Transport in a nearly periodic potential with a magnetic field

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Abstract. A two-dimensional gas of non-interacting quasiparticles in a nearly periodic potential and a perpendicular magnetic field is studied. The potential is a superposition of a periodic potential, induced, for example, by a charge-density wave or a vortex lattice of a type-II superconductor, and a weak random potential due to impurities. Approximating this model by Dirac fermions with random mass, random energy, and random vector potential, we evaluate the density of states and the Hall conductivity using a self-consistent approximation. We obtain a singular average density of states $\rho(E) \approx \rho_0 + |E|^\alpha$, where α decreases with the strength of the randomness. The Hall conductivity has a plateau which is destroyed for strong tunnelling through the saddle points of the nearly periodic potential.

1. Introduction

A two-dimensional system of non-interacting quasiparticles is considered in a periodic potential with a strong perpendicular magnetic field. The periodic potential can be understood as a pinned charge-density wave, formed in a two-dimensional low-density electron gas due to Coulomb interaction. This model is related to a possible formation of a Wigner crystal in a two-dimensional electron gas with a perpendicular magnetic field, a problem which has attracted considerable attention recently [1, 2]. Another possible realization of the model is a system of quasiparticles in a layered type-II superconductor in the presence of a magnetic field, where a periodic structure due to the vortex lattice in the Abrikosov phase arises. Due to impurities, the periodicity of the potential is not perfect in a real system, and we have to incorporate weak random fluctuations of the potential. Therefore, the total potential is a superposition of the periodic potential and weak randomness, creating a nearly periodic potential.

We start with a continuum model which we map to the network model, following the ideas of Chalker and Coddington [3] for the quantum Hall system. The resulting model is defined by an evolution operator for the quasiparticles on a two-dimensional lattice, where the latter is an approximation of the nearly periodic potential. Two types of scattering are possible. One describes the scattering along the equipotential lines, while the other corresponds to the scattering through the saddle points of the potential. This model undergoes a transition between Hall plateaux of the integer quantum Hall effect, the quantum Hall transition (QHT), when the scattering through the saddle points exceeds a critical value. The network model has been very successful as a starting point for numerical studies of the QHT [3–6]. Unfortunately, the model is not easily accessible to analytic methods due to its internal lattice structure. However, a large-scale approximation, for weak disorder,

sufficient for the investigation of transport properties, has been applied to the network model by Ho and Chalker [7]. They found that the network model is equivalent to a model of Dirac fermions. This result indicates that the network model is similar to the tight-binding model with half a flux quantum per plaquette, where the large-scale approximation also leads to Dirac fermions [8–10]. However, the corresponding randomness in the resulting Dirac fermions is different: the tight-binding model with a random potential leads to Dirac fermions with a random mass whereas the network model of the nearly periodic potential leads to Dirac fermions with a random vector potential and a random energy.

Dirac fermions with randomness can be treated by field theoretical methods [9, 11, 12]. A particular case is the purely random vector potential, which can be solved exactly by bosonization [9, 13]. Other types of randomness, to the best of our knowledge, cannot be treated in this way. Therefore, we will apply a self-consistent approximation scheme. In order to control our approximate calculation we will compare it with the exact result of references [9, 13].

The paper is organized as follows. In section 2 we introduce the continuum model, and derive a network model of quasiparticle loops, which is equivalent to the network model of Chalker and Coddington. In section 2.1 we briefly recall a tight-binding approximation of the periodic potential [9, 14]. The large-scale approximation of the network by Dirac fermions is discussed in section 2.2. As physical quantities, we define the average DOS and the Hall conductivity σ_{xy} in terms of the Green's function. The main part of our calculation is presented in section 3, where a self-consistent approximation for the average Green's function is worked out, as well as its consequences for the average DOS and the Hall conductivity. Finally, we discuss our results in section 4.

2. The model and related physical quantities

We consider the dynamics of a particle with charge e in the potential $V(x, y) = V_0 \cos(\pi a x) \cos(\pi a y) + \delta V(x, y)$, where $\delta V(x, y)$ represents a weakly fluctuating disorder potential. We start with a discussion of the classical motion, taking into account the Lorentz force created by the perpendicular field $\boldsymbol{B} = (0, 0, B)$:

$$M\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} + \frac{e}{c}\boldsymbol{B} \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \boldsymbol{\nabla} V = 0 \tag{1}$$

where $\boldsymbol{r} = (x, y, 0)$ is the coordinate vector. For a sufficiently small mass M the first term can be neglected, and the resulting first-order differential equations show that the particle follows the equipotential lines of the potential:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{matrix} x \\ y \end{matrix} = \frac{c}{eB}\begin{matrix} \partial V/\partial y \\ -\partial V/\partial x \end{matrix} \quad . \tag{2}$$

The approximation by the first-order differential equation is equivalent to neglecting the cyclotron motion of the particle and reducing the dynamics to the motion of the guiding centre.

