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Abstract

Elements of Kleene algebras can be used, among others, as abstractions of the input-output semantics of nondeterministic programs or as models for the association of pointers with their target objects. In the first case, one seeks to distinguish the subclass of elements that correspond to deterministic programs. In the second case one is only interested in functional correspondences, since it does not make sense for a pointer to point to two different objects.

We discuss several candidate notions of determinacy and clarify their relationship. Some characterizations that are equivalent in the case where the underlying Kleene algebra is an (abstract) relation algebra are not equivalent for general Kleene algebras.

1 Introduction

Elements of Kleene algebras can be used, among others, as abstractions of the input-output semantics of nondeterministic programs [3] or as models for the association of pointers with their target objects in the style of [9]. In the first case, one seeks to distinguish the subclass of elements that correspond to deterministic programs. In the second case, one is only interested in functional

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correspondences, since it does not make sense for a pointer to point to two different objects.

We discuss several candidate notions of determinacy and show several equivalences. However, it also turns out that some characterizations that are equivalent in the case where the underlying Kleene algebra is an (abstract) relation algebra, are not equivalent for general Kleene algebras.

2 Kleene Algebras

2.1 Definition and Basic Laws

In our definitions we follow [2], since we want to admit general recursive definitions, not just the Kleene star. We are well aware that there are different definitions (see e.g. [6]).

Definition 1 *A* Kleene algebra (KA) *is a sixtuple* $(K, \leq, \top, \cdot, 0, 1)$ *satisfying the following properties:*

(a) $(K, \leq)$ *is a complete lattice with least element* 0 *and greatest element* $\top$*. The supremum of a subset* $L \subseteq K$ *is denoted by* $\sqcup L$*.*
(b) $(K, \cdot, 1)$ *is a monoid.*
(c) The operation $\cdot$ *is universally disjunctive (i.e. distributes through arbitrary suprema) in both arguments.*

The supremum of two elements $x, y \in K$ is given by $x + y \stackrel{\text{def}}{=} \sqcup \{x, y\}$.

Example 2 Perhaps the best-known example of a KA is

$$\mathrm{LAN} \stackrel{\text{def}}{=} (\mathcal{P}(A^*),\ \subseteq, A^*,\ \bullet,\ \emptyset,\ \varepsilon)\ ,$$

the algebra of formal languages over some alphabet A, where A^* is the set of all finite words over A, $\bullet$ denotes concatenation and ε the empty word (as usual, we identify a singleton language with its only element). A related KA is the algebra PAT of path sets in a directed graph under path concatenation (see e.g. [8] for a precise definition).

Another important KA is

$$\mathrm{REL} \stackrel{\text{def}}{=} (\mathcal{P}(M \times M),\ \subseteq, M \times M,\ ;,\ \emptyset,\ I)\ ,$$

the algebra of homogeneous binary relations over some set M under relational composition ;.

Definition 3 *(a) A* Boolean algebra *is a distributive and complemented lattice. The complement of an element x is denoted by $\overline{x}$.*
(b) A Boolean algebra is complete *if its underlying lattice is complete.*
(c) An atom *of a lattice with least element* 0 *is a minimal element in the set of elements different from* 0*. The set of all atoms of a lattice with* 0 *is denoted by* At*. For an element x of such a lattice, $\mathsf{At}(x) \stackrel{\text{def}}{=} \{a \in \mathsf{At} : a \leq x\}$ is the set of* atoms of x.
(d) A lattice with 0 *is* atomic *if for every element x we have $x = \sqcup \mathsf{At}(x)$.*
(e) A KA is called Boolean *if its underlying lattice is a Boolean algebra (and hence a complete Boolean algebra). It is called* atomic *if its underlying lattice is atomic.*

Example 4 More generally than the concrete relation algebra REL, every abstract relation algebra is a KA. Such an *abstract relation algebra* (see e.g. [11]) is a tuple RA $= (N, \leq, ^-, \top, ;, 0, 1, ^{\smile})$, where

(a) $(N, \leq, ^-, 0, \top)$ is a complete Boolean algebra with complement operation $^-$, least element 0 and greatest element $\top$;
(b) $(N, ;, 1)$ is a monoid;
(c) Tarski's rule $x \neq 0 \Rightarrow \top ; x ; \top = \top$ holds;
(d) Dedekind's rule $x ; y \sqcap z \;\leq\; (x \sqcap z ; y^{\smile}) ; (y \sqcap x^{\smile}; z)$ is satisfied.

The elements of N are called *abstract relations*; the operation $^{\smile}$ forms the *converse* of a relation, whereas ; is called *relational composition*. It is customary to use the convention that ; binds tighter than $+$ and $\sqcap$. The reduct $(N, \leq, \top, ;, 0, 1)$ forms a Boolean KA.

Definition 5 *A* right ideal *is an element of the form $a \cdot \top$ with arbitrary a.*

It is straightforward to show that b is a right ideal iff $b = b \cdot \top$.

2.2 Types

A central notion is that of types.

Definition 6 *A* type *of a KA is an element t with $t \leq 1$. We set* TYP $\stackrel{\text{def}}{=} \{t \in K : t \leq 1\}$.

This definition is best illustrated in the KA REL. There, a type corresponds to a subset $T \subseteq M$ and can be represented as the partial identity relation $1_T \stackrel{\text{def}}{=} \{(x, x) : x \in T\}$. Clearly, 1_T is a subidentity, and so there is a one-to-one correspondence between types and subidentities.

Now, restriction of a relation $R \subseteq M \times M$ to arguments of type T, i.e. the relation $R \cap T \times M$, can also be described by composing R with 1_T from the left: $R \cap T \times M = 1_T ; R$. Similarly, co-restriction is modeled by composing a partial identity from the right. Finally, consider types $S, T \subseteq M$ and binary relation $R \subseteq M \times M$. Then $R \subseteq S \times T \Leftrightarrow 1_S ; R ; 1_T = R$. In other words, the "default typing" $M \times M$ of R can be narrowed down to $S \times T$ iff restriction to S and co-restriction to T do not change R.

These observations are the basis for our view of types as subidentities and our algebraic treatment of restriction and co-restriction. For a different, but related, approach see [6].

For the remainder of the paper, we assume KAs to be Boolean.

Definition 7 *The* negation *of a type* $t \leq 1$ *in a KA is* $\neg t \stackrel{\text{def}}{=} \overline{t} \sqcap 1$.

With this definition, $(\mathrm{TYP}, \leq)$ forms again a complete Boolean algebra.

Lemma 8 *Assume a Boolean KA. Then the following hold:*

(a) All types are idempotent, i.e. $t \leq 1 \Rightarrow t \cdot t = t$.
(b) The infimum of two types is their product: $s, t \leq 1 \Rightarrow s \cdot t = s \sqcap t$. *In particular, all types commute under the* $\cdot$ *operation.*
(c) $s, t \leq 1 \Rightarrow (s \sqcap t) \cdot a = s \cdot a \sqcap t \cdot a$.
(d) $t \leq 1 \Rightarrow \overline{t \cdot \top} = \neg t \cdot \top$.
(e) Restriction by a type is equivalent to intersection with a right ideal: $t \leq 1 \Rightarrow t \cdot a = a \sqcap t \cdot \top$.
(f) For all families L *of types,* $(\sqcap L) \cdot \top = \sqcap(L \cdot \top)$.

PROOF. We first note that by monotonicity and neutrality, for $s, t \leq 1$ we have that $s \cdot t \leq s \cdot 1 = s$ and, symmetrically, $s \cdot t \leq t$, so that $s \cdot t \leq s \sqcap t$.

(a)

$$\begin{array}{ll} & t \\ = & \quad \{\!\![\text{ neutrality }]\!\!\} \\ & 1 \cdot t \\ = & \quad \{\!\![\text{ Boolean algebra }]\!\!\} \\ & (t + \neg t) \cdot t \\ = & \quad \{\!\![\text{ disjunctivity }]\!\!\} \\ & t \cdot t + \neg t \cdot t \\ \leq & \quad \{\!\![\text{ above }]\!\!\} \\ & t \cdot t + \neg t \sqcap t \\ = & \quad \{\!\![\text{ Boolean algebra }]\!\!\} \\ & t \cdot t \\ \leq & \quad \{\!\![\text{ by } t \leq 1 \text{ and monotonicity }]\!\!\} \\ & 1 \cdot t \end{array}$$

$$
\begin{array}{ll}
= & \quad \{\!\![\ \text{neutrality}\ ]\!\!\} \\
& t\ .
\end{array}
$$

(b) We have already shown that $s \cdot t$ is a lower bound for s and t. Using (a) and monotonicity, we also have $s \sqcap t = (s \sqcap t) \cdot (s \sqcap t) \le s \cdot t$. Hence $s \cdot t = s \sqcap t$.

