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One-dimensional XXZ model for particles obeying fractional statistics

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We define one-dimensional particles as non-Abelian representations of the symmetric group S_N. The exact solution of an XXZ type Hamiltonian built up with such particles is achieved using the coordinate Bethe ansatz. The Bethe equations show that fractional statistics effectively account for coupling an external gauge field to an integer statistics' system. [S0163-1829(98)52028-0]

The physical behavior of quantum systems is deeply affected by the statistics of the constituting effective degrees of freedom. Quasiparticles and quasiholes in condensed-matter physics may obey statistics interpolating between fermionic and bosonic behavior. Examples are the excitations of two-dimensional electron systems exhibiting the fractional quantum Hall effect.[1] These excitations are called *anyons*. They have been a subject of intense study also in connection with superconductivity[2] and superfluidity.[3] Fractional statistics of such particles arise from the trajectory dependence of the particle exchange procedure in two-dimensional configuration space. This feature makes the concept of anyons purely two dimensional. The Fock space formulation of anyon operator algebras takes into account these characteristics. The creation and annihilation operators [introduced as Jordan-Wigner transforms of the usual fermions on a two-dimensional lattice[4] or as unitary representations of the diffeomorphism group of $\mathbb{R}^2$ (Ref. 5)] obey deformed commutation relations if the exchange involves anyons at different spatial positions (see the Appendix). N-anyon states are Abelian representations of the braid group B_N (Ref. 6) (whereas bosons and fermions furnish, respectively, the identical and alternating Abelian representations of the symmetric group S_N). These features make anyons different from q oscillators, the latter providing a realization of the Gel'fand-Fairlie quantum group, which is a *local* deformation of the Weyl-Heisenberg (bosons) or Clifford algebra (fermions).[7] The path dependence implies that the one-particle state is inextricably related to the complete state of the many-body configuration. This intrinsic nonlocality makes anyon physics very difficult. Even statistical properties of a free anyon gas are only partially established using the virial expansion.[8]

Haldane[9] formulated the notion of fractional statistics without any reference to the spatial dimension D. The generalized Pauli principle is expressed in terms of the reduction of the single-particle Hilbert space when particles are added to a many-body system keeping boundary conditions fixed. Another way to introduce dimensionality-independent fractional statistics has been formulated in Ref. 10, where *quons* have been introduced. Quons' fractional statistics result from the ''superposition'' of statistical properties of bosons and fermions.[11] In $D>2$, this is consistent with spin-statistics theorem. In $2D$, Haldane particles and quons capture the essential features of anyons.[9,5]

Recently, a growing interest has been devoted to generalized statistics in one dimension. A specific way to introduce $D=1$ fractional statistics has been proposed in connection to the quantization of the solutions of the Calogero model.[12] There, the potential $1/x^2$ is interpreted as ''statistics interaction.'' The same notion of fractional statistics applies also to anyons in a strong magnetic field that restricts the allowed energies to the lowest Landau level. The anyon gas, then, is described by an effective-field theory on a ring where the dynamics of particles is one dimensional.[13] It is worthwhile noting that such one-dimensional particles obey fractional statistics, but they are not ''true anyons'' since in $D\neq 2$ trajectories in the particle configuration space have no meaningful braiding property. Instead, nonlocal ''deformations'' of the commutation relations furnish *non-Abelian* representations of the symmetric group S_N.

In this paper, we deal with particles in $D=1$ that preserve the intrinsic nonlocality of two-dimensional anyons, but which are still representations of S_N. This representation is no longer Abelian. The second quantized formalism and the Fock-space representation is developed. The XXZ model for such particles is formulated and solved exactly using coordinate Bethe ansatz[14] (BA) in $D=1$.