For $B > 0$ a particle with energy $E \neq 0$ follows closed anticlockwise orbits around a hill ($E > 0$) or clockwise orbits around a well ($E < 0$) (figure 1). Only a particle with energy $E = 0$ can travel through the whole two-dimensional space because there are extended equipotential lines at $x = (2k + 1)/2a$ or at $y = (2k + 1)/2a$ ($k = 0, 1, \ldots$). The behaviour near $E = 0$ can be quantized using a lattice approximation. This approximation is based on the orbits of quasiparticles at a given energy E which form a network of loops. Quantum effects are described by tunnelling between these loops in regions where the loops

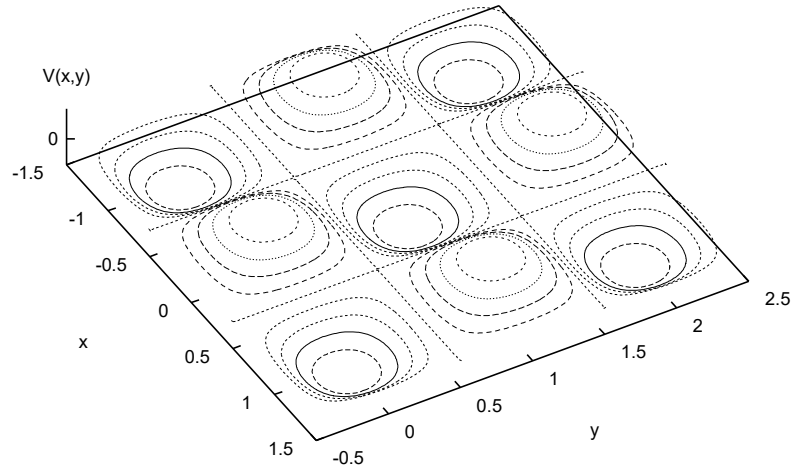

Figure 1. Equipotential lines of $V(x, y) = V_0 \cos(\pi x) \cos(\pi y)$ (arbitrary units).

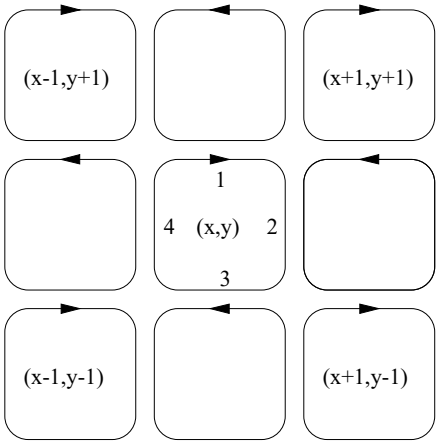

Figure 2. A network of equipotential lines for two fixed energies $\pm E$ close to zero. The arrows indicate the directions of the quasiparticle motion due to the magnetic field $B > 0$.

are close to each other, i.e. at the saddle points of the potential. For $E < 0$ the tunnelling occurs between the loops in figure 2 whose centres (minima of the potential) are located at (x, y) and $(x \pm 1, y \pm 1)$. For $E > 0$ we have a complementary situation in which the preferred loops appear with the opposite current direction. Apart from this difference, the subsequent analysis can be applied to this case as well. We will consider $E < 0$ in the following.

2.1. Lattice approximations

The tunnelling can be expressed in terms of discretized scattering processes of effective quasiparticle positions at the centres of the loop edges, enumerated $1, \ldots, 4$ in figure 2. The scattering is characterized by a scattering parameter β ($0 \leqslant \beta \leqslant \pi/4$), determined by

V_0 and E. As regards the loop at (x, y) in figure 2, there is a quasiparticle state

$$\Psi(t) \equiv (\Psi_1(t, x, y), \Psi_3(t, x, y), \Psi_2(t, x, y), \Psi_4(t, x, y))$$

at time t whose components are associated with the edges j of the loop. At the later time $t + 1$ this state can be scattered along the loop, or can tunnel through one of the four saddle points to the loops at $(x \pm 1, y \pm 1)$. For example, the component $\Psi_1(t, x, y)$ can be scattered either to $\Psi_2(t + 1, x, y)$ with rate $C = \cos\beta$, giving rise to a phase ϕ_2, or can tunnel to $\Psi_4(t + 1, x + 1, y + 1)$ with rate $S = \sin\beta$, giving rise to a phase ϕ_4. All possible processes can be combined as follows:

$$\Psi(t+1) = \begin{matrix} 0 & \mathbf{M} \\ \mathbf{N} & 0 \end{matrix} \Psi(t) \tag{3}$$

