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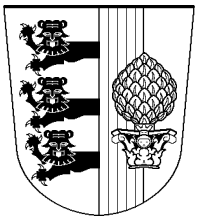

Ideal Stream Algebra

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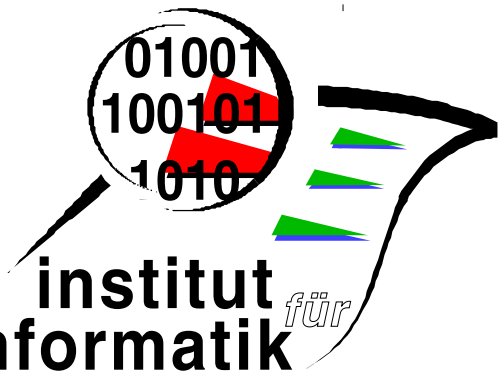

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Ideal Stream Algebra

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Abstract. We provide some mathematical properties of *behaviours* of systems, where the individual elements of a behaviour are modeled by *ideals* of a suitable partial order. It is well-known that the associated ideal completion provides a simple way of constructing algebraic cpos. An ideal can be viewed as a set of consistent finite or compact approximations of an object which itself may even be infinite. A special case is the domain of streams where the finite approximations are the finite prefixes of a stream.

We introduce a special way of characterising behaviours through sets of relevant approximations. This is a generalisation of the technique used earlier for the case of streams. Given a set $P \subseteq M$ of a partial order $(M, \leq)$, we define

$$\mathsf{ide}\, P := \{Q^{\leq} : Q \subseteq P \text{ directed}\}\ ,$$

where $Q^{\leq} := \{x \in M : \exists\, y \in Q : x \leq y\}$ is the downward closure of Q. So $\mathsf{ide}\, P$ is the set of all ideals "spanned" by directed subsets of P. We prove a number of distributivity and monotonicity laws for ide and related operators. They are the basis for correct refinement of specifications into implementations. Various small examples illustrate that the operators lead to very concise while quite clear specifications.

Finally, we give a characterization of safety and liveness and generalize the Alpern/Schneider decomposition lemma to arbitrary domains.

An extended example concerns the specification and transformational development of an asynchronous bounded queue.

Part I: Introduction

1 Origin and Goals

The context of this work is deductive program design, in which implementations are derived from specifications by semantics-preserving deduction rules. Examples of this paradigm are transformational program development (see e.g. [51, 7]) and the refinement calculus (see e.g. [15, 22, 4, 2, 44, 45]). There is a growing conviction that this paradigm is most efficient when based on algebraic rather than purely logical frameworks. For sequential programs this is demonstrated in [7]. The aim there

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is to make program specification and calculation more concise and perspicuous by compacting logic into algebra as much as possible.

In the parallel case, to some extent the work reported in [48, 13] can be viewed as falling into the algebraic realm; purely algebraic approaches are presented in [30, 54]. The present paper presents a particular approach to streams (see e.g. [29, 47, 11] and [60] for a recent survey). It centers around the order-theoretic view of streams and other semantic objects as used in denotational semantics. In addition to order theory we use a suitable algebra of formal languages [37] in reasoning about streams.

2 Streams and Ideals

The basic tool in our approach is the prefix order on words which are considered as system traces. A trace language is directed w.r.t. this order iff it is totally ordered by it. Therefore ideals, i.e., prefix closed directed sets of traces, are a suitable representation of finite and infinite streams.

It is well known that the space of streams under the prefix ordering is isomorphic to the ideal completion of the set of finite streams. Since, however, ideals are just particular trace languages, we can use all operations on formal languages for their manipulation. A large extent of this is covered by conventional regular algebra. Moreover, we can apply the tools developed for quite different purposes in a number of papers on algebraic calculation of graph, pointer and sorting algorithms (see [37, 39] and the references there). Finally, we do not need additional mechanisms for dealing with fairness; rather, fairness is made explicit within the generating expressions for trace languages.

Using regular expressions rather than automata or transition systems gives considerable gain in conciseness and clarity, both in specification and calculation. While this has long been known in the field of syntax analysis, most approaches to the specification of concurrency stay with the fairly detailed level of automata, thus leading to cumbersome and imperspicous expressions. Other approaches use logical formulas for describing sets of traces; these, too, can become very involved. By extracting a few important concepts and coming up with closed expressions for them one can express things in a more structured and concise form. This is done here using regular and regular-like expressions with their strong algebraic properties. The approach can also be nicely tied in with temporal and modal operators (see [43]).

Another advantage of our approach is that we can do with simple set-theoretic notions thus avoiding most of the overhead of domain theory. By this, the approach also is completely orthogonal w.r.t. nesting of data structures, i.e., it admits streams of functions, streams of sets, sets of streams, streams of streams etc. without problems.

3 A Simple Soda Machine

As show the style of our approach and in order to better motivate the technicalities to come we first give a number of examples with informal explanations. The precise definitions will be given in later sections.

We start with the description of a simple soda machine. It accepts half dollars and quarters and emits a can of soda after having received a half dollar's worth in coins. Let h and q denote the events of receiving a half dollar and a quarter, respectively, and c the event of emitting a can of soda. Then the behaviour of that machine is described by the regular-like expression

$$((h \cup q \bullet q) \bullet c)^{\omega} ,$$

where $\bullet$ is concatenation and $\_^{\omega}$ denotes infinite repetition. Each expression of this kind denotes a set of (finite or infinite) streams; in the case of the soda machine all these streams are infinite.

In the above expression, the iterated subexpression $(h \cup q \bullet q) \bullet c$ states the following safety properties: the customer must insert the correct amount of money and is not allowed to insert further money before delivery of the can. The infinite repetition $\_^{\omega}$ combines safety and liveness aspects: it expresses the correct order of insert/deliver cycles, a safety property, and expresses the temporal aspect of eventuality (see e.g. [21]): it guarantees that after insertion of a sufficient amount of money eventually a can is delivered and the machine is ready to accept further orders.

We prefer to leave states implicit as long as possible, since frequently regular expressions are clearer and more concise than the corresponding descriptions by accepting automata (Büchi automata in the case of infinite repetition, see e.g. [50, 61, 62]).

4 Fairness

Other eventuality properties can already be expressed by Kleene's finite repetition operators $\_^{*}$ and $\_^{+}$. To exemplify this, we describe a scheduler for unboundedly fair merging of input from two channels. It is modelled as an infinite stream over the alphabet $\{0, 1\}$, where 0 denotes choice from the left and 1 choice from the right input channel of the merge module. The fact that at least once there is a choice from the left followed eventually by a choice from the right is expressed by the regular expression $0^{+} \bullet 1$. By adding the symmetric requirement and, again infinite repetition to drive the single cycles, we get the following description of the set of streams that model the behaviour of a fair scheduler:

$$\mathcal{SCHED} \stackrel{\text{def}}{=} (0^{+} \bullet 1 \cup 1^{+} \bullet 0)^{\omega} .$$

The "local eventuality" is here expressed by the finiteness of $\_^{+}$, whereas the infinite repetition $\_^{\omega}$ again adds liveness and "global eventuality".

Arbitrary (and hence possibly non-fair) merge would be obtained by replacing this scheduler by $(0 \cup 1)^{\omega}$.

The reason why fairness does not cause problems in our approach is that fairness constraints are expressed using the star operation which has a simple recursive definition using least fixpoints w.r.t. the inclusion ordering on sets of streams, whereas there are continuity problems w.r.t. extensions of the prefix order to sets of streams. This is due to the fact that the prefix order has operational traits and unbounded fairness is operationally not feasible, whereas the inclusion ordering is purely descriptive and hence does not face this problem. It is adequate for proving properties

of sets of streams; when it comes to implementation, of course operationally feasible descendants have to be used.

We prefer to state fairness assumptions explicitly, since this gives much greater flexibility than building them into the underlying semantic framework (such as e.g. in [16]).

5 Channels

Another aspect of fairness and eventuality is exhibited in the description of channels as used in many protocol specifications. The channels are faulty, but fair in the sense that after an unbounded but finite number of faulty transmissions they will at least once transmit correctly.

We will describe their behaviour using streams of functions that model individual transmissions. Let *id* be the identity function which models correct transmission, *fail* the function which transforms any message into an "error element", and *skip* the function transforming a message into empty output. Let in the sequel stand the sub/superscript i for $*$ (unbounded but finite repetition) or $\leq k$ for some $k \in \mathbb{N}$ (bounded repetition). Then the specifications express unbounded or bounded fairness, respectively.

A possibly corrupting but fair channel is described by

$$cchan_i \stackrel{\text{def}}{=} (fail^i \bullet id)^\omega \ ,$$

a possibly lossy but fair channel by

$$lchan_i \stackrel{\text{def}}{=} (skip^i \bullet id)^\omega$$

and a possibly lossy and corrupting but fair channel by

$$lcchan_i \stackrel{\text{def}}{=} ((skip \cup fail)^i \bullet id)^\omega \ .$$

An unfair corrupting channel is

$$arbchan \stackrel{\text{def}}{=} (fail \cup id)^\omega \ .$$

This kind of channel descriptions has been used in [40] for a very concise algebraic correctness proof of the alternating bit protocol.

6 Two Stream-Based Models of Systems

6.1 Modules as Stream-Processing Functions

A stream-processing function (SPF) is a function from tuples of input streams to tuples of output streams (see e.g. [29, 11]). In the case of synchronous systems this may equivalently replaced by a function from a stream of input tuples to a stream of output tuples.

In the SPF view each module is described as an SPF. The advantage of this model is that it allows easy definitions of various composition operations for modules and hence lends itself to a modular structuring of large systems.

The disadvantage in the description of asynchronous systems is that the separation between input and output streams loses causal information, viz. which input triggered which output. This gives rise to the (in)famous merge anomaly [8] which can be fixed by re-introducing time information into the streams. It has to be expressed which elements of a stream are considered to belong to the same time interval. This can be done using explicit time ticks [10] or streams of sequences where each sequence lists the elements belonging to one time interval.

6.2 Trace Models

In the trace view, the overall system is described by the structure of its traces, i.e., possible sequentializations of all system actions (interleaving semantics). In this view, a stream in A^∞ is a complete record of a system run with all channel activities interleaved. This is the view of CCS [34], CSP [25] and process algebra [3]. In the simplest case the trace structure is a *set* of traces (see also [28]).

This view fits directly our notion of behaviour. For a CSP-like view we use the alphabet $A = C \times V$ where C is a set of channel names and V a set of values that are transmitted along the channels. The streams in A^∞ are complete records of system runs with all channel activities interleaved.

The advantage of this view is that it keeps track of the causality between input and output; hence the merge anomaly does not arise.

The disadvantage is the lack of immediate modularity, since the overall system is described. However, modularisation can be re-introduced by restricting attention to subsets of channels.

Part II: Mathematical Background

7 Order-Theoretic Preliminaries

In this section we repeat some basic notions from the theory of partial orders and state some new algebraic properties. The proofs for this section can be found in [40].

For partially ordered set $(M, \leq)$ and $N \subseteq M$ we define the *proper* and *improper downward closure* by

$$N^< \stackrel{\text{def}}{=} \{y \in M : \exists\, x \in N : y < x\}$$
$$N^\leq \stackrel{\text{def}}{=} \{y \in M : \exists\, x \in N : y \leq x\} = N \cup N^<$$

where $y < x \Leftrightarrow y \leq x \wedge x \neq y$. We list some useful properties of these operations:

Lemma 7.1 Consider $N, P \subseteq M$. Then

1. $(N \cup P)^< = N^< \cup P^< \wedge (N \cup P)^\leq = N^\leq \cup P^\leq$ (distributivity).
2. $(N^\leq)^< = N^< \wedge (N^\leq)^\leq = N^\leq$.

The set of *maximal* elements of $N \subseteq M$ is defined by

$$\mathsf{max}\, N \stackrel{\text{def}}{=} N \backslash N^{<} \ .$$

Again, we give some useful properties:

Lemma 7.2 Consider $N, P \subseteq M$. Then

1. $\mathsf{max}\, N = N^{\leq} \backslash N^{<}$.
2. $\mathsf{max}\, N = \mathsf{max}\, N^{\leq}$.
3. $N \subseteq P \Rightarrow N \cap \mathsf{max}\, P \subseteq \mathsf{max}\, N$.
4. $\mathsf{max}\, N \cap P^{<} = \emptyset \Rightarrow \mathsf{max}\,(N \cup P) = \mathsf{max}\, N \cup (\mathsf{max}\, P) \backslash N^{<}$.

We now extend the order $\leq$ to a relation on subsets of M by

$$N \leq P \stackrel{\text{def}}{\Leftrightarrow} N \subseteq P^{\leq} \ .$$

This is the angelic half of the Egli-Milner preorder [52]. In particular, $N^{\leq} \leq N$. Some further useful properties are

Lemma 7.3 Consider $N, P \subseteq M$. Then

1. $N \leq P \Leftrightarrow N \leq P^{\leq}$.
2. $L \subseteq N \wedge N \leq P \wedge P \subseteq Q \Rightarrow L \leq Q$.
3. $N \leq P \Leftrightarrow N^{\leq} \subseteq P^{\leq}$.
4. $N \leq P \Rightarrow \mathsf{max}\,(N \cup P) = \mathsf{max}\, P$.

Since $\leq$ generally is only a preorder between sets, we are interested in the induced equivalence relation

$$N \sim P \stackrel{\text{def}}{\Leftrightarrow} N \leq P \wedge P \leq N \ .$$

For this we have

Lemma 7.4 Consider $N, P \subseteq M$. Then $N \sim P \Leftrightarrow N^{\leq} = P^{\leq}$.

Proof: Immediate from Lemma 7.3.3. ■

A subset $N \subseteq M$ is a *cone* if it is downward closed, i.e., if $N^{\leq} \subseteq N$. Hence on cones $\leq$ and $\subseteq$ coincide; in particular, $\leq$ is a partial order on cones.

Since M is a cone and the intersection of cones is a cone again, the set of all cones forms a complete lattice under inclusion. It is isomorphic to the angelic or Hoare power domain [56] over $(M, \leq)$. However, we are not going to use that domain.

8 Pointwise Extension

In the sequel we will define many functions on single points of M and lift them to subsets of M by *pointwise extension*, i.e., by setting, for $f : M \to M$ and $N \subseteq M$,

$$f(N) \stackrel{\text{def}}{=} \{f(x) : x \in N\} .$$

These pointwise extended functions distribute through arbitrary unions and hence are monotonic w.r.t. inclusion and strict w.r.t. $\emptyset$. We will also use this mechanism to lift these functions a further level to sets of subsets of M.

Pointwise extensions inherit *linear laws*. These are laws of the following form:

- Equational laws in which all variables occur exactly once on both sides of the equality sign. Examples are the laws of neutrality, associativity and commutativity.
- Implications with element relations as atoms in which all variables occur exactly once on both sides of the implication sign. In the inherited form the element relations turn into inclusions. An example is

  $$s \bullet t \in \varepsilon \Rightarrow s \in \varepsilon \wedge t \in \varepsilon$$

  which lifts to

  $$S \bullet T \subseteq \varepsilon \Rightarrow S \subseteq \varepsilon \wedge T \subseteq \varepsilon .$$

9 Directed Sets

A subset $N \subseteq M$ is *directed* if every finite subset of N has an upper bound in N. Equivalently, N is directed if $N \neq \emptyset$ and any two elements in N have a common upper bound in N. Hence every two elements of adirected set are consistent in that they approximate a common element.

For $P \subseteq M$ we denote by $\mathsf{dir}\, P$ the set of all directed subsets of P. Note that the operation dir is monotonic w.r.t. inclusion.

We now study how directedness behaves under union and intersection.

