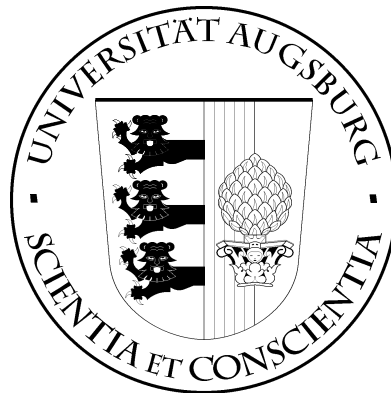


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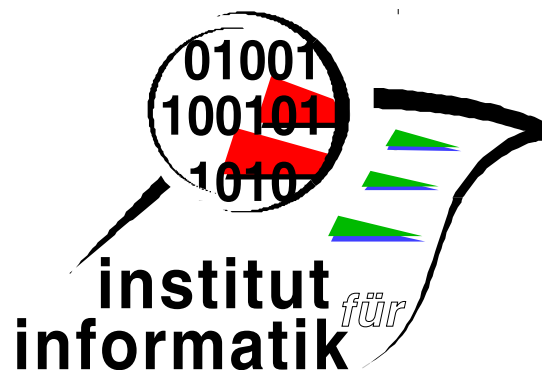


## Characterizing Determinacy in Kleene Algebras

Jules Desharnais and Bernhard Möller

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# Characterizing Determinacy in Kleene Algebras

Jules Desharnais<sup>1</sup>

*Département d'informatique, Université Laval, Québec QC G1K 7P4 Canada*

Bernhard Möller

*Institut für Informatik, Universität Augsburg, D-86135 Augsburg, Germany*

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## Abstract

Elements of Kleene algebras can be used, among others, as abstractions of the input-output semantics of nondeterministic programs or as models for the association of pointers with their target objects. In the first case, one seeks to distinguish the subclass of elements that correspond to deterministic programs. In the second case one is only interested in functional correspondences, since it does not make sense for a pointer to point to two different objects.

We discuss several candidate notions of determinacy and clarify their relationship. Some characterizations that are equivalent in the case where the underlying Kleene algebra is an (abstract) relation algebra are not equivalent for general Kleene algebras.

*Key words:* Keywords: semantics, determinacy, Kleene algebra, relational algebra.

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## 1 Introduction

Elements of Kleene algebras can be used, among others, as abstractions of the input-output semantics of nondeterministic programs [5] or as models for the association of pointers with their target objects in the style of [12]. In the first case, one seeks to distinguish the subclass of elements that correspond to deterministic programs. In the second case, one is only interested in functional

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*Email addresses:* Jules.Desharnais@ift.ulaval.ca (Jules Desharnais), Bernhard.Moeller@informatik.uni-augsburg.de (Bernhard Möller).

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correspondences, since it does not make sense for a pointer to point to two different objects.

We discuss several candidate notions of determinacy and show several equivalences. However, it also turns out that some characterizations that are equivalent in the case where the underlying Kleene algebra is an (abstract) relation algebra, are not equivalent for general Kleene algebras.

## 2 Kleene Algebras

In our definitions we follow [4], since we want to admit general recursive definitions, not just the Kleene star. We are well aware that there are different definitions (see e.g. [9]).

**Definition 1** *A Kleene algebra (KA) is a sextuple  $(K, \leq, 0, \top, \cdot, 1)$  satisfying the following properties:*

- (a)  *$(K, \leq)$  is a complete lattice with least element  $0$  and greatest element  $\top$ . The supremum of a subset  $L \subseteq K$  is denoted by  $\sqcup L$ . The supremum of two elements  $x, y \in K$  is denoted by  $x + y$ .*
- (b)  *$(K, \cdot, 1)$  is a monoid.*
- (c) *The operation  $\cdot$  is universally disjunctive (i.e. distributes through arbitrary suprema) in both arguments.*

This kind of structure is also known as a *quantale* (see e.g. [3]).

**Example 2** Consider an alphabet  $A$ . Then  $A^*$  is the set of all finite words over  $A$ ,  $\bullet$  denotes concatenation, extended pointwise to sets of words, and  $\varepsilon$  the empty word. As customary in formal language theory, we identify a singleton language with its only element. Then perhaps the best-known example of a KA is LAN  $\stackrel{\text{def}}{=} (\mathcal{P}(A^*), \subseteq, \emptyset, A^*, \bullet, \varepsilon)$ , the algebra of formal languages. A related KA is the algebra PAT of path sets in a directed graph under path concatenation (see e.g. [11] for a precise definition).

Another important KA is REL  $\stackrel{\text{def}}{=} (\mathcal{P}(M \times M), \subseteq, \emptyset, M \times M, ;, I)$ , the algebra of homogeneous binary relations over some set  $M$  under relational composition  $;$ .

**Definition 3** (a) *A Boolean algebra is a distributive and complemented lattice. The complement of an element  $x$  is denoted by  $\bar{x}$ .*

- (b) *A Boolean algebra is complete if its underlying lattice is complete.*
- (c) *An atom of a lattice with least element  $0$  is a minimal element in the set of elements different from  $0$ . The set of all atoms of a lattice with  $0$  is denoted by  $\text{At}$ . For an element  $x$  of such a lattice,  $\text{At}(x) \stackrel{\text{def}}{=} \{a \in \text{At} : a \leq x\}$  is the set of atoms of  $x$ .*
- (d) *A lattice with  $0$  is atomic if for every element  $x$  we have  $x = \sqcup \text{At}(x)$ .*

(e) A KA is called *Boolean* if its underlying lattice is a Boolean algebra (and hence a complete Boolean algebra). It is called *atomic* if its underlying lattice is atomic.

**Example 4** More generally than the concrete relation algebra REL, every abstract relation algebra is a KA. Such an *abstract relation algebra* (see e.g. [15]) is a tuple  $RA = (R, \leq, \bar{\cdot}, 0, \top, ;, 1, \smile)$ , where

- (a)  $(R, \leq, \bar{\cdot}, 0, \top)$  is a complete Boolean algebra with complement operation  $\bar{\cdot}$ , least element 0 and greatest element  $\top$ ;
- (b)  $(R, ;, 1)$  is a monoid;
- (c) Tarski's rule  $x \neq 0 \Rightarrow \top ; x ; \top = \top$  holds;
- (d)  $\smile : R \rightarrow R$  is a unary operation such that Dedekind's rule

$$x ; y \sqcap z \leq (x \sqcap z ; y^\smile) ; (y \sqcap x^\smile ; z)$$

is satisfied. Equivalently to this one may postulate the Schröder rule

$$x ; y \leq z \Leftrightarrow x^\smile ; \bar{z} \leq \bar{y} \Leftrightarrow \bar{z} ; y^\smile \leq \bar{x}.$$

The elements of  $R$  are called *abstract relations*; the operation  $\smile$  forms the *converse* of a relation, whereas  $;$  is called *relational composition*. It is customary to use the convention that  $;$  binds tighter than  $+$  and  $\sqcap$ . The reduct  $(R, \leq, 0, \top, ;, 1)$  forms a Boolean KA.

### 3 Types

A central notion is that of types.

**Definition 5** A type of a KA is an element  $t$  with  $t \leq 1$ . We set  $\text{TYP} \stackrel{\text{def}}{=} \{t \in K : t \leq 1\}$ .

Note that by monotonicity, TYP is closed under composition.

This definition is best illustrated in the KA REL. There, a type corresponds to a subset  $T \subseteq M$  and can be represented as the partial identity relation  $1_T \stackrel{\text{def}}{=} \{(x, x) : x \in T\}$ . Clearly,  $1_T$  is a subidentity, and so there is a one-to-one correspondence between types and subidentities.

Now, restriction of a relation  $R \subseteq M \times M$  to arguments of type  $T$ , i.e. the relation  $R \cap T \times M$ , can also be described by composing  $R$  with  $1_T$  from the left:  $R \cap T \times M = 1_T ; R$ . Similarly, co-restriction is modeled by composing a partial identity from the right. Finally, consider types  $S, T \subseteq M$  and binary relation  $R \subseteq M \times M$ . Then  $R \subseteq S \times T \Leftrightarrow 1_S ; R ; 1_T = R$ . In other words, the “default typing”  $M \times M$  of  $R$  can be narrowed down to  $S \times T$  iff restriction to  $S$  and co-restriction to  $T$  do not change  $R$ . These observations

are the basis for our view of types as subidentities and our algebraic treatment of restriction and co-restriction. For a different, but related, approach see [9].

For the remainder of the paper, we assume KAs to be Boolean.

**Definition 6** *The negation of a type  $t \leq 1$  in a KA is  $\neg t \stackrel{\text{def}}{=} \bar{t} \sqcap 1$ .*

With this definition,  $(\text{TYP}, \leq)$  forms again a complete Boolean algebra.