where

$$\mathbf{M} = \begin{matrix} S\mathrm{e}^{\mathrm{i}\phi_1} t_-^x t_+^y & C\mathrm{e}^{\mathrm{i}\phi_1} \\ C\mathrm{e}^{\mathrm{i}\phi_3} & -S\mathrm{e}^{\mathrm{i}\phi_3} t_+^x t_-^y \end{matrix} \qquad \mathbf{N} = \begin{matrix} C\mathrm{e}^{\mathrm{i}\phi_2} & S\mathrm{e}^{\mathrm{i}\phi_2} t_+^x t_+^y \\ -S\mathrm{e}^{\mathrm{i}\phi_4} t_-^x t_-^y & C\mathrm{e}^{\mathrm{i}\phi_4} \end{matrix} \tag{4}$$

and t_+^x (t_-^y), for example, are lattice shift operators in the positive (negative) x- (y-) direction. The phases $\phi_1, \ldots, \phi_4$ are associated with the centres of the loop edges. The weak random potential δV induces weakly fluctuating phases with $\langle\phi_j\rangle = 0$ for $j = 1, 2, 3$ and $\langle\phi_4\rangle = \pi$. This choice ensures the presence of half a flux quantum per current loop on average. (Notice that the original network model [3] has strong disorder due to a uniform distribution of the phases ϕ_j on the interval $[0, 2\pi)$. It is not clear whether our model with weak disorder belongs to the same universality class. Since strong fluctuations of the phase correspond to strong fluctuations of the external magnetic field, the fact that a random magnetic field is a relevant perturbation [15] indicates that the two models may be qualitatively different.) The random potential δV also affects the tunnelling between the loops. This induces randomness in the lattice shift operators $t_\pm^{x,y}$. The latter are statistically independent for different nearest-neighbour pairs on the lattice. We believe that $\beta > \pi/4$ is not realistic for the physical model because this describes a situation where the quasiparticles 'prefer' to tunnel through the saddle points rather than go along the equipotential lines.

The (discrete) time evolution of the quasiparticle state on a loop can also be described by the evolution operator $\mathbf{W}$ as $\Psi(t + 2) = \mathbf{W}\Psi(t)$ [7], with

$$\mathbf{W} = \begin{matrix} 0 & \mathbf{M} \\ \mathbf{N} & 0 \end{matrix}^2 = \begin{matrix} \mathbf{MN} & 0 \\ 0 & \mathbf{NM} \end{matrix} . \tag{5}$$

Assuming (discrete) translational invariance (i.e. $\delta V = 0$ in the original model), we can diagonalize the scattering matrix by applying a Fourier transformation $(x, y) \to (k_1, k_2)$. Thus we find the eigenvalues of $\mathbf{W}$,

$$\lambda_{1/2}(k_1, k_2) = 2SC\cos k_1 \cos k_2 \pm \mathrm{i}\sqrt{1 - 4S^2C^2\cos^2 k_1 \cos^2 k_2}. \tag{6}$$

The corresponding dispersion relation, $E_{1/2}(k_1, k_2)$, obtained via $\lambda_{1/2} = \exp(-\mathrm{i}E_{1/2})$, has a gap around $E = 0$ depending on β (see also [7]). The gap vanishes at $k_1 = k_2 = 0, \pm\pi$ for $\beta = \pi/4$, i.e. for $C = S = 1/\sqrt{2}$. Thus the quantum model reflects the behaviour of the classical model, where infinite equipotential lines exist only at zero energy. The effect of disorder on this behaviour will be studied subsequently.

We briefly discuss an alternative lattice approximation of the almost periodic potential. It is based on the tight-binding representation of the continuous model. For this purpose we regard the minima, $V(x, y) = -V_0$, and the maxima, $V(x, y) = V_0$, of the periodic potential as a lattice—see figure 3. Quasiparticles can hop between neighbouring lattice points with hopping rate t, i.e. for hopping between a maximum and a minimum. Moreover, it is

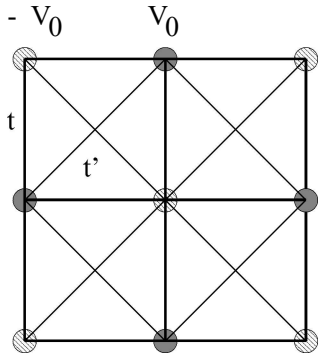

Figure 3. The tight-binding approximation of the periodic potential, where t is the nearest-neighbour and t' the next-nearest-neighbour hopping rate.

reasonable to take the hopping between next-nearest neighbours—which are either pairs of maxima or pairs of minima—into account also and associate a hopping rate t' with this process. The magnetic field leads again to a Peierls phase factor in each of the hopping terms [14]. Disorder can be introduced by adding random fluctuations δV to the lattice points. This lattice approximation differs from the above network model because there are two types of hopping now. The nearest-neighbour hopping between a maximum and a minimum has been excluded in the network approximation, since tunnelling is considered only between states at the same energy.