(c)
$$
\begin{array}{ll}
& (s \sqcap t) \cdot a \\
\le & \quad \{\!\![\ \text{monotonicity}\ ]\!\!\} \\
& s \cdot a \sqcap t \cdot a \\
= & \quad \{\!\![\ \text{neutrality and Boolean algebra}\ ]\!\!\} \\
& (s + \neg s) \cdot (s \cdot a \sqcap t \cdot a) \\
= & \quad \{\!\![\ \text{disjunctivity}\ ]\!\!\} \\
& s \cdot (s \cdot a \sqcap t \cdot a) + \neg s \cdot (s \cdot a \sqcap t \cdot a) \\
\le & \quad \{\!\![\ \text{monotonicity and associativity}\ ]\!\!\} \\
& s \cdot t \cdot a + \neg s \cdot s \cdot a \\
= & \quad \{\!\![\ \text{by strictness, since } \neg s \cdot s \ = \ \neg s \sqcap s \ = \ 0\text{, and Boolean algebra}\ ]\!\!\} \\
& s \cdot t \cdot a \\
= & \quad \{\!\![\ \text{by (b)}\ ]\!\!\} \\
& (s \sqcap t) \cdot a\ .
\end{array}
$$

(d) First, by (c),

$$t \cdot \top \sqcap \neg t \cdot \top = (t \sqcap \neg t) \cdot \top = 0 \cdot \top = 0\ .$$

Second,

$$t \cdot \top + \neg t \cdot \top = (t + \neg t) \cdot \top = 1 \cdot \top = \top\ .$$

(e)
$$
\begin{array}{ll}
& t \cdot \top \sqcap a \\
= & \quad \{\!\![\ \text{Boolean algebra}\ ]\!\!\} \\
& t \cdot (a + \overline{a}) \sqcap a \\
= & \quad \{\!\![\ \text{disjunctivity}\ ]\!\!\} \\
& (t \cdot a + t \cdot \overline{a}) \sqcap a \\
= & \quad \{\!\![\ \text{distributivity}\ ]\!\!\} \\
& (t \cdot a \sqcap a) + (t \cdot \overline{a} \sqcap a) \\
= & \quad \{\!\![\ \text{Boolean algebra, since } t \cdot \overline{a} \le \overline{a} \text{ by monotonicity and neutrality}\ ]\!\!\} \\
& t \cdot a \sqcap a + 0 \\
= & \quad \{\!\![\ \text{Boolean algebra, since } t \cdot a \le a \text{ by monotonicity and neutrality}\ ]\!\!\} \\
& t \cdot a\ .
\end{array}
$$

(f) We show $\overline{(\bigsqcap L) \cdot \top} = \overline{\bigsqcap (L \cdot \top)}$.

$$
\begin{array}{ll}
& \overline{\bigsqcap (L \cdot \top)} \\
= & \quad \{\!\![\ \text{de Morgan}\ ]\!\!\} \\
& \sqcup \{\overline{t \cdot \top} : t \in L\} \\
= & \quad \{\!\![\ \text{by (d)}\ ]\!\!\} \\
& \sqcup \{\neg t \cdot \top : t \in L\} \\
= & \quad \{\!\![\ \text{disjunctivity}\ ]\!\!\} \\
& (\sqcup \{\neg t : t \in L\}) \cdot \top
\end{array}
$$

$$\begin{array}{ll} = & \{\![\text{ de Morgan }]\!\} \\ & (\neg \sqcap \{t : t \in L\}) \cdot \top \\ = & \{\![\text{ by (d) }]\!\} \\ & (\sqcap L) \cdot \top \ . \end{array}$$

2.3 Domain and Codomain

Definition 9 *In a KA* $(K, \leq, \top, \cdot, 0, 1)$*, we can define, for* $a \in K$*, the* domain $\ulcorner a$ *via the Galois connection*

$$\forall\, t : t \leq 1 \;\Rightarrow\; (\ulcorner a \leq t \;\stackrel{\text{def}}{\Leftrightarrow}\; a \leq t \cdot \top)\ .$$

(This is well-defined because of Lemma 8(f), see also [1].) Hence the operation $\ulcorner$ is universally disjunctive and therefore monotonic and strict. Moreover the definition implies $a \leq \ulcorner a \cdot \top$. The *co-domain* $a\urcorner$ is defined symmetrically by the Galois connection: $a\urcorner \leq t \stackrel{\text{def}}{\Leftrightarrow} a \leq \top \cdot t$.

By the Galois connection, the partial orders $(\mathrm{TYP}, \leq)$ and $(\{t \cdot \top : t \in \mathrm{TYP}\}, \leq)$ are isomorphic. Hence we have, for $t \in \mathrm{TYP}$, that $\ulcorner(t \cdot \top) = t$ (which also follows from properties (e) and (h) in Lemma 10 below).

We list a number of useful properties of the domain operation (see again also [1]); analogous ones hold for the co-domain operation.

Lemma 10 *Consider a KA* $(K, \leq, \top, \cdot, 0, 1)$ *and* $a, b, c \in K$.

(a) $\ulcorner a \;=\; \min\{t : t \leq 1 \wedge t \cdot a = a\}$.
(b) $\ulcorner a \cdot a \;=\; a$.
(c) $t \leq 1 \wedge t \cdot a = a \;\Rightarrow\; \ulcorner a \leq t$.
(d) $\ulcorner(a \cdot b) \leq \ulcorner a$.
(e) $t \leq 1 \;\Leftrightarrow\; \ulcorner t \;=\; t$.
(f) $\ulcorner \top \;=\; 1$.
(g) $\ulcorner(\ulcorner a) \;=\; \ulcorner a$.
(h) $\ulcorner(a \cdot \top) \;=\; \ulcorner a$.
(i) $a \cdot \top \leq \ulcorner a \cdot \top$.
(j) $\ulcorner(a \cdot b) \leq \ulcorner(a \cdot \ulcorner b)$.
(k) $a\urcorner \sqcap \ulcorner b \;=\; 0 \;\Rightarrow\; a \cdot b \;=\; 0$.
(l) $\ulcorner a \;=\; 0 \;\Leftrightarrow\; a \;=\; 0$.

According to Lemma 10(l) the domain of an element also decides about its "definedness" if we identify 0 with $\perp$ as used in denotational semantics.

2.4 Locality of Composition

It should be noted that the converse inequation of Lemma 10(j) does not follow from our axiomatization. A counterexample will be given in Section 5.2. Its essence is that composition does not work "locally" in that only the "near end", i.e. the domain, of the right factor of a composition does decide "composability". This observation is the motivation for the term "local composition" defined below.

Definition 11 *A KA has* left-local composition *if it satisfies*

$$\urcorner b \;=\; \ulcorner c \;\Rightarrow\; \ulcorner(a\cdot b) \;=\; \ulcorner(a\cdot c)\ .$$

To check this property, by strictness of $\cdot$ it suffices to consider $a \neq 0$ and, by Lemma 10(l), only b, c with $\urcorner b \;=\; \ulcorner c \neq 0$.

The *right-locality* of composition is defined by the symmetrical property

$$a^{\urcorner} \;=\; b^{\urcorner} \;\Rightarrow\; (a\cdot c)^{\urcorner} \;=\; (b\cdot c)^{\urcorner}\ .$$

A KA has *local composition* if its composition is both left-local and right-local.

Lemma 12 *(a) A KA has left-local composition iff it satisfies*

$$\ulcorner(a\cdot b) \;=\; \ulcorner(a\cdot \urcorner b) \tag{1}$$

(b) If a KA has left-local composition then $\ulcorner(\ulcorner a\cdot b) \;=\; \ulcorner a \sqcap \urcorner b \;=\; \ulcorner a \cdot \urcorner b$.
(c) If a KA has left-local composition then $\urcorner b \leq \ulcorner c \;\Rightarrow\; \ulcorner(a\cdot b) \leq \ulcorner(a\cdot c)$.

PROOF.

(a) ($\Rightarrow$) Immediate from the assumption, since by Lemma 10(g) $\ulcorner(\urcorner b) \;=\; \urcorner b$.
($\Leftarrow$) Assume $\urcorner b = \ulcorner c$.

$$\begin{array}{ll}
 & \ulcorner(a\cdot b) \\
= & \quad \{\!\![\ \text{by (1)}\ ]\!\!\} \\
 & \ulcorner(a\cdot \urcorner b) \\
= & \quad \{\!\![\ \text{assumption}\ ]\!\!\} \\
 & \ulcorner(a\cdot \ulcorner c) \\
= & \quad \{\!\![\ \text{by (1)}\ ]\!\!\} \\
 & \ulcorner(a\cdot c)\ .
\end{array}$$

(b)

$$\begin{array}{ll}
 & \ulcorner(\ulcorner a\cdot b) \\
= & \quad \{\!\![\ \text{by (1)}\ ]\!\!\} \\
 & \ulcorner(\ulcorner a\cdot \urcorner b) \\
= & \quad \{\!\![\ \text{by Lemma 8(b)}\ ]\!\!\} \\
 & \ulcorner(\ulcorner a \sqcap \urcorner b) \\
= & \quad \{\!\![\ \text{by Lemma 10(e)}\ ]\!\!\} \\
 & \ulcorner a \sqcap \urcorner b \\
= & \quad \{\!\![\ \text{by Lemma 8(b)}\ ]\!\!\} \\
 & \ulcorner a\cdot \urcorner b\ .
\end{array}$$

(c)
$$
\begin{array}{cl}
 & \ulcorner(a \cdot b) \\
= & \quad \{\!\![\text{ by (1) }]\!\!\} \\
 & \ulcorner(a \cdot \ulcorner b) \\
\leq & \quad \{\!\![\text{ assumption and monotonicity }]\!\!\} \\
 & \ulcorner(a \cdot \ulcorner c) \\
= & \quad \{\!\![\text{ by (1) }]\!\!\} \\
 & \ulcorner(a \cdot c)\ .
\end{array}
$$

Analogous properties hold for right-locality.

In the sequel we only consider KAs with local composition. All examples given in Section 2.1 satisfy that property.

We conclude this section by establishing a Galois connection between domain and range. It will be useful in Section 3.3 about modal operators.

Lemma 13 *If* $s, t \leq 1$*, then* $\ulcorner(a \cdot t) \leq \neg s \;\Leftrightarrow\; (s \cdot a)\urcorner \leq \neg t$.