For $D=1$ we define a set of creation/annihilation operators $\{f_i^\dagger, f_i\}$ for a spinless particle at site i. They obey the deformed relations

$$f_j^\dagger f_k + q_{j,k} f_k f_j^\dagger = \delta_{j,k}, \tag{1}$$

$$f_j f_k + q_{j,k}^{-1} f_k f_j = 0, \tag{2}$$

where $q_{j,k}^{-1} \doteq (q_{j,k})^{-1}$. Since the operators are path independent in $D\neq 2$ (compare with the Appendix), Eqs. (1) and (2) have to constitute a representation of S_N, and not of B_N. This is ensured by the ''consistency relations''

$$q_{j,k} = q_{k,j}^{-1} = q_{k,j}^\dagger, \tag{3}$$

$$[f_j f_k^\dagger, q_{j,k}] = 0. \tag{4}$$

Such a representation is non-Abelian.[6] For $j=k$, Eq. (3) gives $q_{j,j}=\pm\mathbb{1}$. Hence Eqs. (1) and (2) are an extension of anyon commutation relations[4,5] to $D\neq 2$. In contrast to the true anyonic case (see Appendix), $q_{j,k}$ has no relation with the configuration space geometry, but is a free ''external'' parameter.

Relations (1) and (2) are formally analog to quon commutation rules.[11] Note that the deformation parameter here depends on two indices (j,k), whereas it does not in quon commutation rules. Without this index dependence, relation (3) directly implies $q^2=\mathbb{1}$. As a consequence, quons obey integer statistics in $1D$ if q is a $\mathbb{C}$ number[11] (as an operator, it has eigenvalues ± 1). If $q_{j,k}=\pm\mathbb{1}\ \forall(j,k)$, then Eqs. (1) and (2) describe spinless fermions or bosons, respectively. However, for application we choose $q_{j,k}$ being $\mathbb{C}$ numbers and $q_{j,j}=1$, see Eq. (9), which implies the Pauli exclusion principle, as for spinless electrons or hard-core bosons.

Relations (3) and (4) hold if $q_{j,k}$ is an operator commuting or anticommuting with both f_j and $f_k^\dagger$. For this reason we add this as a postulate[16]

$$[f_k^\dagger, q_{j,k}]=[f_j, q_{j,k}]=0. \tag{5}$$

The introduction of two indices for the deformation parameter allows the construction of consistent commutation relations even for $q_{j,k}$ being $\mathbb{C}$ numbers. We make use of this possibility in Eq. (9).

To develop a Fock representation of the algebra (1) and (2), we take $\nu_j \doteq f_j^\dagger f_j$ as number operators. Relations (3)–(5) yield commutators of ν_j and $f_j^\dagger$, f_j being unaffected by the deformation parameter $q_{j,k}$:

$$[\nu_j,\nu_k]=0, \quad [\nu_j, f_k^\dagger]=\delta_{j,k} f_k^\dagger, \quad [\nu_j, f_k]=-\delta_{j,k} f_k. \tag{6}$$

Moreover, the property $q_{j,j}=\mathbb{1}$ implies that number operators are idempotent: $(\nu_j)^2=\nu_j$. Because of Eq. (6) the one-particle Fock representation of the algebra (1) and (2) is unaffected by $q_{j,k}$. Instead, the action of $f_l, f_l^\dagger, \nu_l$ on the N-particle state $|n_1,\ldots,n_N\rangle$ is deformed according to

$$\begin{aligned} f_l|n_1,\ldots,n_N\rangle &= (-)^{l-1}\delta_{n_l,1}\left[\prod_{k=1}^{l-1} q_{l,k}\right] \\ &\quad\times|n_1,\ldots,n_l-1,\ldots,n_N\rangle, \\ f_l^\dagger|n_1,\ldots,n_N\rangle &= (-)^{l-1}\delta_{n_l,0}\left[\prod_{k=1}^{l-1} q_{k,l}\right] \\ &\quad\times|n_1,\ldots,n_l+1,\ldots,n_N\rangle, \\ \nu_l|n_1,,n_N\rangle &= n_l|n_1,\ldots,n_N\rangle, \end{aligned} \tag{7}$$

where $n_l\in\{0,1\}$. Equations (7) generalize the corresponding relations fulfilled by integer statistics particles[17] characterized by $\Pi_{k=1}^{l-1} q_{l,k}=(\pm)^{l-1}$ (for fermions/bosons). An explicit realization of the operators f_j in terms of spinless fermionic operators a_j is $f_j \doteq a_j \exp(-i\Sigma_l \Phi_l n_l)$, where Φ_l are Hermitian operators commuting with fermionic degrees of freedom. By direct calculation, relations (1) and (2) are obtained by setting $q_{jk}\doteq\exp[i(\Phi_k-\Phi_j)]$. This realization has been suggested in Ref. 15 where $\Phi_k\equiv p_k$, p_k being momenta of a phononic bath coupled to fermionic degrees of freedom.

