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Fast-Diffusion Limit with Large Noise for Systems of Stochastic Reaction-Diffusion Equations

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Abstract

We consider a class of stochastic reaction-diffusion equations with additive noise. We show that in the limit of fast diffusion, one can approximate solutions of the SPDE by the solution of a suitable ordinary differential equation (ODE) only describing the reactions. Also, we show large fluctuations lead in the limit to surprising new terms in the ODE.

We focus on systems with polynomials nonlinearities and give applications to the predator-prey system and a cubic auto-catalytic reaction between two chemicals.

1 Introduction

Reaction diffusion systems are mathematical models that describe how the concentration of one or more substances distributed in space changes. The main features are one one hand the influence of local chemical reactions, in which the substances are converted into each other, and on the other hand diffusion, which causes the substances to spread out in space. We consider here the case of fast diffusion in interaction with large mass-conservative noise.

Reaction–diffusion equations are not limited to the field of chemistry and chemical engineering. They can describe the dynamics of non-chemical systems, and reaction–diffusion equations provide a general theoretical framework for the study of phenomena in areas such as biology, ecology, physics, and materials science.

S. Cerrai [2, 3, 4] and S. Cerrai and M. Freidlin [5] have studied the validity of an averaging principle and the normal deviations of the slow motion from the averaged motion of systems of stochastic reaction-diffusion type perturbed by a noise of multiplicative type. While F. Flandoli [10] has studied the global existence and uniqueness for a stochastic reaction-diffusion equation with polynomial nonlinearity in a bounded domain.

In this article we consider the following system of stochastic reaction-diffusion equations for n species with respect to no flux Neumann boundary conditions on a bounded smooth domain G:

$$\frac{\partial u_i}{\partial t} = \frac{d_i}{\varepsilon^2}\Delta u_i + \mathcal{F}_i(u_1, u_2, ..., u_n) + \sigma_\varepsilon \partial_t W_i(t), \quad \text{for } i = 1, 2, .., n, \tag{1}$$

where $\mathcal{F}_i(u_1, u_2, ..., u_n)$ are polynomials of degree m_i and $W_i(t)$ are finite dimensional Wiener processes.

Our aim is to establish rigorously results for the fast-diffusion limit $\varepsilon \to 0$ in the general class of stochastic reaction diffusion equations given by (1). We study only cases where the noise is not changing the average (i.e., $W_c = 0$) and is sufficiently large, i.e $\sigma_\varepsilon = \varepsilon^{-1}$. In this case, we will show that on any fixed time interval $[0, T_0]$ solutions of Equation (1) are well approximated by

$$u_i(t, x) = b_i(t) + \mathcal{Z}_i(t, x) + \mathcal{O}(\varepsilon^{1-}),$$

where $\mathcal{Z}_i(t, x)$ is an ε-dependent fast Ornstein-Uhlenbeck process corresponding to noise in the limit $\varepsilon \to 0$, which is defined later in (6) and b represents the average concentration of the components of u. The error term has a precise meaning given in Definition 8.

Our main result shows that b is given by the following system of ordinary differential equations

$$\partial_t b_i(t) = \bar{\mathcal{F}}_i(b), \text{ for } i = 1, 2, .., n, \tag{2}$$

with an averaged non-linearity

$$\bar{\mathcal{F}}_i(b) = \int_{\mathcal{H}^\alpha} \mathcal{F}_i(b + z) d\mu(z),$$

where μ is the ε-independent invariant measure of the fast Ornstein-Uhlenbeck process $\mathcal{Z}$. Results of these types without error estimates are well-known in averaging theory. The main novelty of this paper is the explicit error estimate in terms of high moments of the error, while usually only weak convergence towards the approximation is treated.

Equation (2) is an ordinary differential equation and it describes the reaction without diffusion. The equation is without noise, as we consider degenerate noise that has no direct impact on the mean average b. It is surprising that in case $\bar{\mathcal{F}}_i \neq \mathcal{F}_i$ large noise has the potential to generate new effective reaction terms. Also, we investigate the effect of additive noise on the stabilization of the solutions, which we discuss in two examples. This work can be applied to many phenomena in different areas, for instance chemistry and biology, as presented in Section 6.

The reminder of this paper is organized as follows. In Section 2 we state the precise setting for equation (1) and the assumptions that we need. In Section 3 we derive the fast-diffusion limit with error and present the main theorem. In Section 4 we give the proof of the main result, while in Section 5 we give an averaging lemma with error bounds and derive an explicit formula for $\bar{\mathcal{F}}_i$. Finally, we give two examples from biology and chemistry to illustrate the application and implication of our results.

2 Definition and Assumptions

This section states the precise setting for (1) and summarizes all assumptions necessary for our results. For the analysis we work in the separable Hilbert space $\mathcal{H} = \mathcal{L}^2(G)$, where $G \subset \mathbb{R}^d$ is a bounded domain and ∂G is its sufficiently smooth boundary (e.g. Lipschitz), equipped with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

Definition 1 *Define, for $i = 1, 2, ..., n$ and diffusion constants $d_i > 0$*

$$\mathcal{A}_i = d_i \Delta \tag{3}$$

with

$$D(\mathcal{A}_i) = \left\{ u \in \mathcal{H}^2 : \partial_\nu u|_{\partial G} = 0 \right\},$$

where $\partial_\nu u$ is the normal derivative of u on ∂G.

It is well-known that $\mathcal{A}_i$ has an orthonormal base of eigenfunctions $\{e_k\}_{k=1}^{\infty}$. It is the same basis e_k for all i in $\mathcal{H}$ with corresponding eigenvalues $\{d_i \lambda_k\}_{k=0}^{\infty}$ depending on i (cf. Courant and Hilbert [6]).

Suppose that $\mathcal{N} := \ker \mathcal{A}_i = \operatorname{span}\{e_0\}$, where e_0 is constant and $\lambda_0 = 0$. Define $S = \mathcal{N}^\perp$ the orthogonal complement of $\mathcal{N}$ in $\mathcal{H}$, and $P_c u = \frac{1}{|G|} \int_G u dx$ for the projection and define $P_s u := (\mathcal{I} - P_c) u$ for the projection onto the orthogonal complement, where $\mathcal{I}$ is the identity operator on $\mathcal{H}$.