PROOF. By Boolean algebra and Lemma 8(b), the claim is equivalent to

$$s \cdot \ulcorner(a \cdot t) \,=\, 0 \;\;\Leftrightarrow\;\; (s \cdot a)\urcorner \cdot t \,=\, 0\ .$$

We calculate

$$
\begin{array}{cl}
 & s \cdot \ulcorner(a \cdot t) \\
= & \quad \{\!\![\text{ by Lemma 10(e) }]\!\!\} \\
 & \ulcorner(s \cdot \ulcorner(a \cdot t)) \\
= & \quad \{\!\![\text{ local composition }]\!\!\} \\
 & \ulcorner(s \cdot a \cdot t)\ .
\end{array}
$$

Symmetrically, $(s \cdot a)\urcorner \cdot t \,=\, (s \cdot a \cdot t)\urcorner$. Hence

$$
\begin{array}{cl}
 & s \cdot \ulcorner(a \cdot t) \,=\, 0 \\
\Leftrightarrow & \quad \{\!\![\text{ by the above }]\!\!\} \\
 & \ulcorner(s \cdot a \cdot t) \,=\, 0 \\
\Leftrightarrow & \quad \{\!\![\text{ by Lemma 10(l) }]\!\!\} \\
 & s \cdot a \cdot t \,=\, 0 \\
\Leftrightarrow & \quad \{\!\![\text{ by Lemma 10(l) }]\!\!\} \\
 & (s \cdot a \cdot t)\urcorner \,=\, 0 \\
\Leftrightarrow & \quad \{\!\![\text{ by the above }]\!\!\} \\
 & (s \cdot a)\urcorner \cdot t \,=\, 0\ .
\end{array}
$$

3 Candidate Characterizations of Determinacy

3.1 Candidates from Relation Algebra

In an abstract relation algebra, an element R is called a *(partial) function* or a *map* or *deterministic* iff it satisfies $R\breve{}\,; R \subseteq I$. By the Schröder laws this is equivalent to the requirement $R\,; \overline{I} \subseteq \overline{R}$. It is well known that functions are left-distributive through intersection. It is less well known, though, that this property in fact is *equivalent* to the property of being a function. We give a quick proof of this (see also [11]). Assume that R is left-distributive through intersection. Then

$$
\begin{array}{ll}
 & R\,;\overline{I} \cap R \\
= & \quad \{\!\![\ \text{neutrality}\ ]\!\!\} \\
 & R\,;\overline{I} \cap R\,;I \\
= & \quad \{\!\![\ \text{left-distributivity of } R\ ]\!\!\} \\
 & R\,;(\overline{I} \cap I) \\
= & \quad \{\!\![\ \text{Boolean algebra}\ ]\!\!\} \\
 & R\,;0 \\
= & \quad \{\!\![\ \text{strictness}\ ]\!\!\} \\
 & 0\ ,
\end{array}
$$

so that $R\,;\overline{I} \subseteq \overline{R}$ follows by Boolean algebra. While this proof also carries over to Kleene algebras, the reverse implication is not Kleene valid, as we shall show in Section 5.3. In Kleene algebras left-distributivity is only equivalent to a stronger property that results by generalizing the constant I in the above inclusion to a variable (after making it visible in the right hand side). These observations lead to our first three candidates for characterizations of determinate objects (the formula involving converse not being usable in Kleene algebras). We attach names to the characterizing predicates for easier reference.

$$
\begin{array}{rll}
\mathrm{LD}(a) & \overset{\text{def}}{\Leftrightarrow}\ \forall\, b, c : a \cdot (b \sqcap c) = a \cdot b \sqcap a \cdot c & \text{(left-distributivity)} \\
\mathrm{SC}(a) & \overset{\text{def}}{\Leftrightarrow}\ \forall\, b : a \cdot \overline{b} \leq \overline{a \cdot b} & \text{(subsumption of complement)} \\
\mathrm{SC1}(a) & \overset{\text{def}}{\Leftrightarrow}\ a \cdot \overline{1} \leq \overline{a} & \text{(subsumption of complement of 1)}
\end{array}
$$

We have

$$
\mathrm{SC}(a) \Rightarrow \mathrm{SC1}(a) \tag{2}
$$

(set $b = 1$). In Section 5.3 we show that the reverse implication is not valid in all Kleene algebras. However, it holds in LAN, PAT and RA.

To understand this, let us elaborate on the case of the Kleene algebra LAN of formal languages over an alphabet A. There we have $\overline{1} = A^+$, and so for $U \subseteq A^*$ we get $\mathrm{SC1}(U) \Leftrightarrow U \bullet A^+ \subseteq \overline{U}$. In other words, a proper extension of a word in U must not lie in U again. This is equivalent to U being a prefix-free language (i.e. none of the strings of U is a proper prefix of another; in coding theory this is known as the Fano condition). The same applies to the algebra PAT of sets of paths in a graph, modeled as sets of strings of nodes. Assume now $\mathrm{SC1}(U)$ and $x \in U \bullet V \cap U \bullet W$ for $V, W \subseteq A^*$, say $x = u_1 \bullet v = u_2 \bullet w$ for some $u_1, u_2 \in U$, $v \in V$ and $w \in W$. By local linearity of the prefix relation we obtain that u_1 must be a prefix of u_2 or the other way around. By prefix-freeness of U this means $u_1 = u_2$ and cancellativity of $\bullet$ shows $v = w \in V \cap W$. Therefore also $x \in U \bullet (V \cap W)$, i.e. $\mathrm{LD}(U)$ holds. But this is equivalent to $\mathrm{SC}(U)$ as stated in the following lemma.

Lemma 14 $\mathrm{LD}(a) \Leftrightarrow \mathrm{SC}(a)$.

PROOF.

$$
\begin{array}{ll}
 & \forall\, b, c : a \cdot b \ \sqcap\ a \cdot c \ =\ a \cdot (b \sqcap c) \\
\Leftrightarrow & \quad \{\!\![\text{ Boolean algebra }]\!\!\} \\
 & \forall\, b, c : a \cdot ((b \sqcap c) + (b \sqcap \overline{c})) \ \sqcap\ a \cdot ((b \sqcap c) + (\overline{b} \sqcap c)) \ =\ a \cdot (b \sqcap c) \\
\Leftrightarrow & \quad \{\!\![\text{ distributivity, Boolean algebra }]\!\!\} \\
 & \forall\, b, c : a \cdot (b \sqcap c) + (a \cdot (b \sqcap \overline{c}) \sqcap a \cdot (\overline{b} \sqcap c)) \ =\ a \cdot (b \sqcap c) \\
\Leftrightarrow & \quad \{\!\![\text{ Boolean algebra }]\!\!\} \\
 & \forall\, b, c : a \cdot (b \sqcap \overline{c}) \sqcap a \cdot (\overline{b} \sqcap c) \ \leq\ a \cdot (b \sqcap c) \\
\Leftrightarrow & \quad \{\!\![\text{ for proving } \Rightarrow \text{, take } c \stackrel{\text{def}}{=} \overline{b}\text{; the direction } \Leftarrow \text{ is trivial }]\!\!\} \\
 & \forall\, b : a \cdot b \sqcap a \cdot \overline{b} \leq 0
\end{array}
$$

3.2 Domain-Oriented Characterizations

In view of the previous section it appears that LD, SC and SC1 are not appropriate characterizations of (partial) functions. Rather, a function should be characterized in a domain-oriented way: every point in the domain should have a unique "extension". In LAN and PAT this is not guaranteed by SC1 and the equivalent properties SC and LD (recall Lemma 14), since that property there is equivalent to prefix-freeness. So e.g. in PAT, an element might contain both the paths xy and xz starting from graph node x.

Now, in PAT, a node starts a unique path in a path set a iff removal of this path removes that node from the domain of a. This can be captured in a purely order-theoretic way by the property

$$\mathrm{DD}(a) \stackrel{\text{def}}{\Leftrightarrow} \forall\, b : b < a \Rightarrow \ulcorner b < \ulcorner a \qquad \text{(decrease of domain).}$$

Here, $<$ is the strict-order associated with the order $\leq$ underlying the KA under consideration, i.e. $c < d \overset{\text{def}}{\Leftrightarrow} c \leq d \land c \neq d$. Note that all atoms satisfy DD.

This property is equivalent to

$$\mathrm{ED}(a) \overset{\text{def}}{\Leftrightarrow} \forall\, b : b \leq a \land {\ulcorner b} = {\ulcorner a} \Rightarrow b = a \qquad \text{(equality of domain).}$$

PROOF.

$$\begin{array}{ll}
 & b < a \Rightarrow {\ulcorner b} < {\ulcorner a} \\
\Leftrightarrow & \quad \{\!\![\text{ definition of } \Rightarrow \,]\!\!\} \\
 & b \not< a \lor {\ulcorner b} < {\ulcorner a} \\
\Leftrightarrow & \quad \{\!\![\text{ definition of } < \text{ and Boolean algebra }]\!\!\} \\
 & {\ulcorner b} < {\ulcorner a} \lor b \not\leq a \lor b = a \\
\Leftrightarrow & \quad \{\!\![\text{ Boolean algebra and definition of } \Rightarrow \,]\!\!\} \\
 & {\ulcorner b} \not< {\ulcorner a} \land b \leq a \Rightarrow b = a \\
\Leftrightarrow & \quad \{\!\![\text{ definition of } < \text{ and Boolean algebra }]\!\!\} \\
 & ({\ulcorner b} \not\leq {\ulcorner a} \lor {\ulcorner b} = {\ulcorner a}) \land b \leq a \Rightarrow b = a \\
\Leftrightarrow & \quad \{\!\![\text{ distributivity }]\!\!\} \\
 & ({\ulcorner b} \not\leq {\ulcorner a} \land b \leq a) \lor ({\ulcorner b} = {\ulcorner a} \land b \leq a) \Rightarrow b = a \\
\Leftrightarrow & \quad \{\!\![\text{ monotonicity }]\!\!\} \\
 & \mathrm{FALSE} \lor ({\ulcorner b} = {\ulcorner a} \land b \leq a) \Rightarrow b = a \\
\Leftrightarrow & \quad \{\!\![\text{ propositional logic }]\!\!\} \\
 & {\ulcorner b} = {\ulcorner a} \land b \leq a \Rightarrow b = a \;.
\end{array}$$

Another candidate is

$$\mathrm{CD}(a) \overset{\text{def}}{\Leftrightarrow} \forall b : b \leq a \Rightarrow b = {\ulcorner b} \cdot a \qquad \text{(characterization by domain).}$$

Lemma 15 $\mathrm{DD}(a) \Leftrightarrow \mathrm{CD}(a)$.

PROOF. We prove $\mathrm{ED}(a) \Leftrightarrow \mathrm{CD}(a)$. Assume $b \leq a$.

($\Leftarrow$) Suppose ${\ulcorner b} = {\ulcorner a}$. Then by $\mathrm{CD}(a)$ and Lemma 10(b) we get $b = {\ulcorner b} \cdot a = {\ulcorner a} \cdot a = a$.