2.2. Large-scale approximation: Dirac fermions

We are interested in the properties of the quasiparticles on large scales. Therefore, we consider the large-scale approximation of the evolution operator $\mathbf{W}$ by expanding the non-local part in a Taylor expansion $t_{\pm}^{x} f(\boldsymbol{r}) = t_{\pm}^{x}(\boldsymbol{r}) f(\boldsymbol{r} \pm a\boldsymbol{e}_x) \approx t_{\pm}^{x}(\boldsymbol{r})[f(\boldsymbol{r}) \pm a\nabla_x f(\boldsymbol{r})]$; for simplicity we set $a = 1$ in the following. The evolution operator then reads, for $\beta = \pi/4 + \delta\beta$, with $|\delta\beta| \ll 1$,

$$\mathbf{MN} \approx \mathbf{1} + \begin{matrix} \mathrm{i}\bar{\phi} - \nabla_x + \mathrm{i}A_x & 2\,\delta\beta + \nabla_y - \mathrm{i}A_y \\ -2\,\delta\beta + \nabla_y - \mathrm{i}A_y & \mathrm{i}\bar{\phi} + \nabla_x - \mathrm{i}A_x \end{matrix} + \text{random terms from } t_{\pm}^{x,y} \tag{7}$$

and

$$\mathbf{NM} \approx \mathbf{1} + \begin{matrix} \mathrm{i}\bar{\phi} + \nabla_y - \mathrm{i}A_y & -2\,\delta\beta - \nabla_x + \mathrm{i}A_x \\ 2\,\delta\beta - \nabla_x + \mathrm{i}A_x & \mathrm{i}\bar{\phi} - \nabla_y + \mathrm{i}A_y \end{matrix} + \text{random terms from } t_{\pm}^{x,y} \tag{8}$$

with the effective random vector potential components $A_x = (\phi_1 - \phi_3)/2$, $A_y = (\phi_4 - \pi - \phi_2)/2$, and with $\bar{\phi} = (\phi_1 + \cdots + \phi_4 - \pi)/2$. We can express the results (7), (8) using two-dimensional Dirac Hamiltonians H_{MN} and H_{NM}, defined through $\mathbf{MN} \approx \mathbf{1} - \mathrm{i}H_{MN}$ and $\mathbf{NM} \approx \mathbf{1} - \mathrm{i}H_{NM}$, where

$$\begin{aligned} H_{MN} &= -2\,\delta\beta\,\sigma_2 - (\mathrm{i}\nabla_x + A_x)\sigma_3 + (\mathrm{i}\nabla_y + A_y)\sigma_1 + \tilde{H}_{MN} \\ H_{NM} &= 2\,\delta\beta\,\sigma_2 + (\mathrm{i}\nabla_y + A_y)\sigma_3 - (\mathrm{i}\nabla_x + A_x)\sigma_1 + \tilde{H}_{NM} \end{aligned} \tag{9}$$

with Pauli matrices σ_j. The terms $\tilde{H}_{MN}$, $\tilde{H}_{NM}$ are contributions from the random operators $t_{\pm}^{x,y}$. Here $m = 2\,\delta\beta$ appears as a Dirac mass. Since $\delta\beta \leqslant 0$, the Dirac mass cannot be positive. Moreover, it is convenient to rotate the 2×2 matrices H_{MN} and H_{NM} such that

the mass term is in the diagonal. This leads to

$$\begin{aligned} H_{MN} &= m\sigma_3 - (\mathrm{i}\nabla_x + A_x)\sigma_1 - (\mathrm{i}\nabla_y + A_y)\sigma_2 + \tilde{H}_{MN} \\ H_{NM} &= m\sigma_3 + (\mathrm{i}\nabla_x + A_x)\sigma_1 + (\mathrm{i}\nabla_y + A_y)\sigma_2 + \tilde{H}_{NM}. \end{aligned} \tag{10}$$

The random terms, related to the random phases ϕ_j and the random factors $t_{\pm}^{x,y}$, are expanded in terms of Pauli matrices as $\sum_{l=0}^{3} \sigma_l V_l$, where $l = 1, 2$ are contributions to the random vector potential, $l = 0$ is the contribution to the random energy, and $l = 3$ the contribution to the random mass. (σ_0 is the 2×2 unit matrix.) The corresponding expansion for H_{NM} is given by $\sum_{l=0}^{3} \sigma_l V_l'$. It is assumed that V_l and V_l' have a Gaussian distribution with zero mean and correlations

$$\langle V_{l,r} V_{l',r'} \rangle_V = \langle V'_{l,r} V'_{l',r'} \rangle_{V'} = \delta_{rr'} \delta_{ll'} g_l \tag{11}$$

where r, r' denote the lattice points. In principle, V_l and V_l' are correlated. However, these correlations do not play a role here because the Green's function is block diagonal with respect to V_l and V_l'.