Definition 2 *For $\alpha > \frac{1}{2}$ we define the fractional interpolation space $\mathcal{H}^\alpha$ as*

$$\mathcal{H}^\alpha = \left\{ \sum_{k=0}^{\infty} \gamma_k e_k : \sum_{k=1}^{\infty} \gamma_k^2 k^{2\alpha} < \infty \right\} \quad \text{with norm} \left\| \sum_{k=0}^{\infty} \gamma_k e_k \right\|_\alpha^2 = \gamma_0^2 + \sum_{k=1}^{\infty} \gamma_k^2 k^{2\alpha}.$$

Note that $\lambda_k \sim k^{2/d}$ for the eigenvalues of the Laplacian in $G \subset \mathbb{R}^d$, and thus our space $\mathcal{H}^\alpha$ corresponds to the usual Sobolev space $H^{d\alpha}(G)$.

Thus $\alpha > \frac{1}{2}$ implies that $\mathcal{H}^\alpha$ is embedded into $L^\infty(G)$. Moreover, $\|uv\|_\alpha \leq C\|u\|_\alpha\|u\|_\alpha$ (cf. [13]).

The operator $\mathcal{A}_i$ given by Definition 1 generates an analytic semigroup $\{e^{t\mathcal{A}_i}\}_{t\geq 0}$ (cf. Dan Henry [11] or Pazy [14]), on any space $\mathcal{H}^\alpha$ defined by

$$e^{\mathcal{A}_i t}\Big(\sum_{k=1}^{\infty}\gamma_k e_k\Big) = \sum_{k=1}^{\infty} e^{-d_i\lambda_k t}\gamma_k e_k \quad \forall\ t \geq 0.$$

It has the following property: There is an $\omega > 0$ such that for all $t > 0$, $i = 1, 2, ...n$ and all $u \in \mathcal{H}^\alpha$

$$\left\|e^{t\mathcal{A}_i}P_s u\right\|_\alpha \leq e^{-\omega t}\left\|P_s u\right\|_\alpha\ , \tag{4}$$

where ω depends in general on d_i.

For the noise we suppose:

Assumption 3 *For $i = 1, 2, ..., n$, let W_i be a finite dimensional Wiener process on $\mathcal{H}$. Suppose for $t \geq 0$,*

$$W_i(t) = \sum_{k=1}^{N}\alpha_{ik}\beta_{ik}(t)e_k\ \text{, for } i = 1, 2, .., n, \tag{5}$$

where the β_{ik} are $\mathbb{R}$-valued standard Brownian motions and the α_{ik} are real numbers.

Definition 4 *The fast Ornstein-Uhlenbeck process (OU-process, for short) $\mathcal{Z}_i$ is defined as*

$$\mathcal{Z}_i(t) = \sum_{k=1}^{N}\mathcal{Z}_{ik}e_{ik} = \sum_{k=1}^{N}\varepsilon^{-1}\alpha_{ik}\int_0^t e^{\varepsilon^{-2}(t-\tau)\mathcal{A}_s}d\beta_{ik}(\tau)e_k, \tag{6}$$

with

$$\mathcal{Z}_{ik}(t) = \alpha_{ik}\varepsilon^{-1}\int_0^t e^{-\varepsilon^{-2}(t-\tau)d_i\lambda_k}d\beta_{ik}(\tau). \tag{7}$$

For the polynomial $\mathcal{F}_i(u)$, we assume

Assumption 5 *The nonlinearity $\mathcal{F}_i$ is for $i = 1, 2, .., n$, a polynomial of degree m_i. Thus as $\alpha > \frac{1}{2}$ it is bounded by*

$$\|\mathcal{F}_i(u)\|_\alpha \leq C(1 + \|u\|_\alpha^{m_i}) \text{ for all } u \in \mathcal{H}^\alpha. \tag{8}$$

Define m as $m = \max(m_1, ..., m_n)$. Suppose furthermore that there is a polynomial $\bar{\mathcal{F}}_i$ of degree less than or equal m_i such that for some $\kappa > 0$

$$\mathcal{F}_i(u+\mathcal{Z}_i) = \bar{\mathcal{F}}_i(u)+\mathcal{O}(\varepsilon^{1-m_i\kappa}) \text{ for all suitable stochastic processes } u = \mathcal{O}(\varepsilon^{-\kappa}). \tag{9}$$

To be more precise, there is a constant $C > 0$ such that for all stochastic processes with $(du)^2 = 0$ and $\tau^* = \inf\{t > 0 : \|u(t)\|_\alpha > \varepsilon^{-\kappa}\}$, we assume

$$\mathbb{E}\sup_{[0,\tau^*]} \left\|\mathcal{F}_i(u + \mathcal{Z}_i) - \bar{\mathcal{F}}_i(u)\right\|_\alpha^p \le C\varepsilon^{p-pm_i\kappa}.$$

Under Assumption 5 the nonlinearity is a locally Lipschitz-continuous map from $\mathcal{H}^\alpha$ into $\mathcal{H}^\alpha$. Thus it is standard (cf. Da Prato and Zabczyk [7]) to verify that there is a unique local $(\mathcal{H}^\alpha)^n$-valued solution u that is continuous and exists until one of its components blow up.

Assumption 6 *Let $b(t)$ in $\mathcal{N}$ as the solution of (2). Suppose there is $T_1 \le T_0$ and $C > 0$, such that*

$$\sup_{t\in[0,T_1]} \|b(t)\| \le C. \tag{10}$$

The above assumption is usually a lemma that follows directly from the fact that $\bar{\mathcal{F}}_i$ is a polynomial. Note that T_1 in general depends on the initial condition $b(0)$.

For our result we rely on a cut off argument. We consider only solutions (a, ψ) that are not too large, as given by the next definition.

Definition 7 *For the $\mathcal{N} \times S$-valued stochastic process (a, ψ) defined later in (13) we define, for some $T_0 > 0$ and $\kappa \in (0, \frac{1}{2m+1})$, the stopping time τ^* as*

$$\tau^* := T_0 \wedge \inf\left\{t > 0 : \|u(t)\|_\alpha > \varepsilon^{-\kappa}\right\}. \tag{11}$$

We give error estimates in terms of the following $\mathcal{O}$-notation.