**Lemma 7** *Assume a Boolean KA. Then the following hold:*

- (a) *All types are idempotent, i.e.  $t \leq 1 \Rightarrow t \cdot t = t$ .*
- (b) *The infimum of two types is their product:  $s, t \leq 1 \Rightarrow s \cdot t = s \sqcap t$ . In particular, all types commute under the  $\cdot$  operation.*
- (c)  *$s, t \leq 1 \Rightarrow (s \sqcap t) \cdot a = s \cdot a \sqcap t \cdot a$ .*
- (d)  *$t \leq 1 \Rightarrow \overline{t \cdot \top} = \neg t \cdot \top$ .*
- (e) *Restriction by a type can also be expressed as a meet:  $t \leq 1 \Rightarrow t \cdot a = a \sqcap t \cdot \top$ .*
- (f) *Restriction by a type distributes through meet and can be shifted between meet partners:  $t \leq 1 \Rightarrow t \cdot (a \sqcap b) = t \cdot a \sqcap t \cdot b = t \cdot a \sqcap b = a \sqcap t \cdot b$ .*
- (g) *For all families  $L$  of types,  $(\sqcap L) \cdot \top = \sqcap(L \cdot \top)$ .*

**PROOF.** We first note that by monotonicity and neutrality, for  $s, t \leq 1$  we have that  $s \cdot t \leq s \cdot 1 = s$  and, symmetrically,  $s \cdot t \leq t$ , so that  $s \cdot t \leq s \sqcap t$ .

$$\begin{aligned}
\text{(a)} \quad & t \\
&= \quad \{ \text{neutrality} \} \\
&\quad 1 \cdot t \\
&= \quad \{ \text{Boolean algebra} \} \\
&\quad (t + \neg t) \cdot t \\
&= \quad \{ \text{disjunctivity} \} \\
&\quad t \cdot t + \neg t \cdot t \\
&\leq \quad \{ \text{above} \} \\
&\quad t \cdot t + \neg t \sqcap t \\
&= \quad \{ \text{Boolean algebra} \} \\
&\quad t \cdot t \\
&\leq \quad \{ \text{by } t \leq 1 \text{ and monotonicity} \} \\
&\quad 1 \cdot t \\
&= \quad \{ \text{neutrality} \} \\
&\quad t .
\end{aligned}$$

(b) We have already shown that  $s \cdot t$  is a lower bound for  $s$  and  $t$ . Using (a) and monotonicity, we also have  $s \sqcap t = (s \sqcap t) \cdot (s \sqcap t) \leq s \cdot t$ . Hence  $s \cdot t = s \sqcap t$ .

$$\begin{aligned}
\text{(c)} \quad & (s \sqcap t) \cdot a \\
&\leq \quad \{ \text{monotonicity} \} \\
&\quad s \cdot a \sqcap t \cdot a
\end{aligned}$$

$$\begin{aligned}
&= \{ \text{neutrality and Boolean algebra} \} \\
&\quad (s + \neg s) \cdot (s \cdot a \sqcap t \cdot a) \\
&= \{ \text{disjunctivity} \} \\
&\quad s \cdot (s \cdot a \sqcap t \cdot a) + \neg s \cdot (s \cdot a \sqcap t \cdot a) \\
&\leq \{ \text{monotonicity and associativity} \} \\
&\quad s \cdot t \cdot a + \neg s \cdot s \cdot a \\
&= \{ \text{by strictness, since } \neg s \cdot s = \neg s \sqcap s = 0 \} \\
&\quad s \cdot t \cdot a \\
&= \{ \text{by (b)} \} \\
&\quad (s \sqcap t) \cdot a . \\
\text{(d) First, by (c), } &t \cdot \top \sqcap \neg t \cdot \top = (t \sqcap \neg t) \cdot \top = 0 \cdot \top = 0. \text{ Second, } t \cdot \top + \neg t \cdot \top = \\
&(t + \neg t) \cdot \top = 1 \cdot \top = \top. \\
\text{(e) } &t \cdot \top \sqcap a \\
&= \{ \text{Boolean algebra} \} \\
&\quad t \cdot (a + \bar{a}) \sqcap a \\
&= \{ \text{disjunctivity} \} \\
&\quad (t \cdot a + t \cdot \bar{a}) \sqcap a \\
&= \{ \text{distributivity} \} \\
&\quad (t \cdot a \sqcap a) + (t \cdot \bar{a} \sqcap a) \\
&= \{ \text{Boolean algebra, since } t \cdot \bar{a} \leq \bar{a} \text{ by monotonicity and neutrality} \} \\
&\quad (t \cdot a \sqcap a) + 0 \\
&= \{ \text{Boolean algebra, since } t \cdot a \leq a \text{ by monotonicity and neutrality} \} \\
&\quad t \cdot a . \\
\text{(f) By (e) we have } &t \cdot (a \sqcap b) = a \sqcap b \sqcap t \cdot \top, \text{ from which the claim is immediate} \\
&\text{by commutativity, associativity and idempotence of } \sqcap \text{ as well as (e) again.} \\
\text{(g) We show } &\overline{(\sqcap L) \cdot \top} = \overline{\sqcap (L \cdot \top)}. \\
&\overline{\sqcap (L \cdot \top)} \\
&= \{ \text{de Morgan} \} \\
&\quad \sqcup \{ \overline{t \cdot \top} : t \in L \} \\
&= \{ \text{by (d)} \} \\
&\quad \sqcup \{ \neg t \cdot \top : t \in L \} \\
&= \{ \text{disjunctivity} \} \\
&\quad (\sqcup \{ \neg t : t \in L \}) \cdot \top \\
&= \{ \text{de Morgan} \} \\
&\quad (\neg \sqcap \{ t : t \in L \}) \cdot \top \\
&= \{ \text{by (d)} \} \\
&\quad \overline{(\sqcap L) \cdot \top} .
\end{aligned}$$

Concerning property (a), idempotence is also characteristic of so-called retractions and used centrally in Scott's classical paper [16] for the definition of data types: each such type is viewed as the range or, equivalently, the set of fixpoints of a retraction.

## 4 Domain and Codomain

**Definition 8** In a KA  $(K, \leq, \top, \cdot, 0, 1)$ , we can define, for  $a \in K$ , the domain  $\lceil a$  via the Galois connection  $\forall t : t \leq 1 \Rightarrow (\lceil a \leq t \stackrel{\text{def}}{\Leftrightarrow} a \leq t \cdot \top)$ .

This is well-defined because of Lemma 7(g) [1,2]. Hence the operation  $\lceil$  is universally disjunctive and therefore monotonic and strict. Moreover the definition implies  $a \leq \lceil a \cdot \top$ . The *co-domain*  $a^\lceil$  is defined symmetrically by the Galois connection  $a^\lceil \leq t \stackrel{\text{def}}{\Leftrightarrow} a \leq \top \cdot t$ .

By the Galois connection, the partial orders  $(\text{TYP}, \leq)$  and  $(\{t \cdot \top : t \in \text{TYP}\}, \leq)$  are isomorphic. Hence we have, for  $t \in \text{TYP}$ , that  $\lceil(t \cdot \top) = t$  (which also follows from properties (e) and (h) in Lemma 9 below).

We list a number of useful properties of the domain operation (see again also [1] and consider [13] for the proofs); analogous ones hold for the co-domain operation.

**Lemma 9** Consider a KA  $(K, \leq, \top, \cdot, 0, 1)$  and  $a, b, c \in K$ .

- |   |  |
|---|--|
| (a) $\lceil a = \sqcap \{t : t \leq 1 \wedge t \cdot a = a\}$ .   | (g) $\lceil(\lceil a) = \lceil a$ .                            |
| (b) $\lceil a \cdot a = a$ .                                      | (h) $\lceil(a \cdot \top) = \lceil a$ .                        |
| (c) $t \leq 1 \wedge t \cdot a = a \Rightarrow \lceil a \leq t$ . | (i) $a \cdot \top \leq \lceil a \cdot \top$ .                  |
| (d) $\lceil(a \cdot b) \leq \lceil a$ .                           | (j) $\lceil(a \cdot b) \leq \lceil(a \cdot \lceil b)$ .        |
| (e) $t \leq 1 \Leftrightarrow \lceil t = t$ .                     | (k) $a^\lceil \sqcap \lceil b = 0 \Rightarrow a \cdot b = 0$ . |
| (f) $\lceil \top = 1$ .   | (l) $\lceil a = 0 \Leftrightarrow a = 0$ .                     |

## 5 Locality of Composition

It should be noted that the converse inequation of Lemma 9(j) does not follow from our axiomatization. A counterexample will be given in Section 8.2. Its essence is that composition does not work “locally” in that only the “near end”, i.e. the domain, of the right factor of a composition does decide “composability”. This observation is the motivation for the term “local composition” defined below.

**Definition 10** A KA has left-local composition if it satisfies

$$\lceil b = \lceil c \Rightarrow \lceil(a \cdot b) = \lceil(a \cdot c) .$$

Right-locality of composition is defined symmetrically. A KA has local composition if its composition is both left-local and right-local.

**Lemma 11** (a) A KA has left-local composition iff it satisfies

$$\lceil(a \cdot b) = \lceil(a \cdot \lceil b) \tag{1}$$



(b) If a KA has left-local composition then  $\ulcorner(\ulcorner a \cdot b) = \ulcorner a \cdot \ulcorner b = \ulcorner a \sqcap \ulcorner b$ .

**PROOF.**

(a)  $(\Rightarrow)$  Immediate from the assumption, since by Lemma 9(g)  $\ulcorner(\ulcorner b) = \ulcorner b$ .

$(\Leftarrow)$  Assume  $\ulcorner b = \ulcorner c$ .

$$\begin{aligned}
& \ulcorner(a \cdot b) \\
= & \quad \{ \text{by (1)} \} \\
& \ulcorner(a \cdot \ulcorner b) \\
= & \quad \{ \text{assumption} \} \\
& \ulcorner(a \cdot \ulcorner c) \\
= & \quad \{ \text{by (1)} \} \\
& \ulcorner(a \cdot c) . \\
\text{(b)} \quad & \ulcorner(\ulcorner a \cdot b) \\
= & \quad \{ \text{by (1)} \} \\
& \ulcorner(\ulcorner a \cdot \ulcorner b) \\
= & \quad \{ \text{by Lemma 9(e)} \} \\
& \ulcorner a \cdot \ulcorner b \\
= & \quad \{ \text{by Lemma 7(b)} \} \\
& \ulcorner a \sqcap \ulcorner b .
\end{aligned}$$

Analogous properties hold for right-locality. In the sequel we only consider KAs with local composition. All examples given in Section 2 satisfy that property.

We conclude this section by establishing a Galois connection between domain and range. It will be useful in Section 6.3 about modal operators.

