UNIVERSITÄT AUGSBURG

Embedding Constraint Relationships into C-Semirings

Alexander Knapp, Alexander Schiendorfer

Report 2014-03 April 2014

INSTITUT FÜR INFORMATIK
D-86135 AUGSBURG

INTRODUCTION

These notes provide technical details that are required to embed constraint relationships into the c-semiring framework presented in terms of category theory. It contains all steps required to map a dag to a partial order (Section 1), construct the *free* meet monoid from this partial order (Section 3) as well as the *free* c-semiring (Section 4). A constraint solving algorithm based on branch-and-bound search is presented in §34 for c-semirings and in §36 for meet monoids. A concrete instantiation for constraint relationships along with an example soft constraint problem concludes the report in Section 7.

1. PARTIAL ORDERS AND DIRECTED ACYCLIC GRAPHS

1. A *partial order* $(X, \leq)$ is given by a set X and a binary relation $\leq \subseteq X \times X$ such that $\leq$ is reflexive, transitive, and anti-symmetric on X. For $x, y \in X$ we write $x < y$ if $x \leq y$ and $x \neq y$, and $x \geq y$ resp. $x > y$ if $y \leq x$ resp. $y < x$.
A *partial order homomorphism* $\varphi : P \to Q$ from a partial order $P = (|P|, \leq_P)$ to a partial order $Q = (|Q|, \leq_Q)$ is given by a map $\varphi : |P| \to |Q|$ such that $\varphi(p) \leq_Q \varphi(p')$ if $p \leq_P p'$ for all $p, p' \in |P|$.
The category PO of partial orders has the partial orders as objects and the partial order homomorphisms as morphisms.

2. A *directed acyclic graph*, or *dag*, $(X, \to)$ is given by a set X and a binary relation $\to \subseteq X \times X$ such that $\to^+$ is irreflexive. If $x \to y$, then x is a *predecessor* of y, and y is a *successor* of x.
A *dag homomorphism* $\varphi : G \to H$ from a dag $G = (|G|, \to_G)$ to a dag $H = (|H|, \to_H)$ is given by a map $\varphi : |G| \to |H|$ such that $\varphi(g) \to_H \varphi(g')$ if $g \to_G g'$ for all $g, g' \in |G|$.
The category DAG of dags has the dags as objects and the dag homomorphisms as morphisms.

3. Define the functor $PO\langle - \rangle : \mathrm{DAG} \to \mathrm{PO}$ by

$$PO\langle G \rangle = (|G|, \to_G^*) \ ,$$
$$PO\langle \varphi : G \to H \rangle = \varphi \ .$$

Define the functor $DAG : \mathrm{PO} \to \mathrm{DAG}$ by

$$DAG(P) = (|P|, <_P) \ ,$$
$$DAG(\varphi : P \to Q) = \varphi \ .$$

For each $G \in |\mathrm{DAG}|$, define $\eta_G^{\mathrm{PO}} : G \to DAG(PO\langle G \rangle)$ by $\eta_G^{\mathrm{PO}}(g) = g$. Then $\eta^{\mathrm{PO}} = (\eta_G^{\mathrm{PO}})_{G \in |\mathrm{DAG}|}$ is a natural transformation from 1_{DAG} to $DAG \circ PO\langle - \rangle$.
Let $G \in |\mathrm{DAG}|$, $P \in |\mathrm{PO}|$, and $\varphi : G \to DAG(P)$. Define $\varphi^{\sharp_{\mathrm{PO}}} : PO\langle G \rangle \to P$ by

$$\varphi^{\sharp_{\mathrm{PO}}}(g) = \varphi(g) \ .$$

Then $DAG(\varphi^{\sharp_{\mathrm{PO}}})(\eta_G^{\mathrm{PO}}(g)) = \varphi(g)$ and $\varphi^{\sharp_{\mathrm{PO}}}$ is unique with this property.

2. UPPER SEMI-LATTICES

4. A (*bounded*) *upper semi-lattice* $(X, \sqcup, \bot)$ is given by a set X, a binary operation $\sqcup : X \times X \to X$, and a constant $\bot \in X$ such that the following axioms are satisfied for all $x, y, z \in X$:
(1) $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
(2) $x \sqcup y = y \sqcup x$
(3) $x \sqcup x = x$

(4) $x \sqcup \bot = x$

In words, $\sqcup$ is associative, commutative, and idempotent, and has $\bot$ as neutral element.
A (*bounded*) *upper semi-lattice homomorphism* $\varphi : U \to V$ from an upper semi-lattice $U = (|U|, \sqcup_U, \bot_U)$ to an upper semi-lattice $V = (|V|, \sqcup_V, \bot_V)$ is given by a map $\varphi : |U| \to |V|$ such that for all $u_1, u_2 \in |U|$:
(1) $\varphi(u_1 \sqcup_U u_2) = \varphi(u_1) \sqcup_V \varphi(u_2)$
(2) $\varphi(\bot_U) = \bot_V$
The category uSL of upper semi-lattices has the upper semi-lattices as objects and the upper semi-lattice homomorphisms as morphisms.

5. Let P be a partial order. Let $\mathcal{I}_{\text{fin}}(P)$ denote the set of finite subsets of $|P|$ which only contain pairwise incomparable elements w.r.t $\leq_P$. For a subset $S \subseteq |P|$, let $\text{Max}^{\leq_P}(S)$ denote the set of maximal elements of S w.r.t. $\leq_P$.
Define the binary operation $\cup_P : \mathcal{I}_{\text{fin}}(P) \times \mathcal{I}_{\text{fin}}(P) \to \mathcal{I}_{\text{fin}}(P)$ by

$$I \cup_P J = \text{Max}^{\leq_P}(I \cup J) \ .$$

LEMMA. *$(\mathcal{I}_{\text{fin}}(P), \cup_P, \emptyset)$ is an upper semi-lattice.*

Proof. Let $I, J, K \in \mathcal{I}_{\text{fin}}(P)$. For the associativity of $\cup_P$ we have

$$I \cup_P (J \cup_P K) = \text{Max}^{\leq_P}(I \cup \text{Max}^{\leq}(J \cup K)) = \text{Max}^{\leq_P}(I \cup J \cup K) = \text{Max}^{\leq_P}(\text{Max}^{\leq_P}(I \cup J) \cup K) = (I \cup_P J) \cup_P K \ ,$$

since $\text{Max}^{\leq_P}(I \cup \text{Max}^{\leq_P} X) = \text{Max}^{\leq_P}(I \cup X)$ for all $X \in \mathcal{P}_{\text{fin}}|P|$. $\cup_P$ inherits commutativity from $\cup$. For the idempotency of $\cup_P$ we have

$$I \cup_P I = \text{Max}^{\leq_P}(I \cup I) = \text{Max}^{\leq_P} I = I \ ,$$

since $I \in \mathcal{I}_{\text{fin}}(P)$. Finally, we have $I \cup_P \emptyset = I$. □

Define the functor $uSL\langle - \rangle : \text{PO} \to \text{uSL}$ by

$$uSL\langle P \rangle = (\mathcal{I}_{\text{fin}}(P), \cup_P, \emptyset) \ ,$$
$$uSL\langle \varphi : P \to Q \rangle = \lambda\{p_1, \ldots, p_n\} \in \mathcal{I}_{\text{fin}}(P) \, . \, \text{Max}^{\leq_Q}\{\varphi(p_1), \ldots, \varphi(p_n)\} \ .$$

6. Each upper semi-lattice U induces a partial ordering $\leq_U \subseteq |U| \times |U|$ on $|U|$ given by

$$u_1 \leq_U u_2 \iff u_1 \sqcup_U u_2 = u_2 \ .$$

Indeed, $\leq_U$ is reflexive on $|U|$ by the idempotency of $\sqcup_U$, $\leq_U$ is transitive by the associativity of $\sqcup_U$, and $\leq_U$ is anti-symmetric by the commutativity of $\sqcup_U$. Furthermore, $\bot_U$ is the smallest element w.r.t. $\leq_U$, i.e., $\bot_U \leq_U u$ for all $u \in |U|$, by the neutrality of $\bot_U$.
Define the functor $PO : \text{uSL} \to \text{PO}$ by

$$PO(U) = (|U|, \leq_U) \ ,$$
$$PO(\varphi : U \to V) = \varphi \ ,$$

which is well-defined on objects by the remarks above and also morphisms since if $u_1 \leq_U u_2$, i.e., $u_1 \sqcup_U u_2 = u_2$, then $\varphi(u_1) \sqcup_V \varphi(u_2) = \varphi(u_1 \sqcup_U u_2) = \varphi(u_2)$, i.e., $\varphi(u_1) \sqcup_V \varphi(u_2)$.
For each $P \in |\text{PO}|$, define $\eta^{\text{uSL}}_P : P \to PO(uSL\langle P \rangle)$ by $\eta^{\text{uSL}}_P(p) = \{p\}$. Then $\eta^{\text{uSL}} = (\eta^{\text{uSL}}_P)_{P \in |\text{PO}|}$ is a natural transformation from 1_{PO} to $PO \circ uSL\langle - \rangle$.
Let $P \in |\text{PO}|$, $U \in |\text{uSL}|$, and $\varphi : P \to PO(U)$. Define $\varphi^{\sharp_{\text{uSL}}} : uSL\langle P \rangle \to U$ by

$$\varphi^{\sharp_{\text{uSL}}}(\{p_1, \ldots, p_n\}) = \varphi(p_1) \sqcup_U \cdots \sqcup_U \varphi(p_n)$$

for all $\{p_1, \dots, p_n\} \in \mathcal{I}_{\text{fin}}(P)$, where, if $n = 0$, the right hand side is to be understood as $\bot_U$; $\varphi^{\sharp_{\text{uSL}}}$ is indeed an upper semi-lattice homomorphism, since for each $\{p'_1, \dots, p'_n\} \in \mathcal{P}_{\text{fin}}\,|P|$ we have $\varphi^{\sharp_{\text{uSL}}}(\text{Max}^{\leq_P}\{p'_1, \dots, p'_n\}) = \varphi(p'_1) \sqcup_U \cdots \sqcup_U \varphi(p'_n)$: if $p'_i \leq_P p'_j$, then $\varphi(p'_i) \leq_{PO(U)} \varphi(p'_j)$, i.e., $\varphi(p'_i) \sqcup_U \varphi(p'_j) = \varphi(p'_j)$.
Then $PO(\varphi^{\sharp_{\text{uSL}}})(\eta^{\text{uSL}}_P(p)) = \varphi(p)$ and $\varphi^{\sharp_{\text{uSL}}}$ is unique with this property.