Definition 8 *For a real-valued family of processes $\{X_\varepsilon(t)\}_{t\ge0}$ we say that X_ε is of order f_ε, i.e. $X_\varepsilon = \mathcal{O}(f_\varepsilon)$, if for every $p \ge 1$ there exists a constant C_p such that*

$$\mathbb{E}\sup_{t\in[0,\tau^*]} |X_\varepsilon(t)|^p \le C_p f_\varepsilon^p. \tag{12}$$

We use also the analogous notation for time-independent random variables.

Definition 9 *(Multi-Index Notation) Let $\ell \in \mathbb{N}_0^n$, i.e. $\ell = (\ell_1, \ell_2, ..., \ell_n)$ be a vector of nonnegative integers, $u = (u_1, u_2, ...u_n)$. Then we define*

$$\begin{aligned}
|\ell| &= \ell_1 + \ell_2 + ... + \ell_n \\
\ell! &= \ell_1!\ell_2!......\ell_n! \\
u^\ell &= u_1^{\ell_1}u_2^{\ell_2}...u_n^{\ell_n} \\
D^\ell &= \partial_{u_1}^{\ell_1}\partial_{u_2}^{\ell_2}.....\partial_{u_n}^{\ell_n}
\end{aligned}$$

3 The approximation Theorem

Let us first discuss a formal derivation of the limiting equation corresponding to Equation (1). We split the solution u into

$$u_i(t,x) = a_i(t) + \psi_i(t,x), \quad \text{for } i = 1,2,....,n, \tag{13}$$

with $a \in \mathcal{N}$ and $\psi \in \mathcal{S}$. Plugging (13) into (1) and projecting everything onto $\mathcal{N}$ and $\mathcal{S}$ in order to obtain for $i = 1,2,....,n$

$$\partial_t a_i = \mathcal{F}_i^c(a+\psi), \tag{14}$$

and

$$\partial_t \psi_i = \frac{1}{\varepsilon^2}\mathcal{A}\psi_i + \mathcal{F}_i^s(a+\psi) + \frac{1}{\varepsilon}\partial_t W_i(t), \tag{15}$$

where $\mathcal{F}^c = P_c\mathcal{F}$ and $\mathcal{F}^s = P_s\mathcal{F}$. Equations (14) and (15) are stated in the integrated form as

$$a_i(t) = a_i(0) + \int_0^t \mathcal{F}_i^c(a+\psi)d\tau, \tag{16}$$

and as mild formulation as

$$\psi_i(t) = e^{\varepsilon^{-2}t\mathcal{A}}\psi_i(0) + \int_0^t e^{\varepsilon^{-2}(t-\tau)\mathcal{A}_s}\mathcal{F}_i^s(a+\psi)d\tau + \mathcal{Z}_i(t), \tag{17}$$

for $i = 1,..,\ n$, where the fast OU-process $\mathcal{Z}_i$ was defined in Definition 4. Formally, we see that in first approximation ψ equals the OU-process $\mathcal{Z}$ (cf. Lemma 11 in next section for the rigorous approximation). Thus, we can replace ψ by $\mathcal{Z}$ in Eq. (16), in order to obtain

$$a_i(t) = a_i(0) + \int_0^t \mathcal{F}_i^c(a+\mathcal{Z})d\tau.$$

Using Assumption 5, yields the averaged equation with error term

$$a_i(t) = a_i(0) + \int_0^t \bar{\mathcal{F}}_i(a)d\tau + \mathcal{O}(\varepsilon^{1-2m\kappa}).$$

Now the main result of this paper is:

Theorem 10 *(Approximation) Under Assumptions 3, 5 and 6, let u be a solution of (1) with splitting $u = a + \psi$ defined in (13) with the initial condition $u(0) = a(0) + \psi(0)$ with $a(0) \in \mathcal{N}$ and $\psi(0) \in S$ where $a(0)$ and $\psi(0)$ are of order one, and b is a solution of (2) with $b(0) = a(0)$. Then for all $p > 0$, $C_0 > 0$ and $T_0 \geq T_1 > 0$ and all $\kappa \in (0, \frac{1}{2m+1})$, there exists $C > 0$ such that*

$$\mathbb{P}\Big(\sup_{t\in[0,T_1]}\Big\|u(t) - b(t) - \mathcal{Q}(t)\Big\|_\alpha > \varepsilon^{1-2m\kappa-\kappa}\Big) \leq C\varepsilon^p + \mathbb{P}(\sup_{[0,T_1]}\|b\|_\alpha > C_0), \tag{18}$$

where

$$\mathcal{Q}(t) = e^{\varepsilon^{-2}t\mathcal{A}_s}\psi(0) + \mathcal{Z}(t), \tag{19}$$

with fast OU-process $\mathcal{Z}$ defined in (6).

We see that the first part of (19) depending on the initial condition decays exponentially fast on the time-scale of order $\mathcal{O}(\varepsilon^2)$, while $\mathcal{Z}$ is a fast OU-process that approximates noise.

4 Proof of the Main Result

In the first lemma of this section, we see that ψ_i is approximately equal to the fast Ornstein-Uhlenbeck process $\mathcal{Z}_i$ (cf. (6)).

Lemma 11 *Under Assumption 5 there is a constant $C > 0$ such that for $\kappa > 0$ from the definition of τ^* and $p \geq 1$*

$$\mathbb{E} \sup_{t\in[0,\tau^*]} \left\| \psi_i(t) - e^{\varepsilon^{-2}t\mathcal{A}}\psi_i(0) - \mathcal{Z}_i(t) \right\|_\alpha^p \leq C\varepsilon^{2p-m_ip\kappa}, \tag{20}$$

for $i = 1, 2, ..., n$, where m_i is the degree of the polynomial $\mathcal{F}_i$.

Proof. Using triangle inequality for Equation (15) and Assumption 5, to obtain for $\alpha > \frac{1}{2}$

$$\begin{aligned}
\left\| \psi_i(t) - \mathcal{Z}_i(t) - y_i(t) \right\|_\alpha &\leq \left\| \int_0^T e^{\varepsilon^{-2}\mathcal{A}_s(T-\tau)}\mathcal{F}_i^s(a+\psi)d\tau \right\|_\alpha \\
&\leq C \sup_{\tau\in[0,\tau^*]} \|\mathcal{F}_i^s(a+\psi)\|_\alpha \int_0^t e^{-\varepsilon^{-2}\omega(t-\tau)}d\tau \\
&\leq C\varepsilon^2 \sup_{\tau\in[0,\tau^*]} (1 + \|a+\psi\|_\alpha^{m_i}) \\
&\leq C\varepsilon^{2-m_i\kappa}.
\end{aligned}$$

□

We need the following uniform bounds on the OU-process.