We emphasize that a non-compact continuous symmetry exists in the case where $g_0 = g_3 = 0$ (a pure random vector potential) and $m = 0$; that is,

$$H_{MN} = [(1 + \zeta^2)^{1/2}\sigma_0 + \zeta\sigma_3] H_{MN} [(1 + \zeta^2)^{1/2}\sigma_0 + \zeta\sigma_3] \tag{12}$$

and similarly for H_{NM}, with $-\infty < \zeta < \infty$. This symmetry indicates that the pure random vector potential is qualitatively different from the general case with $g_0 + g_3 > 0$, e.g. the symmetry is reduced to a discrete one with $\zeta = \mathrm{i}$ for averages of physical quantities if $m = E = 0$.

The time evolution of the quasiparticles in the network is given by iterating the time step operator $\mathbf{W}$. This requires the evaluation of $\langle \mathbf{W}^n \rangle_V$ for large values of n, since we are interested in the long-time behaviour. We Fourier transform $\mathbf{W}^n$ first, and later average with respect to the random potential V. In a first step,

$$\sum_{n \geqslant 0} \mathbf{W}^n \mathrm{e}^{\mathrm{i}n\omega} = (\mathbf{1} - \mathbf{W}\mathrm{e}^{\mathrm{i}\omega})^{-1} \tag{13}$$

where the frequency ω has an infinitesimal imaginary part $\epsilon > 0$: $\omega = E + \mathrm{i}\epsilon$. Now we return to the Dirac Hamiltonian to determine the large-scale behaviour. For small frequencies we obtain the Green's function of Dirac fermions:

$$(\mathbf{1} - \mathbf{W}\mathrm{e}^{\mathrm{i}\omega})^{-1} \approx \begin{matrix} \mathrm{i}H_{MN} - \mathrm{i}\omega & 0 \\ 0 & \mathrm{i}H_{NM} - \mathrm{i}\omega \end{matrix}^{-1} \equiv \mathrm{i}G(\omega). \tag{14}$$

The evaluation of $\langle G(\omega) \rangle_V$ in a self-consistent approach is described below (section 3).

2.3. *The density of states and Hall conductivity*

According to standard Green's function theory, the average DOS at a given energy E is obtained from the Green's function G as

$$\rho(m, E) = \frac{1}{\pi} \lim_{\epsilon \to 0} \langle \mathrm{Im}[G_{11,rr}(E + \mathrm{i}\epsilon) + G_{22,rr}(E + \mathrm{i}\epsilon)] \rangle_V. \tag{15}$$

The DOS of a pure system ($g_l = 0$) is linear around $E = 0$ [9] and vanishes at $E = 0$ for all values of m.

The current density in a Dirac model can be calculated directly from the Kubo formula [9] or from the response to an external static vector potential q_y [16]. The effect of q_y is a change of the boundary conditions in the y-direction, a concept extensively used in

numerical investigations of Anderson localization to study the existence of delocalized states [17]. The response to the vector potential gives the Hall conductivity σ_{xy} in terms of Green's functions [9, 16]:

$$\sigma_{xy} = \frac{e^2}{h}\frac{\mathrm{i}}{q_y}\int \sum_{r'} \left(\mathrm{Tr}_2\ \sigma_x (H_{MN} - \mathrm{i}z - E)^{-1}_{r,r'} (H_{MN} - \mathrm{i}z - E - q_y\sigma_y)^{-1}_{r',r}\right\}$$
$$+ \mathrm{Tr}_2\ \sigma_x (H_{NM} - \mathrm{i}z - E)^{-1}_{r,r'} (H_{NM} - \mathrm{i}z - E - q_y\sigma_y)^{-1}_{r',r}\}\Big)\ \frac{\mathrm{d}z}{2\pi}. \tag{16}$$

This gives, in the absence of disorder and with periodic boundary conditions, and for $q_y \to 0$,

$$\sigma_{xy} = \frac{e^2}{h}\mathrm{sgn}(-m)\Theta(|m| - |E|) \tag{17}$$

where Θ is the step function. Thus the Hall conductivity has the plateau value e^2/h at $E = 0$, since $m \leqslant 0$ for $\beta \leqslant \pi/4$, while it is undefined at $m = 0$. This behaviour of the transport quantity σ_{xy} is unphysical, and will be replaced by a continuous behaviour when we take disorder into account: then the DOS is found to vanish at $E = 0$ for all m, which indicates that the Hall current flows on the boundary. Therefore, the result (17) depends strongly on the choice of the boundary conditions [5, 7]; conditions other than periodic can suppress σ_{xy}. For the tight-binding model of section 2.1, σ_{xy} was determined for $E = 0$ [14] to be

$$\sigma_{xy} = \frac{e^2}{h}\Theta(t'^2 - V_0^2). \tag{18}$$

Comparing this with the result (17), we see that the network approximation is similar to the tight-binding approximation provided that $t'^2 > V_0^2$.