In the following we consider the $1D$ anisotropic Heisenberg model (XXZ model) of spinless fermions

$$H_{XXZ}=-t\sum_i (f_i^\dagger f_{i+1}+f_{i+1}^\dagger f_i)+U\sum_i \nu_i\nu_{i+1}. \tag{8}$$

The operators f obey relations (1) and (2) with $q_{j,k}$ defined in close analogy with anyonic relations (see the Appendix)

$$q_{j,k}=\begin{cases} q, & j>k, \\ 1, & j=k, \\ q^{-1}, & j<k. \end{cases} \qquad q\in\mathbb{C} \tag{9}$$

We point out that postulate (5) is fulfilled since $q_{j,k}$ is a $\mathbb{C}$ number for arbitrary, fixed (j,k). Relations (3) and (9) imply that q is on the unit circle, i.e., $|q|=1$. Additionally, periodic boundary conditions (PBC) $f_{i+L}\equiv f_i$ are chosen, where L denotes the period. The parameters t and U are the hopping amplitude and the Coulomb interaction strength, respectively.

It is worthwhile mentioning that Eq. (9) implies fixing an order on the infinite periodic chain. This order can only be defined locally on the manifold S^1, which needs two charts for its description. We choose two ''charts'' $C_1\doteq\{j_1,\ldots,j_N\}$ and $C_2\doteq\{j_{N-l},\ldots,j_N,j_1,\ldots,j_{N-l-1}\}$ on the ring thought as a discrete subset of S^1. On each chart, the given order is well defined by interpreting them as ordered sets. The intersections between C_1 and C_2 are $\{j_{N-l},\ldots,j_N\}$ and $\{j_1,\ldots,j_{N-l-1}\}$. In such sets, the orders defined on C_1, C_2 are identical. Now, in $D=1$, only nearest neighbor (n.n.) exchanges can take place. Thus q_{j_k,j_l} is connected to the n.n. exchange $j_k\leftrightarrow j_l$. On the chart C_1, where $j_1<j_N$, the exchange $j_N\leftrightarrow j_1$ is not a n.n. exchange. To allow for n.n. hopping $j_N\leftrightarrow j_1$, we must use C_2 on which $j_N<j_1$. This implies $q_{j_N,j_1}=q^{-1}$. The picture depicted above is equivalent to fixing a period $P_0\doteq\{j_1,\ldots,j_1+L\}$ on the infinite periodic chain. Consistency of the PBC with this induced order is given if the results are independent of P_0. In the following, it will be seen that this condition is fulfilled.

The correspondence between Eq. (8) and the deformed anisotropic Heisenberg model can be established by $S_j^{(+)}\doteq f_j^\dagger$, $S_j^{(-)}\doteq f_j$, and $S_j^{(z)}\doteq 1/2-\nu_j$. On site, the operators $S_j^{(z)}, S_j^{(\pm)}$ generate the fundamental representation (spin $s=1/2$) of $su(2)$, but for $j\neq k$

$$\begin{aligned} [S_j^{(+)}, S_k^{(-)}] &= (1+q_{j,k})S_k^{(+)}S_j^{(-)}-\delta_{j,k}, \\ [S_j^{(-)}, S_k^{(-)}] &= (1+q_{j,k})S_k^{(-)}S_j^{(-)}-\delta_{j,k}, \\ [S_j^{(z)}, S_k^{(\pm)}] &= 0. \end{aligned} \tag{10}$$

We now show that the XXZ model (8) is exactly solvable by means of the coordinate BA. The general N-particle state on a chain with L sites can be written as

$$|\Psi\rangle \doteq \sum_{1\leq j_1<\cdots<j_N\leq L} \psi(j_1,j_2,\ldots,j_N) f_{j_1}^\dagger, f_{j_2}^\dagger,\ldots,f_{j_N}^\dagger|0\rangle. \tag{11}$$