Lemma 12 *Under Assumption 3, for every $\kappa_0 > 0$ and $p \geq 1$ there is a constant C, depending on p, α_k, λ_k, κ_0 and T_0, such that*

$$\mathbb{E} \sup_{t\in[0,T_0]} \|\mathcal{Z}_{ik}(t)\|_\alpha^p \leq C\varepsilon^{-\kappa_0}, \tag{21}$$

and

$$\mathbb{E} \sup_{t\in[0,T_0]} \|\mathcal{Z}_i(t)\|_\alpha^p \leq C\varepsilon^{-\kappa_0}, \tag{22}$$

where $\mathcal{Z}_{ik}(t)$ and $\mathcal{Z}_i(t)$ are defined in (7) and (6), respectively.

Proof. See the proof of Lemma 4.2 in [1]. □

The following corollary states that $\psi(t)$ is with high probability much smaller than $\varepsilon^{-\kappa}$ as assumed the Definition 7 for $t \leq \tau^*$. We show later $\tau^* \geq T_0$ with high probability (cf. proof of Theorem 10).

Corollary 13 *Under the assumptions of Lemmas 11 and 12, if $\psi(0) = \mathcal{O}(1)$, then for $p > 0$ there exist a constant $C > 0$ such that*

$$\mathbb{E} \sup_{t\in[0,\tau^*]} \|\psi_i(t)\|_\alpha^p \leq C\varepsilon^{-\kappa_0} \ . \tag{23}$$

Proof. By triangle inequality and Lemma 12, we obtain from (20) for a fixed $p > 1$ that

$$\mathbb{E} \sup_{t\in[0,\tau^*]} \|\psi_i(t)\|_\alpha^p \leq C + C\varepsilon^{2p-m_ip\kappa} + C\varepsilon^{-\kappa_0} \ ,$$

which implies (23) for $\kappa < \frac{2}{m_i}$. □

Let us now state a result similar to averaging, in case we integrate over the fast decaying contribution of the initial condition in ψ_i. This leads to terms of order $\mathcal{O}(\varepsilon^2)$.

Lemma 14 *For $q \geq 1$ there exists a constant $C > 0$ such that*

$$\int_0^T \|e^{\tau\varepsilon^{-2}\mathcal{A}_s}\psi_0\|_\alpha^q d\tau \leq C\varepsilon^2\|\psi_0\|_\alpha^q \quad \textit{for all } \psi_0 \in \mathcal{H}^\alpha.$$

Proof. Using (4) we obtain

$$\int_0^T \|e^{\varepsilon^{-2}\mathcal{A}_s\tau}\psi_0\|_\alpha^q d\tau \leq c\int_0^T e^{-q\varepsilon^{-2}\omega\tau}\|\psi_0\|_\alpha^q d\tau \leq \frac{\varepsilon^2}{q\omega}\|\psi_0\|_\alpha^q.$$

□

Lemma 15 *Let Assumptions 5 and 3 hold. Then*

$$a_i(t) = a_i(0) + \int_0^t \bar{\mathcal{F}}_i(a)d\tau + \tilde{R}(t), \tag{24}$$

with

$$\tilde{R} = \mathcal{O}(\varepsilon^{1-2m\kappa-\kappa_0}). \tag{25}$$

Proof. From the mild formulation in (17) and Lemma 11, we obtain

$$\psi(t) = \mathcal{Z}(t) + e^{\varepsilon^{-2}t\mathcal{A}}\psi(0) + \mathcal{O}(\varepsilon^{2-m\kappa}) =: \mathcal{Z}(t) + y(t) + R(t), \tag{26}$$

where

$$y(t) = e^{\varepsilon^{-2}t\mathcal{A}}\psi(0) \text{ and } R(t) = \mathcal{O}(\varepsilon^{2-m\kappa}) \ .$$

Substituting from (26) into (16), yields

$$a_i(t) = a_i(0) + \int_0^t \mathcal{F}_i^c(a + \mathcal{Z} + y + R)(\tau)d\tau. \tag{27}$$

Applying Taylor's expansion to the function $\mathcal{F}_i^c : \mathcal{H} \to \mathbb{R}$, yields

$$a_i(t) = a_i(0) + \int_0^t \mathcal{F}_i^c(a + \mathcal{Z})(\tau)d\tau + R_1(t),$$

where $R_1(t)$ is given by

$$R_1(t) = \sum_{|\ell|\geq 1} P_c \int_0^t \frac{D^\ell \mathcal{F}_i(a + \mathcal{Z})}{\ell!}(y + R)^\ell d\tau,$$

and the sum is finite (using that $\mathcal{F}_i^c$ is a polynomial). Using Assumption 5 we obtain the averaged equation with error term

$$a_i(t) = a_i(0) + \int_0^t \bar{\mathcal{F}}_i(a)d\tau + \tilde{R}(t),$$

where

$$\tilde{R}(t) = R_1 + \mathcal{O}(\varepsilon^{1-m\kappa}).$$

To bound $\tilde{R}$ we use Lemmas 14 and 12 and Assumption 5. □

Definition 16 *Define the set $\Omega^* \subset \Omega$ such that all the following estimates hold on Ω^**

$$\sup_{[0,\tau^*]} \|\psi - \mathcal{Q}\|_\alpha < C\varepsilon^{2-m\kappa-\kappa}\ , \tag{28}$$

$$\sup_{[0,\tau^*]} \|\psi\|_\alpha < C\varepsilon^{-\frac{3}{2}\kappa_0}\ , \tag{29}$$

$$\sup_{[0,\tau^*]} \|\tilde{R}\|_\alpha < C\varepsilon^{1-2m\kappa-\kappa}\ , \tag{30}$$

and

$$\sup_{[0,T_1]} \|b\|_\alpha \leq C_0. \tag{31}$$

The set Ω^* has probability close to 1.