Lemma 9.1 Consider $N, P \subseteq M$. Then

1. $N \cup P \in \mathsf{dir}\, M \wedge N \leq P \Rightarrow P \in \mathsf{dir}\, M$.
2. $N \cup P \in \mathsf{dir}\, M \Rightarrow (N \leq P \vee P \leq N) \wedge (N \in \mathsf{dir}\, M \vee P \in \mathsf{dir}\, M)$.
3. $Q^{\leq} \in \mathsf{dir}\, M \Leftrightarrow Q \in \mathsf{dir}\, M$.
4. $N \leq P \wedge P \in \mathsf{dir}\, M \Rightarrow N \cup P \in \mathsf{dir}\, M$.
5. $\mathsf{dir}\,(N \cup P) = \{K \cup L : (K \in \mathsf{dir}\, N \wedge L \subseteq P \wedge L \leq K)\} \cup$
   $\{K \cup L : (L \in \mathsf{dir}\, P \wedge K \subseteq N \wedge K \leq L)\}$.

Proof: 1. Assume $x, y \in P$. By directedness of $N \cup P$ there is a $z \in N \cup P$ with $x \leq z$ and $y \leq z$. If $z \in P$, we are done. Otherwise, by $N \leq P$ there is a $u \in P$ with $z \leq u$ so that by transitivity also $x \leq u$ and $y \leq u$.

2. For $N = \emptyset$ or $P = \emptyset$ the claim is trivial. So consider $N, P \neq \emptyset$ and suppose $N \not\leq P$. Then there is $x \in N$ with $x \not\leq P$. Assume now $y \in P$. By directedness of $N \cup P$ there is a $z \in N \cup P$ with $x, y \leq z$. Since $x \not\leq P$, it follows that $z \in N \backslash P \subseteq N$. Since y was arbitrary, we have shown $P \leq N$.
   The second disjunct is immediate from the first and 1.
3. is immediate from 1 by setting $N = Q^{\leq}$, $P = Q$ and using $Q^{\leq} \leq Q$.
4. Assume $x, y \in N \cup P$. By $N \leq P$ and $P \leq P$ there are $u, v \in P$ with $x \leq u \wedge y \leq v$. Since P is directed, there is $z \in P$ with $u \leq z$ and $v \leq z$. Hence also $x \leq z$ and $y \leq z$ by transitivity.
5. We show ($\subseteq$); the reverse inclusion is immediate from 4.
   Consider $Q \in \mathsf{dir}\,(N \cup P)$. We have $Q = K \cup L$ where $K \stackrel{\text{def}}{=} Q \cap N$ and $L \stackrel{\text{def}}{=} Q \cap P$. By 2 we know $K \leq L \vee L \leq K$. If $K \leq L$ then $L \in \mathsf{dir}\, P$ by 1. If $L \leq K$ then $K \in \mathsf{dir}\, N$ by 1. This shows the claim.

■

10 The Ideal Completion

To tie our approach in with domain-theoretic notions we recall the ideal completion (cf. e.g. [5, 18]). Consider an ordered set $(M, \leq)$. An *ideal* is a directed cone. The set of all ideals is denoted by $I(M)$.

The partial order $(M, \leq)$ is called *Δ-complete* iff every directed set $D \subseteq M$ has a supremum (or least upper bound) $\sqcup M$. An element x of M is *finite* (*compact*) iff for every directed set $D \subseteq M$ with $x \leq \sqcup D$ we have also $x \leq z$ for some $z \in D$. Equivalently, x is finite iff for every ideal $I \subseteq M$ with $x \leq \sqcup I$ we have $x \in I$. $(M, \leq)$ is *algebraic* iff every element of M is the supremum of a directed set of finite elements. A non-finite element of an algebraic set is called a *limit point* or an *infinite element*. With these notions one has

Theorem 10.1 1. The set $(I(M), \subseteq)$ ordered by set inclusion is Δ-complete and algebraic, the finite elements being the *principal ideals* $x^{\leq}$ for $x \in M$. The mapping $\iota : x \mapsto x^{\leq}$ is an embedding of M into $I(N)$.

2. For every monotonic mapping $h : M \to P$ into a Δ-complete set $(P, \leq)$ there is a unique continuous mapping $\overline{h} : I(M) \to P$ extending h, i.e., with $\overline{h}(x^{\leq}) = h(x)$. $\overline{h}$ is given by $\overline{h}(I) = \sqcup\, h(I)$ for $I \in I(M)$; hence $\overline{h}(D^{\leq}) = \sqcup\, h(D)$ for directed $D \subseteq M$.

The ordered set $(I(N), \subseteq)$ is called the *ideal completion* of $(M, \leq)$. We set $M^{\infty} \stackrel{\text{def}}{=} I(M)$. An ideal in M^{∞} is non-compact iff it doesn't have a maximal (and hence greatest) element.

Part III: The Algebra of Ideals

11 Streams as Ideals

We now make our notion of streams precise. Assume an alphabet A of atomic actions or states. Then, as usual, A^* is the set of all finite words over A. By ε we denote the empty word, whereas concatenation is denoted by $\bullet$. A subset of A^* is called a *(formal) language*.

A word u is a *prefix* of a word v, written $u \sqsubseteq v$, iff there is a word w such that $u \bullet w = v$. It is well-known that this defines a partial order on words which is even well-founded. Moreover, ε is the least element in this order. The corresponding strict-order is denoted by $\sqsubset$. A cone of $(A^*, \sqsubseteq)$ is then a prefix-closed language. Note that every non-empty cone contains ε.

A few properties we shall use are the following (where $x, y, u, v, w \in A^*$ and $U, V \subseteq A^*$):

$$v \sqsubseteq w \Leftrightarrow u \bullet v \sqsubseteq u \bullet w \,, \tag{1}$$

$$u \sqsubseteq w \wedge v \sqsubseteq w \Rightarrow u \sqsubseteq v \vee v \sqsubseteq u \,, \tag{2}$$

$$V \neq \emptyset \;\Rightarrow\; (U \bullet V)^{\sqsubseteq} = U^{\sqsubseteq} \cup U \bullet V^{\sqsubseteq} \,. \tag{3}$$

Poperty (2) is also called *local linearity*.

Informally, a stream over A is a finite or infinite sequence of elements of A. The basis of our approach is the observation that such a stream is completely characterized by the set of its finite prefixes. This set is downward closed w.r.t. $\sqsubseteq$, i.e., a cone. Moreover, it is directed, since in the partial order $(A^*, \sqsubseteq)$ by local linearity the directed sets can be characterised another way:

Lemma 11.1 $D \subseteq A^*$ is directed w.r.t. $\sqsubseteq$ iff D is totally ordered by $\sqsubseteq$, i.e., iff for any two elements $u, v \in D$ we have $u \sqsubseteq v$ or $v \sqsubseteq u$.

Hence an ideal of $(A^*, \sqsubseteq)$ is a totally and prefix-closed non-empty language. Note that every ideal contains ε. Therefore an ideal is a set of words of increasing length "growing at the right end". This set may be finite or infinite. A simple example is, for $a \in A$, the infinite ideal

$$a^* = \{\varepsilon,\ a,\ a \bullet a,\ a \bullet a \bullet a,\ a \bullet a \bullet a \bullet a,\ \ldots\} \,.$$

We identify now a stream with the set of its finite prefixes. By the above, this set is an ideal of $(A^*, \sqsubseteq)$. Therefore we call the elements of A^∞ *streams* over A. It should be noted that the compact elements of A^∞ correspond to the elements of A^*; hence, for countable A, the set $(A^\infty, \subseteq)$ has a countable basis of finite elements and therefore is countably algebraic. The *length* of stream S is denoted by $|S|$; it coincides with its cardinality minus one. Let us give a charaterization of infinite streams:

Lemma 11.2 A stream S is infinite iff $\max S = \emptyset$.

Proof: First, by linearity of the prefix order on a stream and by its well-foundedness, an infinite stream cannot have a maximal element. By Lemma 15.1.2 we have also the reverse implication. ■

The compact elements of A^∞ correspond to the elements of A^*, whereas the non-compact elements are precisely the (cardinally) infinite ideals. They correspond to infinite sequences over A and hence we set $A^\omega \stackrel{\text{def}}{=} \{J \in I(A) : \max J = \emptyset\}$.

To resume our previous example, the ideal

$$a^* = \{\varepsilon,\ a,\ a \bullet a,\ a \bullet a \bullet a,\ a \bullet a \bullet a \bullet a,\ \ldots\}$$

is the limit (supremum) of the set of finite ideals

$$\{\{a^i : i \leq n\} : n \in \mathbb{N}\}$$

corresponding to the $\sqsubseteq$-increasing set

$$\{a^n : n \in \mathbb{N}\}$$

of finite words. It may thus be viewed as a representation of the infinite stream of as. This observation is the main motivation for our approach; it allows us to work with infinite streams by manipulating their sets of finite approximations, since in the ideal completion each (finite or infinite) element is *identified* with the set of its finite approximations. This allows carrying over all laws from the algebra of formal languages to streams. Of course, the fact that the set of finite and infinite streams is isomorphic to the ideal completion of the set of finite streams is well-known; what is new here is the direct calculation with the ideals using the underlying algebra.

While our approach was motivated by the particular case of streams, we will perform the mathematical development as far as possible for general ideal completions.

12 A Setting for Non-Interleaving Semantics

To illustrate our approach with a different setting we now sketch how partial-order semantics, allowing true concurrency, can be accomodated in our setting.

Let E be a set of *events*. Then a *history* over E is a partial order $(F, \preceq)$ with a finite set $F \subseteq E$. The order $\preceq$ models temporal/causal dependence. Two events not related by $\preceq$ are considered as parallel/concurrent.

Let now $H(E)$ be the set of all histories over E. We define an approximation ordering $\leq$ on $H(E)$ by

$$\begin{aligned}(F_1, \preceq_1) \leq (F_2, \preceq_2) \stackrel{\text{def}}{\Leftrightarrow}\ & F_1 \subseteq F_2 \wedge \\ & \preceq_1 \ = \ \preceq_2 \cap F_1 \times F_2 \ \wedge \\ & \forall\, x \in F_1 : x^{\preceq_1} = x^{\preceq_2} \ \wedge \\ & \forall\, y \in F_2 : \exists\, x \in F_1 : x \preceq_2 y\ .\end{aligned}$$

This is the appropriate generalization of the prefix relation on words to histories. It means that F_1 is embedded as a cone into F_2 and F_2 may only add "later" events. It is straightforward to check that this indeed defines a partial order. The least element is $(\emptyset, \emptyset)$.

A *chronicle* now is an ideal in $(H(E), \leq)$, and infinite chronicles generalize infinite streams. The case of streams is retrieved if one only considers histories that are linearly ordered by $\preceq$; in that case $\preceq$ corresponds directly to $\sqsubseteq$. In the present paper, we shall not pursue this example further, though.

13 Behaviours and Refinement

Our application of ideals will be the description of systems. To model non-determinacy, we define a *behaviour* to be a set of ideals.

It should be noted that using *sets* of ideals as behaviours allows only "trace-like" semantics in which there is no distinction between internal and external non-determinacy. The algebraic reflection of this is that concatenation, our sequencing operation, distributes through union both from the left and from the right. In algebraic approaches to CCS-like systems (see e.g. [3]) only one of these distributivities holds. This results in models with tree-like objects that reflect the non-deterministic branching structure in time. This detailed record is lost by admitting both distributivities rather than just one.

The set of finite prefixes of a behaviour $\mathcal{B}$ is

$$\mathsf{pref}\,\mathcal{B} \stackrel{\text{def}}{=} \bigcup \mathcal{B} \ .$$

Clearly, pref distributes through union and hence is $\subseteq$-monotonic.

As our refinement relation we choose inclusion, i.e., behaviour $\mathcal{B}$ *refines* behaviour $\mathcal{C}$ if $\mathcal{B} \subseteq \mathcal{C}$. For instance, given a property $P \subseteq M$, the set $\mathsf{ide}\,P$ of ideals satisfying P, is a behaviour. To allow correct local refinements one therefore has to ensure monotonicity of all operations w.r.t. inclusion.

Example 13.1 We resume the example from Section 4 and show that bounded fairness refines unbounded fairness: since all operators involved are monotonic w.r.t. inclusion, we obtain from $0 \bullet 0^{\leq k} \subseteq a^+$ for $a \in A$ that

$$(0 \bullet 0^{\leq k} \bullet 1 \cup 1 \bullet 1^{\leq k} \bullet 0)^\omega \subseteq \mathcal{SCHED} \ .$$

■

14 Describing Behaviours by Properties

We want to characterise ideals by certain sets of "relevant" finite approximations. Such a set, i.e., a subset of our overall partially ordered set M, is called a *property* in this connection.

In the particular case of streams the finite approximations are "snapshots" in the form of finite words in A^*. Assume a set $U \subseteq A^*$ of admissible snapshots. If a stream contains snapshots from a subset $D \subseteq U$ then D has to be directed. However, there may be arbitrary "gaps" between the snapshots in D. To reconstruct the stream we therefore have to "fill in the details" between the snapshots. This is done by taking the prefix closure $D^{\sqsubseteq}$. Hence we define the set of streams, i.e., the behaviour, spanned by snapshot set U as

$$\mathsf{str}\,U \stackrel{\text{def}}{=} \{D^{\sqsubseteq} : D \in \mathsf{dir}\,U\} \ .$$

This is the set of streams that "interpolate" consistent snapshots in VA related notion occurs in [19]; the connection will be made precise in Section 15.

We generalize this to arbitrary partial orders and their ideal completions. Let $(M, \leq)$ be the partial order of finite approximations. For property $P \subseteq M$ we now define by

$$\mathsf{ide}\, P \stackrel{\text{def}}{=} \{D^{\leq} : D \in \mathsf{dir}\, P\}$$

the set of all ideals "spanned" by directed subsets of P. Note that $\mathsf{ide}\, M = I(M)$. Note that ide is monotonic w.r.t. inclusion. A different characterisation of ide is given by

Lemma 14.1 For $I \in I(M)$ and $Q \subseteq M$ the following statements are equivalent:

1. $I \in \mathsf{ide}\, Q$.
2. $I \subseteq (I \cap Q)^{\leq}$.
3. $I = (I \cap Q)^{\leq}$.

Proof: The equivalence of 2 and 3 is obvious by monotonicity of $^{\leq}$ and downward closedness of I.
$(1 \Rightarrow 2)$ Suppose $I = D^{\leq}$ for $D \in \mathsf{dir}\, Q$.

$$\begin{array}{ll} & I \\ = & \quad \{\![\text{ assumption }]\!\} \\ & D^{\leq} \\ = & \quad \{\![\text{ since } D \subseteq Q \ ]\!\} \\ & (D \cap Q)^{\leq} \\ \subseteq & \quad \{\![\text{ monotonicity }]\!\} \\ & (D^{\leq} \cap Q)^{\leq} \\ = & \quad \{\![\text{ assumption }]\!\} \\ & (I \cap Q)^{\leq} \ . \end{array}$$

$(3 \Rightarrow 1)$ Since I is directed, so is $(I \cap Q)^{\leq}$. By Lemma 9.1.3 also $I \cap Q$ is directed and the claim follows. ■

We have the following distributivity property for ide:

Lemma 14.2 Consider $N, P \subseteq M$. Then

$$\mathsf{ide}\,(N \cup P) = \mathsf{ide}\, N \cup \mathsf{ide}\, P \ .$$

Proof:

$$\begin{array}{ll} & I \in \mathsf{ide}\,(N \cup P) \\ \Leftrightarrow & \quad \{\![\text{ by Lemma 14.1 }]\!\} \\ & I = (S \cap (N \cup P))^{\leq} \\ \Leftrightarrow & \quad \{\![\text{ distributivity of } \cap \text{ over } \cup \text{ and Lemma 7.1.1 }]\!\} \\ & I = (S \cap N)^{\leq} \cup (S \cap P)^{\leq} \\ \Rightarrow & \quad \{\![\text{ by directedness of } I, \text{ Lemma 9.1.2 and Lemma 7.3.3 }]\!\} \\ & I = (S \cap N)^{\leq} \vee I = (S \cap P)^{\leq} \end{array}$$

$\Leftrightarrow$ {[by Lemma 14.1]}

$I \in \mathsf{ide}\, N \vee I \in \mathsf{ide}\, P$.

