \;=\; \frac{\pi-\delta}{2}\,\mathbf{P}(\tau_1 > 2\,n) + \int_{\delta/2}^{\pi/2} \sum_{0\le m\neq n} \cos(2(m-n)t)\,\mathbf{P}(\tau_1 > 2\,m)\,\mathrm{d}t$$

$$= \frac{\pi-\delta}{2}\,\mathbf{P}(\tau_1 > 2\,n) - \sum_{0\le m\neq n} \frac{\sin((m-n)\delta)}{2\,(m-n)}\,\mathbf{P}(\tau_1 > 2\,m)\,.$$

It remains to show that the series on the r.h.s. disappears when $\delta \downarrow 0$. To see this, we recall that the sequence of partial sums $s_k(\delta) = \sum_{m=1}^{k} \frac{\sin(m\delta)}{m}$ is bounded for $0 \le \delta \le \pi$ and $s_k(\delta) \to 0$ as $\delta \downarrow 0$ for any integer $k \ge 1$. Therefore, by some obvious rearrangements we find that, for any integer $n \ge 0$,

$$\sum_{m>n} \frac{\sin((m-n)\delta)}{m-n}\,\mathbf{P}(\tau_1 > 2\,m) \;=\; \lim_{N\to\infty} \sum_{m=1}^{N} \frac{\sin(m\delta)}{m}\Big(\sum_{k>N} + \sum_{k=m}^{N}\Big)\mathbf{P}(\tau_1 = 2\,(k+n+1))$$

$$= \sum_{k=1}^{\infty} s_k(\delta)\,\mathbf{P}(\tau_1 = 2\,(k+n+1)) \longrightarrow 0 \;\text{ as }\; \delta \downarrow 0\,,$$

where the last step is justified by Lebesgue's dominated convergence theorem. Hence, we achieve the representation

$$\mathbf{P}(\tau_1 > 2\,n) = \lim_{\delta\downarrow 0} \int_{\delta/2}^{\pi/2} \mathrm{Re}\left[\frac{e^{-2nit}}{(1-e^{2it})\mathsf{K}(e^{it})}\right] \mathrm{d}t = \frac{1}{2}\lim_{\delta\downarrow 0} \int_{\delta}^{\pi} \mathrm{Re}\left[\frac{e^{-int}}{(1-e^{it})\mathsf{K}(e^{it/2})}\right] \mathrm{d}t$$

by an improper Riemann-integral as asserted in (9). □

Next, we recall some analytical properties of the elliptic integral $\mathsf{K}(z)$ the proofs of which can be found in [10] and [1]. According to [1] (p. 15) the function $2\,\mathsf{K}(z)/\pi$ coincides with the hypergeometric function $F(1/2, 1/2, 1; z^2)$, where

$$F(a,b,c;z^2) = 1 + \sum_{n=1}^{\infty} z^{2n} \prod_{i=1}^{n} \frac{(a+i-1)(b+i-1)}{c+i-1} \quad \text{for} \quad a,b,c \in \mathbb{C}^1$$

converges (diverges) for $|z| < (>) 1$ and it also converges for $|z| = 1$ whenever $a,b,c \notin \{0,-1,-2,\ldots\}$ and $\mathrm{Re}(a+b-c) < 0$ or for $|z| = 1, z \neq 1$, whenever $1 > \mathrm{Re}(a+b-c) \ge 0$, see [10] (p. 12). As a direct consequence of this it follows that $\mathsf{K}(z)$ is twice differentiable for $z \in \mathbb{C}^1, |z| \le 1, z \neq 1$, see [1] (p. 282) for explicit expressions of the first and second derivative of $\mathsf{K}(z)$. In particular, for $|z| = 1, z \neq 1$, the following (absolutely convergent) series expansion holds, see e.g. [6] (p. 909):

$$\mathsf{K}(z) = \log\frac{4}{\sqrt{1-z^2}} + \sum_{n=1}^{\infty} \binom{2n}{n}^2 \frac{(1-z^2)^n}{4^{2n}}\Big[\log\frac{1}{\sqrt{1-z^2}} + 2\Big(\log 2 - \sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\Big)\Big] \qquad (11)$$

This means that the mapping $(0,\pi] \ni t \mapsto \mathsf{K}(e^{it/2})$ is continuously differentiable and allows the expansion

$$\begin{aligned} \mathsf{K}(e^{it/2}) &= \log 4 + \Big(1 + \frac{1-e^{it}}{4} + \frac{9(1-e^{it})^2}{64} + \frac{25(1-e^{it})^3}{256}\Big) \log\frac{1}{\sqrt{1-e^{it}}} \\ &+ \frac{2\,\log 2 - 1}{4}\,(1-e^{it}) + \frac{36\,\log 2 - 21}{128}\,(1-e^{it})^2 + c_1\,\theta(t)\,t^3\,, \end{aligned} \tag{12}$$

where the asymptotic order as $t \downarrow 0$ is determined by the last term. In order to express the above expansions in terms of t and $\log(1/t)$ for $t>0$ we use the Taylor expansion $\log(1-z) = -\big(z + z^2/2 + z^3/3 + \cdots\big)$ for $z \in \mathbb{C}^1, |z| < 1$, Euler's formula $e^{it} = \cos(t) + i\,\sin(t)$ and the identity, see [6] (p. 45/46),

$$\begin{aligned} \log\frac{1}{\sqrt{1-e^{it}}} &= -\frac{1}{2}\log(1-e^{it}) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{e^{itk}}{k} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{\cos(tk)}{k} + \frac{i}{2}\sum_{k=1}^{\infty}\frac{\sin(tk)}{k} \\ &= \frac{1}{2}\log\frac{1}{2\,\sin(|t|/2)} + \frac{i}{2}\,\mathrm{sgn}(t)\,\frac{\pi-|t|}{2} \quad \text{for } \ |t| \in (0, 2\,\pi)\,. \end{aligned} \tag{13}$$

Finally, by means of the inequalities $0 \le \sin(t) - t + t^3/6 \le t^5/120$ and $0 \le \cos(t) - 1 + t^2/2 \le t^4/24$ for $t \ge 0$ we have

$$\log\frac{1}{2\,\sin(t/2)} = \log\frac{t}{2\,\sin(t/2)} + \log\frac{1}{t} = \log\frac{1}{t} + \frac{t^2}{24} + \mathcal{O}(t^4) \quad \text{and} \quad 1 - e^{it} = \frac{t^2}{2} - i\,t + \mathcal{O}(t^4)\,,$$

as $t \downarrow 0$. Inserting these relations on the r.h.s of (12) and using the abbreviation $L(t) = \log(1/t)$ for $t>0$ we obtain after elementary calculations the following asymptotic expansions for $R(t) = \mathrm{Re}\big(\mathsf{K}(e^{it/2})\big)$ and $I(t) = \mathrm{Im}\big(\mathsf{K}(e^{it/2})\big)$ as $t \downarrow 0$:

$$\begin{aligned} R(t) &= \frac{1}{2}\,L(t) + \log 4 + \frac{\pi}{16}\,t - \frac{t^2}{128}\,L(t) - \frac{6\,\log 4 + 1}{384}\,t^2 + \mathcal{O}\big(t^3\big) \\ I(t) &= \frac{\pi}{4} - \frac{t}{8}\,L(t) - \frac{\log 4}{4}\,t - \frac{\pi}{256}\,t^2 - \frac{t^3}{1536}\,L(t) + \mathcal{O}\big(t^3\big) \\ R^2(t) + I^2(t) &= \frac{1}{4}\,L^2(t) + \log 4\,L(t) + \frac{\pi^2}{16} + \log^2 4 - \frac{t^2}{128}\,L^2(t) + \mathcal{O}\big(t^2\,L(t)\big)\,. \end{aligned}$$