Proposition 17 *Under Assumptions 3, 5, and 6 the set Ω^* has probability*

$$\mathbb{P}(\Omega^*) \geq 1 - C\varepsilon^p - \mathbb{P}(\sup_{[0,T_1]} \|b\|_\alpha > C_0)\ .$$

Proof. First

$$\mathbb{P}(\Omega^*) \geq 1 - \mathbb{P}(\sup_{[0,\tau^*]} \|\psi - \mathcal{Q}\|_\alpha \geq C\varepsilon^{-3\kappa}) - \mathbb{P}(\sup_{[0,\tau^*]} \|\psi\|_\alpha \geq C\varepsilon^{-\frac{3}{2}\kappa_0})$$

$$-\mathbb{P}(\sup_{[0,\tau^*]} \|\tilde{R}\|_\alpha \geq C\varepsilon^{1-2m\kappa-\kappa}) - \mathbb{P}(\sup_{[0,T_1]} \|b\|_\alpha > C_0).$$

Using Chebychev inequality and Lemmas 11, 12 and 13, we obtain for $\kappa > \kappa_0$ and sufficiently large $q > \frac{2p}{(\kappa-\kappa_0)} > 0$

$$\begin{aligned} \mathbb{P}(\Omega^*) &\geq 1 - C[\varepsilon^{q\kappa} + \varepsilon^{\frac{1}{2}q\kappa} + \varepsilon^{q(\kappa-\kappa_0)}] - \mathbb{P}(\sup_{[0,T_1]} \|b\|_\alpha > C_0) \\ &\geq 1 - C\varepsilon^p - \mathbb{P}(\sup_{[0,T_1]} \|b\|_\alpha > C_0). \end{aligned} \quad (32)$$

□

Theorem 18 *Assume that Assumptions 3, 5, and 6 hold. Suppose* $a(0) = \mathcal{O}(1)$ *and* $\psi(0) = \mathcal{O}(1)$*. Let* b *be a solution of (2) and* a *as defined in (24). If the initial conditions satisfy* $a(0) = b(0)$*, then for* $\kappa < \frac{1}{2m+1}$ *we obtain*

$$\sup_{t\in[0,T_1]} \|a(t) - b(t)\|_\alpha \leq C\varepsilon^{1-2m\kappa-\kappa} \quad on\ \Omega^*, \quad (33)$$

and

$$\sup_{t\in[0,T_1]} \|a(t)\|_\alpha \leq C \quad on\ \Omega^*. \quad (34)$$

Proof. Subtracting (2) from (24) and defining

$$h(t) := a(T) - b(t), \quad (35)$$

we obtain

$$h(t) = \int_0^t [\bar{\mathcal{F}}^c(h+b) - \bar{\mathcal{F}}^c(b)]d\tau + \tilde{R}(t), \quad (36)$$

where the error $\tilde{R}$ is defined as

$$\tilde{R} = \mathcal{O}(\varepsilon^{1-2m\kappa}) \quad (37)$$

Define Q as

$$Q(t) := h(t) - \tilde{R}(t). \quad (38)$$

From Equation (36) we obtain

$$\partial_t Q = \bar{\mathcal{F}}^c(Q + \tilde{R} + b) - \bar{\mathcal{F}}^c(b)$$

Taking the scalar product $\langle Q, \cdot \rangle$ in $\mathcal{H}^\alpha$ on both sides, yields

$$\frac{1}{2}\partial_t \|Q\|_\alpha^2 = \left\langle \bar{\mathcal{F}}^c(Q + \tilde{R} + b) - \bar{\mathcal{F}}^c(b), Q \right\rangle_{\mathcal{H}^\alpha}.$$

Using Young and Cauchy-Schwartz inequalities, where $\bar{\mathcal{F}}^c$ is a polynomial of degree m, we obtain

$$\frac{1}{2}\partial_t \|Q\|_\alpha^2 \leq C\left(1+\|Q\|_\alpha^{m-1}+\|\tilde{R}\|_\alpha^{m-1}+\|b\|_\alpha^{m-1}\right)\left(\|Q\|_\alpha^2+\|\tilde{R}\|_\alpha^2\right).$$

As long as $\|Q(t)\| < 1$, using Equations (30) and (31), we obtain for $\kappa < \frac{1}{2m+1}$

$$\frac{1}{2}\partial_t \|Q(t)\|_\alpha^2 \leq c\,\|Q(t)\|_\alpha^2 + C\varepsilon^{2-2(2m+1)\kappa} \quad \text{on} \quad \Omega^*,$$

Integrating from 0 to t and using Gronwall's lemma, we obtain for $t \leq \tau^* \wedge T_1 \leq T_0$

$$\|Q(t)\|_\alpha \leq C\varepsilon^{1-(2m+1)\kappa}e^{2cT_0},$$

and thus $\|Q(t)\|_\alpha < 1$ for $t \leq \tau^* \wedge T_1$. Taking supremum on $[0,\tau^* \wedge T_1]$

$$\sup_{t\in[0,\tau^*\wedge T_1]} \|Q(t)\|_\alpha \leq C\varepsilon^{1-(2m+1)\kappa} \quad \text{on} \quad \Omega^*.$$

Hence

$$\begin{aligned}\sup_{[0,\tau^*\wedge T_1]} \|a-b\|_\alpha &= \sup_{[0,\tau^*\wedge T_1]} \|Q-\tilde{R}\|_\alpha \leq \sup_{[0,\tau^*\wedge T_1]} \|Q\|_\alpha + \sup_{[0,\tau^*\wedge T_1]} \|\tilde{R}\|_\alpha \\ &\leq C\varepsilon^{1-(2m+1)\kappa} \quad \text{on} \quad \Omega^*. \qquad (39)\end{aligned}$$

We finish the proof by using (35), (38) and

$$\sup_{[0,\tau^*\wedge T_1]} \|a\|_\alpha \leq \sup_{[0,\tau^*\wedge T_1]} \|a-b\|_\alpha + \sup_{[0,\tau^*\wedge T_1]} \|b\|_\alpha \leq C.$$

This implies $T_1 \leq \tau^*$ on Ω^*. Thus, we have on Ω^*

$$\sup_{[0,T_1]} \|a-b\|_\alpha \leq C\varepsilon^{1-(2m+1)\kappa},$$

and

$$\sup_{[0,T_1]} \|a\|_\alpha \leq C.$$

□

Now, we can use the results obtained above to prove the main result from Theorem 10 for the system of SPDE (1).