($\Rightarrow$) Let $c \overset{\text{def}}{=} b + \neg{\ulcorner b} \cdot a$. Because $b \leq a$ and by monotonicity, $c \leq a$. Also,

$${\ulcorner c} = {\ulcorner (b + \neg{\ulcorner b} \cdot a)} = {\ulcorner b} + (\neg{\ulcorner b} \sqcap {\ulcorner a}) = {\ulcorner b} + {\ulcorner a} = {\ulcorner a} \;,$$

where disjunctivity of $\ulcorner$, Lemma 12(b), monotonicity of $\ulcorner$ with the assumption $b \leq a$, and Boolean algebra have been used. Then $c = a$ follows from $\mathrm{ED}(a)$, whence

$${\ulcorner b} \cdot a = {\ulcorner b} \cdot c = {\ulcorner b} \cdot (b + \neg{\ulcorner b} \cdot a) = {\ulcorner b} \cdot b + {\ulcorner b} \cdot \neg{\ulcorner b} \cdot a = b \;.$$

However, CD and DD are not equivalent to LD. In LAN an element a satisfies DD(a) iff it contains at most one word, whereas LD(a) is equivalent to prefix-freeness of a. So in LAN the properties CD and DD imply LD, but not the other way around. We show in section 5.3 that the implication does not hold for arbitrary Kleene algebras.

In REL the properties CD and DD are equivalent to the other characterizations of deterministic relations. However, in the case of an abstract relation algebra in RA, DD does not imply LD, as will be shown in Section 5.4.

However, we can show that LD implies DD in RA. Assume LD(a), which is equivalent to $a^{\smile}\,;a \leq 1$ in RA, and $b \leq a$. Since $b = \ulcorner b \,;b \leq \ulcorner b \,;a$, we only need to prove $\ulcorner b \,;a \leq b$.

$$
\begin{aligned}
& \ulcorner b \,;a \\
= \quad & \qquad \{\!|\ \text{relational algebra}\ |\!\} \\
& b \,;\top \sqcap a \\
\leq \quad & \qquad \{\!|\ \text{Dedekind}\ |\!\} \\
& (b \sqcap a \,;\top^{\smile}) \,;(\top \sqcap b^{\smile}\,;a) \\
= \quad & \qquad \{\!|\ \text{Boolean algebra, since } b \leq a \leq a \,;\top^{\smile} |\!\} \\
& b \,;b^{\smile}\,;a \\
\leq \quad & \qquad \{\!|\ \text{monotonicity}\ |\!\} \\
& b \,;a^{\smile}\,;a \\
\leq \quad & \qquad \{\!|\ \text{assumption } a^{\smile}\,;a \leq 1\ |\!\} \\
& b.
\end{aligned}
$$

Finally, we give another equivalent domain-oriented characterization:

$$\mathrm{SO}(a) \overset{\text{def}}{\Leftrightarrow} \ulcorner : \{b : b \leq a\} \to \{t : t \leq \ulcorner a\} \ \text{ is an order-isomorphism} \qquad \text{(subobject lattice)}$$

Lemma 16 CD(a) $\Leftrightarrow$ SO(a).

PROOF. ($\Rightarrow$) We already know that $\ulcorner$ is monotonic. It is surjective, since for $t \leq \ulcorner a$ we have

$$\ulcorner(t \cdot a) = t \cdot \ulcorner a = t \sqcap \ulcorner a = t\ .$$

Finally, if for $b_1, b_2 \leq a$ we have $\ulcorner b_1 = \ulcorner b_2$, then by CD($a$) we get $b_1 = \ulcorner b_1 \cdot a = \ulcorner b_2 \cdot a = b_2$, so that $\ulcorner$ is injective as well.
($\Leftarrow$) Consider $b \leq a$ and $c \overset{\text{def}}{=} \ulcorner b \cdot a$. We have $\ulcorner c = \ulcorner b \cdot \ulcorner a = \ulcorner b \sqcap \ulcorner a = \ulcorner b$ by monotonicity. Therefore, by injectivity of $\ulcorner$ we get $b = c$.

The modal operators diamond and box are quantifiers about the successor states of a state in a transition system. But they can also be viewed as assertion transformers dealing with sets of states. The (forward) diamond operator assigns to a set of states t the set s of all those states that have a successor in t. The (forward) box operator is the dual of the diamond operator; it assigns to a set of states t the set s of all those states for which all successors lie in t. The backward modal operators are defined symmetrically.

In the setting of Kleene algebras the role of assertions or sets of states is played by types. Hence we can define the modal operators as type transformers. The forward operators of dynamic logic are obtained by setting

$$\begin{aligned} \langle a \rangle t &\stackrel{\text{def}}{=} \ulcorner(a \cdot t) \ , \\ [a]t &\stackrel{\text{def}}{=} \neg\langle a \rangle \neg t \ . \end{aligned}$$

We note that

$$[a]t \ = \ a \to t$$

where

$$a \to b \ \stackrel{\text{def}}{=} \ \neg\ulcorner(a \cdot \neg\ulcorner b)$$

is called *type implication*, an operation which is useful for dealing with assertions in demonic semantics and which enjoys many useful properties, see [3].

Moreover, we can use the Galois connection from Lemma 13 to relate the forward and backward modal operators:

Corollary 17 *For* $s, t \leq 1$,

$$\begin{aligned} s \leq [a]t \ &\Leftrightarrow \ \langle a \rangle^{-} s \leq t \ , \\ s \leq \langle a \rangle t \ &\Leftrightarrow \ [a]^{-} s \leq t \ , \end{aligned}$$

where

$$\begin{aligned} \langle a \rangle^{-} s &\stackrel{\text{def}}{=} (s \cdot a)\urcorner \\ [a]^{-} s &\stackrel{\text{def}}{=} \neg\langle a \rangle^{-} \neg s \ . \end{aligned}$$

PROOF.

$$\begin{array}{ll} & s \leq [a]t \\ \Leftrightarrow & \quad \{\![\text{ definitions, Boolean algebra }]\!\} \\ & \ulcorner(a \cdot \neg t) \leq \neg s \\ \Leftrightarrow & \quad \{\![\text{ by Lemma 13 }]\!\} \\ & (s \cdot a)^\urcorner \leq t \\ \Leftrightarrow & \quad \{\![\text{ definitions }]\!\} \\ & \langle a \rangle^{-} s \leq t \end{array}$$

The second assertion is proved symmetrically.

We now carry over the well-known modal characterization of deterministic relations (see e.g. [10]) to elements of Kleene algebras and call an element a *modally deterministic* iff $\mathrm{MD}(a)$ holds, where

$$\mathrm{MD}(a) \overset{\text{def}}{\Leftrightarrow} \forall\, t : \langle a \rangle t \leq [a]t \ . \tag{3}$$

Note that by Boolean algebra $\mathrm{MD}(a)$ is equivalent to

$$\forall\, t : \ulcorner(a \cdot t) \sqcap \ulcorner(a \cdot \neg t) \ = \ 0 \ .$$

The following properties are easily checked:

Corollary 18 *(a)* $\langle a \rangle 0 = 0$.
(b) $\langle a \rangle 1 = \ulcorner a$.
(c) $[a]0 = \neg\ulcorner a$.
(d) $[a]1 = 1$.
(e) In particular, $\langle a \rangle 0 \leq [a]0$ *and* $\langle a \rangle 1 \leq [a]1$.
(f) Suppose that the only types are 0 *and* 1 *(such as e.g. in the algebra* LAN *of formal languages). Then* $\mathrm{MD}(a)$ *holds for all* a.
(g) $\langle \_ \rangle$ *is monotonic and* $[\_]$ *is antitonic, i.e. for* $a \leq b$ *and* $t \leq 1$ *we have*

$$\langle a \rangle t \leq \langle b \rangle t \ ,$$

$$[b]t \leq [a]t \ .$$

The modal characterization links to the domain-oriented characterizations as follows:

Lemma 19 *We have* $\mathrm{CD}(a) \ \Rightarrow\ \mathrm{MD}(a)$. *The reverse implication is not valid.*

PROOF. Assume CD(a) and consider a type t. We set $d \stackrel{\text{def}}{=} \ulcorner(a \cdot t) \sqcap \ulcorner(a \cdot \neg t)$ and calculate

$$
\begin{array}{ll}
 & d \cdot a \\
= & \quad \{\!\![\text{ definition of } d \text{, distributivity of type restriction (Lemma 8(c)) }]\!\!\} \\
 & \ulcorner(a \cdot t) \cdot a \quad \sqcap \quad \ulcorner(a \cdot \neg t) \cdot a \\
= & \quad \{\!\![\text{ by CD}(a) \text{ and } a \cdot t \leq a \text{ and } a \cdot \neg t \leq a \ ]\!\!\} \\
 & a \cdot t \sqcap a \cdot \neg t \\
= & \quad \{\!\![\text{ distributivity of type restriction (Lemma 8(c)) }]\!\!\} \\
 & a \cdot (t \sqcap \neg t) \\
= & \quad \{\!\![\text{ Boolean algebra and strictness }]\!\!\} \\
 & 0 \ .
\end{array}
$$

Therefore, $d = \ulcorner(d \cdot \ulcorner a) = \ulcorner(d \cdot a) = \ulcorner 0 = 0$, which shows $\langle a \rangle t \leq [a]t$.

To see that the reverse implication fails, consider a Kleene algebra in which the only types are 0 and 1. Then we have DD(a) and hence, by Lemma 15, CD(a) iff a is an atom. However, by Corollary 18(f), MD(a) is always true.

For the relationship with our other characterizations, see Section 5.7.

4 Closure Properties

4.1 Downward Closure

A natural property of functions is that a subobject of a determinate object is determinate again. Here we can show

Lemma 20 *The properties* SC*,* SC1*,* CD *and* MD *are closed under subobjects.*

PROOF. For SC and SC1 this is immediate from monotonicity, since we can restate these properties as $\forall\ b : a \cdot \overline{b} \sqcap a \cdot b = 0$ and $a \cdot \overline{1} \sqcap a = 0$, respectively. For CD, suppose $b \leq a$ and $c \leq b$. Then also $c \leq a$, hence

$$c = \ulcorner c \cdot a = (\ulcorner c \sqcap \ulcorner b) \cdot a = \ulcorner c \cdot \ulcorner b \cdot a = \ulcorner c \cdot b \ .$$

Finally, for MD, the assertion is immediate from Corollary 18(g).

4.2 All Types are Determinate

Based on our original relational motivation, we would like to have that all types are determinate. By the previous section, to ensure this we only need to check that the largest type 1 satisfies all our characterizations. Fortunately, this indeed holds, as the following lemma shows.

Lemma 21 LD(1) $\wedge$ SC1(1) $\wedge$ CD(1) $\wedge$ MD(1).