We end this section by rewriting the integral on the r.h.s. of (10). It is easily seen that

$$\mathrm{Re}\left(\frac{e^{-int}}{1-e^{it}}\right) = \frac{\sin(b_n t)}{2\,\sin(t/2)} \quad \text{and} \quad \mathrm{Im}\left(\frac{e^{-int}}{1-e^{it}}\right) = \frac{\cos(b_n t)}{2\,\sin(t/2)} \quad \text{with} \quad b_n = n + \frac{1}{2}\,.$$

Together with $R(t)$ and $I(t)$ and the formula $\mathrm{Re}(wz^{-1}) = \mathrm{Re}(w)\,\mathrm{Re}(z^{-1}) - \mathrm{Im}(w)\,\mathrm{Im}(z^{-1}) = \big(\mathrm{Re}(w)\,\mathrm{Re}(z) + \mathrm{Im}(w)\,\mathrm{Im}(z)\big)\,|z|^{-2}$ for $w, z \in \mathbb{C}^1, |z| > 0$, we may express the integral on the r.h.s. of (10) as follows:

$$\int_0^\pi \mathrm{Re}\left[\frac{e^{-int}}{(1-e^{it})\mathsf{K}(e^{it/2})}\right]\mathrm{d}t = \int_0^\pi \sin(b_n t)\,S(t)\,\mathrm{d}t + \int_0^\pi \cos(b_n t)\,C(t)\,\mathrm{d}t\,, \tag{14}$$

where

$$S(t) = \frac{R(t)}{2\,\sin(t/2)\,\big(R^2(t)+I^2(t)\big)} \quad \text{and} \quad C(t) = \frac{I(t)}{2\,\sin(t/2)\,\big(R^2(t)+I^2(t)\big)} \quad \text{for} \quad 0 < t \le \pi\,.$$

The following surprising fact facilitates and shortens the approximate computation of $\mathbf{P}(\tau_1 > 2\,n)$ for large n.

Lemma 2. *For any* $n = 0, 1, 2, \ldots$

$$\mathbf{P}(\tau_1 > 2\,n) = \int_0^\pi \sin(b_n t)\,S(t)\,\mathrm{d}t = \int_0^\pi \cos(b_n t)\,C(t)\,\mathrm{d}t \quad \text{with} \quad b_n = n + \frac{1}{2}\,. \tag{15}$$

Proof of Lemma 2: By the identity $2\,\sin(t/2)\,S(t) = \mathrm{Re}\big(1/\mathsf{K}(e^{it/2})\big)$ and relation (8) for $z = e^{it/2}$ we have

$$\int_0^\pi \sin(b_n t)\,S(t)\,\mathrm{d}t = \int_0^\pi \frac{\sin(b_n t)}{2\,\sin(t/2)}\,\mathrm{Re}\left(\frac{1}{\mathsf{K}(e^{it/2})}\right)\,\mathrm{d}t = \frac{2}{\pi}\int_0^\pi \frac{\sin(b_n t)}{2\,\sin(t/2)}\,\mathrm{Re}\big(\,1 - \mathbf{E}e^{it\tau_1/2}\,\big)\,\mathrm{d}t\,.$$

By induction on $n \in \{0, 1, \ldots\}$ and some elementary trigonometric relationships it is easily verified that

$$\frac{\sin(b_n\,t)}{2\,\sin(t/2)} = \frac{1}{2} + \sum_{k=1}^{n}\cos(k\,t) \;\; \text{and} \;\; \int_0^\pi \cos(k\,t)\,\cos(\ell\,t)\,\mathrm{d}t = \begin{cases} \pi/2 & \text{for } k = \ell = 0, 1, \ldots \\ 0 & \text{for integers } k \ne \ell \end{cases}\,.$$

According to these relations we may proceed with

$$\begin{aligned}
\int_0^\pi \sin(b_n t)\,S(t)\,\mathrm{d}t \;&=\; \frac{2}{\pi}\int_0^\pi \Big(\frac{1}{2} + \sum_{k=1}^{n}\cos(k\,t)\Big)\Big(1 - \sum_{k=1}^{\infty}\cos(k\,t)\,\mathbf{P}(\tau_1 = 2k)\,\Big)\,\mathrm{d}t \\
&=\; \frac{2}{\pi}\Big(\,\frac{\pi}{2} - \sum_{k=1}^{n}\int_0^\pi \cos^2(k\,t)\,\mathrm{d}t\,\mathbf{P}(\tau_1 = 2k)\Big) = \mathbf{P}(\tau_1 > 2\,n)\,.
\end{aligned}$$

Finally, combining the just proved relation with (14) and (10) confirms the second equality of (15) completing the proof of Lemma 2. □

3 Proof of Theorem 1

First we recall the *second mean value theorem for integrals* which provides the key to obtain the behaviour of the integrals on the r.h.s. of (15) for $n \to \infty$.

Lemma 3. ([9], p. 447) *If $f(x)$ is a bounded monotonic function on $[a, b]$ and $g(x)$ is continuous on the same interval, then there exists some value $\xi \in [a, b]$ such that*

$$\int_a^b f(x)\, g(x)\, \mathrm{d}x = f(a) \int_a^\xi g(x)\, \mathrm{d}x + f(b) \int_\xi^b g(x)\, \mathrm{d}x\,. \tag{16}$$

A straightforward application of Lemma 3 to an arbitrary nonincreasing function f on $[a, \infty)$ vanishing at ∞ and $g(x) = \sin x$ or $g(x) = \cos x$ with $a > 0$, $b \to \infty$ yields for some $\xi \geq a$ that

$$\left| \int_a^\infty f(x) \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \mathrm{d}x \right| = \lim_{b\to\infty} \left| \int_a^b f(x) \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \mathrm{d}x \right| \leq f(a) \left| \int_a^\xi \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} \mathrm{d}x \right| \leq 2\, f(a)\,. \tag{17}$$

Besides the above-defined $b_n = n + 1/2$ we need two further sequences $\lambda_n := \log(b_n) > 0$ and $a_n := \lambda_n^2 / b_n$ for $n \geq 3$ to shorten notation in what follows. In order to study the asymptotic behaviour of both integrals on the r.h.s. of (14) we summarize the analytic properties of the functions $S(t)$ and $C(t)$ for $t \in (0, \pi]$ in

Lemma 4. *For any $\varepsilon \in (0, \pi)$ there exist nondecreasing, continuously differentiable functions $s_\varepsilon^{(1)}(t), s_\varepsilon^{(2)}(t)$ and $c_\varepsilon^{(1)}(t), c_\varepsilon^{(2)}(t)$ on the interval $[\varepsilon, \pi]$ such that $S(t) = s_\varepsilon^{(1)}(t) - s_\varepsilon^{(2)}(t)$ and $C(t) = c_\varepsilon^{(1)}(t) - c_\varepsilon^{(2)}(t)$ for $\varepsilon \leq t \leq \pi$. Further, there exist a sufficiently small $\varepsilon_0 \in (0, 1)$ and stricIty decreasing functions $S_0(t)$ and $C_0(t)$ on $(0, \varepsilon_0]$ such that, as $t \downarrow 0$,*

$$S(t) \;=\; \frac{2}{t\, L(t)} - \frac{4 \log 4}{t\, L^2(t)} + S_0(t) \;\text{ with }\; S_0(t) = \frac{16 \log^2 4 - \pi^2}{2\, t\, L^3(t)} + \mathcal{O}\Big(\frac{1}{t\, L^4(t)}\Big) \tag{18}$$

$$C(t) \;=\; \frac{\pi}{t\, L^2(t)} - \frac{4\, \pi \log 4}{t\, L^3(t)} + C_0(t) \;\text{ with }\; C_0(t) = \frac{48\, \pi \log^2 4 - \pi^3}{4\, t\, L^4(t)} + \mathcal{O}\Big(\frac{1}{t\, L^5(t)}\Big)\,. \tag{19}$$