**Lemma 12** *If  $s, t \leq 1$ , then  $\ulcorner(a \cdot t) \leq \neg s \Leftrightarrow (s \cdot a)^\ulcorner \leq \neg t$ .*

**PROOF.** By Boolean algebra and Lemma 7(b), the claim is equivalent to  $s \cdot \ulcorner(a \cdot t) = 0 \Leftrightarrow (s \cdot a)^\ulcorner \cdot t = 0$ . We calculate

$$\begin{aligned}
& s \cdot \ulcorner(a \cdot t) \\
= & \quad \{ \text{by Lemma 9(e)} \} \\
& \ulcorner(s \cdot \ulcorner(a \cdot t)) \\
= & \quad \{ \text{local composition} \} \\
& \ulcorner(s \cdot a \cdot t) .
\end{aligned}$$

Symmetrically,  $(s \cdot a)^\ulcorner \cdot t = (s \cdot a \cdot t)^\ulcorner$ . Now the claim is immediate from Lemma 9(1).

## 6 Candidate Characterizations of Determinacy

### 6.1 Candidates from Relation Algebra

In a relation algebra, an element  $R$  is called a (*partial*) *function* or a *map* or *deterministic* iff it satisfies  $R^\smile; R \subseteq I$ . By the Schröder laws this is equivalent to the requirement  $R; \bar{I} \subseteq \bar{R}$ .

It is well known that functions are left-distributive through intersection: First,  $R; (S \cap T) \subseteq R; S \cap R; T$  by monotonicity. For the reverse inclusion we have

$$\begin{aligned}
 & R; S \cap R; T \\
 \subseteq & \quad \{ \text{Dedekind} \} \\
 & (R \cap R; T; S^\smile); (S \cap R^\smile; R; T) \\
 \subseteq & \quad \{ \text{monotonicity} \} \\
 & R; (S \cap R^\smile; R; T) \\
 \subseteq & \quad \{ R^\smile; R \subseteq I, \text{ neutrality and monotonicity} \} \\
 & R; (S \cap T) .
 \end{aligned}$$

It is less well known, though, that this property in fact is *equivalent* to the property of being a function. We give a quick proof of this (see also [15]). Assume that  $R$  is left-distributive through intersection. Then

$$\begin{aligned}
 & R; \bar{I} \cap R \\
 = & \quad \{ \text{neutrality} \} \\
 & R; \bar{I} \cap R; I \\
 = & \quad \{ \text{left-distributivity of } R \} \\
 & R; (\bar{I} \cap I) \\
 = & \quad \{ \text{Boolean algebra} \} \\
 & R; 0 \\
 = & \quad \{ \text{strictness} \} \\
 & 0 ,
 \end{aligned}$$

so that  $R; \bar{I} \subseteq \bar{R}$  follows by Boolean algebra.

However, the equivalence is not Kleene valid, as we shall show in Section 8.3, and so the use of (some form of) Dedekind's rule is essential for that. In Kleene algebras, left-distributivity is only equivalent to a stronger property that results by generalizing the constant  $I$  in the inclusion  $R; \bar{I} \subseteq \bar{R}$  to a variable (after making it visible in the right hand side, see the definition of SC below). These observations lead to our first three candidates for characterizations of determinate objects (the formula involving converse not being usable in Kleene algebras). We attach names to the characterizing predicates

for easier reference.

$$\begin{aligned}
\text{LD}(a) &\stackrel{\text{def}}{\Leftrightarrow} \forall b, c : a \cdot (b \sqcap c) = a \cdot b \sqcap a \cdot c && \text{(left-distributivity)} \\
\text{SC}(a) &\stackrel{\text{def}}{\Leftrightarrow} \forall b : a \cdot \bar{b} \leq \overline{a \cdot b} && \text{(subsumption of complement)} \\
\text{SC1}(a) &\stackrel{\text{def}}{\Leftrightarrow} a \cdot \bar{1} \leq \bar{a} && \text{(subsumption of complement of 1)}
\end{aligned}$$

Concerning the relations between these notions, we first state

**Lemma 13**  $\text{LD}(a) \Leftrightarrow \text{SC}(a)$ .

**PROOF.**  $(\Rightarrow)$   $0 = a \cdot 0 = a \cdot (c \sqcap \bar{c}) = a \cdot c \sqcap a \cdot \bar{c}$  by  $\text{LD}(a)$ .  
 $(\Leftarrow)$

$$\begin{aligned}
&a \cdot b \sqcap a \cdot c \\
= &\quad \{ \text{ Boolean algebra } \} \\
&a \cdot ((b \sqcap c) + (b \sqcap \bar{c})) \sqcap a \cdot c \\
= &\quad \{ \text{ distributivities } \} \\
&(a \cdot (b \sqcap c) \sqcap a \cdot c) + (a \cdot (b \sqcap \bar{c}) \sqcap a \cdot c) \\
= &\quad \{ \text{ lattice algebra } \} \\
&a \cdot (b \sqcap c) + (a \cdot (b \sqcap \bar{c}) \sqcap a \cdot c) \\
= &\quad \{ \text{ since } a \cdot (b \sqcap \bar{c}) \sqcap a \cdot c \leq a \cdot \bar{c} \sqcap a \cdot c = 0 \text{ by } \text{SC}(a) \} \\
&a \cdot (b \sqcap c) .
\end{aligned}$$

Moreover, we clearly have  $\text{SC}(a) \Rightarrow \text{SC1}(a)$  (set  $b = 1$ ).

In Section 8.3, we show that the reverse implication is not valid in all Kleene algebras. However, it holds in LAN, PAT and RA. To understand this, let us elaborate on the case of the Kleene algebra LAN of formal languages over an alphabet  $A$ . There we have  $\bar{1} = A^+$ , and so for  $U \subseteq A^*$  we get  $\text{SC1}(U) \Leftrightarrow U \bullet A^+ \subseteq \bar{U}$ . In other words, a proper extension of a word in  $U$  must not lie in  $U$  again. This is equivalent to  $U$  being a prefix-free language (i.e. none of the strings of  $U$  is a proper prefix of another; in coding theory, this is known as the FANO condition; in process algebra, sets with this property are called *prefix-free* or *prefix antichains*). The same applies to the algebra PAT of sets of paths in a graph, modeled as sets of strings of nodes.

Assume now  $\text{SC1}(U)$  and  $x \in U \bullet V \cap U \bullet W$  for  $V, W \subseteq A^*$ , say  $x = u_1 \bullet v = u_2 \bullet w$  for some  $u_1, u_2 \in U$ ,  $v \in V$  and  $w \in W$ . By local linearity of the prefix relation we obtain that  $u_1$  must be a prefix of  $u_2$  or the other way around. By prefix-freeness of  $U$  this means  $u_1 = u_2$  and cancellativity of  $\bullet$  shows  $v = w \in V \cap W$ . Therefore also  $x \in U \bullet (V \cap W)$ , i.e.  $\text{LD}(U)$  holds. But this is equivalent to  $\text{SC}(U)$  as stated in the above lemma.

## 6.2 Domain-Oriented Characterizations

In view of the previous section it appears that LD, SC and SC1 are not appropriate characterizations of (partial) functions, since in a prefix-free set of paths still different paths may emanate from the same starting node. Therefore, a function should rather be characterized in a domain-oriented way: every point in the domain should have a unique “extension”.

Now, in PAT, a node starts a unique path in a path set  $a$  iff removal of this path removes that node from the domain of  $a$ . This can be captured in a purely order-theoretic way by the property

$$\text{DD}(a) \stackrel{\text{def}}{\Leftrightarrow} \forall b : b < a \Rightarrow \ulcorner b < \ulcorner a \quad (\text{decrease of domain}).$$

Here,  $<$  is the strict-order associated with the order  $\leq$  underlying the KA under consideration, i.e.  $c < d \stackrel{\text{def}}{\Leftrightarrow} c \leq d \wedge c \neq d$ . Note that all atoms satisfy DD.

This property is easily shown to be equivalent to

$$\text{ED}(a) \stackrel{\text{def}}{\Leftrightarrow} \forall b : b \leq a \wedge \ulcorner b = \ulcorner a \Rightarrow b = a \quad (\text{equality of domain}).$$

Another candidate is

$$\text{CD}(a) \stackrel{\text{def}}{\Leftrightarrow} \forall b : b \leq a \Rightarrow b = \ulcorner b \cdot a \quad (\text{characterization by domain}).$$

**Lemma 14**  $\text{DD}(a) \Leftrightarrow \text{CD}(a)$  .

**PROOF.** We prove  $\text{ED}(a) \Leftrightarrow \text{CD}(a)$ .

( $\Leftarrow$ ) Suppose  $b \leq a$  and  $\ulcorner b = \ulcorner a$ . Then by  $\text{CD}(a)$  and Lemma 9(b) we get

$$b = \ulcorner b \cdot a = \ulcorner a \cdot a = a.$$

$$\begin{aligned} (\Rightarrow) \quad & b \leq a \\ & \Rightarrow \quad \{ \text{monotonicity} \} \\ & \quad \ulcorner b \cdot b \leq \ulcorner b \cdot a \quad \wedge \quad \ulcorner b \leq \ulcorner a \\ & \Leftrightarrow \quad \{ \text{by Lemma 9(b)} \} \\ & \quad b \leq \ulcorner b \cdot a \quad \wedge \quad \ulcorner b = \ulcorner b \sqcap \ulcorner a \\ & \Leftrightarrow \quad \{ \text{by Lemma 11(b)} \} \\ & \quad b \leq \ulcorner b \cdot a \quad \wedge \quad \ulcorner b = \ulcorner (\ulcorner b \cdot a) \\ & \Rightarrow \quad \{ \text{by ED}(a) \} \\ & \quad b = \ulcorner b \cdot a . \end{aligned}$$

However, CD and DD are not equivalent to LD. In LAN an element  $a$  satisfies  $\text{DD}(a)$  iff it contains at most one word, whereas  $\text{LD}(a)$  is equivalent to prefix-freeness of  $a$ . So in LAN the properties CD and DD imply LD, but not the

other way around. We show in section 8.3 that the implication does not hold for arbitrary Kleene algebras.

In REL the properties CD and DD are equivalent to the other characterizations of deterministic relations. However, in the case of an abstract relation algebra in RA, DD does not imply LD, as will be shown in Section 8.4.