The action of H_{XXZ} on $|\Psi\rangle$, i.e., the eigenvalue equation, then reads

$$U\sum_{l=1}^{N-1}\delta_{j_l+1,j_{l+1}}\psi(j_1,\ldots,j_N)$$
$$-t\sum_{l=1}^{N} Q_l^{(+)}\psi(j_1,\ldots,j_l-1,\ldots,j_N)$$
$$+Q_l^{(-)}\psi(j_1,\ldots,j_l+1,\ldots,j_N)$$
$$=E\psi(j_1,\ldots,j_N), \qquad (12)$$

where $Q_l^{(\pm)}\doteq\prod_{s=1}^{l-1}q_{j_l,j_s}q_{j_s,j_l\mp 1}$. Using Eq. (9) as the definition for $q_{j,k}$, it follows that

$$Q_l^{(-)}=Q_l^{(+)}=1. \qquad (13)$$

Hence, exactly the same eigenvalue equation as in the ordinary XXZ model is obtained. Consequently, inserting the BA

$$\psi(j_1,\ldots,j_N)=\sum_{\pi\in S_N}A(\pi)\exp\left(i\sum_{m=1}^{N}j_m k_{\pi(m)}\right), \quad (14)$$

the energy E and the two–body scattering matrix $S(k,k')\doteq-\exp[-i\theta(k,k')]$ for the ''deformed'' model are identical with the known terms occurring in the usual XXZ model.[18] Imposing PBC, however, yields

$$\psi(j_2,\ldots,j_N,j_1)=\left[\prod_{l=2}^{N}(-q_{j_1,j_l})\right]\psi(j_1,j_2,\ldots,j_N)$$
$$=(-q)^{N-1}\psi(j_1,j_2,\ldots,j_N). \qquad (15)$$

Equation (15) shows that the fractional statistics produces a twist in the PBC that modifies the periodicity of the Bethe wave function.

Since the twisting factor q^{N-1} does not depend on j_1, the starting point of the chosen period P_0, the boundary condition is consistent with our choice of $q_{j,k}$. The twist q^{N-1} does not depend on the particles' configuration, but on the number of particles only. This is crucial because configuration dependence of the twisting factor would destroy the solvability of the model, since it modifies the structure of the exponential functions in imposing PBC on the BA wave function [making it impossible to extract from Eq. (15) a relation for the amplitudes $A(\pi)$]. So, the coordinate BA solvability of the model (with PBC) demands a careful choice of the functional form of $q_{j,k}$. The choice $q_{j,k}\doteq\exp[i\delta(j-k)]$ for the XXZ model, for instance, leads to the same structure of the S matrix, but produces incompatible boundary conditions.

Since $|q|=1$, $q=\exp[i\arg(q)]$. So a phase shift by multiples of $\arg(q)$ occurs in the BA wave function on the right-hand side of Eq. (15). The Bethe equations (BE) are obtained as

$$k_jL=\arg([-q]^{N-1})+2\pi I_j-\sum_{m=1}^{N}\theta(k_j,k_m), \qquad (16)$$

where $I_j\in\mathbb{Z}$. In the fermionic case, one obtains $k_jL=2\pi[I_j+(N-1)/2]-\sum_{m=1}^{N}\theta(k_j,k_m)$, whereas the hard-core bosonic case yields $k_jL=2\pi I_j-\sum_{m=1}^{N}\theta(k_j,k_m)$. In all cases, $I_j\in\mathbb{Z}$. Equation (16) differs from the BE for the ordinary XXZ model in the additive term $\arg([-q]^{N-1})$. This term has its origin in the fractional statistics of the particles, and vanishes for integer statistics. Equation (16) was obtained for the $1D$ XXZ model on a ring threaded by an external magnetic flux.[19,20] We recover the BE of Ref. 20 identifying $\Phi\equiv\alpha\doteq\arg([-q]^{N-1})$, Φ being the magnetic flux in units of $\hbar c/e$.

The limit $k_N\to\pi-\mu$, where $\cos(\mu)\doteq U/2t$ for $|U|\leq 2|t|$, in the Nth equation of Eq. (16) relates the statistics with the ratio $U/2t$ for the ground state,

$$\alpha=\pi(L-N-2I_N+1)-\mu(L-2N+2). \qquad (17)$$

At half filling $[N/L=1/2,\ I_N=(N-1)/2]$ the energy and the total momentum $P\doteq\sum_{l=1}^{N}k_l/L$ of the ground state are affected by the statistics' factor α (Refs. 20 and 18) as follows:

$$E_0(\alpha)-E_0(0)=\frac{\pi\sin(\mu)}{4\mu(\pi-\mu)L}\,\alpha^2,$$
$$P=\frac{1}{2}\alpha, \qquad (18)$$