The reverse inclusion follows by monotonicity of ide .
Another proof can be given using Lemma 9.1.5. ■

This also shows once again the monotonicity of ide . However, we have even

Corollary 14.3 $N \subseteq P \Leftrightarrow \mathsf{ide}\, N \subseteq \mathsf{ide}\, P$.

Proof: The inclusion from right to left is part of Theorem 10.1.1. ■

It should be noted, however, that ide only distributes through finite unions and hence is not "continuous". For an instance of this see Example 16.3 below.

We have the following properties concerning downward closure:

Lemma 14.4 1. $I \in \mathsf{ide}\,(P^{\leq}) \Leftrightarrow I \subseteq P^{\leq}$.
2. $\mathsf{pref}\,\mathsf{ide}\, P = P^{\leq}$.
3. $\mathsf{ide}\, Q \subseteq \mathsf{ide}\, Q^{\leq}$. The reverse inclusion is not valid.
4. $\mathsf{ide}\, Q \cap \mathsf{ide}\, P \subseteq \mathsf{ide}\,(Q^{\leq} \cap P^{\leq})$.

Proof: 1. ($\Rightarrow$) Assume $I = D^{\leq}$ for $D \in \mathsf{dir}\,(P^{\leq})$. Then, by monotonicity and idempotence of $^{\leq}$ we get $D^{\leq} \subseteq (P^{\leq})^{\leq} = P^{\leq}$, i.e., $I \subseteq P^{\leq}$.
($\Leftarrow$) is straightforward, since $I \subseteq P^{\leq}$ implies $I \in \mathsf{dir}\,(P^{\leq})$ and $I = I^{\leq}$.
2. The inclusion $\subseteq$ is straightforward. For the reverse consider $y \in P^{\leq}$. There is $x \in P$ with $y \leq x$. But then $y \in x^{\leq} \in \mathsf{ide}\, P$.
3. is immediate from $Q \subseteq Q^{\leq}$ and monotonicity of ide . For a counterexample to the reverse inclusion see Example 16.1.
4. immediate from 2.

■

15 Maximal and Infinite Ideals

15.1 Maximal Ideals

Frequently one is interested in processes that continue as long as possible. These are modeled by ideals which are maximal w.r.t. $\leq$ or, equivalently, w.r.t. inclusion. We therefore give a characterisation of maximal ideals. For a behaviour $\mathcal{B}$ we denote the subset of maximal ideals by max ; this agrees with the definition in Section 7, and hence all our laws there apply.

Lemma 15.1 Suppose $I \in I(M)$ and $N \subseteq M$. Then

1. $x \in \mathsf{max}\, I \Leftrightarrow I = x^{\leq}$.
2. $\mathsf{max}\, I = \emptyset \Rightarrow I$ infinite.
3. $\mathsf{max}\, N = \emptyset \wedge I \in \mathsf{max}\,\mathsf{ide}\, N \Rightarrow \mathsf{max}\, I = \emptyset$.

Proof: 1. ($\Rightarrow$) We only need to show $I \subseteq x^{\leq}$; the other inclusion follows from downward closure of I. Suppose $y \in I$. By directedness of I there is $z \in I$ with $x \leq z$ and $y \leq z$. Maximality of x implies $z = x$ and hence $y \leq x$.
($\Leftarrow$)

$$
\begin{aligned}
& \max I \\
= \quad & \{\![\text{ by assumption }]\!\} \\
& \max x^{\leq} \\
= \quad & \{\![\text{ by Lemma 7.2.1 }]\!\} \\
& (x^{\leq})^{\leq} \setminus (x^{\leq})^{<} \\
= \quad & \{\![\text{ by Lemma 7.1.2 }]\!\} \\
& x^{\leq} \setminus x^{<} \\
= \quad & \{\![\text{ by Lemma 7.2.1 }]\!\} \\
& \max x \\
= \quad & \{\![\text{ irreflexivity of } <]\!\} \\
& \{x\} \; .
\end{aligned}
$$

2. Every non-empty finite set has a maximal element.
3. Suppose $\max I \neq \emptyset$, say $x \in \max I$. By 1 then $I = x^{\leq}$ and by $I \in \mathsf{ide}\, N$ we get $x \in N$. Since $\max N = \emptyset$, there is $y \in N$ with $x \leq y$ and $y \neq x$. But then $y^{\leq} \in \mathsf{ide}\, N$ and hence, by Theorem 10.1.1, we have $x^{\leq} \subseteq y^{\leq} \wedge x^{\leq} \neq y^{\leq}$. This is a contradiction to $I \in \max\mathsf{ide}\, N$.

■

15.2 Infinite Ideals

Motivated by 2 we define, for a behaviour $\mathcal{B}$, the set of its infinite ideals as

$$\mathsf{inf}\, \mathcal{B} \stackrel{\text{def}}{=} \{I \in \mathcal{B} : \max I = \emptyset\} \; .$$

For general domains, this is a bit of a misnomer, since there may well be infinite ideals *with* maximal elements. However, we will single out a particular class of domains where this cannot occur and work mostly with these, so that the terminology will be justified. Clearly, inf distributes through arbitrary union and intersection:

$$\mathsf{inf}\,(\bigcup_{i \in I} \mathcal{B}_i) = \bigcup_{i \in I} \mathsf{inf}\, \mathcal{B}_i \; , \tag{4}$$

$$\mathsf{inf}\,(\bigcap_{i \in I} \mathcal{B}_i) = \bigcap_{i \in I} \mathsf{inf}\, \mathcal{B}_i \; . \tag{5}$$

Now Lemma 15.1.3 can be restated as

$$\max N = \emptyset \;\Rightarrow\; \max\mathsf{ide}\, N \subseteq \mathsf{inf}\,\mathsf{ide}\, N \; .$$

The reverse inclusion is generally not valid. For a counterexample choose $M = \mathbb{N} \cup \{\infty\}$ with the usual ordering and consider the ideal $\mathbb{N} \in I(M)$. We have $\max \mathbb{N} = \emptyset$, but $\mathbb{N} \notin \max I(M)$, since $\mathbb{N} \subseteq M \in I(M)$ and $\mathbb{N} \neq M$.

We call a partial order $(M, \leq)$ $\max$-*determined* if

$$\mathsf{inf}\, I(M) \subseteq \max I(M) \; .$$

15.3 Refinement Laws

Now we clarify the relation between $\mathsf{inf\,ide}$ and $\mathsf{max\,ide}$ and investigate monotonicity and distributivity of the $\mathsf{max\,ide}$, $\mathsf{inf\,ide}$ and $\mathsf{max\,inf}$ operations, which is important for refinement. First we note

Lemma 15.2 For $N, P \subseteq M$,

1. $\mathsf{inf\,ide}\, N \cup P = \mathsf{inf\,ide}\, N \cup \mathsf{inf\,ide}\, P$.
   In particular, $\mathsf{inf\,ide}$ is monotonic w.r.t. inclusion.
2. $N = \mathsf{saf}\, N \;\Rightarrow\; \mathsf{inf\,ide}\,(N \cap P) = \mathsf{inf\,ide}\, N \cap \mathsf{inf\,ide}\, P$.

Proof: 1. immediate from Lemma 14.2 and equation (4).
2. immediate from Lemma 24.2 and equation (5). ■

Concerning maximal ideals we have

Lemma 15.3 Let $(M, \leq)$ be max-determined. Then, for $N, P \subseteq M$,

1. $\mathsf{inf\,ide}\, N \;\subseteq\; \mathsf{ide}\, N \cap \mathsf{max}\, I(M) \;\subseteq\; \mathsf{max\,ide}\, N$.
2. $\mathsf{max\,ide}\, N = \mathsf{inf\,ide}\, N \cup \mathsf{ide\,max}\, N$.
3. $\mathsf{max}\, N = \emptyset \Rightarrow \mathsf{inf\,ide}\, N = \mathsf{ide}\, N \cap \mathsf{max}\, I(M) = \mathsf{max\,ide}\, N$.
4. $\mathsf{inf\,ide}\, N \cup P = \mathsf{inf\,ide}\, N \cup \mathsf{inf\,ide}\, P$.
   In particular, $\mathsf{inf\,ide}$ is monotonic w.r.t. inclusion.
5. $N = \mathsf{saf}\, N \;\Rightarrow\; \mathsf{inf\,ide}\,(N \cap P) = \mathsf{inf\,ide}\, N \cap \mathsf{inf\,ide}\, P$.
6. $\mathsf{max}\, N = \emptyset \wedge N \subseteq P \Rightarrow \mathsf{max\,ide}\, N \subseteq \mathsf{max\,ide}\, P$.
7. $\mathsf{max}\, N = \mathsf{max}\, P = \emptyset \Rightarrow \mathsf{max\,ide}\,(N \cup P) = \mathsf{max\,ide}\, N \cup \mathsf{max\,ide}\, P$.
8. If N and P are cones with $\mathsf{max}\, N = \mathsf{max}\, P = \mathsf{max}\,(N \cap P) = \emptyset$ then $\mathsf{max\,ide}\,(N \cap P) = \mathsf{max\,ide}\, N \cap \mathsf{max\,ide}\, P$.

Proof: 1.
$$
\begin{aligned}
& I \in \mathsf{inf\,ide}\, N \\
\Leftrightarrow\quad & \{\![\text{ definition }]\!\} \\
& I \in \mathsf{ide}\, N \wedge \mathsf{max}\, I = \emptyset \\
\Rightarrow\quad & \{\![\text{ since } (M, \leq) \text{ is } \mathsf{max}\text{-determined }]\!\} \\
& I \in \mathsf{ide}\, N \wedge I \in \mathsf{max}\, I(M) \\
\Rightarrow\quad & \{\![\text{ by Lemma 7.2.3, since } \mathsf{ide}\, N \subseteq I(M)]\!\} \\
& I \in \mathsf{max\,ide}\, N\ .
\end{aligned}
$$

2. ($\subseteq$) Suppose $I \in \mathsf{max\,ide}\, N$. If $\mathsf{max}\, I = \emptyset$, then $I \in \mathsf{inf\,ide}\, N$ by definition. Otherwise $\mathsf{max}\, I$ is a singleton, say $\mathsf{max}\, I = \{x\}$, and $I = x^{\leq}$. It follows that $x \in N$. For $y \in N$ with $x \leq y$ we have $x^{\leq} \subseteq y^{\leq} \in \mathsf{ide}\, N$, so that $x^{\leq} = y^{\leq}$ by maximality of $I = x^{\leq}$. Hence also $x = y$. This shows $x \in \mathsf{max}\, N$, so that $I = x^{\leq} \in \mathsf{ide\,max}\, N$.
($\supseteq$) $\mathsf{inf\,ide}\, N \subseteq \mathsf{max\,ide}\, N$ was shown in 1. Suppose now $I \in \mathsf{ide\,max}\, N$, say $I = x^{\leq}$ with $x \in \mathsf{max}\, N$, and $I \subseteq J \in \mathsf{ide}\, N$, say $J = D^{\leq}$ for $D \in \mathsf{dir}\, N$. Consider $y \in J$. By directedness of J there is a $z \in J$ with $x, y \leq z$. By $J = D^{\leq}$ there is a $u \in D$ with $z \leq u$. Hence also $x, y \leq u$. By $D \subseteq N$ and $x \in \mathsf{max}\, N$ we get $x = u$. So $y \leq x$ and hence $y \in x^{\leq} = I$. Altogether, $J \subseteq I$ and hence $J = I$. So $I \in \mathsf{max\,ide}\, N$.

3. Assume $\max N = \emptyset$. Then Lemma 15.1.3 shows $\max \mathsf{ide}\, N \subseteq \inf \mathsf{ide}\, N$ and the equalities follow from 1.
4. $$\begin{array}{ll} & \max \mathsf{ide}\, N \\ = & \quad \{\![\text{ by 3 }]\!\} \\ & \mathsf{ide}\, N \cap \max I(M) \\ \subseteq & \quad \{\![\text{ by assumption } N \subseteq P \text{ and monotonicity of ide }]\!\} \\ & \mathsf{ide}\, P \cap \max I(M) \\ \subseteq & \quad \{\![\text{ by 3 }]\!\} \\ & \max \mathsf{ide}\, P \ . \end{array}$$
5. We aim at an application of Lemma 7.2.4. Suppose therefore that $I \in \max \mathsf{ide}\, N \cap (\mathsf{ide}\, P)^{\subset}$. By 3 we have $I \in \max I(M)$. But by $I \in (\mathsf{ide}\, P)^{\subset}$ there is $J \in \mathsf{ide}\, P$ with $I \subset J$, a contradiction to maximality of I. Hence $\max \mathsf{ide}\, N \cap (\mathsf{ide}\, P)^{\subset} = \emptyset$. By symmetry, also $\max \mathsf{ide}\, P \cap (\mathsf{ide}\, N)^{\subset} = \emptyset$. Now the claim is immediate from Lemma 7.2.4.
6. $(\subseteq)$ follows from 6.
$(\supseteq)$ Assume $I \in \max \mathsf{ide}\, N \cap \max \mathsf{ide}\, P$. Then by 3 we have $I \in \max I(M)$. Hence, again by 3, we only need to show $I \in \mathsf{ide}\,(N \cap P)$. Since N and P are cones we get $I \subseteq N$ and $I \subseteq P$ and hence $I \subseteq N \cap P$ as well, showing the claim.

■

The next lemma allows simplification of the defining property of a behaviour.

Lemma 15.4 Consider $N, P \subseteq M$. Then

$$\max \mathsf{ide}\,(N \cup P) = \max \mathsf{ide}\, P \Leftrightarrow \mathsf{ide}\, N \leq \mathsf{ide}\, P \ .$$

Proof: $(\Leftarrow)$

$$\begin{array}{ll} & \mathsf{ide}\, N \leq \mathsf{ide}\, P \\ \Rightarrow & \quad \{\![\text{ by Lemma 7.3.4 }]\!\} \\ & \max(\mathsf{ide}\, N \cup \mathsf{ide}\, P) = \max \mathsf{ide}\, P \\ \Leftrightarrow & \quad \{\![\text{ by Lemma 14.2 }]\!\} \\ & \max \mathsf{ide}\,(N \cup P) = \max \mathsf{ide}\, P \ . \end{array}$$

$(\Rightarrow)$ If $N = \emptyset$, the claim holds trivially, since $\mathsf{ide}\, \emptyset = \emptyset$. Hence we now assume $N \neq \emptyset$.
We now need the so-called *Maximal Principle* (see e.g. [18]): *Assume a partial order in which every non-empty chain has an upper bound. Then every element has a maximal element above it.*
We apply this to the partial order $(\mathsf{ide}\, N, \subseteq)$. It satisfies the assumption, since $\mathsf{ide}\, N$ is closed under directed unions and hence, in particular, under unions of chains. Consider now $I \in \mathsf{ide}\, N \subseteq \mathsf{ide}\,(N \cup P)$. By the maximal principle there is a $J \in \max \mathsf{ide}\,(N \cup P) = \max \mathsf{ide}\, P$ with $I \leq J$. ■

Under additional assumptions we can simplify the assertion:

Lemma 15.5 Assume $P \in \mathsf{dir}\, M$. Then

$$\mathsf{max}\,\mathsf{ide}\,(N \cup P) \;=\; \mathsf{max}\,\mathsf{ide}\,P \;\Leftrightarrow\; N \leq P\ .$$

Proof: To apply Lemma 15.4 we show that $P \in \mathsf{dir}\, M$ implies

$$\mathsf{ide}\,N \leq \mathsf{ide}\,P \;\Leftrightarrow\; N \leq P\ .$$

($\Rightarrow$) Assume $x \in N$. Then $x^{\leq} \in \mathsf{ide}\,N$ and so there is $I \in \mathsf{ide}\,P$, say $I = D^{\leq}$ for $D \in \mathsf{dir}\,P$, with $x^{\leq} \leq I$. By Lemma 7.3.1-2 we get $x \leq P$.
($\Leftarrow$) For $I \in \mathsf{ide}\,N$ we have $I \leq P \in \mathsf{dir}\,P$ and hence, by Lemma 7.3.1, also $I \leq P^{\leq} \in \mathsf{ide}\,P$. ■

For a counterexample when P is not directed see Example 17.2 in connection with Corollary 15.6.2 below.