**Lemma 15** *LD implies DD in RA.*

**PROOF.** Assume LD( $a$ ), which is equivalent to  $a^\smile; a \leq 1$  in RA, and  $b \leq a$ . We show CD( $a$ ). Since  $b = \ulcorner b; b \leq \ulcorner b; a$ , we only need to prove  $\ulcorner b; a \leq b$ .

$$\begin{aligned}
& \ulcorner b; a \\
= & \quad \{ \text{relational algebra} \} \\
& b; \top \sqcap a \\
\leq & \quad \{ \text{Dedekind} \} \\
& (b \sqcap a; \top^\smile); (\top \sqcap b^\smile; a) \\
= & \quad \{ \text{Boolean algebra, since } b \leq a \leq a; \top^\smile \} \\
& b; b^\smile; a \\
\leq & \quad \{ \text{monotonicity} \} \\
& b; a^\smile; a \\
\leq & \quad \{ \text{assumption } a^\smile; a \leq 1 \} \\
& b.
\end{aligned}$$

Finally, we give another domain-oriented characterization, that is easily shown equivalent to CD:

$$\text{SO}(a) \stackrel{\text{def}}{\cong} \ulcorner : \{b : b \leq a\} \rightarrow \{t : t \leq \ulcorner a\} \text{ is an order-isomorphism}$$

(subobject lattice)

### 6.3 A Modal Characterization

The modal operators diamond and box are quantifiers about the successor states of a state in a transition system. But they can also be viewed as assertion transformers dealing with sets of states. The (forward) diamond operator assigns to a set of states  $t$  the set  $s$  of all those states that have a successor in  $t$ . The (forward) box operator is the dual of the diamond operator; it assigns to a set of states  $t$  the set  $s$  of all those states for which all successors lie in  $t$ . The backward modal operators are defined symmetrically.

In the setting of Kleene algebras, the role of assertions or sets of states is played by types. Hence we can define the modal operators as type transformers. The

operators of dynamic logic are obtained by setting

$$\langle a \rangle t \stackrel{\text{def}}{=} \ulcorner(a \cdot t) \text{ , } \quad [a]t \stackrel{\text{def}}{=} \neg \langle a \rangle \neg t \text{ .}$$

We note that  $[a]t = a \rightarrow t$ , where  $a \rightarrow b \stackrel{\text{def}}{=} \neg \ulcorner(a \cdot \neg b)$  is called *type implication*, an operation which is useful for dealing with assertions in demonic semantics and which enjoys many useful properties, see [5]. Moreover, in relational semantics of imperative programs,  $[a]t = \text{wlp}.a.t$  [2,6].

We now carry over the well-known modal characterization of deterministic relations (see e.g. [14]) to elements of Kleene algebras and call an element  $a$  *modally deterministic* iff  $\text{MD}(a)$  holds, where

$$\text{MD}(a) \stackrel{\text{def}}{\Leftrightarrow} \forall t : \langle a \rangle t \leq [a]t \text{ .} \quad (2)$$

Note that by Boolean algebra  $\text{MD}(a)$  is equivalent to

$$\forall t : \ulcorner(a \cdot t) \sqcap \ulcorner(a \cdot \neg t) = 0 \text{ .} \quad (3)$$

The following properties are easily checked:

**Corollary 16** (a)  $\langle a \rangle 0 = 0$ .

(b)  $\langle a \rangle 1 = \ulcorner a$ .

(c)  $[a]0 = \neg \ulcorner a$ .

(d)  $[a]1 = 1$ .

(e) In particular,  $\langle a \rangle 0 \leq [a]0$  and  $\langle a \rangle 1 \leq [a]1$ .

(f) Suppose that the only types are 0 and 1 (such as e.g. in the algebra LAN of formal languages). Then  $\text{MD}(a)$  holds for all  $a$ .

(g)  $\langle \_ \rangle$  is monotonic and  $[\_]$  is antitonic, i.e. for  $a \leq b$  and  $t \leq 1$  we have  $\langle a \rangle t \leq \langle b \rangle t$  and  $[b]t \leq [a]t$ .

The modal characterization links to the domain-oriented characterizations as follows (for the relationship with our other characterizations see Section 8.7):

**Lemma 17** We have  $\text{CD}(a) \Rightarrow \text{MD}(a)$ . The reverse implication is not valid.

**PROOF.** Assume  $\text{CD}(a)$  and consider a type  $t$ . We set  $d \stackrel{\text{def}}{=} \ulcorner(a \cdot t) \sqcap \ulcorner(a \cdot \neg t)$  and calculate

$$\begin{aligned} & d \cdot a \\ = & \{ \text{definition of } d, \text{ distributivity of type restriction (Lemma 7(c)) } \} \\ & \ulcorner(a \cdot t) \cdot a \sqcap \ulcorner(a \cdot \neg t) \cdot a \\ = & \{ \text{by CD}(a) \text{ and } a \cdot t \leq a \text{ and } a \cdot \neg t \leq a \} \\ & a \cdot t \sqcap a \cdot \neg t \end{aligned}$$

$$\begin{aligned}
&= \{ \text{distributivity of type restriction (Lemma 7(c))} \} \\
&\quad a \cdot (t \sqcap \neg t) \\
&= \{ \text{Boolean algebra and strictness} \} \\
&\quad 0 .
\end{aligned}$$

Therefore,  $d = \ulcorner(d \cdot \ulcorner a) = \ulcorner(d \cdot a) = \ulcorner 0 = 0$ , which shows  $\langle a \rangle t \leq [a]t$ .

To see that the reverse implication fails, consider a Kleene algebra in which the only types are 0 and 1. Then we have  $\text{DD}(a)$  and hence, by Lemma 14,  $\text{CD}(a)$  iff  $a$  is an atom. However, by Corollary 16(f),  $\text{MD}(a)$  is always true.

## 7 Closure Properties

### 7.1 Downward Closure

A natural property of functions is that a subobject of a determinate object is determinate again. Here we can show

**Lemma 18** *The properties SC, SC1, CD and MD are closed under subobjects.*

**PROOF.** For SC and SC1 this is immediate from monotonicity, since we can restate these properties as  $\forall b : a \cdot \bar{b} \sqcap a \cdot b = 0$  and  $a \cdot \bar{1} \sqcap a = 0$ , respectively. For CD, suppose  $b \leq a$  and  $c \leq b$ . Then also  $c \leq a$ , hence

$$c = \ulcorner c \cdot a = (\ulcorner c \sqcap \ulcorner b) \cdot a = \ulcorner c \cdot \ulcorner b \cdot a = \ulcorner c \cdot b .$$

Finally, for MD, the assertion is immediate from Corollary 16(g).

### 7.2 All Types are Determinate

Based on our original relational motivation, we would like to have that all types are determinate. By the previous section, to ensure this we only need to check that the largest type 1 satisfies all our characterizations. Fortunately, this indeed holds, as the following lemma shows.

**Lemma 19**  $\text{LD}(1) \wedge \text{SC1}(1) \wedge \text{CD}(1) \wedge \text{MD}(1)$ .

**PROOF.**  $\text{LD}(1)$  and  $\text{SC1}(1)$  are trivial. For the third assertion, assume  $t \leq 1$ . Then by Lemma 9(e) we have  $t = t \cdot 1 = \ulcorner t \cdot 1$ . Finally, for types  $s, t$  we have, again by Lemma 9(e), that  $[1]t = t = \langle 1 \rangle t$ .

### 7.3 Closure Under Compatible Join

In this section we show that all our characterizations are closed under join of pairwise compatible elements. Here, we call  $a, b$  *compatible* if they agree on the intersection of their domains, ie. if  $\ulcorner b \cdot a = \ulcorner a \cdot b$ . This is trivially satisfied if  $\ulcorner a \cdot \ulcorner b = 0$ . Moreover, every element is compatible with itself. A set  $L \subseteq M$  is *compatible* if the elements of  $L$  are pairwise compatible.

As an auxiliary result we need the following lemma.

**Lemma 20** *Let  $a, b$  be compatible.*

- (a)  $\ulcorner a \cdot b = a \sqcap b = \ulcorner b \cdot a$ .
- (b) If  $\text{LD}(a)$  then  $a \cdot c \sqcap b \cdot d = \ulcorner b \cdot a \cdot (c \sqcap d)$ .
- (c) If  $\text{SC1}(a)$  then  $a \cdot \bar{1} \sqcap b = 0$ .
- (d) If  $\text{CD}(a)$  and  $c \leq a$  then  $c$  and  $b$  are compatible as well.
- (e) If  $\text{MD}(a)$  and  $t \leq 1$  then  $\ulcorner(a \cdot t) \sqcap \ulcorner(b \cdot \neg t) = 0$ .

**PROOF.**

- (a) By Lemma 7(f) we get  $a \sqcap b = a \sqcap \ulcorner a \cdot b = a \sqcap \ulcorner b \cdot a = \ulcorner b \cdot a$ .
- (b)
$$\begin{aligned}
 & a \cdot c \sqcap b \cdot d \\
 &= \{ \text{by } a = \ulcorner a \cdot a \text{ and Lemma 7(f)} \} \\
 & \quad a \cdot c \sqcap \ulcorner a \cdot b \cdot d \\
 &= \{ \text{compatibility} \} \\
 & \quad a \cdot c \sqcap \ulcorner b \cdot a \cdot d \\
 &= \{ \text{by Lemma 7(f)} \} \\
 & \quad \ulcorner b \cdot a \cdot c \sqcap \ulcorner b \cdot a \cdot d \\
 &= \{ \text{by LD}(a) \text{ and downward closure of LD} \} \\
 & \quad \ulcorner b \cdot a \cdot (c \sqcap d) .
 \end{aligned}$$
- (c)
$$\begin{aligned}
 & a \cdot \bar{1} \sqcap b \\
 &= \{ \text{by Lemma 7(f)} \} \\
 & \quad a \cdot \bar{1} \sqcap \ulcorner a \cdot b \\
 &= \{ \text{compatibility} \} \\
 & \quad a \cdot \bar{1} \sqcap \ulcorner b \cdot a \\
 &\leq \{ \text{monotonicity} \} \\
 & \quad a \cdot \bar{1} \sqcap a \\
 &= \{ \text{by SC1}(a) \} \\
 & \quad 0 .
 \end{aligned}$$
- (d)
$$\begin{aligned}
 & \ulcorner c \cdot b \\
 &= \{ \text{by CD}(a) \} \\
 & \quad \ulcorner(\ulcorner c \cdot a) \cdot b \\
 &= \{ \text{local composition} \} \\
 & \quad \ulcorner c \cdot \ulcorner a \cdot b \\
 &= \{ \text{compatibility} \}
 \end{aligned}$$