where $E_0(0)$ denotes the ground-state energy of the undeformed XXZ model. The same structure of BE has been obtained in Refs. 21 and 22. In Ref. 22, the two species of particles, up-spin $(\sigma=+)$ and down-spin $(\sigma=-)$, have dynamics governed by two distinct $XXZ^{(\sigma)}$ Hamiltonians coupled only via a local gauge field, included in $XXZ^{(\sigma)}$ by a Peierls-like substitution $t\to tW_m^{(\sigma)}$, where $W_m^{(\sigma)}\doteq\exp[i\sigma\sum_{l=1}^{L}\alpha_{m-l}n_{l,-\sigma}]$ (determined by the position of all particles of opposite species); $\alpha_l\in\mathbb{R}$, $\alpha_{m+L}\equiv\alpha_m$. A comparison of the BE in Ref. 22 with Eq. (16) shows that our deformation parameter q can be interpreted as the ''global'' coupling constant of the gauge potential by setting $\alpha\sim\sum_{m=1}^{L}\alpha_{m-l}$. Vice versa, such an interaction produces statistics transmutation. In this sense, our deformed XXZ model belongs to the same class of integrable models introduced in Ref. 22.

In conclusion, we have given a formulation of fractional statistics in one dimension realized by an anyonic-type deformation of the second quantized commutation rules. Coordinate BA solvability of the deformed XXZ model demands a proper choice of the functional form of $q_{j,k}$. The statistics we have chosen in the present paper preserves the Yang-Baxter equation as well as the BA solvability of the undeformed model. The resulting BE are, however, modified. They show that fractional statistics plays the same role as a gauge field coupled to the undeformed model. Systematic investigations of fractional statistics seem interesting for at least two reasons. First, fractional statistics may be an alternative approach to handle complicated interactions between particles obeying integer statistics. Such interactions could be modeled deforming the particles' statistics. Second, the study of ''compatible statistics'' could be relevant in order to find integrable Hamiltonians characterized by *braided* Yang–Baxter equations (YBE).[23] Such a feature of the YBE could be closely related to actual braiding of particles in two dimensions.

A further development of the present approach is to take spin into account. A preliminary analysis of the ''deformed'' Hubbard model[15] shows that fractional statistics modify the

S matrix; the R matrix obeys a braided YBE. We will report on this subject in a forthcoming paper.

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APPENDIX

Here we summarize the commutation properties of two-dimensional anyons.[4,5] The creation/annihilation operators obey

$$b^\dagger(\mathbf{x}_C)b(\mathbf{y}_C)+q(\mathbf{x}_C,\mathbf{y}_C)b(\mathbf{y}_C)b^\dagger(\mathbf{x}_C)=\delta_{\mathbf{x}_C,\mathbf{y}_C},$$

$$b(\mathbf{x}_C)b(\mathbf{y}_C)+q^{-1}(\mathbf{x}_C,\mathbf{y}_C)b(\mathbf{y}_C)b(\mathbf{x}_C)=0. \quad \text{(A1)}$$

The operators $b^\dagger(\mathbf{x}_C)$ $[b(\mathbf{x}_C)]$ create (annihilate) an anyon at site $\mathbf{x}_C\doteq(x_1,x_2)$. C denotes the path running from $+\infty$ to $\mathbf{x}_C$ keeping x_2 constant. The relations above hold if $\mathbf{x}_C>\mathbf{y}_C$; in the case $\mathbf{x}_C<\mathbf{y}_C$, they are satisfied substituting $q\leftrightarrow q^{-1}$. Note that $\mathbf{x}_C>\mathbf{y}_C\Leftrightarrow\{x_2>y_2\vee x_1>y_1 \text{ (if } x_2=y_2)\}$. The function $q(\mathbf{x}_C,\mathbf{y}_C)=q(|\mathbf{x}_C-\mathbf{y}_C|)$ can be simplified (see, e.g., Ref. 4) to $q=e^{i\pi\nu}$ $(\nu\in\mathbb{R})$, where ν denotes the *statistics*. If two anyons are at the same position $\mathbf{x}_C=\mathbf{y}_C$, then $q=1$. Otherwise, the standard bosonic or fermionic algebras are deformed by the parameter q.

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