7. The partial ordering of $PO(uSL\langle P\rangle)$ on $\mathcal{I}_{\text{fin}}(P)$ for a partial order P is called the *lower* or *Hoare* ordering on $\mathcal{I}_{\text{fin}}(P)$ which we denote by $\subseteq_P$; it is explicitly given by

$$\begin{aligned} I \subseteq_P J &\iff \text{Max}^{\leq_P}(I \cup J) = J \\ &\iff \forall p \in I \,.\, \exists q \in J \,.\, p \leq_P q \end{aligned}$$

for $I, J \in \mathcal{I}_{\text{fin}}(P)$. It is $\emptyset \subseteq_P I$ for all $I \in \mathcal{I}_{\text{fin}}(P)$.
The dual of the Hoare ordering is the *upper* or *Smyth ordering* $\subseteq^P$ on $\mathcal{I}_{\text{fin}}(P)$ defined by $I \subseteq^P J$ if, and only if, $J \subseteq_{P^{-1}} I$, where $P^{-1} = (|P|, \geq_P)$. Explicitly, the Smyth ordering is given by

$$\begin{aligned} I \subseteq^P J &\iff \text{Min}^{\leq_P}(I \cup J) = I \\ &\iff \forall q \in J \,.\, \exists p \in I \,.\, p \leq_P q \end{aligned}$$

where $\text{Min}^{\leq_P}(S)$ is the set of minimal elements of $S \subseteq |P|$. In particular, the Smyth ordering also induces a binary operation $\cup^P : \mathcal{I}_{\text{fin}}(P) \times \mathcal{I}_{\text{fin}}(P) \to \mathcal{I}_{\text{fin}}(P)$ given by

$$I \cup^P J = \text{Min}^{\leq_P}(I \cup J)\ ,$$

which is also associative, commutative, and idempotent. Here, $I \cup^P \emptyset = I$, i.e., $\emptyset$ is an absorptive element for $\cup^P$, and $I \subseteq^P \emptyset$ for all $I \in \mathcal{I}_{\text{fin}}(P)$, i.e., $\emptyset$ is the greatest element of $\mathcal{I}_{\text{fin}}(P)$ w.r.t. $\subseteq^P$.
The *convex* or *Plotkin ordering* on $\mathcal{I}_{\text{fin}}(P)$ is defined by the intersection of $\subseteq_P$ and $\subseteq^P$, which means

$$I\ (\subseteq_P \cap \subseteq^P)\ J \iff (\forall p \in I \,.\, \exists q \in J \,.\, p \leq_P q) \wedge (\forall q \in J \,.\, \exists p \in I \,.\, p \leq_P q)$$

for $I, J \in \mathcal{I}_{\text{fin}}(P)$.
Finally, $\cup_P$ is monotonic w.r.t. $\subseteq_P$, and $\cup^P$ is monotonic w.r.t. $\subseteq^P$, i.e., for all $I, J, K \in \mathcal{I}_{\text{fin}}(P)$,

$$I \subseteq_P J \text{ implies } I \cup_P K \subseteq_P J \cup_P K\ ,$$
$$I \subseteq^P J \text{ implies } I \cup^P K \subseteq^P J \cup^P K\ .$$

3. Partially ordered monoids

8. A *partially ordered monoid* $(X, \cdot, \varepsilon, \leq)$ is given by a set X, a binary operation $\cdot : X \times X \to X$, a constant $\varepsilon \in X$, and a partial order relation $\leq\, \subseteq X \times X$ such that the following axioms are satisfied for $x, x', y, y', z \in X$:

(1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
(2) $x \cdot y = y \cdot x$
(3) $x \cdot \varepsilon = x$
(4) if $x \leq x'$ and $y \leq y'$, then $x \cdot y \leq x' \cdot y'$

In words, $(X, \cdot, \varepsilon)$ is a commutative monoid with unity ε and $\leq$ is monotone w.r.t. $\cdot$.
A *partially ordered monoid homomorphism* $\varphi : M \to N$ from a partially ordered monoid $M = (|M|, \cdot_M, \varepsilon_M, \leq_M)$ to a partially ordered monoid $N = (|N|, \cdot_N, \varepsilon_N, \leq_N)$ is given by a map $\varphi : |M| \to |N|$ such that for all $m, n \in |M|$:

(1) $\varphi(m \cdot_M n) = \varphi(m) \cdot_N \varphi(n)$
(2) $\varphi(\varepsilon_M) = \varepsilon_N$
(3) if $m \leq_M n$, then $\varphi(m) \leq_N \varphi(n)$

The category poMon of partially ordered monoids has the partially ordered monoids as objects and the partially ordered monoids homomorphisms as morphisms.

9. A partially ordered monoid M is a *join monoid* if for all $m, n \in |M|$

$$m \leq_M m \cdot_M n \,.$$

This requirement is equivalent to requiring that ε_M is the smallest element w.r.t. $\leq_M$. Indeed, if $m \leq_M m \cdot_M n$ holds for all $m, n \in |M|$, then $\varepsilon_M \leq_M \varepsilon_M \cdot_M n = n$ for all $n \in |M|$. Conversely, if $\varepsilon_M \leq_M n$ for all $n \in |M|$, then $m = m \cdot_M \varepsilon_M \leq_M m \cdot_M n$ for all $m, n \in |M|$ by the monotonicity of $\leq_M$.
Dually, a partially ordered monoid M is a *meet monoid* if for all $m, n \in |M|$

$$m \cdot_M n \leq_M m \,,$$

and this requirement is equivalent to requiring that ε_M is the greatest element w.r.t. $\leq_M$.
The full sub-categories of poMon having all join monoids respectively meet monoids as objects are denoted by jMon and mMon, respectively.
There are functors

$$\begin{array}{ll} \mathit{jMon} : \mathrm{mMon} \to \mathrm{jMon} & \mathit{mMon} : \mathrm{jMon} \to \mathrm{mMon} \\ \mathit{jMon}(M) = (|M|, \cdot_M, \varepsilon_M, \leq_M^{-1}) & \mathit{mMon}(M) = (|M|, \cdot_M, \varepsilon_M, \leq_M^{-1}) \\ \mathit{jMon}(\varphi : M \to N) = \varphi & \mathit{mMon}(\varphi : M \to N) = \varphi \end{array}$$

such that $\mathit{jMon} \circ \mathit{mMon} = 1_{\mathrm{jMon}}$ and $\mathit{mMon} \circ \mathit{jMon} = 1_{\mathrm{mMon}}$. Note that meet monoids have also been referred to as partial valuation structures [3] or ic-monoids [4].

10. For a set X let $\mathcal{M}_{\text{fin}}(X)$ be the set of finite multisets over X. We write $\wr x_1, \ldots, x_m \int$ with $x_i \in X$ for $1 \leq i \leq m$ or $\wr l_1 x_1, \ldots, l_n x_n \int$ with $x_i \in X$ and $l_i \in \mathbb{N}$ for $1 \leq i \leq n$ for an element of $\mathcal{M}_{\text{fin}}(X)$, $T \uplus U$ for the multiset union of the multisets T and U, and $T \sqsubseteq U$ for the sub-multiset relation, which is a partial ordering relation on $\mathcal{M}_{\text{fin}}(X)$.
For a partial order P, the *lower* or *Hoare ordering* on $\mathcal{M}_{\text{fin}}\,|P|$ is the binary relation $\sqsubseteq_P \subseteq (\mathcal{M}_{\text{fin}}\,|P|) \times (\mathcal{M}_{\text{fin}}\,|P|)$ given by the transitive closure of

$$T \sqsubseteq U \text{ implies } T \sqsubseteq_P U \,,$$

$$p \leq_P q \text{ implies } T \uplus \wr p \int \sqsubseteq_P T \uplus \wr q \int \,.$$

If $T \sqsubseteq_P U$, then $T \uplus \wr r \int \sqsubseteq_P U \uplus \wr r \int$ for all $r \in X$, since this holds for both defining clauses of the ordering.
For an element $T = \wr l_1 x_1, \ldots, l_n x_n \int \in \mathcal{M}_{\text{fin}}(X)$ with $l_1, \ldots, l_n > 0$, $x_i \neq x_j$ for all $1 \leq i \neq j \leq n$, and $n \geq 0$ let $\mathcal{S}(T) = \bigcup_{1 \leq i \leq n} \{(j, x_i) \mid 1 \leq j \leq l_i\}$.