Recalling the equivalence $\sim$ associated with the preorder $\leq$, we obtain from the previous two lemmata

Corollary 15.6 Consider $N, P \subseteq M$. Then

1. $\mathsf{ide}\,N \sim \mathsf{ide}\,P \;\Rightarrow\; \mathsf{max}\,\mathsf{ide}\,N \;=\; \mathsf{max}\,\mathsf{ide}\,P$.
2. If $N, P \in \mathsf{dir}\, M$ then

$$N \sim P \;\Rightarrow\; \mathsf{max}\,\mathsf{ide}\,N \;=\; \mathsf{max}\,\mathsf{ide}\,P\ .$$

15.4 An Alternative Characterization of Infinite Ideals

We conclude this section by an alternative characterization of the set $\mathsf{inf}\,\mathsf{ide}\,P$ for property $P \subseteq M$. First we define

$$\mathsf{lim}\,P \stackrel{\mathrm{def}}{=} \{I \in I(M) : I \cap P \in \mathsf{dir}\, M \;\wedge\; \mathsf{max}\,(I \cap P) \;=\; \emptyset\}\ .$$

This generalizes the corresponding definition for infinite words or streams in [53, 58, 59, 61, 62] (to cite just a few), which is based on [19]. Other notations for $\mathsf{lim}\,P$ found in the literature are P^{δ} or $\mathbf{P}$. We can then show

Lemma 15.7 1. $\mathsf{inf}\,\mathsf{ide}\,P \;\subseteq\; \mathsf{lim}\,P$.
2. If $(M, \leq)$ is max-determined then the reverse inclusion holds as well.

Proof: We first note that

$$\begin{array}{ll}
& I \in \mathsf{inf}\,\mathsf{ide}\,P \\
\Leftrightarrow & \quad \{\![\ \text{definition}\ ]\!\} \\
& I \in \mathsf{ide}\,P \;\wedge\; \mathsf{max}\,I = \emptyset \\
\Leftrightarrow & \quad \{\![\ \text{by Lemma 14.1}\ ]\!\} \\
& I = (I \cap P)^{\leq} \;\wedge\; \mathsf{max}\,I = \emptyset \\
\Leftrightarrow & \quad \{\![\ \text{equality}\ ]\!\} \\
& I = (I \cap P)^{\leq} \;\wedge\; \mathsf{max}\,(I \cap P)^{\leq} = \emptyset \\
\Leftrightarrow & \quad \{\![\ \text{Lemma 7.2.2}\ ]\!\} \\
& I = (I \cap P)^{\leq} \;\wedge\; \mathsf{max}\,(I \cap P) = \emptyset\ . \qquad (*)
\end{array}$$

Now we prove our claims as follows:

1. $(*)$

   $\Rightarrow$ {[by Lemma 9.1.3]}

   $I \cap P \in \mathsf{dir}\, M \wedge \mathsf{max}(I \cap P) = \emptyset$

   $\Leftrightarrow$ {[definition]}

   $I \in \mathsf{lim}\, P$.

2. Let $(M, \leq)$ be max-determined and assume $I \in \mathsf{lim}\, P$. By $(*)$ it remains to show $I = (I \cap P)^{\leq}$. First, by monotonicity of downward closure we have $(I \cap P)^{\leq} \subseteq I^{\leq} = I$. Using Lemma 7.2.2 we obtain $\mathsf{max}(I \cap P)^{\leq} = \mathsf{max}(I \cap P) = \emptyset$, so that by max-determinedness $(I \cap P)^{\leq} \in \mathsf{max}\, I(M)$ and hence $(I \cap P)^{\leq} = I$.

■

15.5 About max-Determinedness

It remains to investigate under which conditions a partial order is max-determined. To this end we introduce some auxiliary notions. Let $F : \mathcal{P}(M) \to \mathcal{P}(M)$ be some function, such as dir or ide. We say that $N \subseteq M$ *has F-maxima* if every set in $F(N)$ has a maximal element. In addition to the functions mentioned we shall use

$$\mathsf{ne}\, N \stackrel{\text{def}}{=} \{C \subseteq N : C \neq \emptyset\}\ ,$$
$$\mathsf{chai}\, N \stackrel{\text{def}}{=} \{C \subseteq N : C \text{ non-empty chain}\}\ .$$

Lemma 15.8 If $N \subseteq M$ has chai-maxima, then it also has ne-maxima.

Proof: Assume $\emptyset \neq D \subseteq N$ and $\mathsf{max}\, D = \emptyset$. Construct a chain $C \subseteq N$ as follows: Choose $x_0 \in D$ arbitrarily. Assume now that x_i has been found. Since $x_i \notin \emptyset = \mathsf{max}\, D$, there is $x_{i+1} \in D$ with $x_i < x_{i+1}$. Now for $C \stackrel{\text{def}}{=} \{x_i : i \in \mathbb{N}\}$ we have $\mathsf{max}\, C = \emptyset$, a contradiction. ■

Corollary 15.9 If $N \subseteq M$ has chai-maxima, then it also has dir-maxima.

Proof: Every directed set is non-empty. ■

We say that $(M, \leq)$ *separates ideals* if for all $I, J \in I(M)$ with $I \neq J$ the intersection $I \cap J$ has chai-maxima. The connection with max-determinedness is given by

Theorem 15.10 $(M, \leq)$ is max-determined iff $(M, \leq)$ separates ideals.

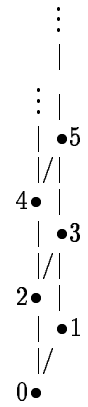
Proof: ($\Rightarrow$) Suppose $I \neq J$ and $C \in \mathsf{chai}\,(I \cap J)$, but $\mathsf{max}\, C = \emptyset$. Then $C^{\leq}$ is an ideal with $\mathsf{max}\, C^{\leq} = \emptyset$. By max-determinedness then $C^{\leq} \in \mathsf{max\,ide}\, M$. Since by downward closedness of I and J we have $C^{\leq} \subseteq I$ and $C^{\leq} \subseteq J$ it follows that $I = C^{\leq} = J$, a contradiction.
($\Leftarrow$) Assume $\mathsf{max}\, I = \emptyset$ and $I \notin \mathsf{max\,ide}\, M$. Then there is $J \neq I$ with $I \subseteq J$. Since $(M, \leq)$ separates ideals and by Corollary 15.9 then $I = I \cap J$ has dir-maxima. In particular, $\mathsf{max}\, I \neq \emptyset$, a contradiction. ■

This has the following surprising consequence:

Corollary 15.11 Let $(M, \leq)$ be max-determined. Then all elements of M are compact.

Proof: By the previous theorem, $(M, \leq)$ separates ideals.
We now first show $\sqcup I \subseteq I$ for all $I \in I(M)$. Assume $y \in \sqcup I$ and set $J \stackrel{\text{def}}{=} y^{\leq}$. We have $I \subseteq J$. If $I \neq J$ then $I = I \cap J$ has a maximal and hence, by directedness, greatest element z. But then $z = \sqcup I = y$ so that $J = I$, a contradiction.
Consider now $x \in M$ and $I \in I(M)$ such that $x \leq \sqcup I \in I$. By downward closedness of I we get $x \in I$ and x is compact. ■

The reverse implication is not valid as the following example shows: Consider



in which all elements are compact. However, for $I \stackrel{\text{def}}{=} \{0, 2, 4, \ldots\}$ we have $\max I = \emptyset$ and $I \subset J \stackrel{\text{def}}{=} \{0, 1, 2, 3, 4, \ldots\}$, i.e., I is not maximal. Concerning separation of ideals, $I = I \cap J$ doesn't have a maximal element.

It will be interesting to find further, more "manageable" characterisations of max-determinedness.

Part IV: A Particular Case: Streams

We now specialise to a particular partial order. We shall represent streams using sets of finite traces. These are finite words over an alphabet A of atomic actions; they are ordered by the prefix relation.

16 Streams and Properties

The set of streams satisfying a property $P \subseteq A^*$ is

$$\mathsf{str}\, P \stackrel{\text{def}}{=} \mathsf{ide}\, P \ .$$

Note that it would not be adequate to work with the set $\mathsf{str}\,(P^{\sqsubseteq})$, the so-called *adherence* of P (see e.g. [47, 58]), instead of $\mathsf{str}\,P$. The reason is that by prefix-closure infinite substreams may "sneak" into a cone although it results from a language of mutually $\sqsubseteq$-incomparable words which represent systems with finite behaviour only.

Example 16.1 The language $L \stackrel{\text{def}}{=} 0^* \bullet 1$ represents a behaviour with arbitrarily long but finite sequences of 0s terminated by the "explicit endmarker" 1. The words in L are mutually incomparable w.r.t. $\sqsubseteq$. Hence all directed subsets of L are singletons and their downward closures are principal ideals and hence finite. So $\mathsf{str}\,L$ consists of finite ideals only. However, the prefix closure $L^{\sqsubseteq}$ contains the infinite ideal 0^* representing the infinite stream 0^ω of 0s. So $\mathsf{str}\,L^{\sqsubseteq} = \mathsf{str}\,L \cup \{0^\omega\}$. ■

Using König's Lemma one can even show that for finite A *every* infinite cone contains an infinite stream. The general definition of ide omits these undesired streams.

So, using ide we can distiguish between erratic and angelic non-determinacy.

Example 16.2 Consider the recursive definition

$$\mathcal{B} = 0 \circ \mathcal{B} \,[]\, 1 \,,$$

where $\circ$ denotes stream concatenation (see Section 18 for a precise definition) and $[]$ denotes non-deterministic choice. In an angelic interpretation of $[]$ always eventually the terminating branch 1 is chosen, and so $\mathcal{B}$ would equal $\mathsf{str}\,L$ of Example 16.1.

In an erratic interpretation of $[]$, on the other hand, no guarantee is given that the terminating branch will ever be chosen, and so $\mathcal{B}$ would equal $\mathsf{str}\,L^{\sqsubseteq}$ of Example 16.1. ■

We want to show now that str (and hence ide) does not distribute through general union:

Example 16.3 Take $U = 0^*$. Then $U = \bigcup_{i \in \mathbb{N}} 0^i$. However, $\mathsf{str}\,U = \{0^*\} \cup \{(0^i)^{\sqsubseteq} : i \in \mathbb{N}\}$, whereas $\bigcup_{i \in \mathbb{N}} \mathsf{str}\,0^i = \{(0^i)^{\sqsubseteq} : i \in \mathbb{N}\}$. ■

17 Maximal and Infinite Streams

As already mentioned, maximal ideals model processes that go on as long as possible. For streams we have a more pleasant situation than for general ideals:

Lemma 17.1 $(A^*, \sqsubseteq)$ is max-determined.

Proof: Assume $I \in \mathsf{ide}\,A^* \wedge \mathsf{max}\,I = \emptyset$ and consider $J \in \mathsf{ide}\,A^*$ with $I \subseteq J$. By Lemma 7.4 and downward closure of I, J it suffices to show $J \leq I$. Consider $y \in J$. Since $\mathsf{max}\,I = \emptyset$ there is some $x \in I \subseteq J$ with $||y|| \leq ||x||$, where $||u||$ denotes the length of word u. Moreover, by directedness of J, there is $z \in J$ with $x \sqsubseteq z \wedge y \sqsubseteq z$. From linearity of $z^{\sqsubseteq}$ it therefore follows that $x \sqsubseteq y \vee y \sqsubseteq x$. However, since $||y|| \leq ||x||$, we must have $y \sqsubseteq x$. ■

This allows us to use all laws from Section 15 for streams. At this point it is also convenient to give the counterexample to the simplified version of Corollary 15.6:

Example 17.2 Set $U \stackrel{\text{def}}{=} 0^* \bullet 1$ and $V \stackrel{\text{def}}{=} U \cup 0^* = U^{\sqsubseteq}$ by (3). Then $U \sim V$, but $\mathsf{max\,str}\, U \neq \mathsf{max\,str}\, V$, since $0^* \in (\mathsf{max\,str}\, V) \backslash (\mathsf{max\,str}\, U)$. ■

Concerning infinite streams, we note that by Lemma 11.2 we have

$$\mathsf{inf\,ide}\, P = \{I \in \mathsf{ide}\, P : I \text{ infinite}\} .$$

To establish the relation with [58] we also show

Lemma 17.3 For $P \subseteq A^*$ we have

$$\lim P = \{I \in A^\infty : I \cap P \text{ infinite}\} .$$

Proof:

$I \in \lim P$

$\Leftrightarrow$ {[definition]}

$I \cap P \in \mathsf{dir}\, M \wedge \mathsf{max}\,(I \cap P) = \emptyset$

$\Leftrightarrow$ {[by $I \cap P \subseteq I$ and Lemma 11.1]}

$I \cap P \neq \emptyset \wedge \mathsf{max}\,(I \cap P) = \emptyset$.

We show now that, for linearly ordered $L \subseteq A^*$,

$$L \text{ infinite} \;\Leftrightarrow\; L \neq \emptyset \wedge \mathsf{max}\, L = \emptyset .$$

($\Rightarrow$) $L \neq \emptyset$ is immediate. Suppose $x \in \mathsf{max}\, L$. By linearity then $L \subseteq x^{\sqsubseteq}$. But then $|L| \leq ||x|| + 1$, a contradiction.
($\Leftarrow$) Every non-empty finite set has a maximal element. ■

To end this section, we write out specializations of some of our laws for the case of streams, since they will be used in the bounded buffer example below:

Corollary 17.4

$$\mathsf{inf\,str}\,(N \cup P) = \mathsf{inf\,str}\, P \Leftarrow N \sqsubseteq P \wedge P \text{ directed}$$
$$\mathsf{inf\,str}\, N = \mathsf{inf\,str}\, P \Leftarrow N \sqsubseteq P \wedge P \sqsubseteq N \wedge N, P \text{ directed}$$

Proof: Immediate from Lemma 17.1, Lemma 15.5 and Corollary 15.6. ■

Example 17.5 $\mathsf{inf\,str}\,((a \bullet b)^* \bullet a = \mathsf{inf\,str}\,(a \bullet b)^*$. ■

18 Stream Concatenation

As a prerequisite for defining infinite repetition we need stream concatenation which, for streams S, T is defined by

$$S \circ T \stackrel{\text{def}}{=} S \cup (\mathsf{max}\, S) \bullet T .$$

Let us explain this definition. If S is finite then $\mathsf{max}\, S$ is a singleton. This part of the overall behaviour then is prefixed to all traces in T to represent the concatenated

behaviour. If S is infinite then $\max S = \emptyset$ and hence, by strictness of $\circ$, we get $S \circ T = S$, as is intuitively expected. We have

$$\max(S \circ T) = (\max S) \bullet (\max T) .$$

It is straightforward to show that $S \circ T$ is indeed a stream and that $(A^\infty, \circ, \varepsilon)$ is a monoid. As a shorthand notation we shall also allow words as first argument of $\circ$. This is made precise by setting

$$u \circ T \stackrel{\text{def}}{=} u^\sqsubseteq \circ T = u^\sqsubseteq \cup u \bullet T .$$

Again, $\circ$ is extended pointwise to behaviours and, in the case of the above shorthand, to languages.