Proof of Theorem 10. For the stopping time we note that (from the previous proof)

$$\Omega \supset \{\tau^* > T_1\} \supseteq \{\sup_{[0,\tau^*\wedge T_1]} \|a\|_\alpha < \varepsilon^{-\kappa}, \ \sup_{[0,\tau^*\wedge T_1]} \|\psi\|_\alpha < \varepsilon^{-\kappa}\} \supseteq \Omega^*.$$

Now let us turn to the approximation result. Using (13) and triangle inequality, we obtain

$$\begin{aligned}\sup_{t\in[0,T_1]} \|u(t)-b(t)-Q(t)\|_\alpha &\le \sup_{[0,T_1]} \|a-b\|_\alpha + \sup_{[0,T_1]} \|\psi - Q\|_\alpha \\ &\le \sup_{[0,T_1]} \|a-b\|_\alpha + \sup_{[0,\tau^*]} \|\psi - Q\|_\alpha .\end{aligned}$$

From (28) and (33), we obtain

$$\sup_{t\in[0,T_1]} \|u(t)-b(t)-Q(t)\|_\alpha \le C\varepsilon^{1-(2m+1)\kappa} \quad \text{on} \quad \Omega^* .$$

Hence,

$$\mathbb{P}\left(\sup_{t\in[0,T_1]} \|u(t)-b(t)-Q(t)\|_\alpha > C\varepsilon^{1-(2m+1)\kappa}\right) \le 1-\mathbb{P}(\Omega^*) \ .$$

Using (32), yields (18). □

5 Averaging

In the first lemma we state the averaging Lemma 5.1 from our paper [1] over the fast OU-process $\mathcal{Z}_k$ (cf. (7)). Actually, this is a slightly generalized version, which we state without proof.

Lemma 19 *Let X be a real valued stochastic process such that for some $r \ge 0$ we have $X(0) = \mathcal{O}(\varepsilon^{-r})$. Suppose $Z = (Z_1, ..., Z_n)$ with*

$$Z_i(t) = \hat{\alpha}_i \varepsilon^{-1} \int_0^t e^{-\varepsilon^{-2}(t-s)\hat{\lambda}_i} dB_i(s),$$

for some $\hat{\alpha}_i, \hat{\lambda}_i \in \mathbb{R}$ and independent Brownian motion B_i, are fast OU-process. Fix $\kappa_0 > 0$. If $dX = GdT$ with $G = \mathcal{O}(\varepsilon^{-r})$, then for $\ell \in \mathbb{N}_0^n$

$$\int_0^t XZ^\ell d\tau = \sum_{i=1}^n \frac{\ell_i(\ell_i-1)\hat{\alpha}_i^2}{2\sum_{j=1}^n \ell_j \hat{\lambda}_j} \int_0^t XZ^{\ell_j} Z_i^{-2} d\tau + \mathcal{O}(\varepsilon^{1-r-\kappa_0}) \ .$$

In the next lemma we state the application of Lemma 19 needed in the proof.

Lemma 20 *Let X be as in Lemma 19. Then, for $\ell \in \mathbb{N}_0^n$ and $|\ell| \ge 1$, we obtain*

1- If one of the ℓ_i is odd, then

$$P_c \int_0^t X\mathcal{Z}^\ell d\tau = \mathcal{O}(\varepsilon^{1-r-\kappa_0}). \tag{40}$$

2-If all ℓ_i are even, then there is a constant C_ℓ such that

$$P_c \int_0^t X \mathcal{Z}^\ell d\tau = C_\ell \int_0^t X d\tau + \mathcal{O}(\varepsilon^{1-r-\kappa_0}). \tag{41}$$

Proof. We study three cases.
First case: Consider single mode forcing for some $k \in \mathbb{N}^n$ with

$$\mathcal{Z}_i(t) = \mathcal{Z}_{ik_i} e_{k_i} = \alpha_{i,k_i} \varepsilon^{-1} \int_0^t e^{-\varepsilon^{-2}(t-\tau) d_i \lambda_{k_i}} d\beta_{k_i}(\tau) e_{k_i}. \tag{42}$$

Thus (recall that X is real valued)

$$P_c \int_0^t X \mathcal{Z}^\ell d\tau = \int_0^t X P_c \left(\prod_{i=1}^n e_{k_i}^{\ell_i} \right) \prod_{i=1}^n \mathcal{Z}_{ik_i}^{\ell_i} d\tau.$$

If one of the ℓ_i is odd, then applying Lemma 19 with $X P_c \prod_{i=1}^n e_{k_i}^{\ell_i}$ instead of X, yields

$$P_c \int_0^t X \mathcal{Z}^\ell d\tau = \mathcal{O}(\varepsilon^{1-r-\kappa_0}).$$

If all ℓ_i are even, we obtain by Lemma 19

$$P_c \int_0^t X \mathcal{Z}^\ell d\tau = \sum_{i=1}^n \frac{\ell_i(\ell_i - 1)\alpha_{i,k_i}^2}{2\sum_{j=1}^n \ell_j d_j \lambda_{k_j}} \int_0^t X P_c \Big(\prod_{i=1}^n e_{k_i}^{\ell_i} \Big) \prod_{j=1}^n \mathcal{Z}_{j,k_j}^{\ell_j} \mathcal{Z}_{i,k_i}^{-2} d\tau + \mathcal{O}(\varepsilon^{1-r-\kappa_0}).$$

Proceeding inductively, we obtain the existence of the constant C_ℓ, once all $\mathcal{Z}$'s are eliminated.