LEMMA. *$T \sqsubseteq_P U$ if, and only if, there is an injective mapping $f : \mathcal{S}(T) \to \mathcal{S}(U)$ with $p \leq_P q$ if $f(j,p) = (k,q)$ for all $(j,p) \in \mathcal{S}(T)$.*

Proof. Let first $T \sqsubseteq_P U$ hold. Then there are an $n > 1$ and $T_1, \ldots, T_n \in \mathcal{M}_{\text{fin}}(X)$ such that $T_1 = T$, $T_n = U$, and either $T_i \sqsubseteq T_{i+1}$ or $T_i = T_i' \uplus \wr p \int$ and $T_{i+1} = T_i' \uplus \wr q \int$ with $p \leq_P q$ for all $1 \leq i < n$. For each $1 \leq i < n$ there is a map $f_i : \mathcal{S}(T_i) \to \mathcal{S}(T_{i+1})$ as required in the claim as follows: If $T_{n-1} \sqsubseteq T_n$, then we choose $f_i = 1_{\mathcal{S}(T_i)}$. If $T_i = T_i' \uplus \wr p \int$ and $T_{i+1} = T_i' \uplus \wr q \int$ with $p \leq_P q$, then we choose $f_i = 1_{\mathcal{S}(T_i')} \cup \{(j,p) \mapsto (k,q)\}$ where $j = |\{l \mid (l,p) \in \mathcal{S}(T_i')\}| + 1$ and $k = |\{l \mid (l,q) \in \mathcal{S}(T_i')\}| + 1$. Then $f_n \circ \ldots \circ f_1 : \mathcal{S}(T) \to \mathcal{S}(U)$ as required in the claim.
For the converse, we prove that if $f : \mathcal{S}(T) \to \mathcal{S}(U)$ is a mapping as required in the claim, then $T \sqsubseteq_P U$ by induction on the cardinality of $\mathcal{S}(T)$. Let $f : \mathcal{S}(T) \to \mathcal{S}(U)$ be given. If $|\mathcal{S}(T)| = 0$, then $\wr\int = T \sqsubseteq U$. Now let $|\mathcal{S}(T)| > 0$ and let $(j,p) \in \mathcal{S}(T)$ such that j is maximal. Then $f(j,p) = (k,q)$

with $p \leq_P q$. Define $g : \mathcal{S}(U) \to \mathcal{S}(U) \setminus \{(k,q)\}$ by $g(l,r) = (l,r)$ if $r \neq q$ or $l < k$, and $g(l,q) = (l-1,q)$ if $l > k$. Let $T', U' \in \mathcal{M}_{\text{fin}}(X)$ be defined by $T = T' \uplus \wr p \int$ and $U = U' \uplus \wr q \int$. Then $\mathcal{S}(T') = \mathcal{S}(T) \setminus \{(j,p)\}$ and $f' : \mathcal{S}(T') \to \mathcal{S}(U')$ defined by $f'(l,r) = g(f(l,r))$ for all $(l,r) \in \mathcal{S}(T')$ is an injective mapping as required in the claim. By induction hypothesis $T' \Subset_P U'$ and thus, by the remark above, $T = T' \uplus \wr p \int \Subset_P U' \uplus \wr p \int \Subset_P U' \uplus \wr q \int = U$. □

We call such a map a *witness* for $T \Subset_P U$.

11. The relation $\Subset_P$ is obviously transitive and reflexive on $\mathcal{M}_{\text{fin}}\,|P|$. It is also antisymmetric: Assume for a contradiction that there are T and U with $T \Subset_P U$ and $U \Subset_P T$, but $T \neq U$ and choose an T with minimal cardinality satisfying this property. Then $T \neq \wr\int$. Let $f : \mathcal{S}(T) \to \mathcal{S}(U)$ and $g : \mathcal{S}(U) \to \mathcal{S}(T)$ be witnessing maps for $T \Subset_P U$ and $U \Subset_P T$. Choose an element $(j,p) \in \mathcal{S}(T)$ such that p is maximal w.r.t. $\leq_P$ in T. Let $f(j,p) = (k,q) \in \mathcal{S}(U)$. Then $p \leq_P q$. If $p \neq q$, there would be a $(j',p') \in \mathcal{S}(T)$ with $p \leq_P q \leq_P p'$ but $p \neq p'$ contradicting the maximality of p in T; thus $f(j,p) = (k,p)$. Assume, without loss of generality, that j and k are maximal. Remove the occurrence of p from T, obtaining T', and from U, obtaining U'. Then $T' \Subset_P U'$ and $U' \Subset_P T'$, since $f' : \mathcal{S}(T') \to \mathcal{S}(U')$ with $f'(x) = f(x)$ if $x \neq (j,p)$ and $g' : \mathcal{S}(U') \to \mathcal{S}(T')$ with $g'(y) = g(y)$ if $y \neq (k,p)$ are witnessing maps, contradicting the minimality of T.
Furthermore, $\uplus$ is associative and commutative; it is also monotonic w.r.t. $\Subset_P$, i.e., for all $T, U, V \in \mathcal{M}_{\text{fin}}\,|P|$,

$$T \Subset_P U \text{ implies } T \uplus V \subseteq_P U \uplus V \ ;$$

and $\wr\int \Subset_P T$ for all $T \in \mathcal{M}_{\text{fin}}\,|P|$. Thus,

LEMMA. *$(\mathcal{M}_{\text{fin}}\,|P|, \uplus, \wr\int, \Subset_P)$ is a join monoid.* □

Define the functor $\mathit{jMon}\langle - \rangle : \mathrm{PO} \to \mathrm{jMon}$ by

$$\mathit{jMon}\langle P \rangle = (\mathcal{M}_{\text{fin}}\,|P|, \uplus, \wr\int, \Subset_P) \ ,$$
$$\mathit{jMon}\langle \varphi : P \to Q \rangle = \lambda \wr p_1, \dots, p_n \int \in \mathcal{M}_{\text{fin}}\,|P| \,.\, \wr \varphi(p_1), \dots, \varphi(p_n) \int \ .$$

12. Dually, the *upper* or *Smyth ordering* on $\mathcal{M}_{\text{fin}}\,|P|$ is the binary relation $\Subset^P \subseteq (\mathcal{M}_{\text{fin}}\,|P|) \times (\mathcal{M}_{\text{fin}}\,|P|)$, defined by $T \Subset^P U$ if, and only if, $U \Subset_{P^{-1}} T$; more explicitly, the Smyth ordering on $\mathcal{M}_{\text{fin}}\,|P|$ is given by the transitive closure of

$$T \Supset U \text{ implies } T \Subset^P U \ ,$$
$$p \leq_P q \text{ implies } T \uplus \wr p \int \Subset^P T \uplus \wr q \int \ ,$$

i.e., $T \Subset^P U$ if, and only if, there is an injective mapping $g : \mathcal{S}(U) \to \mathcal{S}(T)$ with $p \leq_P q$ if $g(k,q) = (j,p)$ for all $(k,q) \in \mathcal{S}(U)$; we call such a map a *witness* for $T \Subset^P U$. The relation $\Subset^P$ is also a partial ordering on $\mathcal{M}_{\text{fin}}\,|P|$, and, again, $\uplus$ is monotonic w.r.t. $\Subset^P$, i.e.,

$$T \Subset^P U \text{ implies } T \uplus V \subseteq^P U \uplus V \ ;$$

and $T \Subset^P \wr\int$ for all $T \in \mathcal{M}_{\text{fin}}\,|P|$. Thus,

LEMMA. *$(\mathcal{M}_{\text{fin}}\,|P|, \uplus, \wr\int, \Subset^P)$ is a meet monoid.* □

This meet monoid over a partial order P does not show suprema of finite sets, in general: Consider the partial order $\mathrm{P} = (\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}, \{\mathrm{a} < \mathrm{c}, \mathrm{b} < \mathrm{c}\})$ (which does show suprema). In the meet monoid $(\mathcal{M}_{\text{fin}}\,|\mathrm{P}|, \uplus, \wr\int, \Subset^{\mathrm{P}})$, we have

$$\wr \mathrm{c} \int \subset^{\mathrm{P}} \wr \mathrm{a} \int, \wr \mathrm{b} \int \ ,$$
$$\wr \mathrm{a}, \mathrm{b} \int \subset^{\mathrm{P}} \wr \mathrm{a} \int, \wr \mathrm{b} \int$$

and no $T \in \mathcal{M}_{\text{fin}}\,|\text{P}|$ exists with $\wr \text{a}, \text{b} \int, \wr \text{c} \int \sqsubset^{\text{P}} T \sqsubset^{\text{P}} \wr \text{a} \int, \wr \text{b} \int$ since, e.g., for $\wr \text{c} \int \sqsubset^{\text{P}} T$, T can only be $\wr \int$ by the first rule (with $\wr \text{a}, \text{b} \int, \wr \text{c} \int \sqsubset^{\text{P}} \wr \int$), or $\wr \text{a} \int$ or $\wr \text{b} \int$ by the second rule; but $\wr \text{a} \int$ and $\wr \text{b} \int$ are incomparable w.r.t. $\sqsubseteq^{\text{P}}$
Define the functor $mMon\langle - \rangle : \text{PO} \to \text{mMon}$ by

$$mMon\langle P \rangle = (\mathcal{M}_{\text{fin}}\,|P|, \uplus, \wr \int, \sqsubseteq^{P}) ,$$
$$mMon\langle \varphi : P \to Q \rangle = \lambda \wr p_1, \dots, p_n \int \in \mathcal{M}_{\text{fin}}\,|P| \,.\, \wr \varphi(p_1), \dots, \varphi(p_n) \int .$$

In particular, $jMon\langle P \rangle = jMon(mMon\langle P^{-1} \rangle)$ and $mMon\langle P \rangle = mMon(jMon\langle P^{-1} \rangle)$.

13. Finally, the *convex* or *Plotkin ordering* on $\mathcal{M}_{\text{fin}}\,|P|$ is the intersection of $\sqsubseteq_P$ and $\sqsubseteq^P$. Then $T\ (\sqsubseteq_P \cap \sqsubseteq^P)\ U$ if, and only if, there is a bijective mapping $h : \mathcal{S}(T) \to \mathcal{S}(U)$ with $p \leq_P q$ if $h(j,p) = (k,q)$ for all $(j,p) \in \mathcal{S}(T)$; we again call such a map a *witness* for $T\ (\sqsubseteq_P \cap \sqsubseteq^P)\ U$. The relation $\sqsubseteq_P \cap \sqsubseteq^P$ is also a partial ordering on $\mathcal{M}_{\text{fin}}\,|P|$, and again $\uplus$ is monotonic w.r.t. this ordering.