PROOF. LD(1) and SC1(1) are trivial. For the third assertion, assume $t \leq 1$. Then by Lemma 10(e)

$$t = t \cdot 1 = \ulcorner t \cdot 1 \ .$$

Finally, for types s, t we have, again by Lemma 10(e),

$$[1]t = t = \langle 1 \rangle t \ .$$

4.3 Closure Under Disjoint Choice

In this section we show that all our characterizations are closed under choice (i.e. join) of determinate objects with pairwise disjoint domains. As an auxiliary result we need the following lemma.

Lemma 22 *Assume* $\ulcorner a \sqcap \ulcorner b = 0$*. Then, for all* c, d*, we have* $a \cdot c \sqcap b \cdot d = 0$*.*

PROOF.

$$\begin{array}{ll} & \ulcorner(a \cdot c \sqcap b \cdot d) \\ \leq & \quad \{\!\!\{ \text{ monotonicity } \}\!\!\} \\ & \ulcorner(a \cdot c) \sqcap \ulcorner(b \cdot d) \\ \leq & \quad \{\!\!\{ \text{ by Lemma 10(d) } \}\!\!\} \\ & \ulcorner a \sqcap \ulcorner b \\ = & \quad \{\!\!\{ \text{ assumption } \}\!\!\} \\ & 0 \ . \end{array}$$

Now the claim follows by Lemma 10(l).

Lemma 23 *Let* P *range over* $\mathrm{LD}, \mathrm{SC}, \mathrm{SC1}, \mathrm{DD}, \mathrm{ED}, \mathrm{CD}, \mathrm{SO}, \mathrm{MD}$*. Let moreover* $L \subseteq K$ *be a set such that* $\forall\ a, b \in L : a \neq b \ \Rightarrow\ \ulcorner a \sqcap \ulcorner b = 0$*. Then*

$$(\forall\ a \in L : \mathrm{P}(a)) \ \Rightarrow\ \mathrm{P}(\sqcup L) \ .$$

PROOF. (LD)

$$
\begin{array}{ll}
 & (\bigsqcup_{a\in L} a)\cdot c \sqcap (\bigsqcup_{b\in L} b)\cdot d \\
= & \quad \{\!\![\ \text{distributivity of } \cdot \text{ over } \sqcup\ ]\!\!\} \\
 & (\bigsqcup_{a\in L} a\cdot c) \sqcap (\bigsqcup_{b\in L} b\cdot d) \\
= & \quad \{\!\![\ \text{distributivity of } \sqcap \text{ over } \sqcup\ ]\!\!\} \\
 & \bigsqcup_{a\in L} \bigsqcup_{b\in L} a\cdot c \sqcap b\cdot d \\
= & \quad \{\!\![\ \text{by assumption and Lemma 22}\ ]\!\!\} \\
 & \bigsqcup_{a\in L} a\cdot c \sqcap a\cdot d \\
= & \quad \{\!\![\ \text{by } \forall\ a \in L : \mathrm{LD}(a)\ ]\!\!\} \\
 & \bigsqcup_{a\in L} a\cdot (c \sqcap d) \\
= & \quad \{\!\![\ \text{distributivity of } \cdot \text{ over } \sqcup\ ]\!\!\} \\
 & (\bigsqcup_{a\in L} a)\cdot (c \sqcap d)\ .
\end{array}
$$

(SC1)

$$
\begin{array}{ll}
 & (\bigsqcup_{a\in L} a)\cdot \overline{1} \sqcap (\bigsqcup_{b\in L} b) \\
= & \quad \{\!\![\ \text{distributivity of } \cdot \text{ over } \sqcup\ ]\!\!\} \\
 & (\bigsqcup_{a\in L} a\cdot \overline{1}) \sqcap (\bigsqcup_{b\in L} b) \\
= & \quad \{\!\![\ \text{distributivity of } \sqcap \text{ over } \sqcup\ ]\!\!\} \\
 & \bigsqcup_{a\in L} \bigsqcup_{b\in L} a\cdot \overline{1} \sqcap b \\
= & \quad \{\!\![\ \text{by assumption and Lemma22}\ ]\!\!\} \\
 & \bigsqcup_{a\in L} a\cdot \overline{1} \sqcap a \\
= & \quad \{\!\![\ \text{by } \forall\ a \in L : \mathrm{SC1}(a)\ ]\!\!\} \\
 & \bigsqcup_{a\in L} 0 \\
= & \quad \{\!\![\ \text{lattice algebra}\ ]\!\!\} \\
 & 0\ .
\end{array}
$$

(CD) Assume $c \leq \sqcup L$.

$$
\begin{array}{ll}
 & \ulcorner c\cdot (\bigsqcup_{b\in L} b) \\
= & \quad \{\!\![\ \text{lattice algebra}\ ]\!\!\} \\
 & \ulcorner (c \sqcap \bigsqcup_{a\in L} a)\cdot (\bigsqcup_{b\in L} b) \\
= & \quad \{\!\![\ \text{distributivity of } \sqcap \text{ and } \ulcorner \text{ over } \sqcup\ ]\!\!\} \\
 & (\bigsqcup_{a\in L} \ulcorner(c \sqcap a))\cdot (\bigsqcup_{b\in L} b) \\
= & \quad \{\!\![\ \text{distributivity of } \cdot \text{ over } \sqcup\ ]\!\!\}
\end{array}
$$

$$
\begin{array}{ll}
& \bigsqcup_{a \in L} \bigsqcup_{b \in L} \ulcorner(c \sqcap a) \cdot b \\
= & \quad \{\!\![\text{ by assumption and } \ulcorner(c \sqcap a) \leq \ulcorner a \ ]\!\!\} \\
& \bigsqcup_{a \in L} \ulcorner(c \sqcap a) \cdot a \\
= & \quad \{\!\![\text{ by } \forall\ a \in L : \mathrm{CD}(a) \ ]\!\!\} \\
& \bigsqcup_{a \in L} c \sqcap a \\
= & \quad \{\!\![\text{ distributivity of } \sqcap \text{ over } \sqcup \ ]\!\!\} \\
& c \sqcap \bigsqcup_{a \in L} a \\
= & \quad \{\!\![\text{ lattice algebra }]\!\!\} \\
& c \ .
\end{array}
$$

(MD)

$$
\begin{array}{ll}
& \ulcorner((\bigsqcup_{a \in L} a) \cdot t) \sqcap \ulcorner((\bigsqcup_{b \in L} b) \cdot \neg t) \\
= & \quad \{\!\![\text{ distributivity of } \cdot \text{ and } \ulcorner \text{ over } \sqcup \ ]\!\!\} \\
& (\bigsqcup_{a \in L} \ulcorner(a \cdot t)) \sqcap (\bigsqcup_{b \in L} \ulcorner(b \cdot \neg t)) \\
= & \quad \{\!\![\text{ distributivity of } \sqcap \text{ over } \sqcup \ ]\!\!\} \\
& \bigsqcup_{a \in L} \bigsqcup_{b \in L} \ulcorner(a \cdot t) \sqcap \ulcorner(b \cdot \neg t) \\
= & \quad \{\!\![\text{ by assumption and } \ulcorner(a \cdot t) \sqcap \ulcorner(b \cdot \neg t) \leq \ulcorner a \sqcap \ulcorner b \ ]\!\!\} \\
& \bigsqcup_{a \in L} \ulcorner(a \cdot t) \sqcap \ulcorner(a \cdot \neg t) \\
= & \quad \{\!\![\text{ by } \forall\ a \in L : \mathrm{MD}(a) \ ]\!\!\} \\
& \bigsqcup_{a \in L} 0 \\
= & \quad \{\!\![\text{ lattice algebra }]\!\!\} \\
& 0 \ .
\end{array}
$$

4.4 Closure under Composition

Another natural property of determinate objects is that they are closed under composition. In this section we investigate which of our candidates for characterization imply this closure.

First, it is straightforward that LD (and hence SC) is closed under composition. Moreover, we have

Lemma 24 MD *is closed under composition.*

PROOF. We first calculate

$$
\begin{array}{ll}
& \langle a \cdot b \rangle t \\
= & \quad \{\![\text{ definition of } \langle \_ \rangle \ ]\!\} \\
& \ulcorner(a \cdot b \cdot t) \\
= & \quad \{\![\text{ local composition }]\!\} \\
& \ulcorner(a \cdot \ulcorner(b \cdot t)) \\
= & \quad \{\![\text{ definition of } \langle \_ \rangle \ ]\!\} \\
& \langle a \rangle (\langle b \rangle t)
\end{array}
$$

from which we also get by duality

$$[a \cdot b]t \ = \ [a]([b]t) \ .$$

Now the claim is immediate by monotonicity (Corollary 18(g)).

Properties SC1 and DD are *not* closed under composition as will be shown in Sections 5.5 and 5.6. However, we can show closure of DD under additional assumptions. To formulate these, we need an auxiliary notion.

Definition 25 *Analogously to the set of subidentities we define the set of* subatoms *as*

$$\mathsf{SAt} \stackrel{\text{def}}{=} \mathsf{At} \cup \{0\} \ ,$$

i.e. as the set of elements that lie below some atom.

Note that all subatoms satisfy DD. Now we can show the following lemma.

Lemma 26 *Suppose that the Boolean algebra underlying our KA is atomic. Denote the set of atoms of an element a by $\mathsf{At}(a)$ and set $\mathsf{SAt}(a) \stackrel{\text{def}}{=} \mathsf{At}(a) \cup \{0\}$.*

(a) *If a is an atom, then $\ulcorner a$ and $a \urcorner$ both are atoms.*
(b) $\mathrm{DD}(a) \Leftrightarrow (\forall\ t \in \mathsf{At}(1) : t \cdot a \in \mathsf{SAt}(a)) \Leftrightarrow (\forall\ t \in \mathsf{At}(\ulcorner a) : t \cdot a \in \mathsf{At}(a))$.
(c) DD *is closed under composition iff the set of subatoms is closed under composition.*

PROOF.

(a) We show this for $\ulcorner a$ (the case of $a \urcorner$ is symmetric). By Lemma 10(l), $\ulcorner a \neq 0$. Assume $t < \ulcorner a$. Then $t \cdot a \leq a$ by $t \leq 1$ and monotonicity. But $t \cdot a = a$ is not possible, because that would imply $\ulcorner a \leq t$, a contradiction. It follows that $t \cdot a = 0$, hence $0 = \ulcorner(t \cdot a) = t \cdot \ulcorner a = t \sqcap \ulcorner a = t$.