Proof of Lemma 4: The smoothness properties of $\mathsf{K}(e^{it/2})$ on $(0, \pi]$ stated in Sect. 2 reveal that $R(t)$ and $I(t)$ are C^1-functions. on $[\varepsilon, \pi]$ for any $\varepsilon > 0$. Since the c.f. $h(t)$ of the lattice r.v. τ_1 with span 2 fulfills $|h(t)| < 1$ for $t \notin \pi\mathbb{Z}^1$, it follows from (8) that $R^2(t) + I^2(t) = |\mathsf{K}(e^{it/2})|^2 = \pi^2/4\, |1 - h(t/2)|^2 \geq c(\varepsilon) > 0$ uniformly for $t \in [\varepsilon, \pi]$. Therefore, also $S(t)$ is a C^1-function and thus of bounded variation on $[\varepsilon, \pi]$. This means that $S(t)$ can be written as difference of nondecreasing C^1-functions $s_\varepsilon^{(1)}(t), s_\varepsilon^{(2)}(t)$ defined on $[\varepsilon, \pi]$. To prove the second part of Lemma 4 we consider only the first relation (18). For this, we use the above-given asymptotic expansions of $R(t)$ and $R^2(t)+I^2(t)$ together with $2\,\sin(t/2) = t - t^3/24 + \mathcal{O}(t^5)$ as $t \downarrow 0$. In this way the denominator of $S(t)$ takes the form

$$2\,\sin(t/2)\,\big(\,R^2(t)+I^2(t)\,\big) = \frac{t\,L^2(t)}{4}\,\big(1+N(t)\big)\,,$$

where

$$N(t) = \frac{4\,\log 4}{L(t)} + \frac{\pi^2+16\,\log^2 4}{4\,L^2(t)} - \frac{7\,t^2}{96} + \mathcal{O}\Big(\frac{t^2}{L(t)}\Big) \longrightarrow 0 \quad \text{as} \quad t\downarrow 0\,.$$

Together with $(1+N(t))^{-1} = 1 - N(t) + 16\,\log^2 4/L^2(t) + \mathcal{O}\big(1/L^3(t)\big)$ and

$$R(t) \;=\; \frac{L(t)}{2}\Big[1+\frac{2\,\log 4}{L(t)}+\frac{\pi}{8}\,\frac{t}{L(t)}-\frac{t^2}{64}-\frac{6\,\log 4+1}{192}\,\frac{t^2}{L(t)}+\mathcal{O}(t^3)\Big] \quad \text{as} \quad t\downarrow 0$$

we find that

$$\begin{aligned}
S(t) \;&=\; \frac{2}{t\,L(t)}\Big(1+\frac{2\,\log 4}{L(t)}+\frac{\pi}{8}\,\frac{t}{L(t)}-\frac{t^2}{64}-\frac{6\,\log 4+1}{192}\,\frac{t^2}{L(t)}+\mathcal{O}(t^3)\Big)\,\big(\,1-N(t)\,\big)\\
&+\; \frac{2}{t\,L(t)}\Big(\frac{16\log^2 4}{L^2(t)}+\mathcal{O}\big(\frac{1}{L^3(t)}\big)\Big)\\
&=\; \frac{2}{t\,L(t)}\Big(1+\frac{2\,\log 4}{L(t)}+\mathcal{O}\big(\frac{t}{L(t)}\big)\Big)+\frac{2}{t\,L(t)}\,\Big(\,\frac{16\log^2 4}{L^2(t)}+\mathcal{O}\big(\frac{1}{L^3(t)}\big)\Big)\\
&-\; \frac{2}{t\,L(t)}\Big(\frac{4\,\log 4}{L(t)}+\frac{\pi^2+16\,\log^2 4}{4\,L^2(t)}+\mathcal{O}(t^2)\Big)-\frac{2}{t\,L(t)}\,\Big(\frac{8\,\log^2 4}{L^2(t)}+\mathcal{O}\big(\frac{1}{L^3(t)}\big)\Big)\\
&=\; \frac{2}{t\,L(t)}\Big(1-\frac{2\,\log 4}{L(t)}+\frac{16\,\log^2 4-\pi^2}{4\,L^2(t)}\,, + \mathcal{O}\big(\frac{1}{L^3(t)}\big)\Big)\,,
\end{aligned}$$

where the last line coincides with (18). The proof of (19) follows in the same manner and is thus omitted.

It remains to show that the function $S_0(t) = S(t) - 2/t\,L(t) + 4\,\log 4/t\,L^2(t)$ is strictly decreasing on $(0,\varepsilon_0)$ for a sufficiently small $\varepsilon_0 > 0$. This is seen by showing that the first derivative $S_0'(t)$ is negative for $(0,\varepsilon_0)$. We only sketch this by using the following representations of $R(t)$ and $I(t)$

$$R(t) = L(t)\,\sum_{k=0}^{\infty}\alpha_k t^{2k} + \sum_{k=0}^{\infty}\beta_k t^k \;\text{ and }\; I(t) = L(t)\,\sum_{k=0}^{\infty}\mu_k t^{2k+1} + \sum_{k=0}^{\infty}\nu_k t^k\,,$$

which can be derived directly from (11) and (13), where the four power series and their derivatives converge absolutely for $|t| \le \pi$. From Sect. 2 we already know the following coefficients

$$\alpha_0 = \frac{1}{2}\,,\;\alpha_1 = -\frac{1}{128}\,,\;\beta_0 = \log 4\,,\;\beta_1 = \frac{\pi}{16}\,,\;\beta_2 = -\frac{6\log 4+1}{384}\,,$$
$$\mu_0 = -\frac{1}{8}\,,\;\mu_1 = -\frac{1}{1536}\,,\;\nu_0 = \frac{\pi}{4}\,,\;\nu_1 = -\frac{\log 4}{4}\,,\;\nu_2 = -\frac{\pi}{256}\,.$$

Without going into details, we may express $S_0(t)$ with $r_0 = 8\,\log^2 4 - \pi^2/2 > 0$ as

$$S_0(t) = \frac{r_0}{t\,L^3(t)}\,\big(1 + r(t)\big)\,,$$

where $r(t)$ has a continuous derivative $r'(t)$ on $(0, \varepsilon_0)$ such that $|r(t)| \le c_1/L(t)$ and $|r'(t)| \le c_2/t\,L^2(t)$ for $0 < t < \varepsilon_0$. A little calculus shows that

$$S_0'(t) = \Big(-\frac{r_0}{t^2\,L^3(t)} + \frac{3\,r_0}{t^2\,L^4(t)}\Big)\,\big(1 + r(t)\big) + \frac{r_0\,r'(t)}{t\,L^3(t)} < 0$$

for any $t \in (0, \varepsilon_0)$, if $\varepsilon_0 > 0$ is chosen sufficiently small. Similar arguments may be applied to show $C_0'(t) < 0$ for any t sufficiently close to 0. Thus, the proof of Lemma 4 is complete. □