LEMMA. *$(\mathcal{M}_{\text{fin}}\,|P|, \uplus, \wr \int, \sqsubseteq_P \cap \sqsubseteq^P)$ is a partially ordered monoid.* □

Define the functor $poMon\langle - \rangle : \text{PO} \to \text{poMon}$ by

$$poMon\langle P \rangle = (\mathcal{M}_{\text{fin}}\,|P|, \uplus, \wr \int, \sqsubseteq_P \cap \sqsubseteq^P) ,$$
$$poMon\langle \varphi : P \to Q \rangle = \lambda \wr p_1, \dots, p_n \int \in \mathcal{M}_{\text{fin}}\,|P| \,.\, \wr \varphi(p_1), \dots, \varphi(p_n) \int .$$

14. Define the functor $PO : \text{poMon} \to \text{PO}$ by

$$PO(M) = (|M|, \leq_M) ,$$
$$PO(\varphi : M \to N) = \varphi .$$

For each $P \in |\text{PO}|$ and each $x \in \{po, j, m\}$ define the partial order homomorphisms $\eta_P^{x\text{Mon}} : P \to PO(xMon\langle P \rangle)$ by $\eta_P^{x\text{Mon}}(p) = \wr p \int$. Then each $\eta^{x\text{Mon}} = (\eta_P^{x\text{Mon}})_{P \in |\text{PO}|}$ is a natural transformation from 1_{PO} to $PO \circ xMon\langle - \rangle$.
Let $x \in \{po, j, m\}$, $P \in |\text{PO}|$, $M \in |x\text{Mon}|$, and $\varphi : P \to PO(M)$. Define $\varphi^{\sharp_{x\text{Mon}}} : xMon\langle P \rangle \to M$ by

$$\varphi^{\sharp_{x\text{Mon}}}(\wr p_1, \dots, p_n \int) = \varphi(p_1) \cdot_M \dots \cdot_M \varphi(p_n)$$

for all $\wr p_1, \dots, p_n \int \in \mathcal{M}_{\text{fin}}\,|P|$, where, if $n = 0$, the right hand side is to be understood as ε_M.
Then for all $x \in \{po, j, m\}$, $PO(\varphi^{\sharp_{x\text{Mon}}})(\eta_P^{x\text{Mon}}(p)) = \varphi(p)$ and all $\varphi^{\sharp_{x\text{Mon}}}$ are unique with this property.

15. Each upper semi-lattice can also be viewed as a join monoid. Indeed, define the functor $jMon : \text{uSL} \to \text{jMon}$ by

$$jMon(U) = (|U|, \sqcup_U, \bot_U, \leq_U) ,$$
$$jMon(\varphi : U \to V) = \varphi .$$

In particular, by §5 and §7, for each partial order P, $jMon(uSL\langle P \rangle) = (\mathcal{I}_{\text{fin}}(P), \cup_P, \emptyset, \subseteq_P)$ is a join monoid.

3.1. *Constructing Meet Monoids from Dags*

16. Let G be a dag. Consider the *single-predecessor lifting* $\rightsquigarrow_G^{\text{SPD}} \subseteq (\mathcal{M}_{\text{fin}}\,|G|) \times (\mathcal{M}_{\text{fin}}\,|G|)$ of G to the finite multisets $\mathcal{M}_{\text{fin}}\,|G|$ over the elements of G, given by

$$T \rightsquigarrow_G^{\text{SPD}} T \uplus \wr g \int ,$$

$g \to_G h$ implies $T \uplus \wr g \int \rightsquigarrow_G^{\mathrm{SPD}} T \uplus \wr h \int$.

Then $G^{\mathrm{SPD}} = (\mathcal{M}_{\mathrm{fin}}\,|G|, \rightsquigarrow_G^{\mathrm{SPD}})$ is a dag, where $T \rightsquigarrow_G^{\mathrm{SPD}} U$ expresses that U is *worse* than T. We write $\leq_G^{\mathrm{SPD}}$ for $\geq_{PO\langle G^{\mathrm{SPD}}\rangle} = ((\rightsquigarrow_G^{\mathrm{SPD}})^*)^{-1}$. Furthermore, $\leq_G^{\mathrm{SPD}}$ is monotonic w.r.t. multiset union and thus $(\mathcal{M}_{\mathrm{fin}}\,|G|, \uplus, \wr\int, \leq_G^{\mathrm{SPD}})$ is a meet monoid. We have

LEMMA. *For each dag G, $mMon\langle PO\langle G\rangle^{-1}\rangle \cong (\mathcal{M}_{\mathrm{fin}}\,|G|, \uplus, \wr\int, \leq_G^{\mathrm{SPD}})$.* □

17. Let G be a dag. Consider the *transitive-predecessors lifting* $\rightsquigarrow_G^{\mathrm{TPD}} \subseteq (\mathcal{M}_{\mathrm{fin}}\,|G|) \times (\mathcal{M}_{\mathrm{fin}}\,|G|)$ of G to the finite multisets $\mathcal{M}_{\mathrm{fin}}\,|G|$ over the elements of G, given by

$$T \rightsquigarrow_G^{\mathrm{TPD}} T \uplus \wr g \int ,$$
$$g_1, \ldots, g_n \to_G^* h \text{ implies } T \uplus \wr g_1, \ldots, g_n \int \rightsquigarrow_G^{\mathrm{TPD}} T \uplus \wr h \int .$$

Then $G^{\mathrm{TPD}} = (\mathcal{M}_{\mathrm{fin}}\,|G|, \rightsquigarrow_G^{\mathrm{TPD}})$ is a dag; we write $\leq_G^{\mathrm{TPD}}$ for $\geq_{PO\langle G^{\mathrm{TPD}}\rangle} = ((\rightsquigarrow_G^{\mathrm{TPD}})^*)^{-1}$. Furthermore, $\leq_G^{\mathrm{TPD}}$ is monotonic w.r.t. multiset union and thus $(\mathcal{M}_{\mathrm{fin}}\,|G|, \uplus, \wr\int, \leq_G^{\mathrm{TPD}})$ is a meet monoid. We have

LEMMA. *Let G be a dag, let $S_1, S_2 \in \mathcal{P}_{\mathrm{fin}}\,|G|$, and let $\bar{S}_1, \bar{S}_2 \in \mathcal{M}_{\mathrm{fin}}\,|G|$ be the finite multisets corresponding to S_1 and S_2, respectively. Then*

$$\bar{S}_1 \leq_G^{\mathrm{TPD}} \bar{S}_2 \textit{ implies } \mathrm{Max}^{PO\langle G\rangle}(S_2) \subseteq_{PO\langle G\rangle} \mathrm{Max}^{PO\langle G\rangle}(S_1) .$$

Proof. Let $\bar{S}_1 \leq_G^{\mathrm{TPD}} \bar{S}_2$ hold. It suffices to prove the claim for the two defining clauses for $\rightsquigarrow_G^{\mathrm{TPD}}$. For the case that there is a $g \in S_1$ such that $S_2 = S_1 \setminus \{g\}$, we have

$$\mathrm{Max}^{PO\langle G\rangle}(S_2) = \mathrm{Max}^{PO\langle G\rangle}(S_1 \setminus \{g\}) \subseteq_{PO\langle G\rangle} \mathrm{Max}^{PO\langle G\rangle}(S_1) \setminus \{g\} \subseteq_{PO\langle G\rangle} \mathrm{Max}^{PO\langle G\rangle}(S_1) .$$

For the case that there is an $h \in S_1$ and $g_1, \ldots, g_n \in S_2$ with $g_1, \ldots, g_n \to_G h$ such that $S_2 = (S_1 \setminus \{h\}) \cup \{g_1, \ldots, g_n\}$, we have

$$\mathrm{Max}^{PO\langle G\rangle}(S_2) = \mathrm{Max}^{PO\langle G\rangle}(S_1 \cup \{g_1, \ldots, g_n\}) \setminus \{h\} \subseteq_{PO\langle G\rangle}$$
$$\mathrm{Max}^{PO\langle G\rangle}(S_1) \setminus \{h\} \subseteq_{PO\langle G\rangle} \mathrm{Max}^{PO\langle G\rangle}(S_1) ,$$

since $h \in S_1$. □

The converse is in general wrong: Let $G = (\{g, h\}, \{(g, h)\})$, i.e., $g \to_G h$; then indeed we have $\mathrm{Max}^{PO\langle G\rangle}(\{g, h\}) \subseteq_{PO\langle G\rangle} \mathrm{Max}^{PO\langle G\rangle}(\{h\})$, but $\wr h \int \not\leq_G^{\mathrm{TPD}} \wr g, h \int$. However, if $S_2 \in \mathcal{I}_{\mathrm{fin}}(PO\langle G\rangle)$, then the converse follows from the map $f : \mathrm{Max}^{PO\langle G\rangle}(S_2) \to \mathrm{Max}^{PO\langle G\rangle}(S_1)$ witnessing $S_2 = \mathrm{Max}^{PO\langle G\rangle}(S_2) \subseteq_{PO\langle G\rangle} \mathrm{Max}^{PO\langle G\rangle}(S_1)$.

3.2. *Lexicographic Product of Meet Monoids*

18. For a meet monoid M, define its set of *collapsing elements* by [3]

$$\mathcal{C}(M) = \{m \in |M| \mid \exists m_1, m_2 \in |M| \,.\, m_1 <_M m_2 \wedge m_1 \cdot_M m = m_2 \cdot_M m\} .$$

EXAMPLE. Let P be a partial order.