Second case: Here we assume $N = 2$. Thus

$$\mathcal{Z}_i = \mathcal{Z}_{i1} e_1 + \mathcal{Z}_{i2} e_2.$$

Using binomial formula $(x+y)^\ell = \sum_{r \in \mathbb{N}_0^n, r \leq \ell} \binom{\ell_i}{r} x^r y^{\ell - r}$ for $\ell \in \mathbb{N}_0^n$, and $x, y \in \mathbb{R}$. We obtain

$$P_c \int_0^t X \mathcal{Z}^\ell d\tau = \sum_{\substack{r \in \mathbb{N}_0^n \\ r \leq \ell}} \int_0^t X P_c \Big(\binom{\ell}{r} e_1^{|r|} e_2^{|\ell - r|} \Big) \mathcal{Z}_{\bullet,1}^r \mathcal{Z}_{\bullet,2}^{\ell - r} d\tau,$$

where $\mathcal{Z}_{\bullet,1} = (\mathcal{Z}_{1,1}, \ldots\ldots, \mathcal{Z}_{n,1})$. Thus using Lemma 19 we can proceed as in the first case.

Third case: For $N > 2$, we can proceed similar to the second case by expanding

$$\Big(\sum_{j=1}^N \mathcal{Z}_{ij} e_j \Big)^\ell.$$

$\square$

Now, let us give an explicit formula for the polynomial $\bar{\mathcal{F}}_i(b)$ which is defined in (2) and Assumption 5. Usually, one would use integration with respect to the invariant measure of the OU-process to determine $\bar{\mathcal{F}}_i$. Here, we give a simple proof for a particular example, which is needed later.

Corollary 21 *Consider the case of single mode forcing as given in Equation (42). Then*

$$\bar{\mathcal{F}}_i(b) = \mathcal{F}_i(b) + \sum_{|\ell|\geq 1} \frac{C_{2\ell}}{(2\ell)!} D^{2\ell}\mathcal{F}_i(b), \tag{43}$$

with

$$C_{2\ell} = \sum_{i=1}^{n} \frac{\ell_i(2\ell_i - 1)\alpha_{i,k_i}^2}{2\sum_{j=1}^{n} \ell_j d_j \lambda_{k_j}} P_c\Big(\prod_{j=1}^{n} e_{k_j}^{2\ell_j}\Big) C_{2\ell-2f_i}, \tag{44}$$

where $f_i = (0, ..., \underbrace{1}_{i-th}, ..., 0)$ *is the standard basis in* $\mathbb{R}^n$ *and* $C_0 = 1$.

Proof. Applying Taylor's expansion for the function $\mathcal{F}_i^c : \mathcal{H} \to \mathbb{R}$, yields

$$\int_0^t \mathcal{F}_i^c(b + \mathcal{Z})d\tau = \sum_{|\ell|\geq 0} P_c \int_0^t \frac{D^\ell \mathcal{F}_i(b)}{\ell!} \mathcal{Z}^\ell d\tau. \tag{45}$$

Applying Lemma 20 to Equation (45), yields

$$\int_0^t \mathcal{F}_i^c(b + \mathcal{Z})d\tau = \sum_{|\ell|\geq 0} \frac{C_\ell}{\ell!} \int_0^t D^\ell \mathcal{F}_i(b)d\tau + \mathcal{O}(\varepsilon^{1-m_i\kappa}), \tag{46}$$

where $C_0 = 1$ and $C_\ell = 0$ if one ℓ_i is odd. Thus

$$\int_0^t \mathcal{F}_i^c(b + \mathcal{Z})d\tau = \sum_{|\ell|\geq 0} \frac{C_{2\ell}}{(2\ell)!} \int_0^t D^{2\ell}\mathcal{F}_i(b)d\tau + \mathcal{O}(\varepsilon^{1-m_i\kappa}).$$

From the first case of the proof of Lemma 20, we obtain (44). $\square$

Example 22 *Let the noise be forcing only one mode, assume* $n = 2$ *and* $d = 1$, *such that the eigenfunction are given by* $e_0 = 1$ *and* $e_k = \sqrt{2}\cos(\pi k x)$ *for* $k > 0$. *We consider several cases.*

First case: *If* $|\ell| = 1$, *then* $(\ell_1, \ell_2) = (1, 0)$ *or* $(0, 1)$ *hence*

$$C_{(2,0)} = \frac{\alpha_{1,k_1}^2}{2d_1\lambda_{k_1}}\Big(P_c e_{k_1}^2\Big)C_{(0,0)} = \frac{\alpha_{1,k_1}^2}{2d_1\lambda_{k_1}}. \tag{47}$$

and

$$C_{(0,2)} = \frac{\alpha_{2,k_2}^2}{2d_2\lambda_{k_2}}. \tag{48}$$

Second case: *If* $|\ell| = 2$, *then* $(\ell_1, \ell_2) = (2,0)$, $(1,1)$ *or* $(0,2)$ *hence*

$$C_{(4,0)} = \frac{9\alpha_{1,k_1}^2}{4d_1\pi^2 k_1^2} C_{(2,0)} = \frac{9\alpha_{1,k_1}^4}{8d_1^2\pi^4 k_1^4},$$

$$C_{(0,4)} = \frac{9\alpha_{2,k_2}^2}{4d_2\pi^2 k_2^2} C_{(2,0)} = \frac{9\alpha_{2,k_2}^4}{8d_2^2\pi^4 k_2^4},$$

and

$$\begin{aligned} C_{(2,2)} &= \frac{1}{2(d_1\lambda_{k_1} + d_2\lambda_{k_2})} P_c\Big(e_{k_1}^2 e_{k_2}^2\Big)[\alpha_{1,k_1}^2 C_{(0,2)} + \alpha_{2,k_2}^2 C_{(2,0)}] \\ &= \frac{\alpha_{1,k_1}^2 \alpha_{2,k_2}^2}{2(d_1\lambda_{k_1} + d_2\lambda_{k_2})} P_c\Big(e_{k_1}^2 e_{k_2}^2\Big), \end{aligned}$$

where we used (47) and (48). Now, we have two different cases, first assume $k_1 \neq k_2$. *In this case* $P_c(e_{k_1}^2 e_{k_2}^2) = 1$, *and*

$$C_{(2,2)} = \frac{\alpha_{1,k_1}^2 \alpha_{2,k_2}^2}{2(d_1\lambda_{k_1} + d_2\lambda_{k_2})}.$$

For the second case consider $k_1 = k_2$. *In this case* $P_c(e_{k_1}^2 e_{k_2}^2) = \frac{3}{2}$, *and*

$$C_{(2,2)} = \frac{3\alpha_{1,k_1}^2 \alpha_{2,k_2}^2}{4(d_1\lambda_{k_1} + d_2\lambda_{k_2})}.$$