$$\begin{aligned}
& \ulcorner c \cdot \ulcorner b \cdot a \\
= & \quad \{ \text{types commute and CD}(a) \} \\
& \ulcorner b \cdot c . \\
(e) \quad & \ulcorner (a \cdot t) \sqcap \ulcorner (b \cdot \neg t) \\
= & \quad \{ \text{by } a \cdot t \leq a \text{ and monotonicity} \} \\
& \ulcorner a \cdot \ulcorner (a \cdot t) \sqcap \ulcorner (b \cdot \neg t) \\
= & \quad \{ \text{by Lemma 7(f)} \} \\
& \ulcorner (a \cdot t) \sqcap \ulcorner a \cdot \ulcorner (b \cdot \neg t) \\
= & \quad \{ \text{by Lemma 11(b)} \} \\
& \ulcorner (a \cdot t) \sqcap \ulcorner (\ulcorner a \cdot b \cdot \neg t) \\
= & \quad \{ \text{compatibility} \} \\
& \ulcorner (a \cdot t) \sqcap \ulcorner (\ulcorner b \cdot a \cdot \neg t) \\
\leq & \quad \{ \text{monotonicity} \} \\
& \ulcorner (a \cdot t) \sqcap \ulcorner (a \cdot \neg t) \\
= & \quad \{ \text{by MD}(a) \} \\
& 0 .
\end{aligned}$$

**Lemma 21** *Let P range over LD, SC, SC1, DD, ED, CD, SO, MD. Let moreover  $L \subseteq K$  be a compatible set. Then  $(\forall a \in L : P(a)) \Rightarrow P(\sqcup L)$ .*

**PROOF.** Due to the equivalences between some of the properties, it suffices to prove the lemma for LD, SC1, CD and MD.

(LD)

$$\begin{aligned}
& (\bigsqcup_{a \in L} a) \cdot c \sqcap (\bigsqcup_{a \in L} a) \cdot d \\
= & \quad \{ \text{distributivity of } \cdot \text{ over } \sqcup \} \\
& (\bigsqcup_{a \in L} a \cdot c) \sqcap (\bigsqcup_{a \in L} a \cdot d) \\
= & \quad \{ \text{distributivity of } \sqcap \text{ over } \sqcup \text{ and renaming} \} \\
& \bigsqcup_{a \in L} \bigsqcup_{b \in L} a \cdot c \sqcap b \cdot d \\
= & \quad \{ \text{by } \forall a \in L : \text{LD}(a) \text{ and Lemma 20(b)} \} \\
& \bigsqcup_{a \in L} \bigsqcup_{b \in L} \ulcorner b \cdot a \cdot (c \sqcap d) \\
= & \quad \{ \text{distributivity of } \cdot \text{ over } \sqcup \} \\
& (\bigsqcup_{a \in L} \bigsqcup_{b \in L} \ulcorner b \cdot a) \cdot (c \sqcap d) \\
= & \quad \{ \text{by } \forall a, b \in L : \ulcorner b \cdot a \leq a \wedge \ulcorner a \cdot a = a \} \\
& (\bigsqcup_{a \in L} a) \cdot (c \sqcap d) .
\end{aligned}$$

(SC1)

$$\begin{aligned}
& (\bigsqcup_{a \in L} a) \cdot \bar{1} \sqcap (\bigsqcup_{a \in L} a) \\
= & \quad \{ \text{distributivity of } \cdot \text{ over } \sqcup \}
\end{aligned}$$

$$\begin{aligned}
& (\bigsqcup_{a \in L} a \cdot \bar{1}) \sqcap (\bigsqcup_{a \in L} a) \\
= & \quad \{ \text{distributivity of } \sqcap \text{ over } \sqcup \text{ and renaming} \} \\
& \bigsqcup_{a \in L} \bigsqcup_{b \in L} a \cdot \bar{1} \sqcap b \\
= & \quad \{ \text{by } \forall a \in L : \text{SC1}(a) \text{ and Lemma 20(c)} \} \\
& 0 .
\end{aligned}$$

(CD) Assume  $c \leq \sqcup L$ . We first show that  $c$  is compatible with all  $b \in L$ :

$$\begin{aligned}
& \ulcorner c \cdot b \\
= & \quad \{ \text{lattice algebra} \} \\
& \ulcorner (c \sqcap \sqcup L) \cdot b \\
= & \quad \{ \text{distributivities} \} \\
& \bigsqcup_{a \in L} \ulcorner (c \sqcap a) \cdot b \\
= & \quad \{ \text{Lemma 20(d)} \} \\
& \bigsqcup_{a \in L} \ulcorner b \cdot (c \sqcap a) \\
= & \quad \{ \text{distributivities} \} \\
& \ulcorner b \cdot (c \sqcap (\bigsqcup_{a \in L} a)) \\
= & \quad \{ \text{lattice algebra} \} \\
& \ulcorner b \cdot c .
\end{aligned}$$

Now

$$\begin{aligned}
& \ulcorner c \cdot (\bigsqcup_{b \in L} b) \\
= & \quad \{ \text{distributivity} \} \\
& \bigsqcup_{b \in L} \ulcorner c \cdot b \\
= & \quad \{ \text{compatibility (result above)} \} \\
& \bigsqcup_{b \in L} \ulcorner b \cdot c \\
= & \quad \{ \text{distributivity} \} \\
& (\bigsqcup_{b \in L} \ulcorner b) \cdot c \\
= & \quad \{ \text{by Lemma 9(a), since } \bigsqcup_{b \in L} \ulcorner b \geq \ulcorner c \text{ by monotonicity} \} \\
& c .
\end{aligned}$$

(MD) We use variant (3) of MD.

$$\begin{aligned}
& \ulcorner ((\bigsqcup_{a \in L} a) \cdot t) \sqcap \ulcorner ((\bigsqcup_{a \in L} a) \cdot \neg t) \\
= & \quad \{ \text{distributivity of } \cdot \text{ and } \ulcorner \text{ over } \sqcup \} \\
& (\bigsqcup_{a \in L} \ulcorner (a \cdot t)) \sqcap (\bigsqcup_{a \in L} \ulcorner (a \cdot \neg t))
\end{aligned}$$

$$\begin{aligned}
&= \quad \{\{ \text{distributivity of } \sqcap \text{ over } \sqcup \text{ and renaming} \}\} \\
&\quad \bigsqcup_{a \in L} \bigsqcup_{b \in L} \ulcorner(a \cdot t) \sqcap \ulcorner(b \cdot \neg t) \\
&= \quad \{\{ \text{by } \forall a \in L : \text{MD}(a) \text{ and Lemma 20(e)} \}\} \\
&\quad 0 .
\end{aligned}$$

For the case of CD we also have the reverse implication:

**Lemma 22** *If  $\text{CD}(\sqcup L)$  then  $\forall a \in L : \text{CD}(a)$  and  $L$  is compatible.*

**PROOF.** The first part of the claim is immediate by downward closure of CD. For the second part, let  $a, b \in L$ . Note that by Lemma 11(b),  $\ulcorner(\ulcorner a \cdot b) = \ulcorner a \cdot \ulcorner b = \ulcorner b \cdot \ulcorner a = \ulcorner(\ulcorner b \cdot a)$ . Moreover,  $\ulcorner a \cdot b \leq a + b$ ,  $\ulcorner b \cdot a \leq a + b$  and  $\text{CD}(a + b)$  by  $a + b \leq \sqcup L$  and downward closure of CD. Hence, using Lemma 11(b), distributivity and Lemma 9(b),

$$\ulcorner a \cdot b = \ulcorner(\ulcorner a \cdot b) \cdot (a + b) = \ulcorner a \cdot b + \ulcorner b \cdot a = \ulcorner(\ulcorner b \cdot a) \cdot (a + b) = \ulcorner b \cdot a .$$

Note that the corresponding property for SC1, LD and MD does not hold. Indeed,  $L \stackrel{\text{def}}{=} \{a, b\}$  in LAN  $\stackrel{\text{def}}{=} (\mathcal{P}(\{a, b\}^*), \subseteq, \emptyset, \{a, b\}^*, \bullet, \varepsilon)$  can be used as a counterexample for all three cases.

#### 7.4 Closure under Composition

Another natural property of determinate objects is that they are closed under composition. In this section we investigate which of our candidates for characterization imply this closure.

First, it is straightforward that LD (and hence SC) is closed under composition. Moreover, we have

**Lemma 23** *MD is closed under composition.*

**PROOF.** We first calculate

$$\begin{aligned}
&\langle a \cdot b \rangle t \\
&= \quad \{\{ \text{definition of } \langle - \rangle \}\} \\
&\quad \ulcorner(a \cdot b \cdot t) \\
&= \quad \{\{ \text{local composition} \}\} \\
&\quad \ulcorner(a \cdot \ulcorner(b \cdot t)) \\
&= \quad \{\{ \text{definition of } \langle - \rangle \}\} \\
&\quad \langle a \rangle (\langle b \rangle t)
\end{aligned}$$

from which by duality we also get  $[a \cdot b]t = [a](\langle b \rangle t)$ . Now the claim is immediate by monotonicity of  $\langle a \rangle$ , which follows directly from its definition.

Properties SC1 and DD are *not* closed under composition as will be shown in Sections 8.5 and 8.6. However, we can show closure of DD under additional assumptions. To formulate these, we need an auxiliary notion.