(1) The set of collapsing elements of the meet monoid $mMon\langle P\rangle$ is empty: If $T \subset^P U$, then $T \uplus \wr p \int = U \uplus \wr p \int$ for some $p \in |P|$ would imply that $T = U$.

(2) The set of collapsing elements of the meet monoid $mMon(jMon(uSL\langle P\rangle))$ is $\mathcal{I}_{\mathrm{fin}}(P) \setminus \{\emptyset\}$: If $\emptyset \neq I \in \mathcal{I}_{\mathrm{fin}}(P)$, then $I \supset_P \emptyset$, but $\emptyset \cup_P I = I = I \cup_P I$. □

Generalising the first example, if M is a *strict* meet monoid, i.e., $m <_M n$ implies $m \cdot_M o <_M n \cdot_M o$ for all $m, n, o \in |M|$, then $\mathcal{C}(M) = \emptyset$. Generalising the second example, all idempotent elements of a meet monoid M which are different from ε_M are collapsing: If $m \in |M|$ such that $m \neq \varepsilon_M$ and $m \cdot_M m = m$, then $m <_M \varepsilon_M$ but $m \cdot_M m = m = \varepsilon_M \cdot_M m$.

LEMMA. *$|M| \setminus \mathcal{C}(M)$ is closed under $\cdot_M$.*

Proof. We show that $m \cdot_M n \in \mathcal{C}(M)$ if, and only if, $m \in \mathcal{C}(M)$ or $n \in \mathcal{C}(M)$: If $m \in \mathcal{C}(M)$ or $n \in \mathcal{C}(M)$, then obviously $m \cdot_M n \in \mathcal{C}(M)$. Conversely, if $m \cdot_M n \in \mathcal{C}(M)$, then there are $m_1, m_2 \in |M|$ with $m_1 <_M m_2$, and $m_1 \cdot_M m \cdot_M n = m_2 \cdot_M m \cdot_M n$; but if $m \notin \mathcal{C}(M)$, then $m_1 \cdot_M m <_M m_2 \cdot_M m$, and if also $n \notin \mathcal{C}(M)$, then $m_1 \cdot_M m \cdot_M n <_M m_2 \cdot_M m \cdot_M n$. ☐

Note that in particular $\varepsilon_M \notin \mathcal{C}(M)$ since $m_1 <_M m_2$ implies $m_1 \cdot_M \varepsilon_M <_M m_2 \cdot_M \varepsilon_M$. Thus, $(|M| \setminus \mathcal{C}(M), \cdot_M, \varepsilon_M, \leq_M)$ forms a meet monoid.

19. A meet monoid M is *bounded* if $|M|$ has a smallest element w.r.t. $\leq_M$; we denote this element by $\bot_M$ if it exists.
In a bounded meet monoid M it holds that $m \cdot_M \bot_M = \bot_M$ for all $m \in |M|$, i.e., $\bot_M$ is an absorbing element. Furthermore, $\bot_M \in \mathcal{C}(M)$. We call a bounded meet monoid *weakly strict* if $m <_M n$ implies $m \cdot_M o <_M n \cdot_M o$ for all $m, n \in |M|$ and $\bot_M \neq o \in |M|$.
Each meet monoid M which is not bounded can be *lifted* into a bounded meet monoid $M_\bot = (|M| \uplus \{\bot\}, \cdot_{M_\bot}, \varepsilon_{M_\bot}, \leq_{M_\bot})$ setting $m \cdot_{M_\bot} \bot = \bot$, $\varepsilon_{M_\bot} = \varepsilon_M$, and $\bot \leq_{M_\bot} m$ for all $m \in |M| \uplus \{\bot\}$.

20. Let M be a meet monoid and let N be a bounded meet monoid. Let

$$L = ((|M| \setminus \mathcal{C}(M)) \times |N|) \cup (\mathcal{C}(M) \times \{\bot_N\}) .$$

Define the binary operation $\cdot_L : L \times L \to L$ by

$$(m_1, n_1) \cdot_L (m_2, n_2) = (m_1 \cdot_M m_1, n_1 \cdot_N n_2) .$$

This is well-defined: If $(m_1, n_1) \in (|M| \setminus \mathcal{C}(M)) \times |N|$ and $(m_2, n_2) \in \mathcal{C}(M) \times \{\bot_N\}$, then $m_1 \cdot_M m_2 \in \mathcal{C}(M)$, but also $n_2 = \bot_N$ and thus $n_1 \cdot_N n_2 = n_1 \cdot_N \bot_N = \bot_N$. $\cdot_L$ inherits associativity and commutativity from M and N.
Further define the element $\varepsilon_L \in L$ by

$$\varepsilon_L = (\varepsilon_M, \varepsilon_N) ,$$

which is also well-defined, since $\varepsilon_M \notin \mathcal{C}(M)$. Also $(m, n) \cdot_L \varepsilon_L = (m, n)$.
Finally, define the *lexicographic ordering* $\leq_L \subseteq L \times L$ on L by

$$(m_1, n_1) \leq_L (m_2, n_2) \iff (m_1 \neq m_2 \text{ and } m_1 \leq_M m_2) \text{ or } (m_1 = m_2 \text{ and } n_1 \leq_N n_2) .$$

LEMMA. *$(L, \cdot_L, \varepsilon_L, \leq_L)$ is a meet monoid.*

Proof. Since $(L, \cdot_L, \varepsilon_L)$ is a commutative monoid, it only remains to show the monotonicity of $\cdot_L$ w.r.t. $\leq_L$. Let $(m_1, n_1) \leq_L (m_2, n_2)$ and an $(m, n) \in L$ be given.
Case $m_1 <_M m_2$: If not $m_1 \cdot_M m <_M m_2 \cdot_M m$, i.e., $m_1 \cdot_M m = m_2 \cdot_M m$, then $m \in \mathcal{C}(M)$ and thus $n = \bot_N$. Hence

$$(m_1, n_1) \cdot_L (m, n) = (m_1 \cdot_M m, n_1 \cdot_N n) = (m_1 \cdot_M m, \bot_N) = \\ (m_2 \cdot_M m, \bot_N) = (m_2 \cdot_M m, n_2 \cdot_N n) = (m_2, n_2) \cdot_L (m, n) .$$

Case $m_1 = m_2$ and $n_1 \leq_N n_2$: Then $m_1 \cdot_M m = m_2 \cdot_M m$ and $n_1 \cdot_N n \leq_N n_2 \cdot_N n$. ☐

Let us write $M \ltimes N$ for $(L, \cdot_L, \varepsilon_L, \leq_L)$.

21. Let M and N be meet monoids such that N is bounded. Then $\mathcal{C}(M \ltimes N) = (\mathcal{C}(M) \times \{\bot_N\}) \cup ((|M| \setminus \mathcal{C}(M)) \times \mathcal{C}(N))$. Indeed, let $(m, n) \in \mathcal{C}(M \ltimes N)$. Then there are $(m_1, n_1), (m_2, n_2) \in |M \ltimes N|$ with $(m_1, n_1) <_{M \ltimes N} (m_2, n_2)$, i.e., $m_1 <_M m_2$ or $m_1 = m_2$ and $n_1 <_N n_2$, and $(m_1, n_1) \cdot_{M \ltimes N} (m, n) = (m_2, n_2) \cdot_{M \ltimes N} (m, n)$, i.e., $m_1 \cdot_M m = m_2 \cdot_M m$ and $n_1 \cdot_N n = n_2 \cdot_N n$. If $m_1 <_M m_2$, then $m \in \mathcal{C}(M)$ and therefore $n = \bot_N$; if $m_1 = m_2$, then $n \in \mathcal{C}(N)$. Conversely, let first $(m, n) \in \mathcal{C}(M) \times \{\bot_N\}$; then there are $m_1, m_2 \in |M|$ with $m_1 <_M m_2$ and $m_1 \cdot_M m = m_2 \cdot_M m$, hence $(m_1, \bot_N) <_{M \ltimes N} (m_2, \bot_N)$ with $(m_1, \bot_N) \cdot_{M \ltimes N} (m, n) = (m_2, \bot_N) \cdot_{M \ltimes N} (m, n)$. Now let $(m, n) \in (|M| \setminus \mathcal{C}(M)) \times \mathcal{C}(N)$; then there are $n_1, n_2 \in |N|$ with $n_1 <_N n_2$ and $n_1 \cdot_N n = n_2 \cdot_N n$, hence $(m, n_1) <_{M \ltimes N} (m, n_2)$ with $(m, n_1) \cdot_{M \ltimes N} (m, n) = (m, n_2) \cdot_{M \ltimes N} (m, n)$.
Abbreviate $|M| \setminus \mathcal{C}(M)$ by $R(M)$. Then

$$
\begin{aligned}
&|M \ltimes N| \setminus \mathcal{C}(M \ltimes N) = R(M \ltimes N) = \\
&\quad ((R(M) \times |N|) \cup (\mathcal{C}(M) \times \{\bot_N\})) \setminus ((R(M) \times \mathcal{C}(N)) \cup (\mathcal{C}(M) \times \{\bot_N\})) = \\
&\quad R(M) \times R(N) = (|M| \setminus \mathcal{C}(M)) \times (|N| \setminus \mathcal{C}(N)) \,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|(M \ltimes N) \ltimes O| = \\
&\quad (R(M \ltimes N) \times |O|) \cup (\mathcal{C}(M \ltimes N) \times \{\bot_O\}) = \\
&\quad (R(M) \times R(N) \times |O|) \cup (R(M) \times \mathcal{C}(N) \times \{\bot_O\}) \cup (\mathcal{C}(M) \times \{(\bot_N, \bot_O)\}) = \\
&\quad (R(M) \times ((R(N) \times |O|) \cup (\mathcal{C}(N) \times \{\bot_O\}))) \cup (\mathcal{C}(M) \times \{\bot_{N \ltimes O}\}) = \\
&\quad (R(M) \times |N \ltimes O|) \cup (\mathcal{C}(M) \times \{\bot_{N \ltimes O}\}) = \\
&\quad |M \ltimes (N \ltimes O)| \,,
\end{aligned}
$$

from which it follows that $\ltimes$ is associative.