Now we turn to prove the asymptotic estimate

$$\mathbf{P}(\tau_1 > 2n) = \frac{\pi}{\log n} - \frac{\pi\,(\,\gamma + 4\,\log 2\,)}{\log^2 n} + \mathcal{O}\Big(\frac{1}{\log^3 n}\Big) \quad \text{as} \quad n \to \infty\,, \tag{20}$$

where evidently $\log n$ can be replaced by λ_n. At the end of this section we will show that (20) is equivalent to (1). In order to verify (20) we have the choice to use the first or the second integral of (15). We choose to examine the first one. By applying Lemma 3 to the nondecreasing functions $f(t) = s_\varepsilon^{(1)}(t)$ resp. $f(t) = s_\varepsilon^{(2)}(t)$ (occurring in Lemma 4) and $g(t) = \sin(b_n\,t)$ for $\varepsilon \le t \le \pi$ we may write

$$\int_\varepsilon^\pi \sin(b_n\,t)\,s_\varepsilon^{(i)}(t)\,\mathrm{d}t = s_\varepsilon^{(i)}(\varepsilon)\int_\varepsilon^{\xi_i} \sin(b_n\,t)\,\mathrm{d}t + s_\varepsilon^{(i)}(\pi)\int_{\xi_i}^\pi \sin(b_n\,t)\,\mathrm{d}t = \mathcal{O}(b_n^{-1})$$

with certain intermediate values $\xi_i \in [\varepsilon, \pi]$ for $i = 1, 2$ so that

$$\int_\varepsilon^\pi \sin(b_n\,t)\,S(t)\,\mathrm{d}t = \int_\varepsilon^\pi \sin(b_n\,t)\,\big(s_\varepsilon^{(1)}(t) - s_\varepsilon^{(2)}(t)\big)\,\mathrm{d}t = \mathcal{O}(b_n^{-1}) \quad \text{as} \quad n \to \infty\,, \tag{21}$$

where $\varepsilon > 0$ can be chosen arbitrarily small. We may put $\varepsilon = \varepsilon_0$ with ε_0 from Lemma 4. Next we treat the remaining part of the first integral on the r.h.s. of (14). Obviously, we may split this integral as follows:

$$\int_0^{\varepsilon_0} \sin(b_n\,t)\,S(t)\,\mathrm{d}t = 2\,\Bigg(\int_0^{a_n} + \int_{a_n}^{\varepsilon_0}\Bigg)\frac{\sin(b_n\,t)}{t\,L(t)}\,\mathrm{d}t - 4\,\log 4\Bigg(\int_0^{\lambda_n/b_n} + \int_{\lambda_n/b_n}^{\varepsilon_0}\Bigg)\frac{\sin(b_n\,t)}{t\,L^2(t)}\,\mathrm{d}t + I_n(S_0)\,,$$

where, in view of the monotonicity of $S_0(t)$ and $|\,S_0(t)\,| \le c_3/t\,L^3(t)$ for $0 < t \le \varepsilon_0$, Lemma 3 gives

$$
\begin{aligned}
I_n(S_0) &= \Bigg(\int\limits_0^{1/b_n} + \int\limits_{1/b_n}^{\varepsilon_0}\Bigg) \sin(b_n\,t)\,S_0(t)\,\mathrm{d}t = \frac{1}{b_n}\int\limits_0^1 \sin(x)\,S_0\Big(\frac{x}{b_n}\Big)\,\mathrm{d}x + S_0(b_n^{-1})\int\limits_{1/b_n}^{\xi} \sin(b_n\,x)\,\mathrm{d}x \\
&+ \;\; S_0(\varepsilon_0)\int\limits_{\xi}^{\varepsilon_0} \sin(b_n\,x)\,\mathrm{d}x = c_2\,\theta\,\Big(\int\limits_0^1 \frac{\sin(x)}{x\,(\lambda_n - \log x)^3}\,\mathrm{d}x + \frac{2}{\lambda_n^3}\Big) + \mathcal{O}(b_n^{-1}) = \mathcal{O}(\lambda_n^{-3})
\end{aligned}
$$

as $n \to \infty$. Again by Lemma 3 and $a_n = \lambda_n^2/b_n$ there exist intermediate values ξ_3, ξ_4 such that

$$
\int\limits_{a_n}^{\varepsilon_0} \frac{\sin(b_n\,t)}{t\,L(t)}\,\mathrm{d}t \;=\; \frac{b_n}{\lambda_n^2\,(\lambda_n - 2\,\log\lambda_n)}\int\limits_{a_n}^{\xi_3} \sin(b_n\,x)\,\mathrm{d}x + \frac{1}{\varepsilon_0\,L(\varepsilon_0)}\int\limits_{\xi_3}^{\varepsilon_0} \sin(b_n\,x)\,\mathrm{d}x = \mathcal{O}(\lambda_n^{-3})
$$

$$
\int\limits_{\lambda_n/b_n}^{\varepsilon_0} \frac{\sin(b_n\,t)}{t\,L^2(t)}\,\mathrm{d}t \;=\; \frac{b_n}{\lambda_n\,(\lambda_n - \log\lambda_n)^2}\int\limits_{\lambda_n/b_n}^{\xi_4} \sin(b_n\,x)\,\mathrm{d}x + \frac{1}{\varepsilon_0\,L^2(\varepsilon_0)}\int\limits_{\xi_4}^{\varepsilon_0} \sin(b_n\,x)\,\mathrm{d}x = \mathcal{O}(\lambda_n^{-3})
$$

as $n \to \infty$. For $p \in \{1, 2, 3\}$ we have to express the integral

$$
\int\limits_0^{\lambda_n^{3-p}/b_n} \frac{\sin(b_n\,t)}{t\,L^p(t)}\,\mathrm{d}t \overset{x=t\,b_n}{=} \int\limits_0^{\lambda_n^{3-p}} \frac{\sin(x)\,\mathrm{d}x}{x\,\log^p(b_n/x)} = \Bigg(\int\limits_{\lambda_n^{p-3}}^{\lambda_n^{3-p}} + \int\limits_0^{\lambda_n^{p-3}}\Bigg)\frac{\sin(x)\,\mathrm{d}x}{x\,(\lambda_n - \log x)^p}\,.
$$

as linear combination of λ_n^{-1} and λ_n^{-2} with a remainder term of magnitude $\mathcal{O}(\lambda_n^{-3})$. Since $\sin(x) \le x$ for $x \ge 0$, the second integral is easily seen to be less than λ_n^{-3}. Hence, it remains to treat the first integral just for $p \in \{1, 2\}$. By the algebraic relation $(1 \mp \varepsilon)^{-p} = 1 \pm p\,\varepsilon + \Theta_p(\mp\varepsilon)\,\varepsilon^2$, where $\Theta_p(-\varepsilon) \ge 0$ and $|\Theta_p(\pm\varepsilon)|$ is bounded by $2^{p+2} - 2(p+2)$ for $0 \le \varepsilon \le 1/2$, the first integral can be decomposed as follows:

$$
\begin{aligned}
\int\limits_{\lambda_n^{p-3}}^{\lambda_n^{3-p}} \frac{\sin(x)\,\mathrm{d}x}{x\,(\lambda_n - \log x)^p} &= \frac{1}{\lambda_n^p}\int\limits_{\lambda_n^{p-3}}^{\lambda_n^{3-p}} \frac{\sin(x)\,\mathrm{d}x}{x\,\big(1 - \frac{\log x}{\lambda_n}\big)^p} = \frac{1}{\lambda_n^p}\int\limits_{\lambda_n^{p-3}}^{\lambda_n^{3-p}} \frac{\sin(x)}{x}\,\mathrm{d}x - \frac{p}{\lambda_n^{p+1}}\int\limits_{\lambda_n^{p-3}}^{\lambda_n^{3-p}} \frac{\sin(x)}{x}\,L(x)\,\mathrm{d}x \\
&+ \frac{1}{\lambda_n^{p+2}}\Bigg(\int\limits_{\lambda_n^{p-3}}^{x_p} + \int\limits_{x_p}^{\lambda_n^{3-p}}\Bigg)\frac{\sin(x)}{x}\,\Theta_p\Big(-\frac{\log x}{\lambda_n}\Big)\,\log^2 x\,\mathrm{d}x\,,
\end{aligned}
\tag{22}
$$