(b) We prove the first equivalence only, the second one being trivial.

($\Rightarrow$) Suppose $\mathrm{DD}(a)$ and let $t \in \mathsf{At}(1)$. Assume $t \cdot a \neq 0$ and let $b < t \cdot a$. By Lemmas 15 and 20, $\mathrm{DD}(t \cdot a)$ holds, so that $\ulcorner b < \ulcorner (t \cdot a) \leq \ulcorner t$, whence $\ulcorner b = 0$ since $t \in \mathsf{At}(1)$. By Lemma 10(l), $b = 0$, so that $t \cdot a \in \mathsf{At}(a)$.

($\Leftarrow$) We first note that in an atomic KA, all elements c are such that $c = \sqcup\{t \cdot c : t \in \mathsf{At}(1)\}$ $(*)$. Suppose $b < a$. By $(*)$ there must be a $t \in \mathsf{At}(1)$ such that $t \cdot b < t \cdot a$. But since $t \cdot a \in \mathsf{At}(a)$, we must have $t \cdot b = 0$. By Lemma 10(l), $\ulcorner (t \cdot b) = 0$; however, $\ulcorner (t \cdot a) \neq 0$, since $t \cdot a \in \mathsf{At}(a)$. Hence, $t \cdot \ulcorner b = \ulcorner (t \cdot b) < \ulcorner (t \cdot a) = t \cdot \ulcorner a$, from which $\ulcorner b < \ulcorner a$ follows again by $(*)$.

(c) ($\Rightarrow$) Let a, b be atoms. Then $\mathrm{DD}(a)$ and $\mathrm{DD}(b)$, whence $\mathrm{DD}(a \cdot b)$, by assumption. If $a \cdot b \neq 0$, there exists $c < a \cdot b$. By $\mathrm{DD}(a \cdot b)$, $\ulcorner c < \ulcorner (a \cdot b) \leq \ulcorner a$, whence $\ulcorner c = 0$, since $\ulcorner a$ is an atom, by part (a). By Lemma 10(l), $c = 0$, so that $a \cdot b$ is an atom.

($\Leftarrow$) Suppose $\mathrm{DD}(a)$ and $\mathrm{DD}(b)$. For any $t \in \mathsf{At}(1)$, $t \cdot a$ is a subatom, by part (b) and $\mathrm{DD}(a)$. By part (a), $(t \cdot a)\urcorner$ is a subatom. Hence, by part (b) and $\mathrm{DD}(b)$, $(t \cdot a)\urcorner \cdot b$ is a subatom. Now, $t \cdot a \cdot b = (t \cdot a) \cdot ((t \cdot a)\urcorner \cdot b)$ and thus, by assumption, $t \cdot a \cdot b$ is a subatom, being a composition of two subatoms. Because t is arbitrary, part (b) implies $\mathrm{DD}(a \cdot b)$.

4.5 Determinacy of Loops

In this section we apply our previous results to the semantics of while loops. Classically, a loop of the form

$$\mathsf{while}\ G\ \mathsf{do}\ B\ \mathsf{od}$$

with guard G and body B is modeled in Kleene algebra as follows (see e.g. [6]). The guard is represented by a type g characterizing all states that satisfy G. The semantics of the body B is given by an element b of the underlying KA. Then the loop itself is described by the semantical value

$$(g \cdot b)^* \cdot \neg g\ .$$

This represents the informal view that the loop repeats the body B as long as the guard G stays true and terminates as soon as a state is reached in which G becomes false.

We want to show now that the semantics of a loop with determinate body is determinate again. We perform this using the predicates MD and CD:

Lemma 27 *(a) If* $\mathrm{MD}(b)$ *holds then the elements* $(g \cdot b)^i \cdot \neg g$ $(i \in \mathbb{N})$ *have pairwise disjoint domains.*

(b) $\mathrm{MD}(b) \Rightarrow \mathrm{MD}((g\cdot b)^*\cdot\neg g)$.

(c) Assume that in the KA under consideration CD *is closed under composition. Then* $\mathrm{CD}(b) \Rightarrow \mathrm{CD}((g\cdot b)^*\cdot\neg g)$.

PROOF.

(a) An easy induction shows $\mathrm{MD}((g\cdot b)^i)$ and $\mathrm{MD}((g\cdot b)^i\cdot t)$ for all $i\in\mathbb{N}$ and all $t\leq 1$. Now consider two elements $(g\cdot b)^i\cdot\neg g$ and $(g\cdot b)^j\cdot\neg g$ where, without loss of generality, $j>i$.

$$\begin{array}{cl}
 & \ulcorner((g\cdot b)^i\cdot\neg g)\sqcap\ulcorner((g\cdot b)^j\cdot\neg g)\\
= & \quad\{\!\![\ \text{arithmetic}\ ]\!\!\}\\
 & \ulcorner((g\cdot b)^i\cdot\neg g)\sqcap\ulcorner((g\cdot b)^i\cdot g\cdot b\cdot(g\cdot b)^{j-i-1}\cdot\neg g)\\
= & \quad\{\!\![\ \text{local composition}\ ]\!\!\}\\
 & \ulcorner((g\cdot b)^i\cdot\neg g)\sqcap\ulcorner((g\cdot b)^i\cdot\ulcorner(g\cdot b\cdot(g\cdot b)^{j-i-1}\cdot\neg g))\\
\leq & \quad\{\!\![\ \text{by Lemma 10(d) and 10(e)}\ ]\!\!\}\\
 & \ulcorner((g\cdot b)^i\cdot\neg g)\sqcap\ulcorner((g\cdot b)^i\cdot g)\\
= & \quad\{\!\![\ \text{by }\mathrm{MD}((g\cdot b)^i)\ ]\!\!\}\\
 & 0\ .
\end{array}$$

(b) The claim follows from (a) and Lemma 23 together with the fact that

$$(g\cdot b)^*\cdot\neg g \;=\; \sqcup\{(g\cdot b)^i\cdot\neg g : i\in\mathbb{N}\}\ .$$

(c) By Lemma 19 CD implies MD. Now the claim follows analogously to (b).

5 Counterexamples

5.1 A Technique for Constructing Kleene Algebras

In this section, various finite Kleene algebras are constructed in the following way. We head for algebras in which the underlying lattice is an *atomic* Boolean algebra. In each case we list the set At of atoms; the other elements are then given by all possible joins of atoms (including the empty join). If there are n atoms, the algebra thus has 2^n elements. The Boolean operations are defined as follows:

$$\begin{aligned}
p+q &\overset{\text{def}}{=} \sqcup\{x\in\mathsf{At} : x\leq p \vee x\leq q\},\\
p\sqcap q &\overset{\text{def}}{=} \sqcup\{x\in\mathsf{At} : x\leq p \wedge x\leq q\},\\
\overline{p} &\overset{\text{def}}{=} \sqcup\{x\in\mathsf{At} : x\not\leq p\},\\
0 &\overset{\text{def}}{=} \sqcup\emptyset,\\
\top &\overset{\text{def}}{=} \sqcup\mathsf{At}.
\end{aligned}$$

The meet of two atoms is of course 0. Obviously, this defines an atomic Boolean algebra.

Composition $\cdot$ is given by a table for the atoms only. Composition of the other elements is obtained through disjunctivity, thus satisfying this axiom by construction. E.g., for atoms a, b, c, d we set

$$(a+b)\cdot(c+d) \stackrel{\text{def}}{=} a\cdot c + a\cdot d + b\cdot c + b\cdot d\ .$$

If the composition of atoms is associative, by disjunctivity this propagates to sums of atoms, i.e. to the other elements. In the same way, neutrality of 1 propagates from atoms to sums of atoms.

5.2 Concerning Local Composition

Here we present a KA that does not have local composition. It has two atoms 1 and a (and thus four elements). Its composition table is

$\cdot$	1	a
1	1	a
a	a	0

This composition is associative. There are only two types, viz. 0 and 1. Hence by Lemma 10(l) we have $\ulcorner a = 1$.

Now, $\ulcorner(a\cdot a) = \ulcorner 0 = 0$, but $\ulcorner(a\cdot \ulcorner a) = \ulcorner(a\cdot 1) = \ulcorner a = 1$.

This algebra is a special case of a whole class of algebras similar to LAN, but with words of bounded length. Specifically, let A be any set and, for $i \in \mathbb{N}$, let

$$S_n \stackrel{\text{def}}{=} \{w \in A^* : |w| \leq n\}\ ,$$

where $|w|$ is the length of word w. For $U, V \subseteq S_n$, define *bounded concatenation* by

$$U \odot V \stackrel{\text{def}}{=} \{u \bullet v : u \in U\ \wedge\ v \in V\ \wedge\ |u \bullet v| \leq n\}\ .$$

Then,

$$\mathrm{LAN}_n \stackrel{\text{def}}{=} (\mathcal{P}(S_n), \subseteq, S_n,\ \odot\ ,\ \emptyset,\ \varepsilon)$$

is a Kleene algebra where locality of composition does not hold. The example given above is obtained by starting from a set A with a single element and setting $n = 1$.

5.3 *Concerning* SC1*,* SC *and* DD

In this section we show that it is not the case that

$$\mathrm{SC1}(a) \Rightarrow \mathrm{SC}(a) \tag{4}$$

and that DD does not imply either of SC1 and SC.

The counter-example consists of a finite Kleene algebra with three atoms $1, a, b$ and the following composition table (which obviously is associative and satisfies locality of composition):

$\cdot$	1	a	b
1	1	a	b
a	a	b	b
b	b	b	b

This algebra is isomorphic to the algebra generated by the following concrete relations under relational composition:

$$\begin{aligned} 1 &= \{(0,0),(1,1),(2,2)\} \\ a &= \{(0,1),(1,2),(2,2)\} \\ b &= \{(0,2),(1,2),(2,2)\} \end{aligned}$$

In this algebra we have $\mathrm{SC1}(a)$ and $\mathrm{DD}(a)$, but not $\mathrm{SC}(a)$ (and hence not $\mathrm{LD}(a)$). Moreover, $\mathrm{DD}(b)$ holds (since b is an atom), but $\mathrm{SC1}(b)$ does not.