4. C-Semirings

22. A *c-semiring* [1] $(X, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ is given by a set X, two binary operations $\otimes, \oplus : X \times X \to X$, and two constants $\mathbf{0}, \mathbf{1} \in X$ such that the following axioms are satisfied for all $x, y, z \in X$:

(1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
(2) $x \oplus y = y \oplus x$
(3) $x \oplus \mathbf{1} = \mathbf{1}$
(4) $x \oplus \mathbf{0} = x$
(5) $(x \otimes y) \otimes z = x \otimes (y \otimes z)$
(6) $x \otimes y = y \otimes x$
(7) $x \otimes \mathbf{0} = \mathbf{0}$
(8) $x \otimes \mathbf{1} = x$
(9) $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$

In words, $\oplus$ is associative and commutative, has $\mathbf{1}$ as annihilator and $\mathbf{0}$ as neutral element; $\otimes$ is associative and commutative, has $\mathbf{0}$ as annihilator and $\mathbf{1}$ as neutral element; and $\otimes$ distributes over $\oplus$.
A *c-semiring homomorphism* $\varphi : A \to B$ from a c-semiring $A = (|A|, \oplus_A, \otimes_A, \mathbf{0}_A, \mathbf{1}_A)$ to a c-semiring $B = (|B|, \oplus_B, \otimes_B, \mathbf{0}_B, \mathbf{1}_B)$ is given by a map $\varphi : |A| \to |B|$ such that for all $a_1, a_2 \in |A|$:

(1) $\varphi(a_1 \oplus_A a_2) = \varphi(a_1) \oplus_B \varphi(a_2)$
(2) $\varphi(a_1 \otimes_A a_2) = \varphi(a_1) \otimes_B \varphi(a_2)$
(3) $\varphi(\mathbf{0}_A) = \mathbf{0}_B$
(4) $\varphi(\mathbf{1}_A) = \mathbf{1}_B$

The category cSRng of c-semirings has the c-semirings as objects and the c-semiring homomorphisms as morphisms.

23. In a c-semiring $(X, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ the operation $\oplus$ is idempotent:

$$x \oplus x = (x \otimes \mathbf{1}) \oplus (x \otimes \mathbf{1}) = x \otimes (\mathbf{1} \oplus \mathbf{1}) = x \otimes \mathbf{1} = x \;.$$

Thus, there is a functor $uSL : \mathrm{cSRng} \to \mathrm{uSL}$, defined by

$$uSL(A) = (|A|, \oplus_A, \mathbf{0}) \;,$$
$$uSL(\varphi : A \to B) = \varphi \;.$$

For a c-semiring A, the thereby induced ordering $\leq_{uSL(A)}$, explicitly given by $a \leq_{uSL(A)} b$ if, and only if, $a \oplus_A b = b$, will be written as $\preceq_A$.
With this definition, for all $a, b, c \in |A|$ it holds that

(1) $\mathbf{0} \preceq_A a \preceq_A \mathbf{1}$;
(2) $a \preceq_A a \oplus_A b$ and $b \preceq_A a \oplus_A b$;
(3) if $a \preceq_A c$ and $b \preceq_A c$, then $a \oplus_A b \preceq_A c$.

Also $\oplus_A$ is monotone w.r.t. $\preceq_A$ in both arguments, i.e.,

$$a \preceq_A a' \text{ and } b \preceq_A b' \text{ implies } a \oplus_A b \preceq_A a' \oplus_A b' \;.$$

24. In a c-semiring $(X, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ the operation $\otimes$ is monotone w.r.t. the induced ordering $\preceq$, since if $x \preceq x'$, i.e., $x \oplus x' = x'$, then

$$(x \otimes y) \oplus (x' \otimes y) = (x \oplus x') \otimes y = x' \otimes y \;,$$

i.e., $x \otimes y \preceq x' \otimes y$, from which it follows that

$$x \preceq x' \text{ and } y \preceq y' \text{ implies } x \otimes y \preceq x' \otimes y' \;.$$

Furthermore, for all $x, y \in X$

$$x \otimes y \preceq x \text{ and } x \otimes y \preceq y \;,$$

since

$$(x \otimes y) \oplus x = (x \otimes y) \oplus (x \otimes \mathbf{1}) = x \otimes (y \oplus \mathbf{1}) = x \otimes \mathbf{1} = x \;.$$

Thus, there is a functor $mMon : \mathrm{cSRng} \to \mathrm{mMon}$, given by

$$mMon(A) = (|A|, \otimes_A, \mathbf{1}_A, \preceq_A) \;,$$
$$mMon(\varphi : A \to B) = \varphi \;.$$

Note that $mMon(A)$ is a bounded meet monoid with $\bot_{mMon(A)} = \mathbf{0}_A$.

25. Consider the c-semiring of *boolean values* $\mathrm{B} = (\{\bot, \top\}, \vee, \wedge, \bot, \top)$ where $\vee$ and $\wedge$ have their usual meaning.

LEMMA. B *is initial in* cSRng. □

26. Consider the c-semiring $\mathrm{T} = (\{*\}, \cdot, \cdot, *, *)$ with $* \cdot * = *$.

LEMMA. T *is terminal in* cSRng. □

27. Let $A = (|A|, \oplus_A, \otimes_A, \mathbf{0}_A, \mathbf{1}_A)$ and $B = (|B|, \oplus_B, \otimes_B, \mathbf{0}_B, \mathbf{1}_B)$ be c-semirings. Define $\oplus_{A\times B}$, $\otimes_{A\times B} : (|A| \times |B|) \times (|A| \times |B|) \to |A| \times |B|$ by

$$(a_1, b_1) \oplus_{A\times B} (a_2, b_2) = (a_1 \oplus_A a_2, b_1 \oplus_B b_2)$$
$$(a_1, b_1) \otimes_{A\times B} (a_2, b_2) = (a_1 \otimes_A a_2, b_1 \otimes_B b_2)$$

Then $A \times B = (|A| \times |B|, \oplus_{A\times B}, \otimes_{A\times B}, (\mathbf{0}_A, \mathbf{0}_B), (\mathbf{1}_A, \mathbf{1}_B))$ is a c-semiring.
Define $\pi_1 : A \times B \to A$ by $\pi_1(a, b) = a$ and $\pi_2 : A \times B \to B$ by $\pi_2(a, b) = b$. Then π_1 and π_2 are c-semiring homomorphisms. Furthermore, for any c-semiring $C = (|C|, \oplus_C, \otimes_C, \mathbf{0}_C, \mathbf{1}_C)$ and two c-semiring homomorphisms $\varphi_1 : C \to A$ and $\varphi_2 : C \to B$, the c-semiring homomorphism $\langle \varphi_1, \varphi_2 \rangle : C \to A \times B$ defined by $\langle \varphi_1, \varphi_2 \rangle(c) = (\varphi_1(c), \varphi_2(c))$ is unique for the property $\varphi_1 = \langle \varphi_1, \varphi_2 \rangle \,;\, \pi_1$ and $\varphi_2 = \langle \varphi_1, \varphi_2 \rangle \,;\, \pi_2$.

LEMMA. cSRng *has finite products.* □

28. The *collapsing elements* $\mathcal{C}(A)$ of a c-semiring A are the collapsing elements of $mMon(A)$.
A c-semiring A is *total* if for all $a, a_1, a_2 \in |A|$:

$$a_1 \prec_A a_2 \text{ or } a_1 = a_2 \text{ or } a_2 \prec_A a_1 \,.$$

Let A and B be c-semirings where A is total. Let $L = ((|A| \setminus \mathcal{C}(A)) \times |B|) \cup (\mathcal{C}(A) \times \{\mathbf{0}_B\})$. Define $\oplus_{A\ltimes B}, \otimes_{A\ltimes B} : L \times L \to L$ by

$$(a_1, b_1) \oplus_{A\ltimes B} (a_2, b_2) = \begin{cases} (a_1, b_1) & \text{if } a_2 \prec_A a_1 \\ (a_2, b_2) & \text{if } a_1 \prec_A a_2 \\ (a_1, b_1 \oplus_B b_2) & \text{if } a_1 = a_2 \end{cases},$$
$$(a_1, b_1) \otimes_{A\ltimes B} (a_2, b_2) = (a_1 \otimes_A a_2, b_1 \otimes_B b_2) \,.$$

Then $(a, b) \otimes_{A\ltimes B} ((a_1, b_1) \oplus_{A\ltimes B} (a_2, b_2)) = ((a, b) \otimes_{A\ltimes B} (a_1, b_1)) \oplus_{A\ltimes B} ((a, b) \otimes_{A\ltimes B} (a_2, b_2))$ for all $(a, b), (a_1, b_1), (a_2, b_2) \in L$, where the right hand side is $(a \otimes_A a_1, b \otimes_B b_1) \oplus_{A\ltimes B} (a \otimes a_2, b \otimes_B b_2)$: If $a_1 = a_2$, then $(a_1, b_1) \oplus_{A\ltimes B} (a_2, b_2) = (a_1, b_1 \oplus_B b_2)$ and $a \otimes_A a_1 = a \otimes_A a_2$, from which the claim follows by the distributivity of $\otimes_B$ over $\oplus_B$. If $a_2 \prec_A a_1$ (the case $a_1 \prec_A a_2$ is symmetric), then $(a_1, b_1) \oplus_{A\ltimes B} (a_2, b_2) = (a_1, b_1)$. If additionally $a \notin \mathcal{C}(A)$, then $a \otimes_A a_2 \prec_A a \otimes_A a_1$; if $a \in \mathcal{C}(A)$, then $b = \mathbf{0}_B$, and in both cases the claim follows.
Thus $A \ltimes B = (L, \oplus_{A\ltimes B}, \otimes_{A\ltimes B}, (\mathbf{0}_A, \mathbf{0}_B), (\mathbf{1}_A, \mathbf{1}_B))$ is a c-semiring, the *lexicographic product* of A and B.