where $x_p > 1$ is chosen (independently of n) in such a way that the function $x \mapsto \Theta_p\big(-\frac{\log x}{\lambda_n}\big)\,\frac{\log^2 x}{x} = \frac{\lambda_n^2}{x}\,\Big(\big(1 - \frac{\log x}{\lambda_n}\big)^{-p} - \big(1 + \frac{p\,\log x}{\lambda_n}\big)\Big)$ has a negative first derivative and is therefore decreasing on the whole interval $[x_p, \lambda_n^{3-p}]$. This enables us to apply Lemma 3 in an obvious way giving

$$
\Bigg|\int\limits_{x_p}^{\lambda_n^{3-p}} \sin(x)\,\Theta_p\Big(-\frac{\log x}{\lambda_n}\Big)\,\frac{\log^2 x}{x}\,\mathrm{d}x\Bigg| \le 4\,(\,2^{p+1} - p - 2\,)\,\Big(\frac{\log^2 x_p}{x_p} + \frac{\log^2(\lambda_n^{3-p})}{\lambda_n^{3-p}}\Big)\,,
$$

whereas the corresponding integral over $[\lambda_n^{p-3}, x_p]$ is bounded by $2 + \log^3 x_p/3$. This means that the last integral in (22) is bounded. Combining the well-known improper integrals

$$\int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x = \frac{\pi}{2} \quad \text{and} \quad \int_0^\infty \frac{\sin(x)}{x}\,L(x)\,\mathrm{d}x = \frac{\pi\,\gamma}{2} \quad , \quad \text{see [6]} \quad (\text{p. } 626)\ ,$$

with the subsequent two estimates (both follow from (17) for large enough n) for $q = 0$ and $q = 1$

$$\int_0^{\lambda_n^{p-3}} \frac{\sin(x)}{x}\,L^q(x)\,\mathrm{d}x = \int_{\log(\lambda_n^{3-p})}^{\infty} \sin(e^{-y})\,y^q\,\mathrm{d}y \ \le \int_{\log(\lambda_n^{3-p})}^{\infty} e^{-y}\,y^q\,\mathrm{d}y \le c_3\,\frac{\log^q(\lambda_n)}{\lambda_n^{3-p}}$$

and

$$\left|\ \int_{\lambda_n^{3-p}}^{\infty} \sin(x)\,\frac{\log^q x}{x}\,\mathrm{d}x\right| \le 2\,(3-p)^q\,\frac{\log^q(\lambda_n)}{\lambda_n^{3-p}}$$

we deduce from (22) that

$$\int_0^{\lambda_n^{3-p}/b_n} \frac{\sin(b_n\,t)}{t\,L^p(t)}\,\mathrm{d}t = \frac{\pi}{2\,\lambda_n^p} - \frac{p\,\gamma\,\pi}{2\,\lambda_n^{p+1}} + \mathcal{O}(\lambda_n^{-3}) \quad \text{for} \quad p \in \{1,2,3\}\,.$$

Finally, after summarizing the above asymptotic relations we arrive at

$$\begin{aligned}\int_0^{\pi} \sin(b_n\,t)\,S(t)\,\mathrm{d}t \ &= \ 2\int_0^{a_n} \frac{\sin(b_n\,t)}{t\,L(t)}\,\mathrm{d}t - 4\,\log 4 \int_0^{\lambda_n/b_n} \frac{\sin(b_n\,t)}{t\,L^2(t)}\,\mathrm{d}t + \mathcal{O}(\lambda_n^{-3}) \\ &= \ \frac{\pi}{\lambda_n} - \frac{\pi\,(\gamma + 4\,\log 2)}{\lambda_n^2} + \mathcal{O}(\lambda_n^{-3}) \quad \text{as} \quad n \to \infty\,. \end{aligned} \tag{23}$$

Finally, by inserting the sum of this and relation (23) on the r.h.s. of (14) and having in mind (9) we arrive at the desired estimate (20).

To accomplish the proof of Theorem 1 we show that (20) implies (1) and vice versa. Since $\mathbf{P}(\,\log\tau_1 > x\,) = \mathbf{P}(\,\tau_1 > e^x\,) = \mathbf{P}(\,\tau_1 > 2\,\lfloor e^x/2\rfloor\,)$ for $x > \log 2$ we can use (20) for $n = \lfloor e^x/2\rfloor$ so that

$$\mathbf{P}(\,\log\tau_1 > x\,) = \frac{\pi}{\log\lfloor e^x/2\rfloor} - \frac{\pi\,(\,\gamma + 4\,\log 2\,)}{\big(\log\lfloor e^x/2\rfloor\big)^2} + \mathcal{O}\Big(\frac{1}{x^3}\Big) \tag{24}$$

as $x \to \infty$. By definition of the floor function $\lfloor\cdot\rfloor$ we have $\log\big(e^x/2 - 1\big) \ge \log\big(\lfloor e^x/2\rfloor\big) > \log\big(e^x/2\big)$ for $x > \log 2$. Thus, by standard arguments, $\log\big(\lfloor e^x/2\rfloor\big) = x - \log 2 - 4\,\theta\,e^{-x}$ for $x \ge \log 4$ showing that $\big(\log(\lfloor e^x/2\rfloor)\big)^{-1} = x^{-1} + \log 2\,x^{-2} + \log^2 2\,x^{-3} + \mathcal{O}(x^{-4})$ as $x \to \infty$. Obviously, (1) follows by inserting the latter relation on the r.h.s. of (24). Using the same arguments we get (20) by setting $x = \log(2\,n) = \log n + \log 2$ in (1). This completes the proof of Theorem 2. □

Remark 2. The approximation of $\mathbf{P}(\tau_1 > 2\,n)$ by linear combinations of powers $(\log n)^{-k}$, $k = 1, 2, ...,$ in (20) can be improved by a higher-order expansion of the integrals in (22). A rather lengthy computation using the improper integral

$$\int_0^\infty \frac{\sin(x)}{x}\, L^2(x)\,\mathrm{d}x = \frac{\pi}{24}\left(12\,\gamma^2 + \pi^2\right) \quad , \quad \text{see} \quad [6] \quad (\text{p. } 627)$$

enables us to replace the remainder term $\mathcal{O}\big((\log n)^{-3}\big)$ by $\pi\,\big(\,(\gamma + 4\,\log 2)^2 - \pi^2/6\big)(\log n)^{-3} + \mathcal{O}\big((\log n)^{-4}\big)$.

4 Proof of Theorem 2

The proof of Theorem 2 is based on the following famous result due to C.-G. Esseen which allows to estimate the Kolmogorov distance between distribution functions by the nearness of their characteristic functions, see [4] (pp. 510 - 512) for details and proof.

Lemma 5. (Esseen's smoothness lemma) *Let F be a non-decreasing function on $\mathbb{R}^1$ and let G be a function of bounded variation with bounded derivative G' on $\mathbb{R}^1$ such that $F(\pm\infty) = G(\pm\infty)$. Then for arbitrary $T > 0$*

$$\sup_{x\in\mathbb{R}^1} |\,F(x) - G(x)\,| \le \frac{1}{\pi}\int_{-T}^{T} \Big|\,\frac{f(t) - g(t)}{t}\,\Big|\,\mathrm{d}t + \frac{24}{\pi\,T}\sup_{x\in\mathbb{R}^1}|G'(x)|\,, \tag{25}$$

where f and g denote the corresponding Fourier-Stieltjes transforms of F and G, respectively.