5.4 *Concerning* DD *and* LD

We show that DD does *not* imply LD, even for RAs. The counterexample is McKenzie's non-representable 16-element RA [7] (see also Appendix A in [11] [2]).

[2] The entry for $c \cdot b$ in Fig. A.2.3 of [11] should be changed to z.

The algebra has four atoms $1, a, b, c$ (and thus sixteen elements). The composition table of the atoms is

;	1	a	b	c
1	1	a	b	c
a	a	a	$a+b$	$\top$
b	b	$a+b$	$\overline{b}$	$b+c$
c	c	$\top$	$b+c$	c

Since a is an atom, it satisfies DD. Now, $a\,;(b \sqcap c) = 0$, but $a\,;b \sqcap a\,;c = (a+b) \sqcap \top = a+b$.

5.5 Non-Closure of SC1 *under Composition*

Consider again the algebra of Section 5.3. There, SC1 is not closed under composition, since a satisfies SC1, but the composition $b = a \cdot a$ does not.

Let us mention that the algebras from Sections 5.2 and 5.6 below cannot be used as counterexamples. There, all atoms satisfy SC1. Moreover, in the latter we have for all $x \in \{b, c, d, e, f\}$ that $x \cdot \overline{1} = 0$ which makes SC1 closed under composition in that algebra.

5.6 Non-Closure of DD *under Composition*

To show that DD (and hence CD) is not closed under composition we use a KA with nine atoms, a, b, c, d, e, f, i, j, k. Composition of the atoms is given by the following table:

$\cdot$	a	b	c	d	e	f	i	j	k
a	0	$d+e$	$d+f$	0	0	0	0	a	0
b	0	0	0	0	0	0	0	0	b
c	0	0	0	0	0	0	0	0	c
d	0	0	0	0	0	0	0	0	d
e	0	0	0	0	0	0	0	0	e
f	0	0	0	0	0	0	0	0	f
i	a	0	0	d	e	f	i	0	0
j	0	b	c	0	0	0	0	j	0
k	0	0	0	0	0	0	0	0	k

The identity of composition is given by $1 \stackrel{\text{def}}{=} i+j+k$.

It is slightly tedious to verify that composition is associative, although this is facilitated by the fact that most entries of the composition table are 0. Another manner to verify associativity is through the following relational model on the set $\{0,1,2,3,4\}$, for which standard relational composition gives the above table (in fact, we started from this concrete model rather than from the abstract one). Associativity follows from the fact that relational composition is associative.

$$\begin{array}{lll}
a = \{(0,1)\} & d = \{(0,2)\} & i = \{(0,0)\} \\
b = \{(1,2),(1,3)\} & e = \{(0,3)\} & j = \{(1,1)\} \\
c = \{(1,2),(1,4)\} & f = \{(0,4)\} & k = \{(2,2),(3,3),(4,4)\}
\end{array}$$

From the composition table and the definition of domain, one obtains

$$\begin{array}{lll}
\ulcorner a = \ulcorner d = \ulcorner e = \ulcorner f = \ulcorner i = i \ , & \ulcorner b = \ulcorner c = \ulcorner j = j \ , & \ulcorner k = k \ , \\
i\urcorner = i \ , \quad a\urcorner = j\urcorner = j \ , & b\urcorner = c\urcorner = d\urcorner = e\urcorner = f\urcorner = k\urcorner = k \ . &
\end{array}$$

It is then easy to check that locality of composition is satisfied.

Now the question is: if

$$\forall\, c : c < a \Rightarrow \ulcorner c < \ulcorner a \quad \text{and} \quad \forall\, c : c < b \Rightarrow \ulcorner c < \ulcorner b \ ,$$

do we have

$$\forall\, c : c < a \cdot b \;\Rightarrow\; \ulcorner c < \ulcorner (a \cdot b) \ ?$$

Because a and b are atoms, they satisfy DD. Also, $a \cdot b = d + e$, so that $d < a \cdot b$. But $\ulcorner d = \ulcorner (a \cdot b) = i$.

The same algebra is another counterexample to implication (4), since

$$\begin{aligned} a \cdot \overline{1} &= a \cdot (a + b + c + d + e + f) \\ &= d + e + f \leq b + c + d + e + f + 1 = \overline{a}, \end{aligned}$$

while

$$a \cdot \overline{b} = a \cdot (a + c + d + e + f + 1) = a + d + f$$

and

$$\overline{a \cdot b} = \overline{d + e} = a + b + c + f + 1,$$

so that implication (4) does not hold. By Lemma 14, this is also shown by the fact that $a \cdot (b \sqcap c) \neq a \cdot b \sqcap a \cdot c$.

Another counter-example to (4) is obtained from the above one by replacing the three atomic subidentities i, j, k by a single atomic identity 1. However, the resulting algebra does not satisfy locality of composition.

The following properties were useful in finding the counter-example. It is easy to see that if implication (4) holds for every atom b, then it holds for every element. Thus, it suffices to examine atoms. Moreover, implication (4) holds if $b \leq 1$ and also if $b = \overline{1}$. Thus, to find the counter-example, we had to look for an atom $b < \overline{1}$. Also, using the technique of Jónsson and Tarski concerning Boolean algebras with operators [4,5], it became apparent that we had to invent two atoms whose intersection in the relational model is not empty.

The same algebra can be used to give a counterexample showing that DD (or CD) does not imply LD. The element a is an atom and thus satisfies DD. However,

$$a \cdot (b \sqcap c) = a \cdot 0 = 0 \neq d = (d + e) \sqcap (d + f) = a \cdot b \sqcap a \cdot c.$$

First we note that MD does not imply SC, since otherwise, by Lemmas 15 and 19, and transitivity of implication, we would obtain that DD implies SC, in contradiction to Section 5.4 and Lemma 14.

Second, MD does not imply SC1 either. The algebra in Section 5.5 has 0 and 1 as its only types. Hence, by Corollary 18(f), we have MD(b), but SC1(b) does not hold.

Concerning the reverse implications, let us first see the informal meaning of MD in the KA PAT. Consider a graph node y, viewed as an atomic type, and a set of paths a. Then $\langle a \rangle y$ is the set of all nodes from which *some* path in a leads to y, whereas a node x is in $[a]y$ iff *all* paths in a that start in x end in y. So MD(a) holds iff all paths in a that start in the same node also end in the same node. However, a may contain several different paths between two nodes.

Now, as we have seen in Section 3.1, in PAT the properties SC(a) and SC1(a) are equivalent to prefix-freeness of a. Hence for different nodes x, y, z the set $a = \{xy, xz\}$ of paths satisfies SC(a) and SC1(a) but not MD(a). Therefore neither SC nor SC1 implies MD.

A consequence of the last paragraph is that SC1 does not imply CD; indeed, if this were the case, we would have that SC1 implies MD, because of Lemma 19. By Lemma 15, SC1 does not imply DD either.

6 Linking the Views

It turns out that in the case of an atomic KA K, we can set up a homomorphism from K into a concrete KA of type REL. This will allow us to link the domain-oriented and the relational characterizations of functions.

6.1 A Homomorphism

Consider an atomic KA K.

Definition 28 *For element $a \in K$, we define a relation $[\![a]\!]$ between atomic types by setting for $s, t \in \mathsf{At}(1)$*

$$s \, [\![a]\!] \, t \ \stackrel{\text{def}}{\Leftrightarrow} \ s \cdot a \cdot t \neq 0 \ .$$

Before we come to the main result of this section, we need an auxiliary property:

Corollary 29 *If* $u \in \mathsf{At}(1)$, *then* $u \cdot a \neq 0 \Leftrightarrow u = \ulcorner(u \cdot a) \Leftrightarrow u \leq \ulcorner a$.

PROOF.

$$\begin{array}{ll}
 & u \cdot a \neq 0 \\
\Leftrightarrow & \quad \{\!\![\text{ by Lemma 10(l) }]\!\!\} \\
 & \ulcorner(u \cdot a) \neq 0 \\
\Leftrightarrow & \quad \{\!\![\text{ by Lemmas 10(d) and 10(e), } \ulcorner(u \cdot a) \leq \ulcorner u = u \ ]\!\!\} \\
 & \ulcorner(u \cdot a) \neq 0 \wedge \ulcorner(u \cdot a) \leq u \\
\Leftrightarrow & \quad \{\!\![\ u \text{ is an atom }]\!\!\} \\
 & u = \ulcorner(u \cdot a) \\
\Leftrightarrow & \quad \{\!\![\text{ locality of composition and Lemma 10(e) }]\!\!\} \\
 & u = u \cdot \ulcorner a \\
\Leftrightarrow & \quad \{\!\![\text{ by Lemma 8(b), Boolean algebra }]\!\!\} \\
 & u \leq \ulcorner a
\end{array}$$

Lemma 30 *The mapping* $[\![\_]\!]$ *is a universally disjunctive homomorphism from* K *to* REL.

PROOF.

(a) By strictness, $s\,[\![0]\!]\,t \Leftrightarrow s \cdot 0 \cdot t \neq 0 \Leftrightarrow$ FALSE. Hence $[\![0]\!] = \emptyset$.

(b)
$$\begin{array}{ll}
 & s\,[\![1]\!]\,t \\
\Leftrightarrow & \quad \{\!\![\text{ definition of } [\![_]\!], \text{ neutrality }]\!\!\} \\
 & s \cdot t \neq 0 \\
\Leftrightarrow & \quad \{\!\![\text{ since } s, t \text{ are atoms }]\!\!\} \\
 & s = t\ .
\end{array}$$

Hence $[\![1]\!] = I$.

(c)
$$\begin{array}{ll}
 & s\,[\![\sqcup\{a_j : j \in J\}]\!]\,t \\
\Leftrightarrow & \quad \{\!\![\text{ definition of } [\![_]\!], \text{ distributivity }]\!\!\} \\
 & \sqcup\{s \cdot a_j \cdot t : j \in J\} \neq 0 \\
\Leftrightarrow & \quad \{\!\![\text{ Boolean algebra }]\!\!\} \\
 & \exists\, j \in J : s \cdot a_j \cdot t \neq 0 \\
\Leftrightarrow & \quad \{\!\![\text{ definition of } [\![_]\!]\ ]\!\!\} \\
 & \exists\, j \in J : s[\![a_j]\!]t \\
\Leftrightarrow & \quad \{\!\![\text{ definition of union }]\!\!\} \\
 & s\,(\bigcup_{j \in J} [\![a_j]\!])\,t\ .
\end{array}$$

Hence $[\![\sqcup\{a_j : j \in J\}]\!] = \bigcup_{j \in J} [\![a_j]\!]$.