19 Infinite Repetition

We now give the usual greatest fixpoint definition of the set U^ω of streams that result from infinite repetition of words from a language $U \subseteq A^*$:

$$\begin{array}{l} U^\omega = U \circ U^\omega \ \wedge \\ \mathcal{X} \subseteq U \circ \mathcal{X} \ \Rightarrow \ \mathcal{X} \subseteq U^\omega . \end{array}$$

According to the Knaster-Tarski fixpoint theorem this is well-defined by monotonicity of $\circ$. Note that by this definition $\emptyset^\omega = \emptyset$. However, if $\varepsilon \in U$ then $U^\omega = A^\infty$. For that reason, U^ω is usually considered only for $\varepsilon \notin U$.

It should be noted that for $|U| \geq 2$ and $U \cap \varepsilon = \emptyset$ there are nontrivial solutions of $\mathcal{X} = U \circ \mathcal{X}$ properly less than U^ω. As an example consider the behaviour $U^* \circ \bigcup_{u \in U} u^\omega$ of all eventually periodic streams.

To tie this in with the str-operation, we quote [58], p. 433:

$$\varepsilon \notin U \ \Rightarrow \ \lim U^* = U^\omega \cup U^* \circ \lim U ,$$

or, using Lemma 15.7 and $\max$-determinedness,

$$\varepsilon \notin U \ \Rightarrow \ \inf \mathsf{str}\, U^* = U^\omega \cup U^* \circ \inf \mathsf{str}\, U .$$

From this, by strictness of $\circ$ it is immediate that

$$\varepsilon \notin U \ \Rightarrow \ \inf \mathsf{str}\, U = \emptyset \ \Rightarrow \ U^\omega = \inf \mathsf{str}\, U^* . \tag{6}$$

A sufficient condition to establish the premise is given by

Lemma 19.1 If $U \subseteq A^* \backslash \varepsilon$ satisfies the Fano condition, i.e., the words in U are mutually incomparable w.r.t. $\sqsubseteq$, then

$$U^\omega = \inf \mathsf{str}\, U^* .$$

Proof: By the Fano condition, all directed subsets of U are singletons. Hence $\mathsf{str}\, U = \{u^\leq : u \in U\}$ consists of finite streams only. ■

Note that if $\varepsilon \in U$ then U satisfies the Fano condition iff $U = \varepsilon$; for this case the above equation doesn't hold, since then $\mathsf{inf}\,\mathsf{str}\,U^* = \emptyset$. It should also be mentioned that U satisfies the Fano condition iff $U = \mathsf{max}\,U$. To see what happens if the Fano condition is not satisfied, consider

Example 19.2 Let $A = \{a, b\}$ and $U \stackrel{\text{def}}{=} \{a \bullet b^n : n \in \mathbb{N}\} \subseteq A^*$. Then $U \in \mathsf{dir}\,U^*$, since $U \subseteq U^*$ and U is directed. Hence $U^{\sqsubseteq} = \varepsilon \cup U \in \mathsf{str}\,U^*$ and, since $U^{\sqsubseteq}$ is infinite, even $U^{\sqsubseteq} \in \mathsf{inf}\,\mathsf{str}\,U^*$. Now, $U^{\sqsubseteq}$ represents an a followed by infinitely many bs; but this behaviour clearly does not arise from repeated concatenation of words in U. It is "sneaked in" by the fact that simply considering directed subsets of U^* throws away too much structural information. ■

To allow a characterization of U^ω for languages that do not satisfy the Fano condition, one can artificially enforce it by attaching a special endmarker to all words in U and remove it after singling out the infinite streams. Let $\# \notin A$ be a new letter and consider streams over the extended alphabet $A \cup \#$. Moreover, denote by $A \lhd u$ the word that results from u by removing all occurrences of $\#$ and extend the operation $A\lhd$ pointwise to languages and behaviours. Then we have

Lemma 19.3 For $U \subseteq A^*\backslash\varepsilon$,

$$U^\omega \stackrel{\text{def}}{=} A \lhd \mathsf{inf}\,\mathsf{str}\,(U \bullet \#)^* \ .$$

For the somewhat tedious proof see [41].

The streams in $\mathsf{str}\,(U \bullet \#)^*$ correspond to finite and infinite sequences that result from concatenating arbitrary elements of U with the separator $\#$ in between. The operation max then selects the prefix-maximal ones of these; if $\varepsilon \notin U$ these are precisely the infinite words resulting from repeatedly concatenating words from U. The separators are used to record the "construction history" of the streams; they are finally thrown away again by the filter $A\lhd$. In this way subsets of U^* which are directed "by accident" are ignored. A similar mechanism for defining iteration is employed in [48] in the finite case and in [10] in the infinite case.

20 Streams of Functions

We have made no assumptions about our alphabet A. Hence it may even be a set of functions. Then streams over A model components with time-dependent behaviour. We have seen examples of this in the description of various faulty channels in Section 5.

A stream S of arguments is fed into a stream F of functions by $S >> F$. A wordwise definition of this is

$$\begin{aligned} \varepsilon >> w &\stackrel{\text{def}}{=} \varepsilon \ , \\ s >> \varepsilon &\stackrel{\text{def}}{=} \varepsilon \ , \\ a \bullet s >> f \bullet w &\stackrel{\text{def}}{=} f(a) \bullet (s >> w) \ . \end{aligned}$$

This operation is extended pointwise to languages and behaviours. In index notation we have

$$(S >> F)_i \;=\; F_i(S_i)\ ,$$

provided $i \leq \mathsf{min}\,\{|S|, |F|\}$.

Example 20.1 For finite stream S we have

$$(S \bullet T) >> cchan_* \;=\; (S >> arbchan) \bullet (T >> cchan_*)\ .$$

This reflects the unbounded fainess of $cchan_*$: we have no guarantee *when* correct transmission occurs, and hence the elements of S may or may not be transmitted correctly. ■

With bound assumptions one gets more precise information:

Example 20.2 We have

$$m > k \;\Rightarrow\; (a^m \bullet T) >> cchan_{\leq k} \;\in\; A^{\leq k} \bullet a \bullet A^\infty\ .$$

A channel with fairness bound k *must* transmit a correctly at least once if it receives more than k copies of a. ■

21 Feedback and State-Based Systems

21.1 The Feedback Operation

An essential operation on SPFs is *feedback* of some outputs to the inputs. Assume an SPF $F : A^\infty \times B^\infty \;\to\; A^\infty \times C^\infty$. Then its feedback $feedF : B^\infty \;\to\; C^\infty$ is given by

$$(feedF)(S) \;=\; T\ \mathsf{where}\ (Z, T) = F(Z, S)\ .$$

For an illustration see Figure 1.

The semantics of this recursive declaration is the usual least-fixpoint one. This version of the feedback operator hides the feedback stream. If this is to made visible one simply copies it and feeds one copy back whereas the other is transmitted to the outside.

21.2 State-Based Systems and Automata

This operation together with streams of functions allows a very convenient and concise description of state-based systems.

Assume a set Q of states, an input alphabet A and an output alphabet B. Then a *time-dependent automaton* is given by a stream $H \in (Q \times A \;\to\; Q \times B)^\infty$.

We may now feed this automaton with a starting state $q_0 \in Q$ and a stream $S \in A^\infty$ of input values to produce a stream of output values in B^∞. The stream of states entered during the processing of the input is constructed by a feedback and hidden from the outside. This is described by

$$auto(H, q_0, S) = T\ \mathsf{where}\ (Z, T) = (q_0 \bullet Z, S) >> H\ .$$

By placing various restrictions on the entities involved, we can distinguish a hierarchy of automata:

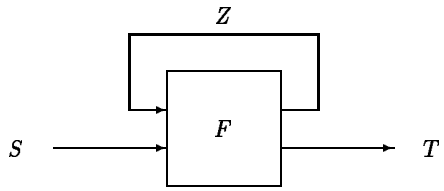


Fig. 1. The feedback operation

- If no further restrictions are made, we obtain *timed and state-dependent* automaton.
- If we require $|Q| = 1$ then we have a *timed and state-independent* automaton.
- If we take $H = f^\omega$ for some $f : Q \times A \to Q \times B$, we obtain a *timeless and state-dependent* automaton.
- If we again take $H = f^\omega$ but also require $|Q| = 1$, we have a *timeless and state-independent* automaton.

Fo example, an easy proof by induction over the structure of the finite words shows

Lemma 21.1 If $|Q| = 1$ then

$$auto(omf, q_0, S) = S >> g^\omega \ ,$$

where $g(i) = \pi_2(f(q_0, i))$.

We now illustrate the general case by the following

Example 21.2 We give a description of a one-place asynchronous buffer. The example is taken fom [9]. Consider a set D of data. The input alphabet is

$$A \stackrel{\text{def}}{=} D \cup \{!\} \ .$$

An input $d \in D$ means that d is to be stored in the buffer, whereas ! means a request for the current contents of the buffer.

At each time point the buffer may accept or reject its input which is shown by a boolean value. In addition to that the buffer will output data if it accepts the request signal. So we choose the output alphabet

$$B \stackrel{\text{def}}{=} (D \cup \{\varepsilon\}) \times \mathbb{B} \ ,$$

where ε models the case of no proper output.

As the set of states we choose

$$Q \stackrel{\text{def}}{=} D \cup \{\varepsilon\}$$

where ε models the state of being empty whereas $d \in D$ models the state of containing value d.

Now we define two transition functions

$$acc, rej : Q \times A \to Q \times B$$

which model acceptance and rejection of the input. We have

$$\begin{aligned} acc(q,d) &= (\text{if } q = \varepsilon \text{ then } d \text{ else } q, (\varepsilon, q = \varepsilon)) \ , \\ acc(q,!) &= (\varepsilon, (q, q \neq \varepsilon)) \ , \\ rej(q,x) &= (q, (\varepsilon, \text{false})) \ . \end{aligned}$$

The behaviour of a fair buffer, i.e., one which rejects inputs only finitely many times before eventually accepting one is the specified as

$$auto((rej^* \bullet acc)^\omega, \varepsilon) \ .$$

In particular, we can avoid the use of prophecy variables (see e.g. [9]) in this style. ■

22 Processes and Synchronised Parallel Composition

While the previous two sections are appropriate for the SPF view of distributed systems, we now define operators that are adequate for the trace view (cf. Section 6). The particular definitions given here draw strongly on the corresponding ones in [24].

Assume an overall alphabet A for our streams. A *process* is a pair $(B, \mathcal{B})$ where $B \subseteq A$ is the *alphabet* of the process and $\mathcal{B} \subseteq B^\infty$ is a behaviour. We set

$$\alpha(B,\mathcal{B}) \stackrel{\text{def}}{=} B \ , \qquad \beta(B,\mathcal{B}) \stackrel{\text{def}}{=} \mathcal{B} \ .$$

An axiliary operation is the *projection* of words to an alphabet $B \subseteq A$, It is defined inductively as follows:

$$\begin{aligned} \varepsilon \dagger B &\stackrel{\text{def}}{=} \varepsilon \\ (a \bullet s) \dagger B &\stackrel{\text{def}}{=} \begin{cases} a \bullet (s \dagger B) & \text{if } a \in B \\ s \dagger B & \text{otherwise.} \end{cases} \end{aligned}$$

Projection is extended pointwise to languages and behaviours. The projection of a stream is a stream again.

Using projection we can characterize processes in another way: the pair $(B, \mathcal{B})$ is a process iff $\forall\, S \in \mathcal{B} : S \dagger B = S$.

We need to lift the notion of refinement to processes. We allow that a process is refined by another one that has addtional "internal" actions. Since then refinement amounts to inclusion of (the projection of) the behaviour, we abuse notation and write again $\subseteq$ for the refinement relation:

$$P \subseteq Q \stackrel{\text{def}}{\Leftrightarrow} \alpha P \supseteq \alpha Q \wedge (\beta P) \dagger \alpha Q \subseteq \beta Q \ .$$

In this case we say that P *refines* Q. It is easily checked that $\subseteq$ is a partial order on processes.

If behaviours are "loose enough" in that they allow arbitrary actions in between the "interesting" ones, one can model synchronised parallel composition very simply by intersection (see e.g. [25]). For general behaviours this works well only if they are "loosened" by interspersing arbitrary actions between the proper ones; this is again taken from [24]. The intersection then allows only traces in which the actions interesting to both partners occur in a sequence that is acceptable to both partners (i.e., allowed in both behaviours) whereas the private actions of each partner are not constrained by the other partner.

Hence, for processes P, Q, we define the parallel composition $P||Q$ by setting

$$\begin{aligned} \alpha(P||Q) &\stackrel{\text{def}}{=} \alpha P \cup \alpha Q \ , \\ S \in \beta(P||Q) &\stackrel{\text{def}}{\Leftrightarrow} S = S \dagger \alpha(P||Q) \wedge \\ &\quad\ S \dagger \alpha P \in \beta P \wedge \\ &\quad\ S \dagger \alpha Q \in \beta Q \ . \end{aligned}$$

Note, in particular, that $||$ is commutative, associative and idempotent then. Moreover,

$$P \subseteq Q \ \Leftrightarrow \ P||Q = P \ .$$

If $\alpha P = \alpha Q$ then $\beta(P||Q) = \beta P \cap \beta Q$.

This parallel composition operator will be used in our extended example in Section VI.

Part V: Safety and Liveness

We have already informally discussed safety and liveness (see e.g. [31, 1, 20]). We want to show how these notions can be expressed algebraically. In [1] and subsequent papers, a *property* is a set of infinite sequences of states. The appropriate counterpart in our setting is therefore a set of streams, more generally, ideals, i.e., a *behaviour*.

23 Safety

23.1 Definition and Topological Properties

In [1] a behaviour $\mathcal{B} \subseteq A^\omega$ over infinite streams is called *safe* if the following holds:

$$\forall\, S \in A^\omega : S \notin \mathcal{B} \Rightarrow (\exists\, s \in S : \forall\, T \in A^\omega : s \circ T \notin \mathcal{B}) \ .$$

This means that for every stream not in the behaviour there is a decisive finite prefix s where something went "irrepairably wrong" in that *no* continuation of s can bring the computation back to the "good path".