6 Applications

In this section we consider all examples with homogeneous Neumann boundary condition on the interval $[0,1]$. Suppose the eigenfunctions are

$$e_k = \begin{cases} 1 & \text{if } k = 0 \\ \sqrt{2}\cos(\pi k x) & \text{if } \ k > 0, \end{cases} .$$

and

$$\mathcal{N} = span\{1\}.$$

The eigenvalues of the operator $-\mathcal{A}_i = -d_i\partial_x^2$ are $\lambda_k = \pi^2 k^2$. We apply our results to some real-life applications from biology and chemistry as follows:

6.1 Biological Application

We consider a simple predator–prey system with diffusion which is a modified Lotka–Volterra system with logistic growth of the prey and with both predator and prey dispersing by diffusion (cf. J. Murray [12]). Also, Dunbar

[8, 9] discussed this model in detail. The model with additive noise takes the form

$$\begin{aligned} \partial_t u_1 &= \frac{d_1}{\varepsilon^2}\frac{\partial^2 u_1}{\partial x^2} + Au_1(1-\frac{u_1}{K}) - Bu_1u_2 + \varepsilon^{-1}\partial_t W_1(t), \\ \partial_t u_2 &= \frac{d_2}{\varepsilon^2}\frac{\partial^2 u_2}{\partial x^2} + Cu_1u_2 - Du_2 + \varepsilon^{-1}\partial_t W_2(t), \end{aligned}$$

where u_1 is the prey, u_2 is the predator, A, B, C, D and K, the prey carrying capacity, are positive constants. We assume very fast diffusion $\frac{d_i}{\varepsilon^2}$, and in addition large fluctuation of order ε^{-1}. If we take

$$W_1(t) = \alpha_1\beta_1(t)\cos(\pi x) \quad \text{and} \quad W_2(t) = \alpha_2\beta_2(t)\cos(\pi x),$$

then the noise is large, but it does not change the average population size. Our main theorem states that

$$u(t) = b(t) + \mathcal{Z}(t) + \mathcal{O}(\varepsilon^{1-}),$$

on finite time horizons, as long as the approximation b remains bounded. Here

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \text{ and } \mathcal{Z} = \begin{pmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2 \end{pmatrix}$$

where b_1 and b_2 are the solutions of

$$\begin{aligned} \partial_t b_1 &= Ab_1(1-\frac{b_1}{K}) - Bb_1b_2 - \frac{A\alpha_1^2}{2d_1\pi^2 K}, \\ \partial_t b_2 &= Cb_1b_2 - Db_2, \end{aligned}$$

and

$$\mathcal{Z}_i = \varepsilon^{-1}\alpha_i \int_0^t e^{-\varepsilon^{-2}(t-s)d_1\pi^2} d\beta_i(t)\cos(\pi x) \quad \text{for } i = 1, 2.$$

The biological effect of the $\frac{A\alpha_1^2}{2d_1\pi^2 K}$ term can be understood by neglecting b_2, i.e assuming no predators. In this case one gets

$$\partial_t b_1 = Ab_1(1-\frac{b_1}{K}) - \frac{A\alpha_1^2}{2d_1\pi^2 K}.$$

There appears a surprising effective extra drift term from the combination of fast diffusion and large fluctuations, although both terms individually do not change the average population size. If

$$b_1 < \frac{\alpha_1^2}{2d_1\pi^2 K}, \tag{49}$$

then $\frac{db_1}{dt} < 0$, i.e. it decreases. Since the effect of b_2 is to further decrease b_1 then Equation (49) implies the annihilation of the system.

6.2 Chemical Application

A simple archetype example for a reaction-diffusion system is a cubic autocatalytic reaction between two chemicals according to the rule $A + B \to 2B$ with rate $r = \rho u_1 u_2^2$.

Denoting by u_1 and u_2 the concentration of A and B, respectively. The two species satisfy the equations:

$$\begin{aligned} \partial_t u_1 &= \frac{d_1}{\varepsilon^2}\frac{\partial^2 u_1}{\partial x^2} - \rho u_1 u_2^2 + \varepsilon^{-1}\partial_t W_1(t), \\ \partial_t u_2 &= \frac{d_2}{\varepsilon^2}\frac{\partial^2 u_2}{\partial x^2} + \rho u_1 u_2^2 + \varepsilon^{-1}\partial_t W_2(t), \end{aligned}$$

We consider again large diffusion and large noise which both might be introduced by fast stirring. If we take

$$\begin{aligned} W_1(t) &= \alpha_1 \beta_1(t)\cos(\pi x) \\ W_2(t) &= \alpha_2 \beta_2(t)\cos(\pi x) \end{aligned}$$

then our main theorem states that

$$u(t) = b(t) + \mathcal{Z}(t) + \mathcal{O}(\varepsilon^{1-}),$$

with

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \text{ and } \mathcal{Z} = \begin{pmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2 \end{pmatrix}$$

where b_1 and b_2 are the solutions of

$$\begin{aligned} \partial_t b_1 &= -\rho b_1 b_2^2 - \frac{\rho\alpha_2^2}{2d_2\pi^2} b_1, \\ \partial_t b_2 &= \rho b_1 b_2^2 + \frac{\rho\alpha_2^2}{2d_2\pi^2} b_1. \end{aligned}$$

We note that high fluctuations in combination with fast diffusion lead to effective new terms describing the transformation of b_1 to b_2. Although both terms individually do not change the average $\int u_i dx = b_i$, or there nonlinear combination does.

Let us check the bound on b. We note that

$$\sum_{i=1}^{n} \partial_t b_i = 0,$$

integrating from 0 to t

$$\sum_{i=1}^{n} b_i(t) = \sum_{i=1}^{n} b_i(0) = C_0.$$

As $b_i \geq 0$, we have

$$0 \leq b_i(t) \leq \sum_{i=1}^{n} b_i(t) \leq C_0.$$

Hence, we obtain for all times $t > 0$

$$\|b(t)\| = \left(\sum_{i=1}^{2} b_i^2(t) \right)^{1/2} \leq C_0\sqrt{2}.$$

Thus the approximation is valid on any fixed finite time interval.

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