We will apply Lemma 5 in case of the distribution functions

$$F(x) = \mathbf{P}\Big(\frac{1}{n}\sum_{i=1}^{n}\log\tau_i - \pi\,\log n - \kappa \le x\Big) \quad \text{and} \quad G(x) = \mathbf{P}(S \le x) = \int_{-\infty}^{x} p(u)\,\mathrm{d}u$$

Since $G'(x) = p(x)$ is bounded, we put $T := \delta\,n/\log^2 n$ for any integer $n \ge 2$ with an appropriately chosen number $\delta > 0$ (not depending on n), and then it remains to show that $\int_{-T}^{T} |\,f(t) - g(t)\,|\,|t|^{-1}\mathrm{d}t \le c_4\,T^{-1}$ for some constant $c_4 > 0$, where, using the notation introduced in Sect. 1,

$$f(t) = e^{-it\,(\,\pi\,\log n + \kappa\,)}\Big(\mathbf{E}\exp\Big\{\frac{it}{n}\,\log\tau_1\Big\}\Big)^n \quad \text{and} \quad g(t) = \mathbf{E}e^{it\,S} = e^{-\pi^2\,|t|/2 - i\,\pi\,t\,\log|t|}\,.$$

In order to apply the estimate $|\log(z) - (z-1)| \le |z-1|^2$, which holds for $z \in \mathbb{C}^1$ satisfying $|z - 1| \le 1/2$, we next examine the term $\mathbf{E}\exp\big\{\frac{it}{n}\,\log\tau_1\big\} - 1$ for $|t| \le T$.

By applying the partial integration formula and Lemma 1 we may carry out the following decomposition:

$$\begin{aligned}\mathbf{E}e^{it\,\log\tau_1}-1 &= -\int\limits_{\log 2-0}^{\infty}\big(e^{itx}-1\big)\,\mathrm{d}\mathbf{P}(\log\tau_1>x)=e^{it\log 2}-1+it\int\limits_{\log 2}^{\infty}e^{itx}\,\mathbf{P}(\log\tau_1>x)\mathrm{d}x\\ &= e^{it\log 2}-1+it\int\limits_{\log 2}^{\infty}\big(e^{itx}-1\big)\big(\mathbf{P}(\log\tau_1>x)-\pi/x-c_0/x^2\big)\,\mathrm{d}x\\ &+ it\pi\int\limits_{\log 2}^{\infty}\Big(\frac{\cos(tx)}{x}+i\,\frac{\sin(tx)}{x}\Big)\,\mathrm{d}x+itc_0\int\limits_{\log 2}^{\infty}\frac{e^{itx}-1}{x^2}\,\mathrm{d}x+it\,\mu\end{aligned}$$

with the finite number $\mu:=\int_{\log 2}^{\infty}\big(\mathbf{P}(\log\tau_1>x)-\pi/x\big)\mathrm{d}x$, which can be considered as some kind of first-order pseudomoment, see [2] (p. 25) for details. By means of the inequalities $|e^{ix}-1|\le|x|$ and $|e^{ix}-1-ix|\le x^2/2$ for $x\in\mathbb{R}^1$ combined with (1) we may proceed with

$$\begin{aligned}\mathbf{E}e^{it\,\log\tau_1}-1 &= it\,\big(\log 2+\mu\big)+it\,\pi\int\limits_{|t|\log 2}^{\infty}\Big(\frac{\cos(x)}{x}+i\,\mathrm{sgn}(t)\,\frac{\sin(x)}{x}\Big)\,\mathrm{d}x-t^2\,c_0\int\limits_{\log 2}^{1/|t|}\frac{\mathrm{d}x}{x}\\ &- t\,c_0\int\limits_{\log 2}^{1/|t|}\frac{\sin(tx)-tx}{x^2}\,\mathrm{d}x-t\,c_0\int\limits_{1/|t|}^{\infty}\frac{\sin(tx)}{x^2}\,\mathrm{d}x+c_5\,t^2\,\theta\,.\end{aligned}$$

Now, we quote from [6] (p. 936) and [8] (p. 53) the relation

$$\int_y^{\infty}\frac{\cos(x)}{x}\,\mathrm{d}x=\int_0^y\frac{1-\cos(x)}{x}\,\mathrm{d}x-\gamma-\log y\quad\text{for}\quad y>0\,,$$

which, for $y=|t|\,\log 2$, yields that $\int_{|t|\log 2}^{\infty}\frac{\cos(x)}{x}\,\mathrm{d}x=-\gamma-\log\log 2-\log|t|+\theta\,\frac{\log 4}{4}\,t^2$. Obviously, both integrals in the last line are bounded by $c_6\,t^2$ so that together with $\int_{|t|\log 2}^{\infty}\frac{\sin(x)}{x}\,\mathrm{d}x=\frac{\pi}{2}+|t|\,\theta$ we arrive at

$$\begin{aligned}\mathbf{E}e^{it\,\log\tau_1}-1 &= it\big(\log 2-\pi\,\gamma-\pi\,\log\log 2+\mu\big)-\frac{\pi^2}{2}\,|t|-it\,\pi\,\log|t|+c_0\,t^2\,\log|t|+c_7\,t^2\,\theta\\ &= \log g(t)+it\,\kappa+c_0\,t^2\,\log|t|+c_7\,t^2\,\theta\,.\end{aligned}\tag{26}$$

Here we have additionally used the relation $\mu=\kappa-\log 2+\pi(\gamma+\log\log 2)$ which follows directly from (28) to be shown in the next Sect. 5. Since the mapping $x\mapsto x\,\log(1/x)$ is stricly increasing for $x\in(0,1/e]$, it readily seen from (26) that, for $|t|\le\varepsilon\,n$,

$$\big|\,\mathbf{E}e^{\frac{it}{n}\,\log\tau_1}-1\,\big|\;\le\;\big(\pi^2/2+\kappa+\pi\,\log(1/\varepsilon)\big)\,\varepsilon+\big(c_{13}+c_0\,\log(1/\varepsilon)\big)\,\varepsilon^2\le 1/2\,,$$

where $\varepsilon \in (0,1/e)$ is chosen small enough to achieve the right-hand uniform bound. Hence, we obtain that

$$\begin{aligned}\log f(t) &= -it\,(\,\pi\,\log n + \kappa\,) + n\,\big(\,\mathbf{E}e^{\frac{it}{n}\,\log\tau_1} - 1\,\big) + n\,\theta\,\big|\,\mathbf{E}e^{\frac{it}{n}\,\log\tau_1} - 1\,\big|^2 \\ &= -\frac{\pi^2}{2}|t| - it\pi\,\log|t| - c_0\,\frac{t^2}{n}\,\log\frac{n}{|t|} + c_8\,\theta\,\frac{t^2}{n}\,\log^2\frac{n}{|t|} \quad \text{for} \quad |t| \le \varepsilon\, n\,.\end{aligned}$$

The latter estimate implies that

$$|\,\log f(t) - \log g(t)\,| \le c_9\,\frac{t^2}{n}\,\big(\,\log^2 n + \log^2|t|\,\big) \le \frac{\pi^2\,|t|}{4} \quad \text{for} \quad |t| \le T = \delta\, n/\log^2 n$$

for any $n \ge 2$ and $\delta > 0$ chosen sufficiently small. Finally, $|g(t)| = \exp\{-\pi^2\,|t|/2\}$ and an application of the inequality $|e^z - 1| \le |z|\,e^{|z|}$ for $z = \log f(t) - \log g(t)$ yield

$$\frac{|f(t) - g(t)|}{|t|} = e^{-\pi^2|t|/2}\,\frac{|\exp\{\log f(t) - \log g(t)\} - 1|}{|t|} \le c_9\,\frac{|t|}{n}\,\big(\,\log^2 n + \log^2|t|\,\big)\,e^{-\pi^2|t|/4}$$

for $|t| \le T$. This estimate shows us that the integral over $t \in [-T,T]$ is bounded by $c_4\,T^{-1}$ as announced at the beginning of the proof. Thus, by means of Lemma 3 the proof of (2) is finished. In view of the monotonicity of the mapping $x \mapsto e^x$ we may rewrite the above distribution functions F and G as follows:

$$F(x) = \mathbf{P}\Big(\,\frac{\sqrt[n]{\tau_1\cdots\tau_n}}{e^\kappa\,n^\pi} \le e^x\,\Big) \quad \text{and} \quad G(x) = \mathbf{P}(e^S \le e^x) = \int_{-\infty}^{e^x} p(\log u)\,\frac{\mathrm{d}u}{u}\,.$$