29. Let M be meet monoid. We write $\mathcal{I}_{\text{fin}}(M)$ for $\mathcal{I}_{\text{fin}}(PO(M))$. Define the operations $\tilde{\cdot}_M, \tilde{\cup}_M : \mathcal{I}_{\text{fin}}(M) \times \mathcal{I}_{\text{fin}}(M) \to \mathcal{I}_{\text{fin}}(M)$ by

$$I \mathbin{\tilde{\cdot}_M} J = \mathrm{Max}^{\leq_M}\{m \cdot_M n \mid m \in I,\ n \in J\} \,,$$
$$I \mathbin{\tilde{\cup}_M} J = \mathrm{Max}^{\leq_M}(I \cup J) \,.$$

LEMMA. $(\mathcal{I}_{\text{fin}}(M), \tilde{\cup}_M, \tilde{\cdot}_M, \emptyset, \{\varepsilon_M\})$ *is a c-semiring.*

Proof. Let $I, J, K \in \mathcal{I}_{\text{fin}}(M)$.
The operation $\tilde{\cup}_M$ is associative and commutative and has $\emptyset$ as neutral element by §5. Furthermore, $I \mathbin{\tilde{\cup}_M} \{\varepsilon_M\} = \{\varepsilon_M\}$, since ε_M is the greatest element of $|M|$ w.r.t. $\leq_M$.
For the associativity of $\tilde{\cdot}_M$ we have

$$I \mathbin{\tilde{\cdot}_M} (J \mathbin{\tilde{\cdot}_M} K) =$$
$$\mathrm{Max}^{\leq_M}\{m_I \cdot_M m_{JK} \mid m_I \in I,\ m_{JK} \in \mathrm{Max}^{\leq_M}\{m_J \cdot_M m_K \mid m_J \in J,\ m_K \in K\}\} =$$

$$\mathrm{Max}^{\leq_M}\{m_I \cdot_M m_J \cdot_M m_K \mid m_I \in I,\ m_J \in J,\ m_K \in K\}\} =$$
$$\mathrm{Max}^{\leq_M}\{m_{IJ} \cdot_M m_K \mid m_{IJ} \in \mathrm{Max}^{\leq_M}\{m_I \cdot_M m_J \mid m_I \in I,\ m_J \in J\},\ m_K \in K\} =$$
$$(I \tilde{\cdot}_M J) \tilde{\cdot}_M K\ ,$$

since

$$\mathrm{Max}^{\leq_M}\{m \cdot_M n \mid m \in I,\ n \in \mathrm{Max}^{\leq_M}(X)\} = \mathrm{Max}^{\leq_M}\{m \cdot_M n \mid m \in I,\ n \in X\}$$

for all $X \in \mathcal{P}_{\text{fin}}\,|M|$. Assume $n \in X$ but $n \notin \mathrm{Max}^{\leq_M}(X)$; then there exists some (maximal) $n' \in X$ such that $n <_M n'$; Since $n \leq n'$ implies $m \cdot_M n \leq m \cdot_M n'$ by the monotonicity of $\cdot_M$, $m \cdot_M n'$
Also $\tilde{\cdot}_M$ inherits commutativity from $\cdot_M$; $I \tilde{\cdot}_M \emptyset = \emptyset$ by definition; and $I \tilde{\cdot}_M \{\varepsilon_M\} = I$, since ε_M is neutral in M.
Finally, $\tilde{\cdot}_M$ distributes over $\tilde{\cup}_M$:

$$I \tilde{\cdot}_M (J \tilde{\cup}_M K) =$$
$$\mathrm{Max}^{\leq_M}\{m_I \cdot_M m_{JK} \mid m_I \in I,\ m_{JK} \in \mathrm{Max}^{\leq_M}(J \cup K)\} =$$
$$\mathrm{Max}^{\leq_M}\{m_I \cdot_M m_{JK} \mid m_I \in I,\ m_{JK} \in J \cup K\} =$$
$$\mathrm{Max}^{\leq_M}(\{m_I \cdot_M m_J \mid m_I \in I,\ m_J \in J\} \cup \{m_I \cdot_M m_K \mid m_I \in I,\ m_K \in K\}) =$$
$$\mathrm{Max}^{\leq_M}(\mathrm{Max}^{\leq_M}\{m_I \cdot_M m_J \mid m_I \in I,\ m_J \in J\} \cup$$
$$\mathrm{Max}^{\leq_M}\{m_I \cdot_M m_K \mid m_I \in I,\ m_K \in K\}) =$$
$$(I \tilde{\cdot}_M J) \tilde{\cup}_M (I \tilde{\cdot}_M K)\ ,$$

since

$$\mathrm{Max}^{\leq_M}(I \cup \mathrm{Max}^{\leq_M}(X)) = \mathrm{Max}^{\leq_M}(I \cup X)$$

for all $X \in \mathcal{P}_{\text{fin}}\,|M|$. □

Let $\varphi : M \to N$ be a meet monoid homomorphism. For $X \in \mathcal{P}_{\text{fin}}\,|M|$, we have

$$\mathrm{Max}^{\leq_N}(\varphi(\mathrm{Max}^{\leq_M}(X))) = \mathrm{Max}^{\leq_N}(\varphi(X))\ .$$

Indeed, on the one hand, $\mathrm{Max}^{\leq_N}(\varphi(\mathrm{Max}^{\leq_M}(X))) \subseteq \mathrm{Max}^{\leq_N}(\varphi(X))$, since $\mathrm{Max}^{\leq_M}(X) \subseteq X$. For $\mathrm{Max}^{\leq_N}(\varphi(X)) \subseteq \mathrm{Max}^{\leq_N}(\varphi(\mathrm{Max}^{\leq_M}(X)))$ it suffices to show that for each $n \in \varphi(X)$ there is an $n' \in \varphi(\mathrm{Max}^{\leq_M}(X))$ such that $n \leq_M n'$. Thus, let $n \in \varphi(X)$, i.e., $n = \varphi(m)$ for some $m \in X$. Then there is an $m' \in \mathrm{Max}^{\leq_M}(X)$ with $m \leq_M m'$, hence $n = \varphi(m) \leq_N \varphi(m')$, and $\varphi(m') \in \varphi(\mathrm{Max}^{\leq_M}(X))$.
Define the functor $cSRng\langle - \rangle : \mathrm{mMon} \to \mathrm{cSRng}$ by

$$cSRng\langle M \rangle = (\mathcal{I}_{\text{fin}}(M), \tilde{\cup}_M, \tilde{\cdot}_M, \emptyset, \{\varepsilon_M\})\ ,$$
$$cSRng\langle \varphi : M \to N \rangle = \lambda\{m_1, \dots, m_k\} \in \mathcal{I}_{\text{fin}}(M)\,.\ \mathrm{Max}^{\leq_N}\{\varphi(m_1), \dots, \varphi(m_k)\}\ .$$

Indeed, $cSRng\langle \varphi : M \to N \rangle$ is a c-semiring homomorphism from $cSRng\langle M \rangle$ to $cSRng\langle N \rangle$:

$$cSRng\langle \varphi \rangle(\emptyset) = \emptyset\ ,$$
$$cSRng\langle \varphi \rangle(\{\varepsilon_M\}) = \{\varphi(\varepsilon_M)\} = \{\varepsilon_N\}\ ,$$
$$cSRng\langle \varphi \rangle(I_1 \tilde{\cup}_M I_2) = cSRng\langle \varphi \rangle(\mathrm{Max}^{\leq_M}(I_1 \cup I_2)) =$$
$$\mathrm{Max}^{\leq_N}(\varphi(\mathrm{Max}^{\leq_M}(I_1 \cup I_2))) = \mathrm{Max}^{\leq_N}(\varphi(I_1 \cup I_2)) = \mathrm{Max}^{\leq_N}(\varphi(I_1) \cup \varphi(I_2)) =$$
$$cSRng\langle \varphi \rangle(I_1) \tilde{\cup}_N cSRng\langle \varphi \rangle(I_2)\ ,$$
$$cSRng\langle \varphi \rangle(I_1 \tilde{\cdot}_M I_2) = cSRng\langle \varphi \rangle(\mathrm{Max}^{\leq_M}\{m_1 \cdot_M m_2 \mid m_1 \in I_1,\ m_2 \in I_2\})) =$$
$$\mathrm{Max}^{\leq_N}(\varphi(\mathrm{Max}^{\leq_M}\{m_1 \cdot_M m_2 \mid m_1 \in I_1,\ m_2 \in I_2\})) =$$

$$\begin{aligned}&\mathrm{Max}^{\leq_N}\{\varphi(m_1 \cdot_M m_2) \mid m_1 \in I_1,\ m_2 \in I_2\} =\\&\mathrm{Max}^{\leq_N}\{\varphi(m_1) \cdot_N \varphi(m_2) \mid m_1 \in I_1,\ m_2 \in I_2\} =\\&\mathrm{Max}^{\leq_N}\{n_1 \cdot_N n_2 \mid n_1 \in \varphi(I_1),\ n_2 \in \varphi(I_2)\} =\\&cSRng\langle\varphi\rangle(I_1) \,\tilde{\cdot}_N\, cSRng\langle\varphi\rangle(I_2)\ .\end{aligned}$$