**Definition 24** *Suppose that the Boolean algebra underlying our KA is atomic with atom set  $\text{At}$ . Analogously to the set of subidentities we define the set of subatoms as  $\text{SAt} \stackrel{\text{def}}{=} \text{At} \cup \{0\}$ , i.e. as the set of elements that lie below some atom. Moreover, we set  $\text{SAt}(a) \stackrel{\text{def}}{=} \text{At}(a) \cup \{0\}$ .*

Note that all subatoms satisfy DD. Now we can show the following lemma.

**Lemma 25** (a) *If  $a$  is an atom, then  $\ulcorner a$  and  $a^\top$  are atoms as well.*  
 (b)  $\text{DD}(a) \Leftrightarrow (\forall t \in \text{At}(1) : t \cdot a \in \text{SAt}(a)) \Leftrightarrow (\forall t \in \text{At}(\ulcorner a) : t \cdot a \in \text{At}(a)).$   
 (c) *DD is closed under composition iff the set of subatoms is closed under composition.*

**PROOF.**

- (a) We show this for  $\ulcorner a$  (the case of  $a^\top$  is symmetric). By Lemma 9(1),  $\ulcorner a \neq 0$ . Assume  $t < \ulcorner a$ . Then  $t \cdot a \leq a$  by  $t \leq 1$  and monotonicity. But  $t \cdot a = a$  is not possible, because by Lemma 9(a) that would imply  $\ulcorner a \leq t$ , a contradiction. Since  $a$  is an atom, this implies  $t \cdot a = 0$ , hence  $0 = \ulcorner(t \cdot a) = t \cdot \ulcorner a = t \sqcap \ulcorner a = t$ .
- (b) We prove the first equivalence only, the second one being trivial.  
 $(\Rightarrow)$  Suppose  $\text{DD}(a)$  and let  $t \in \text{At}(1)$ . Assume  $t \cdot a \neq 0$  and let  $b < t \cdot a$ . By Lemmas 14 and 18,  $\text{DD}(t \cdot a)$  holds, so that  $\ulcorner b < \ulcorner(t \cdot a) \leq \ulcorner t$ , whence  $\ulcorner b = 0$  since  $t \in \text{At}(1)$ . By Lemma 9(1),  $b = 0$ , so that  $t \cdot a \in \text{At}(a)$ .  
 $(\Leftarrow)$  By  $1 = \sqcup \text{At}(1)$  and distributivity we have  $c = \sqcup \{t \cdot c : t \in \text{At}(1)\}$  for all  $c$ . Hence, if  $c \leq d$  then

$$c < d \Leftrightarrow \exists t \in \text{At}(1) : t \cdot c < t \cdot d. \quad (*)$$

- Suppose  $b < a$ . By (\*) there must be a  $t \in \text{At}(1)$  such that  $t \cdot b < t \cdot a$ . In particular,  $t \cdot a \neq 0$ . But since  $t \cdot a \in \text{SAt}(a)$ , we must have  $t \cdot b = 0$ . By Lemma 9(1),  $\ulcorner(t \cdot b) = 0$ ; however,  $\ulcorner(t \cdot a) \neq 0$ , since  $t \cdot a \neq 0$ . Hence,  $t \cdot \ulcorner b = \ulcorner(t \cdot b) < \ulcorner(t \cdot a) = t \cdot \ulcorner a$ , from which  $\ulcorner b < \ulcorner a$  follows again by (\*).
- (c)  $(\Rightarrow)$  Let  $a, b$  be atoms. Then  $\text{DD}(a)$  and  $\text{DD}(b)$ , whence  $\text{DD}(a \cdot b)$ , by assumption. If  $a \cdot b \neq 0$ , there exists  $c < a \cdot b$ . By  $\text{DD}(a \cdot b)$ ,  $\ulcorner c < \ulcorner(a \cdot b) \leq \ulcorner a$ , whence  $\ulcorner c = 0$ , since  $\ulcorner a$  is an atom, by part (a). By Lemma 9(1),  $c = 0$ , so that  $a \cdot b$  is an atom.  
 $(\Leftarrow)$  Suppose  $\text{DD}(a)$  and  $\text{DD}(b)$ . For any  $t \in \text{At}(1)$ ,  $t \cdot a$  is a subatom, by part (b) and  $\text{DD}(a)$ . By part (a),  $(t \cdot a)^\top$  is a subatom. Hence, by part (b) and  $\text{DD}(b)$ ,  $(t \cdot a)^\top \cdot b$  is a subatom. Now,  $t \cdot a \cdot b = (t \cdot a) \cdot ((t \cdot a)^\top \cdot b)$  and thus, by assumption,  $t \cdot a \cdot b$  is a subatom, being a composition of two subatoms. Because  $t$  is arbitrary, part (b) implies  $\text{DD}(a \cdot b)$ .

## 7.5 Determinacy of Loops

In this section we apply our previous results to the semantics of while loops. Classically, a loop of the form `while  $G$  do  $B$  od` with guard  $G$  and body  $B$  is modeled in Kleene algebra as follows (see e.g. [9]). The guard is represented by a type  $g$  characterizing all states that satisfy  $G$ . The semantics of the body  $B$  is given by an element  $b$  of the underlying KA. Then the loop itself is described by the semantical value

$$(g \cdot b)^* \cdot \neg g .$$

This represents the informal view that the loop repeats the body  $B$  as long as the guard  $G$  stays true and terminates as soon as a state is reached in which  $G$  becomes false.

We want to show now that the semantics of a loop with determinate body is determinate again. We perform this using the predicates MD and CD:

**Lemma 26** (a) *If MD( $b$ ) holds then the elements  $(g \cdot b)^i \cdot \neg g$  ( $i \in \mathbb{N}$ ) have pairwise disjoint domains.*

(b)  $\text{MD}(b) \Rightarrow \text{MD}((g \cdot b)^* \cdot \neg g)$ .

(c) *Assume that in the KA under consideration CD is closed under composition. Then  $\text{CD}(b) \Rightarrow \text{CD}((g \cdot b)^* \cdot \neg g)$ .*

### PROOF.

(a) An easy induction shows  $\text{MD}((g \cdot b)^i)$  and  $\text{MD}((g \cdot b)^i \cdot t)$  for all  $i \in \mathbb{N}$  and all  $t \leq 1$ . Now consider two elements  $(g \cdot b)^i \cdot \neg g$  and  $(g \cdot b)^j \cdot \neg g$  where, without loss of generality,  $j > i$ .

$$\begin{aligned} & \ulcorner (g \cdot b)^i \cdot \neg g \urcorner \sqcap \ulcorner (g \cdot b)^j \cdot \neg g \urcorner \\ = & \quad \{ \text{arithmetic} \} \\ & \ulcorner (g \cdot b)^i \cdot \neg g \urcorner \sqcap \ulcorner (g \cdot b)^i \cdot g \cdot b \cdot (g \cdot b)^{j-i-1} \cdot \neg g \urcorner \\ = & \quad \{ \text{local composition} \} \\ & \ulcorner (g \cdot b)^i \cdot \neg g \urcorner \sqcap \ulcorner (g \cdot b)^i \cdot \ulcorner g \cdot b \cdot (g \cdot b)^{j-i-1} \cdot \neg g \urcorner \urcorner \\ \leq & \quad \{ \text{by Lemma 9(d) and 9(e)} \} \\ & \ulcorner (g \cdot b)^i \cdot \neg g \urcorner \sqcap \ulcorner (g \cdot b)^i \cdot g \urcorner \\ = & \quad \{ \text{by MD}((g \cdot b)^i) \} \\ & 0 . \end{aligned}$$

(b) The claim follows from (a) and Lemma 21 together with the fact that

$$(g \cdot b)^* \cdot \neg g = \sqcup \{ (g \cdot b)^i \cdot \neg g : i \in \mathbb{N} \} .$$

(c) By Lemma 17, CD implies MD, and the claim follows analogously to (b).

## 8 Counterexamples

### 8.1 A Technique for Constructing Kleene Algebras

In this section, various finite Kleene algebras are constructed in the following way. We head for algebras in which the underlying lattice is an *atomic* Boolean algebra. In each case we list the set  $\mathbf{At}$  of atoms; the other elements are then given by all possible joins of atoms (including the empty join). If there are  $n$  atoms, the algebra thus has  $2^n$  elements. The Boolean operations are defined via the atom sets of the resulting elements:

$$\begin{aligned} \mathbf{At}(p + q) &\stackrel{\text{def}}{=} \mathbf{At}(p) \cup \mathbf{At}(q) , & \mathbf{At}(p \sqcap q) &\stackrel{\text{def}}{=} \mathbf{At}(p) \cap \mathbf{At}(q) , \\ \mathbf{At}(\bar{p}) &\stackrel{\text{def}}{=} \mathbf{At} \setminus \mathbf{At}(p) , & \mathbf{At}(0) &\stackrel{\text{def}}{=} \emptyset , & \mathbf{At}(\top) &\stackrel{\text{def}}{=} \mathbf{At} . \end{aligned}$$

The meet of two atoms is of course 0. Obviously, this defines an atomic Boolean algebra.

Composition  $\cdot$  is given by a table for the atoms only. Composition of the other elements is obtained through disjunctivity, thus satisfying this axiom by construction. E.g., for atoms  $a, b, c, d$  we set

$$(a + b) \cdot (c + d) \stackrel{\text{def}}{=} a \cdot c + a \cdot d + b \cdot c + b \cdot d .$$

If the composition of atoms is associative, by disjunctivity this propagates to sums of atoms, i.e. to the other elements. In the same way, neutrality of 1 propagates from atoms to sums of atoms.

### 8.2 Concerning Local Composition

Here we present a KA that does not have local composition. It has two atoms 1 and  $a$  (and thus four elements). Its composition table is shown in Fig. 1.

$\cdot$	1	$a$
1	1	$a$
$a$	$a$	0

Fig. 1. An algebra without local composition

There are only two types, viz. 0 and 1. Hence by Lemma 9(1) we have  $\ulcorner a = 1$ . Now,  $\ulcorner(a \cdot a) = \ulcorner 0 = 0$ , but  $\ulcorner(a \cdot \ulcorner a) = \ulcorner(a \cdot 1) = \ulcorner a = 1$ .