We want to simplify the formal definition above by moving from logic to algebra. First, using contraposition, the formula can be transformed to

$$\forall\, S \in A^\omega : (\forall\, s \in S : \exists\, T \in A^\omega : s \circ T \in \mathcal{B}) \Rightarrow S \in \mathcal{B} \ .$$

Now, recalling the definition $\mathsf{pref}\,\mathcal{B} = \cup\mathcal{B}$ from Section 13, we have

$$\exists\, T \in A^\omega : s \circ T \in \mathcal{B} \;\Leftrightarrow\; s \in \mathsf{pref}\,\mathcal{B}\ . \tag{7}$$

Hence the safety condition reduces to

$$\begin{array}{ll} & \forall\, S \in A^\omega : (\forall\, s \in S : s \in \mathsf{pref}\,\mathcal{B}) \;\Rightarrow\; S \in \mathcal{B} \\ \Leftrightarrow & \quad \{\![\text{ set theory }]\!\} \\ & \forall\, S \in A^\omega : S \subseteq \mathsf{pref}\,\mathcal{B} \;\Rightarrow\; S \in \mathcal{B} \\ \Leftrightarrow & \quad \{\![\text{ by prefix-closedness of } \mathsf{pref}\,\mathcal{B} \text{ and Lemma 14.4.1 }]\!\} \\ & \forall\, S \in A^\omega : S \in \mathsf{str}\,\mathsf{pref}\,\mathcal{B} \;\Rightarrow\; S \in \mathcal{B} \\ \Leftrightarrow & \quad \{\![\text{ defining } \mathsf{should}\,\mathcal{B} \stackrel{\text{def}}{=} \mathsf{str}\,\mathsf{pref}\,\mathcal{B} \ ]\!\} \\ & \mathsf{should}\,\mathcal{B} \subseteq \mathcal{B}\ . \end{array}$$

This simplified form involves only order-theoretic notions and hence generalizes easily to arbitrary ideal completions. Consider a partial order $(M, \leq)$ and a behaviour $\mathcal{B} \subseteq M^\infty$. Then we call $\mathcal{B}$ *safe* iff $\mathsf{should}\,\mathcal{B} \subseteq \mathcal{B}$, where

$$\mathsf{should}\,\mathcal{B} \stackrel{\text{def}}{=} \mathsf{ide}\,\mathsf{pref}\,\mathcal{B}\ .$$

By $\subseteq$-monotonicity of pref and ide also should is $\subseteq$-monotonic. Note that for all $\mathcal{B} \subseteq M^\infty$ we have $\mathcal{B} \subseteq \mathsf{should}\,\mathcal{B}$. So a behaviour $\mathcal{B} \subseteq M^\infty$ is safe iff $\mathcal{B} = \mathsf{should}\,\mathcal{B}$. Moreover,

Lemma 23.1 1. Safe behaviours are closed under arbitrary intersections and finite unions.
2. should is idempotent.
3. $\mathsf{should}\,\mathcal{B}$ is the least safe behaviour containing $\mathcal{B}$.

Proof: 1. Assume a family $(\mathcal{B}_j)_{j\in J}$ of safe behaviours. Then for all $j \in J$ we have by monotonicity of should and safety of $\mathcal{B}_j$ that

$$\mathsf{should}\,(\bigcap_{j\in J} \mathcal{B}_j) \subseteq \mathsf{should}\,\mathcal{B}_j \subseteq \mathcal{B}_j\ ,$$

so that

$$\mathsf{should}\,(\bigcap_{j\in J} \mathcal{B}_j) \subseteq \bigcap_{j\in J} \mathcal{B}_j\ ,$$

i.e., $\bigcap_{j\in J} \mathcal{B}_j$ is safe again.

For union we calculate, for $I \in M^\infty$,

$$\begin{array}{ll} & I \in \mathsf{should}\,(\mathcal{B} \cup \mathcal{C}) \\ \Leftrightarrow & \quad \{\![\text{ definition and distributivity of } \mathsf{pref} \ ]\!\} \\ & I \subseteq \mathsf{pref}\,\mathcal{B} \cup \mathsf{pref}\,\mathcal{C} \\ \Leftrightarrow & \quad \{\![\text{ boolean algebra }]\!\} \\ & I = (I \cap \mathsf{pref}\,\mathcal{B}) \cup (I \cap \mathsf{pref}\,\mathcal{C})\ . \end{array}$$

Set now $I_{\mathcal{B}} \stackrel{\text{def}}{=} I \cap \mathsf{pref}\,\mathcal{B}$ and $I_{\mathcal{C}} \stackrel{\text{def}}{=} I \cap \mathsf{pref}\,\mathcal{C}$. Since I is directed, by Lemma 9.1.2 we have $I_{\mathcal{B}} \leq I_{\mathcal{C}}$ or $I_{\mathcal{C}} \leq I_{\mathcal{B}}$. So by downward closedness of $I_{\mathcal{B}}$ and $I_{\mathcal{C}}$ and Lemma 7.3.3 we have $I_{\mathcal{B}} \subseteq I_{\mathcal{C}}$ or $I_{\mathcal{C}} \subseteq I_{\mathcal{B}}$ and hence $I = I_{\mathcal{B}}$ or $I = I_{\mathcal{C}}$. But then, by boolean algebra and the definition, we get $I \in \mathsf{should}\,\mathcal{B} \vee I \in \mathsf{should}\,\mathcal{C}$, so that by safety of $\mathcal{B}$ and $\mathcal{C}$ also $I \in \mathcal{B} \cup \mathcal{C}$.

2. For $I \in M^\infty$ we have

$$
\begin{array}{ll}
 & I \in \mathsf{should}\,\mathsf{should}\,\mathcal{B} \\
\Leftrightarrow & \quad \{\![\text{ definitions }]\!\} \\
 & I \subseteq \bigcup\{J \in M^\infty : J \subseteq \mathsf{pref}\,\mathcal{B}\} \\
\Leftrightarrow & \quad \{\![\text{ using the principal ideals } J = x^{\leq} \text{ for } x \in \mathsf{pref}\,\mathcal{B} \,]\!\} \\
 & I \subseteq \mathsf{pref}\,\mathcal{B} \\
\Leftrightarrow & \quad \{\![\text{ definitions }]\!\} \\
 & I \in \mathsf{should}\,\mathcal{B}\ .
\end{array}
$$

3. Let $\mathcal{C}$ be safe with $\mathcal{B} \subseteq \mathcal{C}$. Then

$$
\begin{array}{ll}
 & \mathsf{should}\,\mathcal{B} \\
\subseteq & \quad \{\![\text{ monotonicity }]\!\} \\
 & \mathsf{should}\,\mathcal{C} \\
= & \quad \{\![\text{ safety of } \mathcal{C} \,]\!\} \\
 & \mathcal{C}\ .
\end{array}
$$

But $\mathsf{should}\,\mathcal{B}$ is safe by 2.

■

By these properties, the safe behaviours coincide with the closed sets of a topology on M^∞ (cf. e.g. [57]) and should is the topological closure operator.

23.2 Safety and Snapshot sets

Let us now study how safety is reflected in snapshot sets. In other words, we want to know when for $P \subseteq M$ the behaviour $\mathsf{ide}\,P$ is safe. We calculate, for $I \in M^\infty$,

$$
\begin{array}{ll}
 & I \in \mathsf{should}\,\mathsf{ide}\,P \\
\Leftrightarrow & \quad \{\![\text{ definition of should }]\!\} \\
 & I \in \mathsf{ide}\,\mathsf{pref}\,\mathsf{ide}\,P \\
\Leftrightarrow & \quad \{\![\text{ by Lemma 14.4.2 }]\!\} \\
 & I \in \mathsf{ide}\,(P^{\leq}) \\
\Leftrightarrow & \quad \{\![\text{ by Lemma 14.4.1 }]\!\} \\
 & I \subseteq P^{\leq}
\end{array}
$$

and hence

$\mathsf{ide}\, P$ is safe

$\Leftrightarrow$ $\quad \{\![$ by the above $]\!\}$

$\forall\, I \in M^\infty : I \subseteq P^\leq \Rightarrow I \in \mathsf{ide}\, P$

$\Rightarrow$ $\quad \{\![\, \forall\, u \in P : u^\leq \subseteq P^\leq \,]\!\}$

$\forall\, u \in P^\leq : u^\leq \in \mathsf{ide}\, P$

$\Rightarrow$ $\quad \{\![$ since for $D \in \mathsf{dir}\, P$ we have $D^\leq = u^\leq \Leftrightarrow u \in D \,]\!\}$

$P^\leq \subseteq P$.

On the other hand,

$$P^\leq \subseteq P \Rightarrow \forall\, I \in M^\infty : I \subseteq P^\leq \Rightarrow I \subseteq P\ .$$

Altogether we have shown

Lemma 23.2 The behaviour $\mathsf{ide}\, P$ is safe iff $P^\leq \subseteq P$, i.e., iff P is downward closed.

For that reason we call a snapshot set $P \subseteq M$ a *safety property* iff it is downward closed. We have

Corollary 23.3 If $I \in M^\infty$ and P is a safety property, then $I \in \mathsf{ide}\, P \Leftrightarrow I \subseteq P$.

Proof: immediate from Lemma 14.4.1. ■

For a safety property P the behaviour $\mathsf{ide}\, P$ is closed under unions (i.e., suprema) of $\subseteq$-ascending chains of streams. In the special case of streams, safety properties are simply prefix-closed subsets of A^*.

24 Continual Satisfaction

24.1 The General Case

In connection with safety issues one is interested in the set of all objects that satisfy a property also in all their finite approximations. Given a property $P \subseteq M$ we define the property $\mathsf{saf}\, P$ by

$$\mathsf{saf}\, P \stackrel{\mathrm{def}}{=} \{x \in M : x^\leq \subseteq P\}\ .$$

The set $\mathsf{saf}\, P$ has also been termed the *prefix kernel* of P in [48, 63]. We have

Lemma 24.1 1. $\mathsf{saf}\, P \subseteq P$.
2. $\mathsf{saf}\, P = P$ iff P is a safety property.
3. $\mathsf{saf}\, P$ is the greatest safety property contained in P.
4. saf is monotonic and strict w.r.t. $\emptyset$.
5. $\mathsf{saf}\,(P \cap Q) = \mathsf{saf}\, P \cap \mathsf{saf}\, Q$.
6. $I \in \mathsf{ide}\,\mathsf{saf}\, P \Leftrightarrow I \subseteq P$.

Proof: 1.

$$
\begin{array}{ll}
 & x \in \mathsf{saf}\, P \\
\Leftrightarrow & \quad \{\![\text{ definition }]\!\} \\
 & x^{\leq} \subseteq P \\
\Rightarrow & \quad \{\![\; x \in x^{\leq} \;]\!\} \\
 & x \in P \; .
\end{array}
$$

2. ($\Rightarrow$)

$$
\begin{array}{ll}
 & x \in P \\
\Leftrightarrow & \quad \{\![\text{ assumption }]\!\} \\
 & x \in \mathsf{saf}\, P \\
\Leftrightarrow & \quad \{\![\text{ definition }]\!\} \\
 & x^{\leq} \subseteq P \; .
\end{array}
$$

($\Leftarrow$)

$$
\begin{array}{ll}
 & x \in P \\
\Rightarrow & \quad \{\![\text{ assumption }]\!\} \\
 & x^{\leq} \subseteq P \\
\Leftrightarrow & \quad \{\![\text{ definition }]\!\} \\
 & x \in \mathsf{saf}\, P
\end{array}
$$

so $P \subseteq \mathsf{saf}\, P$; the reverse inclusion was shown in 1.

3. It is obvious that $\mathsf{saf}\, P$ is a safety property. Let $Q \subseteq P$ be a safety property and $x \in Q$. By definition then $x^{\leq} \subseteq Q \subseteq P$ and hence $x \in \mathsf{saf}\, P$.
4. is immediate from the definition.
5.

$$
\begin{array}{ll}
 & x \in \mathsf{saf}\, (P \cap Q) \\
\Leftrightarrow & \quad \{\![\text{ definition }]\!\} \\
 & x^{\leq} \subseteq P \cap Q \\
\Leftrightarrow & \quad \{\![\text{ infimum property of intersection }]\!\} \\
 & x^{\leq} \subseteq P \wedge x^{\leq} \subseteq Q \\
\Leftrightarrow & \quad \{\![\text{ definition }]\!\} \\
 & x \in \mathsf{saf}\, P \wedge x \in \mathsf{saf}\, Q \; .
\end{array}
$$

6.

$$
\begin{array}{ll}
 & I \in \mathsf{ide}\, \mathsf{saf}\, P \\
\Leftrightarrow & \quad \{\![\text{ by Lemma 14.1 }]\!\} \\
 & I \subseteq (I \cap \mathsf{saf}\, P)^{\leq} \\
\Leftrightarrow & \quad \{\![\text{ since } \mathsf{saf}\, P \subseteq P \;]\!\} \\
 & I \subseteq \mathsf{saf}\, P^{\leq} \\
\Leftrightarrow & \quad \{\![\text{ by downward closedness of } \mathsf{saf}\, P \;]\!\}
\end{array}
$$

$$
\begin{array}{ll}
& I \subseteq \mathsf{saf}\, P \\
\Leftrightarrow & \quad \{\![\text{ by downward closedness of } I \,]\!\} \\
& I \subseteq P \ .
\end{array}
$$

■

Note that saf does not distribute through union. We can now state a further distributivity property for ide:

Lemma 24.2 Consider $N, P \subseteq M$. Then

$$N = \mathsf{saf}\, N \Rightarrow \mathsf{ide}\,(N \cap P) = \mathsf{ide}\, N \cap \mathsf{ide}\, P \ .$$

Proof: We only need to show ($\supseteq$), since the reverse inclusion follows from monotonicity of ide.
Assume $S \in \mathsf{ide}\, N \cap \mathsf{ide}\, P$, say $S = D^{\leq} = E^{\leq}$ with $D \in \mathsf{dir}\, N \wedge E \in \mathsf{dir}\, P$. By Lemma 7.4 then $E \leq D$, and by Lemma 7.3.2 we get $E \leq N$, since $D \subseteq N$. Now $N = N^{\leq}$ shows $E \subseteq N$. Since $E \subseteq P$ we get $E \subseteq N \cap P$ and, since E is directed, even $E \in \mathsf{dir}\,(N \cap P)$. This shows that $S = E^{\leq}$ and hence $S \in \mathsf{ide}\,(N \cap P)$. ■

24.2 Deriving a Recursion for saf

Next, for the particular case of streams we want to derive a grammar-like or automaton-like representation for safety properties of the form $\mathsf{saf}\, P$ for some $P \subseteq A^*$. We use induction on the words involved. For the induction base we calculate

$$
\begin{array}{ll}
& \varepsilon \in \mathsf{saf}\, P \\
\Leftrightarrow & \quad \{\![\text{ definition }]\!\} \\
& \varepsilon^{\sqsubseteq} \subseteq P \\
\Leftrightarrow & \quad \{\![\varepsilon^{\sqsubseteq} = \varepsilon \,]\!\} \\
& \varepsilon \in P \ .
\end{array}
$$

For the induction step, we have, for arbitrary $c \in A$,

$$
\begin{array}{ll}
& c \bullet s \in \mathsf{saf}\, P \\
\Leftrightarrow & \quad \{\![\text{ definition }]\!\} \\
& (c \bullet s)^{\sqsubseteq} \subseteq P \\
\Leftrightarrow & \quad \{\![\text{ by }]\!\} \\
& c^{\sqsubseteq} \cup c \bullet s^{\sqsubseteq} \subseteq P \\
\Leftrightarrow & \quad \{\![\text{ set theory }]\!\} \\
& c^{\sqsubseteq} \subseteq P \wedge c \bullet s^{\sqsubseteq} \subseteq P \\
\Leftrightarrow & \quad \{\![c^{\sqsubseteq} = \varepsilon \cup c \,]\!\} \\
& \varepsilon \in P \wedge c \in P \wedge c \bullet s^{\sqsubseteq} \subseteq P \ .
\end{array}
$$

We assume now that P itself is already given in the form of an automaton-like recursion. Then there is a systematic way for passing from that to a recursion for $\mathsf{saf}\, P$. Suppose that P satisfies, for all $c \in A$ and $U \subseteq A^*$,

$$c \bullet U \subseteq P \Leftrightarrow U \subseteq F_c(P) \tag{8}$$

for some function $F : A \to (\mathcal{P}(A^*) \to \mathcal{P}(A^*))$. In other words, we assume that the "recursive call" $F_c(P)$ depends only on the first symbol of the word to be analysed. Not that this assumption means a Galois connection between $c\bullet$ and F_c.