Substituting e^x for $x \in \mathbb{R}^1$ by $y > 0$ reveals that the l.h.s. of (3) is just equal to $\sup\{\,|\,F(x) - G(x)\,|\,:\,x \in \mathbb{R}^1\,\}$. This completes the proof of Theorem 2. $\square$

5 Concluding Remarks

We first express the pseudomoment-like term μ in terms of the probabilities $Q_{2k} = \mathbf{P}(\tau_1 = 2k)$ and $T_{2k} = \mathbf{P}(\tau_1 > 2k)$, respectively. For this end we rewrite the integral $\int_{\log 2}^{\infty}$ as infinite sum as the follows

$$
\begin{aligned}
\mu &= \sum_{k=1}^{\infty} \int\limits_{\log(2k)}^{\log(2k+2)} \big(\mathbf{P}(\log\tau_1 > x) - \pi/x\big)\mathrm{d}x = \sum_{k=1}^{\infty} \int\limits_{\log(2k)}^{\log(2k+2)} \big(\mathbf{P}(\log\tau_1 > \log(2k)) - \pi/x\big)\mathrm{d}x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n-1} \Big(T_{2k}\,\big(\log(k+1) - \log(k)\big) - \pi\,\big(\,\log\log(2k+2) - \log\log(2k)\,\big)\Big) \\
&= \lim_{n\to\infty} \Big(\sum_{k=1}^{n-1} T_{2k}\, \log\big(1 + \frac{1}{k}\big) - \pi\, \log\log n \Big) + \pi\, \log\log 2 \qquad (27) \\
&= \lim_{n\to\infty} \Big(\sum_{k=2}^{n-1} Q_{2k}\, \log k + \mathbf{P}(\,\tau_1 \geq 2n\,)\, \log n - \pi\, \log\log n \Big) + \pi\, \log\log 2\,.
\end{aligned}
$$

From (1) we deduce that

$$
\mu \;=\; \lim_{n\to\infty} \Big(\sum_{k=2}^{n} Q_{2k}\, \log k - \pi\, \log\log n \Big) + \pi\,(1 + \log\log 2) \qquad (28)
$$

Combining (28) and the definition of κ with the well-known limit $\gamma^* := \lim_{n\to\infty}\big[\,\sum_{k=2}^{n}(k\,\log k)^{-1} - \log\log n\,\big] \approx 0.7946786$, see [5] (p.32), leads to the series representation $\kappa = \sum_{k=2}^{\infty}\big[\mathbb{P}(\tau_1 = 2k)\,\log k - \pi\,(k\,\log k)^{-1}\big] + \log 2 + \pi\,(1 - \gamma + \gamma^*)$. It should be noted that the convergence of the last series is very slowly. In view of (1) , $T_2 = 3/4$ and the definition of γ^* we may rewrite the r.h.s. of (28) as follows:

$$
\begin{aligned}
\mu &= \lim_{n\to\infty} \Big(\sum_{k=2}^{n} \Big\{ T_{2k}\, \log\big(1 + \frac{1}{k}\big) - \frac{\pi}{k\,\log k} \Big\} \Big) + \frac{3}{4}\, \log 2 + \pi\,(\gamma^* + \log\log 2) \\
&= \sum_{k=1}^{\infty} T_{2k} \Big[\log\big(1 + \frac{1}{k}\big) - \frac{1}{k} \Big] + \sum_{k=2}^{\infty} \frac{1}{k} \Big(T_{2k} - \frac{\pi}{\log k} \Big) + \frac{3}{4} + \pi\,(\gamma^* + \log\log 2)\,.
\end{aligned}
$$

Note that (1) ensures the convergence of the both series and by $\mu = \kappa - \log 2 + \pi(\gamma + \log\log 2)$ we get the alternative representation $\kappa = \sum_{k=1}^{\infty} T_{2k}\,\big[\log\big(1 + \frac{1}{k}\big) - \frac{1}{k}\big] + \sum_{k=2}^{\infty} \frac{1}{k}\big(\,T_{2k} - \frac{\pi}{\log k}\,\big) + 0.75 + \log 2 + \pi\,(\gamma^* - \gamma)$.

The computation of the constant κ consists in evaluating the partial sums $s_n = \sum_{k=2}^{n} Q_{2k}\,\log k$, where the ${Q_{2k}}'$s can be determined from the recursive relation (4) or from the closed formula (6). The approximate value $\kappa \approx -6.6947$ given in Theorem 2 is obtained via computing $s_{200.000}$, however, without numerical error analysis. Alternatively, one can determine the tail probabilities T_{2n} successively by the recursive equation $T_{2n} = 1 - \sum_{k=1}^{n} P_{2k}\, T_{2n-2k}$ (which can be found in analogy to (4)) with $T_0 = 1$ for $n = 1, 2, \ldots$. This yields $T_2 = 3/4, T_4 = 43/64$, etc. and allows to evaluate the above series representations of κ derived from (28).

6 Appendix: Asymptotic behaviour of the second integral of (15)

In this supplementary section we derive (20) once more by investigating the second integral $\int_0^\pi \cos(b_n\,t)\,C(t)\,\mathrm{d}t$ on the r.h.s. of (14). By the same arguments which have been used to show (21) we obtain that

$$\int\limits_\varepsilon^\pi \cos(b_n\,t)\,C(t)\,\mathrm{d}t = \int\limits_\varepsilon^\pi \cos(b_n\,t)\,\big(c_\varepsilon^{(1)}(t) - c_\varepsilon^{(2)}(t)\big)\,\mathrm{d}t = \mathcal{O}(b_n^{-1}) \quad \text{as} \quad n \to \infty \tag{29}$$

for any fixed $\varepsilon > 0$. We may put $\varepsilon = \varepsilon_0$ with ε_0 from Lemma 4. Next we treat the remaining part of the second integral occurring in (14) and (15), respectively. For this purpose we split $\int_0^{\varepsilon_0} \cos(b_n\,t)\,C(t)\,\mathrm{d}t$ according to the representation (19) of $C(t)$ as follows:

$$\begin{aligned}\int\limits_0^{\varepsilon_0} \cos(b_n\,t)\,C(t)\,\mathrm{d}t \;&=\; \pi\Bigg(\int\limits_0^{\lambda_n b_n^{-1}} + \int\limits_{\lambda_n b_n^{-1}}^{\varepsilon_0}\Bigg)\frac{\cos(b_n\,t)}{t\,L^2(t)}\,\mathrm{d}t - 4\,\pi\,\log 4\Bigg(\int\limits_0^{b_n^{-1}} + \int\limits_{b_n^{-1}}^{\varepsilon_0}\Bigg)\frac{\cos(b_n\,t)}{t\,L^3(t)}\,\mathrm{d}t \\ &+\; \Bigg(\int\limits_0^{(\lambda_n b_n)^{-1}} + \int\limits_{(\lambda_n b_n)^{-1}}^{\varepsilon_0}\Bigg)\cos(b_n\,t)\,C_0(t)\,\mathrm{d}t\,.\end{aligned}$$