This algebra is a special case of a whole class of algebras similar to LAN, but with words of bounded length. Specifically, let  $A$  be any set and, for  $i \in \mathbb{N}$ ,

define  $S_n \stackrel{\text{def}}{=} \{w \in A^* : |w| \leq n\}$ , where  $|w|$  is the length of word  $w$ . For  $U, V \subseteq S_n$ , define *bounded concatenation* by  $U \odot V \stackrel{\text{def}}{=} \{u \bullet v : u \in U \wedge v \in V \wedge |u \bullet v| \leq n\}$ . Then  $\text{LAN}_n \stackrel{\text{def}}{=} (\mathcal{P}(S_n), \subseteq, S_n, \odot, \emptyset, \varepsilon)$  is a Kleene algebra in which locality of composition does not hold. The example given above is obtained by starting from a set  $A$  with a single element and setting  $n = 1$ .

### 8.3 Concerning SC1, SC and DD

In this section we show that  $\text{SC1}(a) \not\Rightarrow \text{SC}(a)$  and that DD does not imply either of SC1 and SC. The counter-example consists of a finite Kleene algebra with three atoms  $1, a, b$  and the composition table shown in Fig. 2 (which obviously is associative and satisfies locality of composition). This algebra is

$\cdot$	1	a	b
1	1	a	b
a	a	b	b
b	b	b	b

Fig. 2.  $\text{SC1}(a)$  does not imply  $\text{SC}(a)$

isomorphic to the algebra generated by the following concrete relations under relational composition:  $a = \{(0, 1), (1, 2), (2, 2)\}$ ,  $b = \{(0, 2), (1, 2), (2, 2)\}$  and  $1 = \{(0, 0), (1, 1), (2, 2)\}$ . In this algebra we have  $\text{SC1}(a)$  and  $\text{DD}(a)$ , but not  $\text{SC}(a)$  (and hence not  $\text{LD}(a)$ ). Moreover,  $\text{DD}(b)$  holds (since  $b$  is an atom), but  $\text{SC1}(b)$  does not.

### 8.4 Concerning DD and LD

We show that DD does *not* imply LD, even for RAs. The counterexample is McKenzie's non-representable 16-element RA [10] (see also Appendix A in [15]<sup>2</sup>).

The algebra has four atoms  $1, a, b, c$  (and thus sixteen elements). The composition table of the atoms is shown in Fig. 3. Since  $a$  is an atom, it satisfies DD. Now,  $a ; (b \sqcap c) = 0$ , but  $a ; b \sqcap a ; c = (a + b) \sqcap \top = a + b$ .

### 8.5 Non-Closure of SC1 under Composition

Consider again the algebra of Section 8.3. There, SC1 is not closed under composition, since  $a$  satisfies SC1, but the composition  $b = a \cdot a$  does not.

Let us mention that the algebras from Sections 8.2 and 8.6 below cannot be used as counterexamples. There, all atoms satisfy SC1. Moreover, in the latter

<sup>2</sup> The entry for  $c \cdot b$  in Fig. A.2.3 of [15] should be changed to  $z$ .

;	1	$a$	$b$	$c$
1	1	$a$	$b$	$c$
$a$	$a$	$a$	$a + b$	$\top$
$b$	$b$	$a + b$	$\bar{b}$	$b + c$
$c$	$c$	$\top$	$b + c$	$c$

Fig. 3. DD does not imply LD

we have for all  $x \in \{b, c, d, e, f\}$  that  $x \cdot \bar{1} = 0$  which makes SC1 closed under composition in that algebra.

### 8.6 Non-Closure of DD under Composition

To show that DD (and hence CD) is not closed under composition we use a KA with nine atoms,  $a, b, c, d, e, f, i, j, k$ . Composition of the atoms is given by the table in Fig. 4. The identity of composition is given by  $1 \stackrel{\text{def}}{=} i + j + k$ .

·	$a$	$b$	$c$	$d$	$e$	$f$	$i$	$j$	$k$
$a$	0	$d + e$	$d + f$	0	0	0	0	$a$	0
$b$	0	0	0	0	0	0	0	0	$b$
$c$	0	0	0	0	0	0	0	0	$c$
$d$	0	0	0	0	0	0	0	0	$d$
$e$	0	0	0	0	0	0	0	0	$e$
$f$	0	0	0	0	0	0	0	0	$f$
$i$	$a$	0	0	$d$	$e$	$f$	$i$	0	0
$j$	0	$b$	$c$	0	0	0	0	$j$	0
$k$	0	0	0	0	0	0	0	0	$k$

Fig. 4. Non-closure of DD under composition

It is slightly tedious to verify that composition is associative, although this is facilitated by the fact that most entries of the composition table are 0. Another manner to verify associativity is through the following relational model on the set  $\{0, 1, 2, 3, 4\}$ , for which standard relational composition gives the above table (in fact, we started from this concrete model rather than from the abstract one). Associativity follows from the fact that relational composition



is associative.

$$\begin{aligned}
a &= \{(0, 1)\} & d &= \{(0, 2)\} & i &= \{(0, 0)\} \\
b &= \{(1, 2), (1, 3)\} & e &= \{(0, 3)\} & j &= \{(1, 1)\} \\
c &= \{(1, 2), (1, 4)\} & f &= \{(0, 4)\} & k &= \{(2, 2), (3, 3), (4, 4)\}
\end{aligned}$$

From the composition table and the definition of domain, one obtains

$$\begin{aligned}
\lceil a &= \lceil d = \lceil e = \lceil f = \lceil i = i, & \lceil b &= \lceil c = \lceil j = j, & \lceil k &= k, \\
\bar{i} &= i, & \bar{a} &= \bar{j} = j, & \bar{b} &= \bar{c} = \bar{d} = \bar{e} = \bar{f} = \bar{k} = k.
\end{aligned}$$

It is then easy to check that locality of composition is satisfied.

Now the question is: if  $\forall c : c < a \Rightarrow \lceil c < \lceil a$  and  $\forall c : c < b \Rightarrow \lceil c < \lceil b$ , do we have  $\forall c : c < a \cdot b \Rightarrow \lceil c < \lceil(a \cdot b)$ ? Because  $a$  and  $b$  are atoms, they satisfy DD. Also,  $a \cdot b = d + e$ , so that  $d < a \cdot b$ . But  $\lceil d = \lceil(a \cdot b) = i$ .

The same algebra is another counterexample to the implication  $\text{SC1}(a) \Rightarrow \text{SC}(a)$ , since

$$\begin{aligned}
a \cdot \bar{1} &= a \cdot (a + b + c + d + e + f) \\
&= d + e + f \leq b + c + d + e + f + 1 = \bar{a},
\end{aligned}$$

while  $a \cdot \bar{b} = a \cdot (a + c + d + e + f + 1) = a + d + f$  and  $\overline{a \cdot b} = \overline{d + e} = a + b + c + f + 1$ . By Lemma 13, the implication  $\text{SC1}(a) \Rightarrow \text{SC}(a)$  is also violated; this is illustrated by  $a \cdot (b \sqcap c) \neq a \cdot b \sqcap a \cdot c$ .

Another counter-example to  $\text{SC1}(a) \Rightarrow \text{SC}(a)$  is obtained from the above one by replacing the three atomic subidentities  $i, j, k$  by a single atomic identity 1. However, the resulting algebra does not satisfy locality of composition.

The same algebra can be used to give a counterexample showing that DD (or CD) does not imply LD. The element  $a$  is an atom and thus satisfies DD. However,  $a \cdot (b \sqcap c) = a \cdot 0 = 0 \neq d = (d + e) \sqcap (d + f) = a \cdot b \sqcap a \cdot c$ .

### 8.7 Concerning SC, SC1 and MD

First we note that MD does not imply SC, since otherwise, by Lemmas 14 and 17, and transitivity of implication, we would obtain that DD implies SC, in contradiction to Section 8.4 and Lemma 13.

Second, MD does not imply SC1 either. The algebra in Section 8.3 has 0 and 1 as its only types. Hence, by Corollary 16(f), we have MD( $b$ ), but SC1( $b$ ) does not hold.

Concerning the reverse implications, let us first see the informal meaning of MD in the KA PAT. Consider a graph node  $y$ , viewed as an atomic type, and a set of paths  $a$ . Then  $\langle a \rangle y$  is the set of all nodes from which *some* path in  $a$  leads to  $y$ , whereas a node  $x$  is in  $[a]y$  iff *all* paths in  $a$  that start in  $x$  end in  $y$ . So MD( $a$ ) holds iff all paths in  $a$  that start in the same node also end in the same node. However,  $a$  may contain several different paths between two nodes.

Now, as we have seen in Section 6.1, in PAT the properties SC( $a$ ) and SC1( $a$ ) are equivalent to prefix-freeness of  $a$ . Hence for different nodes  $x, y, z$  the set  $a = \{xy, xz\}$  of paths satisfies SC( $a$ ) and SC1( $a$ ) but not MD( $a$ ). Therefore neither SC nor SC1 implies MD.

A consequence of the last paragraph is that SC1 does not imply CD; indeed, if this were the case, we would have that SC1 implies MD, because of Lemma 17. By Lemma 14, SC1 does not imply DD either.

## 9 Linking the Views

It turns out that in the case of an atomic KA  $K$ , we can set up a homomorphism from  $K$  into a concrete KA of type REL. This will allow us to link the domain-oriented and the relational characterizations of functions.