Under this assumption we can continue as follows:

$$\begin{array}{ll} & c \bullet s^{\sqsubseteq} \subseteq P \\ \Leftrightarrow & \quad \{\![\text{ by assumption (8) }]\!\} \\ & s^{\sqsubseteq} \subseteq F_c(P) \\ \Leftrightarrow & \quad \{\![\text{ definition }]\!\} \\ & s \in \mathsf{saf}\, F_c(P) \ . \end{array}$$

Note that a bi-implication linear in s results. To sum up, we have shown

Lemma 24.3 Suppose property $P \in \mathcal{P}(A^*)$ satisfies

$$c \bullet U \subseteq P \Leftrightarrow U \subseteq F_c(P) \ .$$

Then

$$\begin{array}{r} \varepsilon \in \mathsf{saf}\, P \Leftrightarrow \varepsilon \in P \ , \\ c \bullet U \subseteq \mathsf{saf}\, P \Leftrightarrow \varepsilon \in P \wedge c \in P \wedge U \subseteq \mathsf{saf}\, F_c(P) \ . \end{array}$$

Assume now that we are given two properties P and Q and seek a recursion for $\mathsf{saf}\, P \cap \mathsf{saf}\, Q = \mathsf{saf}\,(P \cap Q)$. The following result is immediate from Lemma 24.1.5 and Lemma 24.3:

Lemma 24.4 Suppose P, Q satisfy

$$(c \bullet U \subseteq P \Leftrightarrow U \subseteq F_c(P)) \wedge (c \bullet U \subseteq Q \Leftrightarrow U \subseteq G_c(P)) \ .$$

Then

$$\begin{array}{r} \varepsilon \in \mathsf{saf}\,(P \cap Q) \Leftrightarrow \varepsilon \in P \cap Q \ , \\ c \bullet U \subseteq \mathsf{saf}\,(P \cap Q) \Leftrightarrow c^{\leq} \subseteq P \cap Q \wedge U \subseteq \mathsf{saf}\,(F_c(P) \cap G_c(Q)) \ . \end{array}$$

This corresponds to the construction of a product automaton.

25 Liveness

25.1 Definition and Topological Properties

Following again [1] we call a behaviour $\mathcal{B}$ over streams *live* iff

$$\forall\, s \in A^* : \exists\, T \in A^\omega : s \circ T \in \mathcal{B}\ .$$

Using again (7) we can reduce this to

$$\forall\, s \in A^* : s \in \mathsf{pref}\, \mathcal{B}$$

and hence to

$$A^* \subseteq \mathsf{pref}\, \mathcal{B}\ .$$

Since A^* is the set of compact elements of A^∞ we can again easily generalize this to arbitrary ideal completions. Consider a partial order $(M, \leq)$ and a behaviour $\mathcal{B} \subseteq M^\infty$. Then we call $\mathcal{B}$ is called *live* iff

$$M \subseteq \mathsf{pref}\, \mathcal{B}\ .$$

We show now (see again [1])

Lemma 25.1 $\mathcal{B}$ is live iff it is topologically dense in M^∞, i.e., iff $\mathsf{should}\, \mathcal{B} = M^\infty$.

Proof:

$M \subseteq \mathsf{pref}\, \mathcal{B}$

$\Leftrightarrow$ {[for ($\Rightarrow$) use transitivity of inclusion, for ($\Leftarrow$) the principal ideals $J = x^{\leq}$ for $x \in \mathsf{pref}\, \mathcal{B}$]}

$\forall\, J \in M^\infty : J \subseteq \mathsf{pref}\, \mathcal{B}$

$\Leftrightarrow$ {[by Corollary 23.3 and the definition of should]}

$\forall\, J \in M^\infty : J \in \mathsf{should}\, \mathcal{B}$

$\Leftrightarrow$ {[set theory]}

$M^\infty \subseteq \mathsf{should}\, \mathcal{B}$

$\Leftrightarrow$ {[set theory]}

$M^\infty = \mathsf{should}\, \mathcal{B}$.

■

Now we obtain

Lemma 25.2 Every behaviour is the intersection of a live and a safe behaviour.

Proof: We could copy the proof of the respective theorem in [1] verbatim, since it proceeds purely in topological terms. However, we give a simpler proof that avoids most of the topological reasoning in [1].
Assume $\mathcal{B} \subseteq M^\infty$. We have

$\mathcal{B}$

$=$ {[since $\mathcal{B} \subseteq \mathsf{should}\,\mathcal{B}$]}

$\mathsf{should}\,\mathcal{B} \backslash (\mathsf{should}\,\mathcal{B} \backslash \mathcal{B})$

$=$ {[definition of $\backslash$, where $\overline{\mathcal{C}}$ denotes the complement of $\mathcal{C}$ w.r.t. M^∞]}

$\mathsf{should}\,\mathcal{B} \cap \overline{\mathsf{should}\,\mathcal{B} \cap \overline{\mathcal{B}}}$

$=$ {[de Morgan and double complement]}

$\mathsf{should}\,\mathcal{B} \cap (\overline{\mathsf{should}\,\mathcal{B}} \cup \mathcal{B})$.

Since $\mathsf{should}\,\mathcal{B}$ is safe, the claim is shown if $\overline{\mathsf{should}\,\mathcal{B}} \cup \mathcal{B}$ is live. We calculate

$\mathsf{should}\,(\overline{\mathsf{should}\,\mathcal{B}} \cup \mathcal{B})$

$\supseteq$ {[since $\subseteq$-monotonic and hence superdistributive over $\cup$]}

$\mathsf{should}\,(\overline{\mathsf{should}\,\mathcal{B}}) \cup \mathsf{should}\,\mathcal{B}$

$\supseteq$ {[since should is extensive]}

$\overline{\mathsf{should}\,\mathcal{B}} \cup \mathsf{should}\,\mathcal{B}$

$=$ {[definition of complement]}

M^∞ ,

so that we are done by Lemma 25.1. ■

Inspection of the proof leads to the following abstraction. Consider a boolean algebra $(K, \leq)$ with greatest element $\top$. Call a function $f : K \to K$ a *pre-closure* if it is extensive, i.e., satisfies $\forall\, x : x \leq f(x)$, and monotonic. Next, say that $y \in K$ is *f-dense* if $f(y) = \top$. Then we have

Corollary 25.3 Every element x of K is the meet of an f-image and an f-dense element, viz.

$$x = f(x) \sqcap (\overline{f(x)} \sqcup x) .$$

Another way of replacing the topological proof of Lemma 25.2 in [1] by a proof over boolean algebras is presented in [23]. However, our proof is simpler still.

25.2 Liveness and Snapshot Sets

As in the case of safety, we now investigate when a property P spans a live behaviour. We calculate

$\mathsf{ide}\,P$ is live

$\Leftrightarrow$ {[definition]}

$M \subseteq \mathsf{pref}\,\mathsf{ide}\,P$

$\Leftrightarrow$ {[by Lemma 14.4.2]}

$$
\begin{array}{ll}
& M \subseteq P^{\leq} \\
\Leftrightarrow & \quad \{\![\text{ definition of } \leq]\!\} \\
& M \leq P \ .
\end{array}
$$

Hence we call $P \subseteq M$ a *liveness property* iff $M \leq P$.

25.3 Spanning Infinite Behaviours by Snapshot Sets

We now define that part of a snapshot set that is relevant for the infinite streams. We call a set $Q \subseteq M$ *lively* iff $Q \neq \emptyset \wedge \max Q = \emptyset$.

Lemma 25.4 1. If Q is lively and $x \in Q$ then there is an $I \in \inf \mathsf{ide}\, Q$ with $x \in I$.
2. If Q is lively then $\inf \mathsf{ide}\, Q \neq \emptyset$.
3. If M itself is lively then for every $\mathcal{B}$ we have $M^{\infty} \subseteq \mathcal{B}$ iff $\inf M^{\infty} \subseteq \inf \mathcal{B}$.

Proof: 1. We construct a chain $(x_i)_{i \in \mathbb{N}}$ as follows. Choose $x_0 \stackrel{\text{def}}{=} x$. Assume now that x_i has been chosen. Since $x_i \notin \max Q = \emptyset$, there is an $x_{i+1} \in Q$ with $x_i < x_{i+1}$.
By construction then $K \stackrel{\text{def}}{=} \{x_i : i \in \mathbb{N}\} \in \mathsf{dir}\, Q$ and hence $I \stackrel{\text{def}}{=} K^{\leq} \in \mathsf{ide}\, Q$. Moreover, $\max I = \max K = \emptyset$, i.e, $I \in \inf \mathsf{ide}\, Q$.
2. Immediate from 1.
3. Immediate from 1.

■

In connection with the results below, property 3. will allow easier liveness proofs. Note that this is particularly relevant for the case of streams, since the set A^* of compact elements itself is lively.

Now we define the *live part* of $P \subseteq M$ as

$$\mathsf{liv}\, P \stackrel{\text{def}}{=} \bigcup \mathcal{L}_P$$

where

$$\mathcal{L}_P \stackrel{\text{def}}{=} \{Q \subseteq P : Q \text{ lively}\} \ .$$

This operation enjoys the following properties:

Lemma 25.5 1. $\mathsf{liv}\, P \subseteq P$.
2. $\max \mathsf{liv}\, P = \emptyset$.
3. P is lively iff $P \neq \emptyset \wedge P = \mathsf{liv}\, P$.
4. liv is $\subseteq$-monotonic.
5. $\mathsf{liv}\, \mathsf{liv}\, P = \mathsf{liv}\, P$.
6. $\mathsf{liv}\, P \neq \emptyset \Rightarrow \inf \mathsf{ide}\, P \neq \emptyset$.
7. $\mathcal{L}_P \neq \emptyset \Rightarrow \inf \mathsf{ide}\, P \neq \emptyset$.
8. $\inf \mathsf{ide}\, P = \inf \mathsf{ide}\, \mathsf{liv}\, P$.
9. $\mathsf{pref}\, \inf \mathsf{ide}\, P = (\mathsf{liv}\, P)^{\leq}$.

Proof: 1. is clear from the definition.
2. Assume $x \in \max \mathsf{liv}\, P \subseteq \mathsf{liv}\, P$. Then there is a $Q \in \mathcal{L}_P$ with $x \in Q$. Since $x \notin \emptyset = \max Q$, there is $y \in Q \subseteq P$ with $x < y$. Contradiction!

3. The implication ($\Rightarrow$) is clear. For the converse we use that $\max P = \max \mathsf{liv}\, P = \emptyset$ by 2.
4. We have $P \subseteq Q \Rightarrow \mathcal{L}_P \subseteq \mathcal{L}_Q \Rightarrow \bigcup \mathcal{L}_P \subseteq \bigcup \mathcal{L}_Q$.
5. By 1 and 4 we have $\mathsf{liv}\,\mathsf{liv}\, P \subseteq \mathsf{liv}\, P$. For the converse we calculate

$$\begin{array}{ll} & Q \subseteq P \wedge Q \text{ lively} \\ \Rightarrow & \quad \{\![\text{ definition of } \mathsf{liv}\, P \,]\!\} \\ & Q \subseteq \mathsf{liv}\, P \wedge Q \text{ lively} \\ \Rightarrow & \quad \{\![\text{ definition of } \mathcal{L} \,]\!\} \\ & \mathcal{L}_P \subseteq \mathcal{L}_{\mathsf{liv}\, P} \\ \Rightarrow & \quad \{\![\text{ monotonicity of } \bigcup \text{ and definition of } \mathsf{liv} \,]\!\} \\ & \mathsf{liv}\, P \subseteq \mathsf{liv}\,\mathsf{liv}\, P \ . \end{array}$$

6. By 6 we have $\max \mathsf{liv}\, P = \emptyset$. Now from Lemma 25.4 and using $\subseteq$-monotonicity of the $\mathsf{inf}\,\mathsf{ide}$-operation we get $\emptyset \neq \mathsf{inf}\,\mathsf{ide}\,\mathsf{liv}\, P \subseteq \mathsf{inf}\,\mathsf{ide}\, P$.
7. First note that $\emptyset \notin \mathcal{L}_P$. But then $\bigcup \mathcal{L}_P \neq \emptyset$ iff $\mathcal{L}_P \neq \emptyset$. Now apply 6.
8. By 1 and $\subseteq$-monotonicity of $\mathsf{inf}\,\mathsf{ide}$ we get $\supseteq$. Assume, conversely, $I \in \mathsf{inf}\,\mathsf{ide}\, P$. Then there is a $D \in \mathsf{dir}\, P$ with $I = D^{\leq}$. Since D is directed, we have $D \neq \emptyset$. Moreover, $\max D = \max I = \emptyset$. So $D \in \mathcal{L}_P$ and hence also $D \subseteq \mathsf{liv}\, P$, i.e., $D \in \mathsf{dir}\,\mathsf{liv}\, P$. So $I \in \mathsf{inf}\,\mathsf{str}\,\mathsf{liv}\, P$ as well.
9. $$\begin{array}{ll} & \mathsf{pref}\,\mathsf{inf}\,\mathsf{ide}\, P \\ = & \quad \{\![\text{ definitions }]\!\} \\ & \bigcup \{D^{\leq} : D \in \mathsf{dir}\, P \wedge \max D = \emptyset\} \\ = & \quad \{\![\text{ distributivity }]\!\} \\ & (\bigcup \{D : D \in \mathsf{dir}\, P \wedge \max D = \emptyset\})^{\leq} \\ \subseteq & \quad \{\![\text{ definition of } \mathcal{L} \,]\!\} \\ & (\bigcup \mathcal{L}_P)^{\leq} \\ = & \quad \{\![\text{ definition of } \mathsf{liv} \,]\!\} \\ & (\mathsf{liv}\, P)^{\leq} \ . \end{array}$$

   Assume conversely $y \in \mathsf{liv}\, P^{\leq}$. There is a $Q \in \mathcal{L}_P$ and an $x \in Q$ with $y \leq x$. By 25.4.1 there is an $I \in \mathsf{inf}\,\mathsf{ide}\, Q \subseteq \mathsf{inf}\,\mathsf{ide}\, P$ with $x \in I$.

■

So in particular, liv is again a kernel operator. Moreover, by 7, to show that a snapshot set P spans infinite ideals, it suffices to exhibit a lively $Q \subseteq P$. Such Qs can frequently be constructed by induction.

Part VI: Extended Example: Buffers and Queues

26 Specification of a Bounded Buffer

As an example of the use of our constructs, we give a specification of bounded buffer and queue modules. For this we use the particular domain $A^\infty = A^* \cup A^\omega$ of finite and infinite streams over a set A of atomic actions. This example uses the trace view (cf. Section 6) of streams. It was motivated by the asynchronous bounded queue implementation in the collection of the IFIP WG10.5 benchmark problems in hardware verification [27].

The buffer module has one input and one output port. In describing such modules, we choose the letters a for the action of inputting and b for outputting and set $A \stackrel{\text{def}}{=} \{a, b\}$. Boundedness of a module can be enforced by requiring the number of input actions to exceed the number of output actions by at most some $n \in \mathbb{N}$ which then is the *capacity* of the device.