For the special case of a random vector potential we can evaluate the dissipative conductivity and the localization length in a simple way. Using the identity (which holds for $m = 0$)

$$G(-\omega) = -\sigma_3 G(\omega)\sigma_3 \tag{19}$$

we may express the two-particle Green's function $\langle G(\omega)G(\omega^*)\rangle$ as a product of one-particle Green's functions on the *same* complex half-plane. As a consequence of this property, the critical properties of the two-particle Green's function are identical to those of the one-particle Green's function. This explains the finding that the critical exponent of the localization length is identical to the decay exponent of $\langle G(\omega)\rangle$ in an explicit calculation [9].

3. Self-consistent approximation

The full Green's function $\langle G\rangle_V$ can be related to the Green's function G_0 in the absence of randomness

$$G_0(\omega) = \begin{matrix} \omega - \langle H_{MN}\rangle_V & 0 \\ 0 & \omega - \langle H_{NM}\rangle_V \end{matrix}^{-1} \tag{20}$$

through Dyson's equation:

$$\langle G\rangle_V = G_0 + G_0\Sigma\langle G\rangle_V. \tag{21}$$

The self-energy Σ, in lowest-order perturbation theory, is given by $\Sigma \approx \langle V G_0 V\rangle_V$. Expanding Σ in terms of Pauli matrices, $\Sigma = \Sigma_0 + \Sigma_1\sigma_1 + \Sigma_2\sigma_2 + \Sigma_3\sigma_3$ and, taking into account that $\langle G\rangle_V^{-1} = G_0^{-1} - \Sigma$, we see that Σ_1 and Σ_2 shift the gradient operators

in $\langle H_{MN}\rangle_V$ and $\langle H_{NM}\rangle_V$. Since the parity is conserved for G_0 as well as for $\langle G\rangle_V$, these contributions must vanish. The remaining contributions, $\Sigma_0 \equiv -\mathrm{i}\eta$ and $\Sigma_3 \equiv -m_s$, can be evaluated in a self-consistent approximation, where we use the second order in perturbation theory, and replace G_0 in the expression for Σ by $(G_0^{-1} - \Sigma)^{-1}$, with the result

$$\Sigma = \langle V(G_0^{-1} - \Sigma)^{-1} V\rangle_V. \tag{22}$$

With $g_{ij} = g_i + g_j$, this self-consistent equation leads to two coupled equations:

$$m_s = m\frac{(g_{12} - g_{03})I}{1 - (g_{12} - g_{03})I} \tag{23}$$

and

$$\eta = (\eta - \mathrm{i}\omega)gI \tag{24}$$

with the integral

$$I = 2\int [(m + m_s)^2 + (\eta - \mathrm{i}\omega)^2 + k^2]^{-1}\ \frac{\mathrm{d}^2k}{(2\pi)^2} = \frac{1}{2\pi}\ln\ \ 1 + \frac{\lambda^2}{(m + m_s)^2 + (\eta - \mathrm{i}\omega)^2}\ \ . \tag{25}$$

We introduced λ as an ultra-violet cut-off, and $g = g_0 + \cdots + g_3$.

For $\omega = 0$, there are two solutions of (24): one with $\eta = 0$, and a second one with $\eta \neq 0$ and $I = g^{-1}$. The latter implies with (23) a renormalization of the Dirac mass,

$$\bar{m} \equiv m + m_s = m\frac{g}{2g_{03}} \tag{26}$$

and with (25), we find

$$\eta^2 = \mathrm{e}^{-2\pi/g} - m^2(g/2g_{03})^2. \tag{27}$$

The multiplicative renormalization of the Dirac mass in equation (26) diverges as $g_{03} \to 0$. This indicates a special behaviour for a pure random vector potential. Although this case is not interesting in terms of the network model, it has been discussed extensively in the literature [9, 11, 18].

For $m = 0$ ($\omega \neq 0$) we obtain with equations (24) and (25) an equation which determines η:

$$\mathrm{e}^{2\pi\eta/g(\eta - \mathrm{i}\omega)} = 1 + \frac{\lambda^2}{(\eta - \mathrm{i}\omega)^2}. \tag{28}$$

Given η, we can evaluate the DOS as

$$\rho(m = 0, E) = \frac{1}{g}\mathrm{Re}(\eta) \tag{29}$$

which for $E = 0$ vanishes if $m^2 \geqslant m_c^2$, where

$$m_c = \pm\frac{2g_{03}}{g}\mathrm{e}^{-\pi/g} \tag{30}$$

where we set $\lambda = 1$ for simplicity. In this case the DOS as a function of m follows a semi-circle (equation (27)), multiplied by $g/2g_{03}$, with the maximum at $m = 0$ given by $\rho(m = 0, E = 0) = \mathrm{e}^{-\pi/g}/g$. The radius of the semi-circle is $2(g_{03}/g)\mathrm{e}^{-\pi/g}$, and thus vanishes for $g_{03} \to 0$. This means that the DOS vanishes for a pure random vector potential ($g_{03} = 0$) for any $m \neq 0$, but jumps to $\rho(m = 0) = \mathrm{e}^{-\pi/g_{12}}/g_{12}$ at $m = 0$. This again indicates a singular behaviour (instability) for the pure random vector potential, at least in terms of our self-consistent approximation [19]. Such behaviour was also found in