**Definition 27** Consider an atomic KA  $K$ . For element  $a \in K$ , we define a relation  $\llbracket a \rrbracket$  between atomic types by setting for  $s, t \in \text{At}(1)$

$$s \llbracket a \rrbracket t \stackrel{\text{def}}{\iff} s \cdot a \cdot t \neq 0 .$$

**Lemma 28** If  $u \in \text{At}(1)$ , then  $u \cdot a \neq 0 \iff u = \ulcorner(u \cdot a) \iff u \leq \ulcorner a$ .

**PROOF.**

$$\begin{aligned} & u \cdot a \neq 0 \\ \iff & \{ \text{by Lemma 9(1)} \} \\ & \ulcorner(u \cdot a) \neq 0 \\ \iff & \{ \text{by Lemmas 9(d) and 9(e), } \ulcorner(u \cdot a) \leq \ulcorner u = u \} \\ & \ulcorner(u \cdot a) \neq 0 \wedge \ulcorner(u \cdot a) \leq u \\ \iff & \{ u \text{ is an atom} \} \\ & u = \ulcorner(u \cdot a) \\ \iff & \{ \text{locality of composition, Lemma 9(e) and Lemma 11(b)} \} \\ & u = u \cdot \ulcorner a \\ \iff & \{ \text{by Lemma 7(b), Boolean algebra} \} \end{aligned}$$

$$u \leq \ulcorner a$$

**Lemma 29** *The mapping  $\llbracket - \rrbracket$  is a universally disjunctive monoid homomorphism from the  $(\cdot, 1)$  reduct of  $K$  to the  $(;, I)$  reduct of  $\text{REL}$ .*

**PROOF.**

$$\begin{aligned}
\text{(a)} \quad & s \llbracket \sqcup \{a_j : j \in J\} \rrbracket t \\
& \Leftrightarrow \{ \text{definition of } \llbracket - \rrbracket, \text{ distributivity} \} \\
& \sqcup \{s \cdot a_j \cdot t : j \in J\} \neq 0 \\
& \Leftrightarrow \{ \text{Boolean algebra} \} \\
& \exists j \in J : s \cdot a_j \cdot t \neq 0 \\
& \Leftrightarrow \{ \text{definition of } \llbracket - \rrbracket \} \\
& \exists j \in J : s \llbracket a_j \rrbracket t \\
& \Leftrightarrow \{ \text{definition of union} \} \\
& s \left( \bigcup_{j \in J} \llbracket a_j \rrbracket \right) t .
\end{aligned}$$

$$\text{Hence } \llbracket \sqcup \{a_j : j \in J\} \rrbracket = \bigcup_{j \in J} \llbracket a_j \rrbracket .$$

$$\begin{aligned}
\text{(b)} \quad & s \llbracket 1 \rrbracket t \\
& \Leftrightarrow \{ \text{definition of } \llbracket - \rrbracket, \text{ neutrality} \} \\
& s \cdot t \neq 0 \\
& \Leftrightarrow \{ \text{since } s, t \text{ are atoms} \} \\
& s = t .
\end{aligned}$$

Hence  $\llbracket 1 \rrbracket = I$ .

(c) We first show  $\llbracket a \cdot b \rrbracket \subseteq \llbracket a \rrbracket ; \llbracket b \rrbracket$ .

$$\begin{aligned}
& s \llbracket a \cdot b \rrbracket t \\
& \Leftrightarrow \{ \text{definition of } \llbracket - \rrbracket, \text{ Lemma 9(b)} \} \\
& s \cdot a \cdot a^\top \cdot \ulcorner b \cdot b \cdot t \neq 0 \\
& \Leftrightarrow \{ a^\top \cdot \ulcorner b = \sqcup \{u : u \in \text{At}(a^\top \cdot \ulcorner b)\}, \text{ distributivity} \} \\
& (\sqcup \{s \cdot a \cdot u \cdot b \cdot t : u \in \text{At}(a^\top \cdot \ulcorner b)\}) \neq 0 \\
& \Leftrightarrow \{ \text{Boolean algebra} \} \\
& \exists u \in \text{At}(a^\top \cdot \ulcorner b) : s \cdot a \cdot u \cdot b \cdot t \neq 0 \\
& \Rightarrow \{ \text{strictness} \} \\
& \exists u \in \text{At}(a^\top \cdot \ulcorner b) : s \cdot a \cdot u \neq 0 \wedge u \cdot b \cdot t \neq 0 \\
& \Leftrightarrow \{ \text{definition of } \llbracket - \rrbracket \} \\
& \exists u \in \text{At}(a^\top \cdot \ulcorner b) : s \llbracket a \rrbracket u \wedge u \llbracket b \rrbracket t \\
& \Rightarrow \{ \text{the atoms of a type are atomic types} \} \\
& s (\llbracket a \rrbracket ; \llbracket b \rrbracket) t .
\end{aligned}$$

Now we show the reverse inclusion. Assume  $s (\llbracket a \rrbracket ; \llbracket b \rrbracket) t$ , say  $s \cdot a \cdot u \neq 0$  and  $u \cdot b \cdot t \neq 0$  for some atomic type  $u$ . Then by strictness  $a^\top \cdot u \neq 0$  and  $u \cdot \ulcorner b \neq 0$ . Since  $u$  is an atom, we get  $u \leq a^\top$  and  $u \leq \ulcorner b$  by Lemma 28. Therefore  $u \leq a^\top \cdot \ulcorner b$ , and hence

$$s \cdot a \cdot b \cdot t = s \cdot a \cdot a^\top \cdot \ulcorner b \cdot b \cdot t \geq s \cdot a \cdot u \cdot b \cdot t .$$

Now, by local composition, again Lemma 28 and Lemma 9(1),

$$\ulcorner(s \cdot a \cdot u \cdot b \cdot t) = \ulcorner(s \cdot a \cdot \ulcorner(u \cdot b \cdot t)) = \ulcorner(s \cdot a \cdot u) \neq 0 .$$

Therefore, by Lemma 9(1), also  $s \cdot a \cdot u \cdot b \cdot t \neq 0$  and by monotonicity, since  $u \leq 1$ , also  $s \cdot a \cdot b \cdot t \neq 0$ , i.e.  $s(\llbracket a \cdot b \rrbracket) t$ .

By universal disjunctivity we get from this also

**Corollary 30**  $\llbracket a^* \rrbracket = \llbracket a \rrbracket^*$  and  $\llbracket a^+ \rrbracket = \llbracket a \rrbracket^+$ .

The link between the views is now established by

**Lemma 31** *In an atomic KA  $K$ ,  $\text{MD}(a) \Leftrightarrow \llbracket a \rrbracket^\vee; \llbracket a \rrbracket \subseteq I$ .*

**PROOF.** In this derivation,  $s, t, u \in \text{At}(1)$ .

$$\begin{aligned} & \llbracket a \rrbracket^\vee; \llbracket a \rrbracket \subseteq I \\ \Leftrightarrow & \quad \{ \text{definition of } \llbracket \_ \rrbracket, I = \llbracket 1 \rrbracket \text{ by Lemma 29} \} \\ & \forall s, t : (\exists u : u \llbracket a \rrbracket s \wedge u \llbracket a \rrbracket t) \Rightarrow s \llbracket 1 \rrbracket t \\ \Leftrightarrow & \quad \{ \text{definition of } \llbracket \_ \rrbracket \} \\ & \forall s, t : (\exists u : u \cdot a \cdot s \neq 0 \wedge u \cdot a \cdot t \neq 0) \Rightarrow s \cdot 1 \cdot t \neq 0 \\ \Leftrightarrow & \quad \{ \text{Lemma 28} \} \\ & \forall s, t : (\exists u : u \leq \ulcorner(a \cdot s) \wedge u \leq \ulcorner(a \cdot t)) \Rightarrow s \cdot t \neq 0 \\ \Leftrightarrow & \quad \{ \text{Boolean algebra} \} \\ & \forall s, t : (\exists u : u \leq \ulcorner(a \cdot s) \sqcap \ulcorner(a \cdot t)) \Rightarrow s \cdot t \neq 0 \\ \Leftrightarrow & \quad \{ u \in \text{At}(1), \text{Lemma 7(b)} \} \\ & \forall s, t : \ulcorner(a \cdot s) \sqcap \ulcorner(a \cdot t) \neq 0 \Rightarrow s \sqcap t \neq 0 \\ \Leftrightarrow & \quad \{ \text{contrapositive} \} \\ & \forall s, t : s \sqcap t = 0 \Rightarrow \ulcorner(a \cdot s) \sqcap \ulcorner(a \cdot t) = 0 \\ \Leftrightarrow & \quad \{ \text{Boolean algebra} \} \\ & \forall s, t : t \leq \neg s \Rightarrow \ulcorner(a \cdot s) \sqcap \ulcorner(a \cdot t) = 0 \\ \Leftrightarrow & \quad \{ \text{for proving } \Rightarrow, \text{ take } t \stackrel{\text{def}}{=} \neg s; \\ & \quad \text{the direction } \Leftarrow \text{ is trivial by monotonicity} \} \\ & \forall s : \ulcorner(a \cdot s) \sqcap \ulcorner(a \cdot \neg s) = 0 \\ \Leftrightarrow & \quad \{ \text{definition of MD, } \langle \rangle \text{ and } \llbracket \_ \rrbracket \} \\ & \text{MD}(a) \end{aligned}$$

## 10 Summary of the Results

To give the reader a survey of what has been achieved in this paper, we summarize our results in a table of mutual (non-)implications and (non-)closure properties (see Figure 5). Equivalent properties are in the same line/column. When there is a  $\Rightarrow$  or a  $\Leftarrow$ , the reverse implication does not hold. When an

	SC1	SC LD	ED SO CD DD	MD	closed under subobjects	closed under composition	closed under compatible join
SC1	$\Leftrightarrow$	$\Leftarrow$			yes	no	yes
SC,LD	$\Rightarrow$	$\Leftrightarrow$			yes	yes	yes
ED,SO,CD,DD			$\Leftrightarrow$	$\Rightarrow$	yes	cond.	yes
MD			$\Leftarrow$	$\Leftrightarrow$	yes	yes	yes

Fig. 5. Relationship between properties

entry is blank, it means that no implication holds. The entry “cond.” means “conditionally valid”; see Lemma 25 for the precise result.

## 11 Conclusion

The theory of Kleene algebras offers surprising new views on the notion of a function. Characterizations that are equivalent in relation algebras [15] differ in this generalized setting.

However, the second author has shown that the characterization CD is sufficient to reprove (in a simpler fashion!) all properties of overwriting that were shown relationally in [12]. So it seems that the generalized setting indeed has its merits.

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