We denote by s_c the number of occurrences of $c \in A$ in $s \in A^*$. Formally,

$$\begin{aligned} \varepsilon_c &\stackrel{\text{def}}{=} 0 \ , \\ (a \bullet s)_c &\stackrel{\text{def}}{=} \delta_{ac} + s_c \ , \end{aligned}$$

where δ is the Kronecker symbol defined by

$$\delta_{xy} \stackrel{\text{def}}{=} \begin{cases} 1 \text{ if } x = y \ , \\ 0 \text{ otherwise} \ . \end{cases}$$

Generalising the above informal description slightly, we define, for $n \in \mathbb{Z}$ and $a, b \in A$, the set

$$\mathrm{EX}_n^{ab} \stackrel{\text{def}}{=} \{s \in A^* : s_a \leq s_b + n\}$$

of snapshots. Then $s \in \mathrm{EX}_n^{ab}$ max be pronounced as "a exceeds b by at most n in s". The specification is, however, very loose in that the balance between as and bs might be struck only at the very end of a word. For instance, $a^{k+n} \bullet b^k \in \mathrm{EX}_{ab}^n$. So the restriction may be violated in prefixes and only established in the end. For bounded devices, this is not possible. They need a stronger specification. Therefore we strengthen our snapshot set to the safety property

$$\mathrm{B}_n^{ab} \stackrel{\text{def}}{=} \mathsf{saf}\, \mathrm{EX}_n^{ab} \ .$$

Now $\mathsf{ide}\, \mathrm{B}_n^{ab}$ is the set of all finite and infinite streams that satisfy EX_n^{ab} in all prefixes. However, we are interested in devices that work for an unbounded time. This is specified by considering as overall behaviour of such a device the set

$$\mathcal{B}_n^{ab} \stackrel{\text{def}}{=} \mathsf{inf}\, \mathsf{str}\, \mathrm{B}_n^{ab}$$

consisting only of infinite admissible streams.

A buffer is a device in which the number of outputs must not exceed the number of inputs. Hence we define

$$\mathcal{BF}^{ab} \stackrel{\text{def}}{=} \mathcal{B}_0^{ba} \ .$$

Note the reversal of the arguments in the superscript. The finitary property B_0^{ba} spells out to $s_b \leq s_a$, as required. This describes an unbounded buffer. A bounded buffer of capacity n then is described by

$$\mathcal{B}_n^{ab} \stackrel{\text{def}}{=} \mathcal{BF}^{ab} \cap \mathcal{B}_n^{ab} \ .$$

This specifies the set of all infinite streams for which in all finite prefixes, the number of outputs does not exceed the number of inputs and there must be no more than n inputs between adjacent outputs.

27 Transformation to Automaton Form

We consider again the buffer example in the introduction. We recall the family of properties

$$\mathrm{EX}_n^{ab} \stackrel{\text{def}}{=} \{s \in A^* : s_a \leq s_b + n\} \ ,$$

where $n \in \mathbb{Z}$ and $A = \{a, b\}$. From this predicative, implicit definition we want to calculate a more explicit description corresponding to a generating grammar or accepting automaton. This can be done by a simple unfold/fold transformation using induction on the words in A^*. For the induction basis we calculate

$$\begin{array}{ll} & \varepsilon \in \mathrm{EX}_n^{ab} \\ \Leftrightarrow & \quad \{\![\text{ definition of EX }]\!\} \\ & \varepsilon_a \leq \varepsilon_b + n \\ \Leftrightarrow & \quad \{\![\text{ definition of count }]\!\} \\ & 0 \leq 0 + n \\ \Leftrightarrow & \quad \{\![\text{ arithmetic }]\!\} \\ & 0 \leq n \ . \end{array}$$

For the induction step, we consider an arbitrary $c \in A$:

$$\begin{array}{ll} & c \bullet s \in \mathrm{EX}_n^{ab} \\ \Leftrightarrow & \quad \{\![\text{ definition of EX }]\!\} \\ & (c \bullet s)_a \leq (c \bullet s)_b + n \\ \Leftrightarrow & \quad \{\![\text{ definition of count }]\!\} \\ & \delta_{ca} + s_a \leq \delta_{cb} + s_b + n \\ \Leftrightarrow & \quad \{\![\text{ arithmetic }]\!\} \\ & s_a \leq s_b + n + \delta_{cb} - \delta_{ca} \\ \Leftrightarrow & \quad \{\![\text{ definition of EX }]\!\} \\ & s \in \mathrm{EX}_{n+\delta_{cb}-\delta_{ca}}^{ab} \ , \end{array}$$

Note that the recursion relations are linear bi-implications. Therefore we obtain

$$\begin{aligned} \varepsilon \in \mathrm{EX}_n^{ab} &\Leftrightarrow 0 \leq n \ , \\ c \bullet U \subseteq \mathrm{EX}_n^{ab} &\Leftrightarrow U \subseteq \mathrm{EX}_{n+\delta_{cb}-\delta_{ca}}^{ab} \ . \end{aligned}$$

This corresponds to an infinite grammar with nonterminals EX_n^{ab} or an infinite automaton with states EX_n^{ab} $(n \in \mathbb{Z})$.

28 Counting Resumed

Next we want a similar representation for

$$\mathrm{B}_n^{ab} \stackrel{\mathrm{def}}{=} \mathsf{saf}\, \mathrm{EX}_n^{ab} \ .$$

This can be done quite systematically using Lemma 24.3. We obtain

$$\begin{aligned} \varepsilon \in \mathrm{B}_n^{ab} &\Leftrightarrow 0 \leq n \ , \\ c \bullet U \subseteq \mathrm{B}_n^{ab} &\Leftrightarrow 0 \leq n \wedge 0 \leq n \wedge U \subseteq \mathrm{B}_n^{ab} \text{ for } c \in A \backslash \{a, b\} \ , \\ a \bullet U \subseteq \mathrm{B}_n^{ab} &\Leftrightarrow 0 \leq n \wedge 0 \leq n-1 \wedge U \subseteq \mathrm{B}_{n-1}^{ab} \ , \\ b \bullet U \subseteq \mathrm{B}_n^{ab} &\Leftrightarrow 0 \leq n \wedge 0 \leq n+1 \wedge U \subseteq \mathrm{B}_{n+1}^{ab} \ . \end{aligned}$$

which simplifies to

$$\begin{aligned} \varepsilon \in \mathrm{B}_n^{ab} &\Leftrightarrow 0 \leq n \ , \\ c \bullet U \subseteq \mathrm{B}_n^{ab} &\Leftrightarrow 0 \leq n \wedge U \subseteq \mathrm{B}_n^{ab} \text{ for } c \in A \backslash \{a, b\} \ , \\ a \bullet U \subseteq \mathrm{B}_n^{ab} &\Leftrightarrow 0 \leq n-1 \wedge U \subseteq \mathrm{B}_{n-1}^{ab} \ , \\ b \bullet U \subseteq \mathrm{B}_n^{ab} &\Leftrightarrow 0 \leq n \wedge U \subseteq \mathrm{B}_{n+1}^{ab} \ . \end{aligned}$$

In particular, $\mathrm{B}_n^{ab} = \emptyset$ for $n < 0$.

Now we consider the bounded buffer behaviour. We calculate:

$$\begin{aligned} & \mathcal{B}_n^{ab} \\ = & \quad \{\![\text{ definition }]\!\} \\ & \mathcal{BF}^{ab} \cap \mathcal{B}_n^{ab} \\ = & \quad \{\![\text{ definition }]\!\} \\ & \mathcal{B}_0^{ba} \cap \mathcal{B}_n^{ab} \\ = & \quad \{\![\text{ definition }]\!\} \\ & \mathsf{inf}\, \mathsf{str}\, \mathrm{B}_0^{ba} \cap \mathsf{inf}\, \mathsf{str}\, \mathrm{B}_n^{ab} \\ = & \quad \{\![\text{ by Lemma 15.3.5, since the B sets are specified as safety properties }]\!\} \\ & \mathsf{inf}\, \mathsf{str}\, (\mathrm{B}_0^{ba} \cap \mathrm{B}_n^{ab}) \ . \end{aligned}$$

So the problem has been reduced to finding an explicit representation for $\mathrm{B}_0^{ba} \cap \mathrm{B}_n^{ab}$, which is a simple product automaton construction. It is a special case of the automaton for

$$\mathrm{G}_{mn} \stackrel{\mathrm{def}}{=} \mathrm{B}_m^{ba} \cap \mathrm{B}_n^{ab} \ .$$

29 Decomposition

Let us now define a buffer process by setting

$$\mathrm{BB}_m^{ab} \stackrel{\mathrm{def}}{=} (\{a,b\}, \mathcal{B}_m^{ab}) \ .$$

Then using parallel composition we can state the following nice decomposition properties:

Lemma 29.1 1. $\mathrm{EX}_m^{ab} \cap \mathrm{EX}_n^{bc} \subseteq \mathrm{EX}_{m+n}^{ac}$.
2. $\mathrm{B}_m^{ab} \cap \mathrm{B}_n^{bc} \subseteq \mathrm{B}_{m+n}^{ac}$.
3. $S \dagger \{a,b\} \in \mathcal{B}_m^{ab} \wedge S \dagger \{b,c\} \in \mathcal{B}_n^{bc} \Rightarrow S \dagger \{a,c\} \in \mathcal{B}_{m+n}^{ac}$.
4. $\mathrm{BB}_m^{ab} \,||\, \mathrm{BB}_n^{bc} \subseteq \mathrm{BB}_{m+n}^{ac}$.

Proof: 1.

$$\begin{array}{ll} & s \in \mathrm{EX}_m^{ab} \cap \mathrm{EX}_n^{bc} \\ \Leftrightarrow & \quad \{\!\![\text{ definition }]\!\!\} \\ & s_a \leq s_b + m \wedge s_b \leq s_c + n \\ \Rightarrow & \quad \{\!\![\text{ transitivity and monotonicity }]\!\!\} \\ & s_a \leq s_c + m + n \\ \Leftrightarrow & \quad \{\!\![\text{ definition }]\!\!\} \\ & s \in \mathrm{EX}_{m+n}^{ac} \ . \end{array}$$

2. immediate from 1.,Lemma 24.1.5 and Lemma 24.1.4.
3. immediate from 2.
4. immediate from 3.

■

This allows decomposing a buffer of capacity n into a parallel composition of n buffers of capacity 1. Of course, it needs to be shown that the intersections/parallel compositions are non-empty. This follows from our results in Section 25: first, it is easy to show that

$$\begin{array}{rcl} (a \bullet b)^* \subseteq \mathrm{EX}_n^{ab} & \Leftarrow & n \geq 0 \ , \\ \mathsf{inf\ str}\,(a \bullet b)^* \subseteq \mathcal{B}_n^{ab} & \Leftarrow & n \geq 1 \ . \end{array}$$

From this we get

$$\mathsf{inf\ str}\,(a \bullet b \bullet c)^* \subseteq \beta(\mathrm{BB}_m^{ab} \,||\, \mathrm{BB}_n^{bc}) \quad \Leftarrow \quad m,n \geq 1 \ .$$

Since $(a \bullet b \bullet c)^*$ is lively, $\mathsf{inf\ str}\,(a \bullet b \bullet c)^*$ and hence $\mathrm{BB}_m^{ab} \,||\, \mathrm{BB}_n^{bc}$ are non-empty.

30 The One-Place Buffer

For the special case of $n = 1$ we have

$$\mathrm{BB}_1^{ab} = \mathrm{G}_{01} \ ,$$

where

$$\begin{array}{ll} \varepsilon \in \mathrm{G}_{01} \Leftrightarrow \mathrm{TRUE} \ , & \varepsilon \in \mathrm{G}_{10} \Leftrightarrow \mathrm{TRUE} \ , \\ a \bullet U \subseteq \mathrm{G}_{01} \Leftrightarrow U \subseteq \mathrm{G}_{10} \ , & a \bullet U \subseteq \mathrm{G}_{10} \Leftrightarrow \mathrm{FALSE} \ , \\ b \bullet U \subseteq \mathrm{G}_{01} \Leftrightarrow \mathrm{FALSE} \ , & b \bullet U \subseteq \mathrm{G}_{10} \Leftrightarrow U \subseteq \mathrm{G}_{01} \ . \end{array}$$

This corresponds to a two-state accepting automaton for the bounded buffer property, which is sufficient for purposes of implementation.

However, the above can also be seen as a regular grammar or system of equations for languages. If desired, we can calculate from it a regular expression for $\mathrm{BB}_1^{ab} = \mathrm{G}_{01}$ using twice

$$\textbf{(Arden's Rule)}\ \frac{U \cap \varepsilon = \emptyset,\ X = V \cup U \bullet X}{X = U^* \bullet V\ .}$$

This gives

$$\mathrm{BB}_1^{ab} = (a \bullet b)^* \bullet (\varepsilon \cup a)\ .$$

Using

$$(a \bullet b)^* \bullet a \sim (a \bullet b)^*$$

and Corollary 17.4, we obtain

$$\mathcal{B}_1^{ab} = \mathsf{inf}\,\mathsf{str}\,(a \bullet b)^*\ .$$

Finally we use the fact that the language $a \bullet b$ as a singleton trivially satisfies the Fano condition, so that Lemma 19.1 gives

$$\mathcal{B}_1^{ab} = (a \bullet b)^\omega\ ,$$

as expected.

31 From Buffers to Queues

So far we have only talked about the relative order of input and output *events*. For queues also the relative order of input and output *values* is relevant. We use now the refined alphabet $A = C \times V$ where C is the set of channel names and V the set of values. An element of A will be denoted as $c\langle v\rangle$. As a shorthand we introduce

$$c = \{c\langle x\rangle : x \in V\}\ . \tag{9}$$

For a word $w \in A^*$ we define the word $chans(w)$ of channels on which activity occurred and for each $c \in C$ the word $vals_c(W)$ of values transmitted along c. Their inductive definitions read

$$\begin{aligned} chans(\varepsilon) &= \varepsilon\ , \\ chans(b\langle x\rangle \bullet w) &= b \bullet chans(w)\ , \end{aligned}$$

$$\begin{aligned} vals_c(\varepsilon) &= \varepsilon\ , \\ vals_c(b\langle x\rangle \bullet w) &= \begin{cases} x \bullet vals_c(w) & \text{if } b = c\ , \\ vals_c(w) & \text{otherwise}\ . \end{cases} \end{aligned}$$

These operations are extended pointwise to languages and behaviours.

With these operations we may specify the behaviour of a *faithful* component, i.e., a component which does not re-order or lose messages when transmitting from channel a to channel b, as

$$\mathcal{FA}^{ab} \stackrel{\text{def}}{=} \{S : vals_a(S) = vals_b(S)\}\ .$$

A bounded queue is then specified as a faithful bounded buffer:

$$\mathcal{BQ}_n^{ab} \stackrel{\text{def}}{=} \mathcal{FA}^{ab} \cap \mathcal{B}_n^{ab} \ .$$

Here a, b in $\mathcal{B}_n^{ab}$ are to be understood according to abbreviation (9).

The decomposition properties for buffers carry over to queues, so that again a queue of capacity n can be refined into the parallel composition of n queues of capacity 1. Moreover, a similar calculation as before, using again Arden's rule, yields for the refinement

$$\mathcal{BQ}_1^{ab} = (\bigcup_{x \in V} a\langle x\rangle \bullet b\langle x\rangle)^\omega \ .$$

Part VII: Conclusion

We have introduced some algebraic operators and laws that can be used in the specification and derivation of systems. By abstracting from the domain of streams for which most of the notions were coined originally, we have obtained a rich set of laws which hold for a wide variety of domains. The order-theoretic approach lends itself well to an algebraic treatment. The point-free formulation eases and compactifies specifications, proofs of the basic properties and the actual derivations. Further research along these lines should search for similar algebraic characterisations of other important notions about systems and to explore their algebraic properties.

Concerning the underlying theory, our domain-thoretic notions should be tied in more closely with the topological view (see e.g. [53, 57]). Moreover, in the stream domain there obviously is a close connection with temporal operators: $\mathsf{str}\ P$ is related to intermittent assertions [14] and hence the formula $\Box\Diamond P$ (always eventually P) in temporal logic, while $\mathsf{saf}\ P$ corresponds to $\boxed{\mathrm{i}}\, P$ (P holds in all initial subintervals [46]). These connections are made precise and carried over to arbitrary domains in [42, 43]. The resulting "modal algebra" as well as the ideal and stream algebra developed in the present paper need to be tried out in larger and more realistic case studies of deductive design of parallel systems.

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