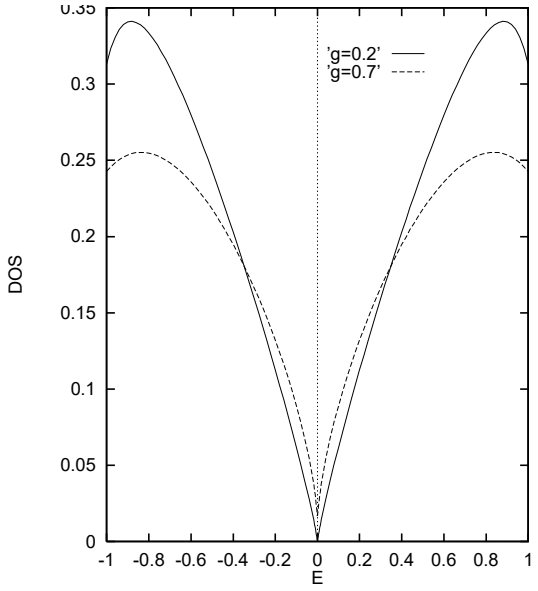

Figure 4. The density of states for weak ($g = 0.2$) and intermediate ($g = 0.7$) disorder with the scattering parameter $\beta = \pi/4$. The minimum at $E = 0$ is $e^{-\pi/g}/g$ (see the text).

perturbation theory around the (integrable) case (a random vector potential only), where infinitely many relevant operators appear due to the contribution of the random Dirac mass [11].

The density of states $\rho(m = 0, E)$ behaves smoothly for $E \neq 0$, as shown in figure 4. It shows an effective power law, $e^{-\pi/g}/g + |E|^{\alpha}$ with a cut-off at large E (depending on λ), except for very small and very large energies. As can be seen in figure 4, the exponent decreases, starting at $\alpha = 1$ for $g = 0$, with increasing g.

The Hall conductivity can also be calculated within the self-consistent approximation. For this purpose we replace the Green's function in equation (16) by $(G_0^{-1} - \Sigma)^{-1}$. This gives an expression identical to that found for the random Dirac mass [10], except that m_c is replaced by the new critical mass (30):

$$\sigma_{xy} \approx \frac{e^2}{h}\mathrm{sgn}(-m)\ \ 1 - (2/\pi)\arctan\ \ \sqrt{m_c^2/m^2 - 1}\ \ \Theta(m_c^2 - m^2)\ . \qquad (31)$$

This describes a Hall plateau with a continuous decrease to zero near the critical scattering parameter $\beta_c = \pi/4$, which implies that disorder changes the (unphysical) discontinuity of the pure system (17) to a more realistic behaviour. However, for $g_{03} = 0$, $g > 0$, the critical mass m_c vanishes. Thus σ_{xy} is undefined for $m = m_c$, like for the pure case. Consequently, the random vector potential alone is not sufficient to create a continuous behaviour of σ_{xy}. However, since there exist exact calculations for the random vector potential [9] and numerical results [18], we wish to compare our self-consistent approach with those. In order to avoid problems with singularities at $m = 0$ we assume $m > 0$ and set $m \to 0$ only at the end. In particular, we have to choose that solution of equation (24) which vanishes with vanishing frequency ω. Going back to the self-consistent equation (23), we can express I of equation (25) in terms of the renormalized mass $\bar{m}$ as

$$I = (1 - 1/x)/g_{12}$$

where $x = \bar{m}/m$. Consequently, η—compare with equation (24)—is given by

$$\eta = \mathrm{i}\omega(1 - x).$$

From equations (23), (25), we obtain

$$x = \left(1 - \frac{g_{12}}{2\pi} \ln\left(1 + \frac{1}{x^2(m^2 - \omega^2)}\right)\right)^{-1}. \tag{32}$$

This implies, with $\omega^2 \lesssim m^2$,

$$x \sim \left(1 + \frac{g_{12}}{2\pi} \ln[x^2(m^2 - \omega^2)]\right)^{-1}. \tag{33}$$