(d) We first show $[\![a \cdot b]\!] \subseteq [\![a]\!] \,;\, [\![b]\!]$.

$$
\begin{array}{ll}
 & s\, [\![a \cdot b]\!]\, t \\
\Leftrightarrow & \quad \{\!\![\text{ definition of } [\![\_]\!], \text{ Lemma 10(b) }]\!\!\} \\
 & s \cdot a \cdot a^{\urcorner} \cdot {}^{\ulcorner}b \cdot b \cdot t \neq 0 \\
\Leftrightarrow & \quad \{\!\![\; a^{\urcorner} \cdot {}^{\ulcorner}b = \sqcup \{u : u \in \mathsf{At}(a^{\urcorner} \cdot {}^{\ulcorner}b)\}, \text{ distributivity }]\!\!\} \\
 & (\sqcup \{s \cdot a \cdot u \cdot b \cdot t : u \in \mathsf{At}(a^{\urcorner} \cdot {}^{\ulcorner}b)\} \neq 0 \\
\Leftrightarrow & \quad \{\!\![\text{ Boolean algebra }]\!\!\} \\
 & \exists\, u \in \mathsf{At}(a^{\urcorner} \cdot {}^{\ulcorner}b) : s \cdot a \cdot u \cdot b \cdot t \neq 0 \\
\Rightarrow & \quad \{\!\![\text{ strictness }]\!\!\} \\
 & \exists\, u \in \mathsf{At}(a^{\urcorner} \cdot {}^{\ulcorner}b) : s \cdot a \cdot u \neq 0 \wedge u \cdot b \cdot t \neq 0 \\
\Leftrightarrow & \quad \{\!\![\text{ definition of } [\![\_]\!] \;]\!\!\} \\
 & \exists\, u \in \mathsf{At}(a^{\urcorner} \cdot {}^{\ulcorner}b) : s\, [\![a]\!]\, u \wedge u\, [\![b]\!]\, t \\
\Rightarrow & \quad \{\!\![\text{ the atoms of a type are atomic types }]\!\!\} \\
 & s\, ([\![a]\!] \,;\, [\![b]\!])\, t \; .
\end{array}
$$

Now we show the reverse inclusion. Assume $s\,([\![a]\!] \,;\, [\![b]\!])\,t$, say $s \cdot a \cdot u \neq 0$ and $u \cdot b \cdot t \neq 0$ for some atomic type u. Then by strictness $a^{\urcorner} \cdot u \neq 0$ and $u \cdot {}^{\ulcorner}b \neq 0$. Since u is an atom, we get $u \leq a^{\urcorner}$ and $u \leq {}^{\ulcorner}b$ by Corollary 29. Therefore $u \leq a^{\urcorner} \cdot {}^{\ulcorner}b$, and hence

$$s \cdot a \cdot b \cdot t = s \cdot a \cdot a^{\urcorner} \cdot {}^{\ulcorner}b \cdot b \cdot t \geq s \cdot a \cdot u \cdot b \cdot t \; .$$

Now, by local composition, again Corollary 29 and Lemma 10(l),

$${}^{\ulcorner}(s \cdot a \cdot u \cdot b \cdot t) = {}^{\ulcorner}(s \cdot a \cdot {}^{\ulcorner}(u \cdot b \cdot t)) = {}^{\ulcorner}(s \cdot a \cdot u) \neq 0 \; .$$

Therefore, by Lemma 10(l), also $s \cdot a \cdot u \cdot b \cdot t \neq 0$ and by monotonicity, since $u \leq 1$, also $s \cdot a \cdot b \cdot t \neq 0$, i.e. $s\,([\![a \cdot b]\!])\,t$.

Corollary 31 $[\![a^*]\!] = [\![a]\!]^*$ *and* $[\![a^+]\!] = [\![a]\!]^+$.

PROOF.

$$
\begin{array}{ll}
 & [\![a^*]\!] \\
= & \quad \{\!\![\text{ definition of }^* \;]\!\!\} \\
 & [\![\sqcup \{a^i : i \in \mathbb{N}\}]\!] \\
= & \quad \{\!\![\text{ Lemma 30 }]\!\!\} \\
 & \bigcup\limits_{i \in \mathbb{N}} [\![a^i]\!] \\
= & \quad \{\!\![\text{ Lemma 30 and induction }]\!\!\} \\
 & \bigcup\limits_{i \in \mathbb{N}} [\![a]\!]^i \\
= & \quad \{\!\![\text{ definition of }^* \;]\!\!\} \\
 & [\![a]\!]^* \; .
\end{array}
$$

We can now show the following lemma.

Lemma 32 *In an atomic KA K,* $\mathrm{MD}(a) \Leftrightarrow [\![a]\!]^{\smile} ; [\![a]\!] \subseteq I$ *.*

PROOF. In this derivation, $s, t, u \in \mathsf{At}(1)$.

$$
\begin{array}{ll}
 & [\![a]\!]^{\smile} ; [\![a]\!] \subseteq I \\
\Leftrightarrow & \quad \{\![\text{ definition of } [\![\_]\!], \text{ definition of REL, } I = [\![1]\!] \text{ by Lemma 30 }]\!\} \\
 & \forall\, s, t : (\exists\, u : u[\![a]\!]s \wedge u[\![a]\!]t) \Rightarrow s[\![1]\!]t \\
\Leftrightarrow & \quad \{\![\text{ definition of } [\![\_]\!]]\!\} \\
 & \forall\, s, t : (\exists\, u : u \cdot a \cdot s \neq 0 \wedge u \cdot a \cdot t \neq 0) \Rightarrow s \cdot 1 \cdot t \neq 0 \\
\Leftrightarrow & \quad \{\![\text{ Corollary 29 }]\!\} \\
 & \forall\, s, t : (\exists\, u : u \leq \ulcorner(a \cdot s) \wedge u \leq \ulcorner(a \cdot t)) \Rightarrow s \cdot t \neq 0 \\
\Leftrightarrow & \quad \{\![\text{ Boolean algebra }]\!\} \\
 & \forall\, s, t : (\exists\, u : u \leq \ulcorner(a \cdot s) \sqcap \ulcorner(a \cdot t)) \Rightarrow s \cdot t \neq 0 \\
\Leftrightarrow & \quad \{\![\; u \in \mathsf{At}(1), \text{ Lemma 8(b) }]\!\} \\
 & \forall\, s, t : \ulcorner(a \cdot s) \sqcap \ulcorner(a \cdot t) \neq 0 \Rightarrow s \sqcap t \neq 0 \\
\Leftrightarrow & \quad \{\![\text{ contrapositive }]\!\} \\
 & \forall\, s, t : s \sqcap t = 0 \Rightarrow \ulcorner(a \cdot s) \sqcap \ulcorner(a \cdot t) = 0 \\
\Leftrightarrow & \quad \{\![\text{ Boolean algebra }]\!\} \\
 & \forall\, s, t : t \leq \neg s \Rightarrow \ulcorner(a \cdot s) \sqcap \ulcorner(a \cdot t) = 0 \\
\Leftrightarrow & \quad \{\![\text{ for proving } \Rightarrow, \text{ take } t \stackrel{\text{def}}{=} \neg s; \\
 & \quad\quad \text{the direction } \Leftarrow \text{ is trivial by monotonicity }]\!\} \\
 & \forall\, s : \ulcorner(a \cdot s) \sqcap \ulcorner(a \cdot \neg s) = 0 \\
\Leftrightarrow & \quad \{\![\text{ definition of MD, } \langle\,\rangle \text{ and } [\,] \;]\!\} \\
 & \mathrm{MD}(a)
\end{array}
$$

This leaves us with the somewhat paradoxical situation that the modal characterization MD is the only “really relational” one: since we have shown LD, SC, SC1, CD and DD to be non-equivalent to MD, they cannot enjoy the property of this lemma.

7 Summary of the Results

To give the reader a survey of what has been achieved in this paper, we first recall the definitions of all characterizations that have been investigated:

$$\begin{aligned}
\mathrm{LD}(a) &\Leftrightarrow \forall\, b, c : a \cdot (b \sqcap c) = a \cdot b \sqcap a \cdot c \\
\mathrm{SC}(a) &\Leftrightarrow \forall\, b : a \cdot \overline{b} \leq \overline{a \cdot b} \\
\mathrm{SC1}(a) &\Leftrightarrow a \cdot \overline{1} \leq \overline{a} \\
\mathrm{DD}(a) &\Leftrightarrow (\forall\, b : b < a \Rightarrow \ulcorner b < \ulcorner a) \\
\mathrm{CD}(a) &\Leftrightarrow (\forall b : b \leq a \Rightarrow b = \ulcorner b \cdot a) \\
\mathrm{SO}(a) &\Leftrightarrow \ulcorner : \{b : b \leq a\} \to \{t : t \leq \ulcorner a\} \text{ is an order-isomorphism} \\
\mathrm{MD}(a) &\Leftrightarrow \forall\, t : \langle a \rangle t \leq [a] t
\end{aligned}$$

The following table shows mutual (non-)implications as well as (non-)closure properties.

	SC1	SC LD	SO CD DD	MD	closed under subobjects	closed under composition	closed under disjoint choice
SC1	$\Leftrightarrow$	$\Leftarrow$			yes	no	yes
SC, LD	$\Rightarrow$	$\Leftrightarrow$			yes	yes	yes
SO, CD, DD			$\Leftrightarrow$	$\Rightarrow$	yes	no	yes
MD			$\Leftarrow$	$\Leftrightarrow$	yes	yes	yes

Equivalent properties are in the same line/column. When there is a $\Rightarrow$ or a $\Leftarrow$, the reverse implication does not hold. When an entry is blank, it means that no implication holds.

8 Conclusion

The theory of Kleene algebras offers surprising new views on the notion of a function. Characterizations that are equivalent in relation algebras [11] differ in this generalized setting.

However, the second author has shown that the characterization CD is sufficient to reprove (in a simpler fashion!) all properties of overwriting that were shown relationally in [9]. So it seems that the generalized setting indeed has its merits.

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