From Lemma 4 we see that $|\,C_0(t)\,| \le c_{10}/t\,L^4(t)$ for $0 < t \le \varepsilon_0$ and this in turn implies that

$$\Bigg|\int\limits_0^{(\lambda_n b_n)^{-1}} \cos(b_n\,t)\,C_0(t)\,\mathrm{d}t\Bigg| \le c_{10}\int\limits_0^{\lambda_n^{-1}} \frac{\mathrm{d}t}{t\,(\lambda_n + L(t))^4} \overset{x=L(t)}{=} c_{10}\int\limits_{\log\lambda_n}^{\infty} \frac{\mathrm{d}x}{(\lambda_n + x)^4} = \frac{c_{10}}{3\,(\lambda_n + \log\lambda_n)^3}\,.$$

Since, in addition, $C_0(t)$ is monotonic for $0 < t \le \varepsilon_0$ it follows by applying the above Lemma 2 that

$$\int\limits_{(\lambda_n b_n)^{-1}}^{\varepsilon_0} \cos(b_n\,t)\,C_0(t)\mathrm{d}t = \frac{C_0((\lambda_n b_n)^{-1})}{b_n}\int\limits_{\lambda_n^{-1}}^{\xi b_n} \cos(x)\mathrm{d}x + \frac{C_0(\varepsilon_0)}{b_n}\int\limits_{\xi b_n}^{\varepsilon_0 b_n} \cos(x)\mathrm{d}x = \frac{2\,c_{10}\,\theta}{\lambda_n^3} + \mathcal{O}(b_n^{-1})\,.$$

Once more applying (16) with $f(t) = 1/(t\,L^p(t))$, $g(t) = \cos(b_n\,t)$ and $a = \lambda_n^{3-p}/b_n$, $b = \varepsilon_0$ for $p \in \{2, 3\}$ yields with some $\xi \in [\lambda_n^{3-p}/b_n, \varepsilon_0]$ that

$$\int\limits_{\lambda_n^{3-p}/b_n}^{\varepsilon_0} \frac{\cos(b_n\,t)}{t\,L^p(t)}\,\mathrm{d}t = \frac{b_n}{\lambda_n^{3-p}(\lambda_n - \log(\lambda_n^{3-p}))^p}\int\limits_{\lambda_n^{3-p}/b_n}^{\xi} \cos(b_n\,t)\,\mathrm{d}t + \mathcal{O}\big(\frac{1}{b_n}\big) = \mathcal{O}(\lambda_n^{-3})$$

as $n\to\infty$. Hence, by combining (29)with the other just proved estimates we have

$$\int_0^{\pi}\cos(b_n\,t)\,C(t)\,\mathrm{d}t \;=\; \pi\int_0^{\lambda_n/b_n}\frac{\cos(b_n\,t)}{t\,L^2(t)}\,\mathrm{d}t-4\,\pi\,\log 4\int_0^{1/b_n}\frac{\cos(b_n\,t)}{t\,L^3(t)}\,\mathrm{d}t+\mathcal{O}(\lambda_n^{-3}) \qquad (30)$$

as $n\to\infty$. It remains to evaluate the two integrals in the last line up the asymptotic order $\mathcal{O}(\lambda_n^{-3})$. It is readily seen that

$$\begin{aligned}\int_0^{1/b_n}\frac{\cos(b_n\,t)}{t\,L^3(t)}\,\mathrm{d}t &\overset{s=b_n\,t}{=} -\int_0^{1}\frac{\cos(s)}{(\lambda_n+L(s))^3}\,\mathrm{d}L(s) \overset{s=e^{-x}}{=} \int_0^{\infty}\frac{\cos(e^{-x})\,\mathrm{d}x}{(\,\lambda_n+x\,)^3}\\ &= \int_{\lambda_n}^{\infty}\frac{\mathrm{d}x}{x^3}-\int_0^{\infty}\frac{(1-\cos(e^{-x}))\,\mathrm{d}x}{(\,\lambda_n+x\,)^3}=\frac{1}{2\,\lambda_n^2}+\mathcal{O}(\lambda_n^{-3})\,. \qquad (31)\end{aligned}$$

After splitting the first integral $\int_0^{\lambda_n/b_n}=\int_0^{1/b_n}+\int_{1/b_n}^{\lambda_n/b_n}$ we use the previous substitutions leading to

$$\int_0^{\lambda_n/b_n}\frac{\cos(b_n\,t)}{t\,L^2(t)}\,\mathrm{d}t \;=\; \frac{1}{\lambda_n}-\int_0^{\infty}\frac{(1-\cos(e^{-x}))\,\mathrm{d}x}{(\,\lambda_n+x\,)^2}+\int_1^{\lambda_n}\frac{\cos(t)}{t\,\log^2(b_n/t)}\,\mathrm{d}t$$

By the inequality $1-\cos(e^{-x})\le e^{-2\,x}/2$ it follows that

$$\begin{aligned}\int_0^{\infty}\frac{(1-\cos(e^{-x}))\,\mathrm{d}x}{(\,\lambda_n+x\,)^2} &= \frac{1}{\lambda_n^2}\int_0^{\log\lambda_n}\frac{(1-\cos(e^{-x}))\,\mathrm{d}x}{(\,1+x/\lambda_n\,)^2}+\mathcal{O}(\lambda_n^{-3})\\ &= \frac{1}{\lambda_n^2}\int_0^{\log\lambda_n}(1-\cos(e^{-x}))\,\mathrm{d}x+\mathcal{O}(\lambda_n^{-3})=\frac{1}{\lambda_n^2}\int_0^{1}\frac{1-\cos(x)}{x}\,\mathrm{d}x+\mathcal{O}(\lambda_n^{-3})\,.\end{aligned}$$

Using the expansion $(1-\log(t)/\lambda_n)^{-2}=1+2\,\log(t)/\lambda_n+\mathcal{O}(\log^2\lambda_n/\lambda_n^2)$ for $1\le t\le\log\lambda_n$ we find in analogy to (22) that

$$\int_1^{\lambda_n}\frac{\cos(t)}{t\,\log^2(b_n/t)}\,\mathrm{d}t \;=\; \frac{1}{\lambda_n^2}\int_1^{\lambda_n}\frac{\cos(t)\,\mathrm{d}t}{t\,(\,1-\log(t)/\lambda_n\,)^2}=\frac{1}{\lambda_n^2}\int_1^{\lambda_n}\frac{\cos(t)}{t}\,\mathrm{d}t+\mathcal{O}(\lambda_n^{-3})\,,$$

where λ_n in the integral on the r.h.s. can be replaced by ∞. Thus, by combining the above estimates with the identity

$$\int_y^{\infty}\frac{\cos(x)}{x}\,\mathrm{d}x=\int_0^{y}\frac{1-\cos(x)}{x}\,\mathrm{d}x-\gamma-\log y\quad,\quad\text{see [6] (p. 936) and [8] (p. 53)}$$

at $y = 1$ we are directly led to

$$\int_0^{\lambda_n/b_n} \frac{\cos(b_n\, t)}{t\, L^2(t)}\, \mathrm{d}t = \frac{1}{\lambda_n} - \frac{\gamma}{\lambda_n^2} + \mathcal{O}(\lambda_n^{-3})\,,$$

whence together with (31) we obtain from (30) that

$$\int_0^{\pi} \cos(b_n\, t)\, C(t)\, \mathrm{d}t = \frac{\pi}{\lambda_n} - \frac{\pi\,(\gamma + 4\,\log 2)}{\lambda_n^2} + \mathcal{O}(\lambda_n^{-3}) \quad \text{as} \quad n \to \infty\,,$$

which confirms (20) once more.

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