Assuming weak disorder ($g_{12} \ll 1$) we exponentiate the right-hand side to get

$$x \sim [x^2(m^2 - \omega^2)]^{-g_{12}/2\pi} \tag{34}$$

and

$$x \sim (m^2 - \omega^2)^{-g_{12}/[2\pi(1+g_{12}/\pi)]}. \tag{35}$$

The decay length ξ of the average Green's function reads in terms of the self-consistent approximation (in units of the lattice constant)

$$\xi = [\bar{m}^2 + (\eta - \mathrm{i}\omega)^2]^{-1/2}. \tag{36}$$

Thus, from (35) and inserting $\bar{m}$, $\eta = \mathrm{i}\omega(1 - x)$, we find

$$\xi \sim (m^2 - \omega^2)^{-1/2(1+g_{12}/\pi)}. \tag{37}$$

This agrees with the earlier results [9, 13], obtained from a renormalization group argument and a bosonization approach, respectively. In the latter it is not restricted to weak disorder, but holds true for all values of g_{12}.

4. Discussion and conclusions

Starting from a two-dimensional system of non-interacting quasiparticles in an almost periodic potential with a magnetic field, we have derived an effective model for large-scale properties. The latter is a model of 2D Dirac fermions with random mass, random energy and random vector potential. Our derivation is analogous to that of Ho and Chalker for a random potential [7]. This model shows a gap due to the periodic potential, which disappears if the rates for clockwise and anticlockwise scattering of quasiparticles are equal. The gap is proportional to the Dirac mass. The average Green's function of the random Dirac model has been evaluated in a self-consistent approximation. The latter is equivalent to the saddle-point (or large-N) approximation of the model [20]. It gives a self-energy which consists of a multiplicative renormalization of the average Dirac mass, $m \to mg/2g_{03}$, and a spontaneous creation of a complex self-energy.

In terms of the self-consistent approximation, only the combinations g_{03} and g_{12} enter the results. This supports the conjecture [4] that the randomness of the scattering rates at the saddle points is irrelevant (i.e. there is no qualitative difference between $g_3 > 0$ and $g_3 = 0$) as long as $g_0 > 0$. In other words, the randomness of the phases is sufficient for the qualitative description of the QHT, at least for the DOS and Hall conductivity.

The mass renormalization is only a factor $1/2$ in the absence of a random vector potential ($g_{12} = 0$), but it grows with g_{12}, and η is a function of the renormalized mass and the (real) energy E. The average DOS is proportional to the real part of η. It shows a semi-circular behaviour with width m_c, given in equation (30). This width vanishes if only a random

vector potential is present, indicating a qualitatively different situation. As a function of E, it describes an effective power law, unless E is very small. The exponent of the power law α decreases with disorder strength, starting at $\alpha = 1$ for the pure system. In the case of very weak disorder ($g \sim 0$), we take the gth power of equation (28), and expand the result in powers of g, with the result

$$\eta = \frac{Eg}{2\pi \mathrm{i}} \ln[1 - \lambda^2/(E + \mathrm{i}\epsilon)^2] + \mathrm{O}(g^2). \tag{38}$$

This yields with equation (29) the well-known linear behaviour of the DOS for $E^2 < \lambda^2$:

$$\rho(m = 0, E) = \frac{|E|}{2}\Theta(\lambda^2 - E^2) + \mathrm{O}(g). \tag{39}$$

The overall behaviour of the DOS is in very good agreement with a numerical result for a similar Dirac model [18]. Furthermore, a non-zero DOS near the QHT, where σ_{xy} deviates from the Hall plateau value e^2/h, as found by us, also agrees with numerical observations [21, 22]. The Hall conductivity σ_{xy} has a plateau in the regime where scattering is dominated by states localized on the loops. However, in the absence of disorder, σ_{xy} is not defined if scattering is equally probable along the loops and through the saddle points ($\beta = \pi/4$). This problem disappears in the presence of disorder, since σ_{xy} then changes continuously from the plateau value $\sigma_{xy} = e^2/h$ to $\sigma_{xy} = 0$ at $\beta = \pi/4$.

We conclude that the special properties of the pure system are cured by the randomness: the vanishing average DOS at $E = 0$ is elevated to a non-zero value. Although this value is exponentially small, it indicates that a band of states exist where the quasiparticle can tunnel with high probability through the saddle points of the periodic potential (i.e. near $\beta = \pi/4$). The tunnelling, which can be understood as quantum percolation, obviously destroys the plateau of the Hall conductivity. At least for the case of a random Dirac mass, it is known that part of this band consists of delocalized states [12], leading to a non-zero conductivity $\sigma_{xx} \approx (e^2/h\pi)/(1 + g/2\pi)$ [23]. Numerical calculations [3–5] indicate that delocalized states also exist in the network model. Moreover, for the random vector potential we found a divergent localization length when $m^2 \to \omega^2$ within our self-consistent approach. All these results are strong hints that there are delocalized states for the fully random Dirac Hamiltonian, in the regime of strong tunnelling through the saddle points of the nearly periodic potential.

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