30. For each $M \in |\mathrm{mMon}|$, define $\eta_M^{\mathrm{cSRng}} : M \to mMon(cSRng\langle M\rangle)$ by $\eta_M^{\mathrm{cSRng}}(m) = \{m\}$. Then $\eta^{\mathrm{cSRng}} = (\eta_M^{\mathrm{cSRng}})_{M\in|\mathrm{mMon}|}$ is a natural transformation from 1_{mMon} to $mMon \circ cSRng\langle-\rangle$.
Let $M \in |\mathrm{mMon}|$, $A \in |\mathrm{cSRng}|$, and $\varphi : M \to mMon(A)$. Define $\varphi^{\sharp_{\mathrm{cSRng}}} : cSRng\langle M\rangle \to A$ by

$$\varphi^{\sharp_{\mathrm{cSRng}}}(\{m_1, \ldots, m_n\}) = \varphi(m_1) \oplus_A \cdots \oplus_A \varphi(m_n)$$

for all $\{m_1, \ldots, m_n\} \in \mathcal{I}_{\mathrm{fin}}(M)$, where, if $n = 0$, the right hand side is to be understood as $\mathbf{0}_A$; $\varphi^{\sharp_{\mathrm{cSRng}}}$ is indeed a c-semiring homomorphism, since for each $\{m'_1, \ldots, m'_n\} \in \mathcal{P}_{\mathrm{fin}}\,|M|$ we have $\varphi^{\sharp_{\mathrm{cSRng}}}(\mathrm{Max}^{\leq_M}\{m'_1, \ldots, m'_n\}) = \varphi(m'_1) \oplus_A \cdots \oplus_A \varphi(m'_n)$: if $m'_i \leq_M m'_j$ then $\varphi(m'_i) \leq_{mMon(A)} \varphi(m'_j)$, i.e., $\varphi(m'_i) \preceq_A \varphi(m'_j)$, and thus $\varphi(m'_i) \oplus_A \varphi(m'_j) = \varphi(m'_j)$.
Then $mMon(\varphi^{\sharp_{\mathrm{cSRng}}})(\eta_M^{\mathrm{cSRng}}(m)) = \varphi(m)$ and $\varphi^{\sharp_{\mathrm{cSRng}}}$ is unique with this property.

5. Soft Constraints

31. We redefine essential notions introduced in [5]. A *constraint domain* (X, D) is given by a set X of *variables* and a family $D = (D_x)_{x\in X}$ of *variable domains* where each D_x is a set. A constraint domain (X, D) is *finite* if X and $\bigcup_{x\in X} D_x$ are finite.
A *valuation* for a constraint domain (X, D) is a dependent map $v \in \Pi x \in X \,.\, D_x$, i.e., $v(x) \in D_x$; we abbreviate $\Pi x \in X \,.\, D_x$ by $[X \to D]$.
A *constraint* c over a constraint domain (X, D), or (X, D)-*constraint*, is given by a map $c : [X \to D] \to \mathbb{B}$. We also write $v \models c$ for $c(v) = tt$.

32. Given a constraint domain (X, D) and a c-semiring G, a G-*soft constraint* γ over (X, D), or (X, D)-G-*soft constraint*, is given by a map $\gamma : [X \to D] \to |G|$. In particular, a constraint over (X, D) can be considered a B-soft constraint over (X, D).
Let Γ be a finite set of (X, D)-G-soft constraints. For a $v \in [X \to D]$ let the *solution degree* for Γ of v be

$$\Gamma(v) = \textstyle\bigotimes_G\{\gamma(v) \mid \gamma \in \Gamma\}\ .$$

Define a binary relation $\leq_\Gamma \subseteq [X \to D] \times [X \to D]$ by

$$v \leq_\Gamma w \iff \Gamma(v) \preceq_G \Gamma(w)\ ,$$

meaning that valuation w is a *better solution* for Γ than the valuation v.
The *maximum solution degrees* of Γ are given by

$$\Gamma^* = \mathrm{Max}^{\preceq_G}\{\Gamma(v) \mid v \in [X \to D]\}\ ,$$

and the *maximum solutions* by

$$\mathrm{Max}^{\leq_\Gamma}[X \to D] = \{v \in [X \to D] \mid \Gamma(v) \in \Gamma^*\}\ .$$

33. For a constraint domain (X, D) we fix an *extended* constraint domain $(X, D^?)$ setting $D^? = (D^?_x)_{x \in X}$ with $D^?_x = D_x \uplus \{?\}$, where ? is fresh.
A valuation $p \in \Pi x \in X \,.\, D^?_x = [X \to D^?]$ is called a *partial valuation* for (X, D).
The *domain of definition* $\mathrm{def}(p)$ of a partial valuation p for (X, D) is the set $\{x \in X \mid p(x) \neq ?\}$. For $p, q \in [X \to D^?]$, we write $p \sqsubseteq q$ if $x \in \mathrm{def}(p)$ implies $x \in \mathrm{def}(q)$ and $q(x) = p(x)$ for each $x \in X$; by $p\uparrow$ we denote the set $\{v \in [X \to D] \mid p \sqsubseteq v\}$ of (X, D)-valuations.

34. An $(X, D^?)$-G-soft constraint $\omega : [X \to D^?] \to |G|$ is *bounding* if $\omega(v) \preceq_G \omega(p)$ for all $p \in [X \to D^?]$ and $v \in p\uparrow$. A bounding $(X, D^?)$-G-soft constraint ω is *tight* for a finite set of (X, D)-G-soft constraints Γ if $\omega(v) = \Gamma(v)$ for all $v \in [X \to D]$.
A pair (π, ω) of $(X, D^?)$-G-soft constraints forms a *bounding pair* if ω is bounding and for each $p\{x \mapsto d\} \in [X \to D^?]$ there is a $v \in p\uparrow$ with $\pi(p\{x \mapsto d\}) \preceq_G \omega(v)$; a bounding pair (π, ω) is *tight* for a finite set of (X, D)-G-soft constraints Γ if ω is tight for Γ.
For a bounding pair (π, ω) of $(X, D^?)$-G-soft constraints the following "branch & bound" algorithm $\mathsf{maxSolDegs}_{(\pi,\omega)}$ computes, given a partial valuation $p \in [X \to D^?]$ and a finite set of lower bounds $L \subseteq |G|$ (which we assume to contain only elements which are pairwise incomparable w.r.t. $\preceq_G$), the maximum degrees in $L \cup \{\omega(v) \mid v \in p\uparrow\}$ w.r.t. $\preceq_G$, i.e., in particular, if $p = (\lambda x \in X \,.\, ?)$ and $L = \emptyset$, the maximum degrees in $\{\omega(v) \mid v \in [X \to D]\}$:

Assume: – (X, D) finite constraint domain
– G c-semiring
– (π, ω) bounding pair of $(X, D^?)$-G-soft constraints
In: – $p \in [X \to D^?]$ partial valuation for (X, D)
– $L \subseteq |G|$ finite and pairwise incomparable w.r.t. $\preceq_G$
Return: $\mathrm{Max}^{\preceq_G}(L \cup \omega(p\uparrow))$

$\mathsf{maxSolDegs}_{(\pi,\omega)}(p, L) \equiv$
if $\forall x \in X \,.\, p(x) \neq ?$
then return $\mathrm{Max}^{\preceq_G}(L \cup \{\omega(p)\})$
$x \leftarrow$ **choose** $\{x \in X \mid p(x) = ?\}$
$L \leftarrow \mathrm{Max}^{\preceq_G}(L \cup \{\pi(p\{x \mapsto d\}) \mid d \in D_x\})$
for $d \in D_x$
do if $\neg\exists l \in L \,.\, \omega(p\{x \mapsto d\}) \preceq_G l$
then $L \leftarrow \mathsf{maxSolDegs}_{(\pi,\omega)}(p\{x \mapsto d\}, L)$ **fi od**
return L

We prove the claim that

$$\mathsf{maxSolDegs}_{(\pi,\omega)}(p, L) = \mathrm{Max}^{\preceq_G}(L \cup \omega(p\uparrow))$$

by a first induction on the cardinality n of $\{x \in X \mid p(x) = ?\}$. If $n = 0$, i.e., $p \in [X \to D]$, then $\mathsf{maxSolDegs}_{(\pi,\omega)}(p, L) = \mathrm{Max}^{\preceq_G}(L \cup \{\omega(p)\})$ and $\{\omega(p)\} = \omega(p\uparrow)$. If $n > 0$, then let $x \in X$ with $p(x) = ?$, and let $d_1, \ldots, d_r$ be an enumeration of D_x. Let $P = \{\pi(p\{x \mapsto d_i\}) \mid 1 \leq i \leq r\}$ and define

$$L_0 = \mathrm{Max}^{\preceq_G}(L \cup P)\ ,$$

and inductively

$$L_k = \begin{cases} L_{k-1} & \text{if } \exists l \in L_{k-1} \,.\, \omega(p\{x \mapsto d_k\}) \preceq_G l \\ \mathsf{maxSolDegs}_{\omega}(p\{x \mapsto d_k\}, L_{k-1}) & \text{otherwise} \end{cases}$$

for $1 \leq k \leq r$. We prove the sub-claim that

$$L_k = \mathrm{Max}^{\preceq_G}(L \cup P \cup \bigcup_{1 \leq j \leq k} \omega(p\{x \mapsto d_j\}\uparrow))$$

for all $0 \leq k \leq r$ by a second induction on k: For $k = 0$, $L_0 = \mathrm{Max}^{\preceq_G}(L \cup P)$ by definition. For $k > 0$, let there first be an $l \in L_{k-1}$ with $\omega(p\{x \mapsto d_k\}) \preceq_G l$. Since $\omega(v) \preceq_G \omega(p\{x \mapsto d_k\}) \preceq_G l$ for all $v \in p\{x \mapsto d_k\}\uparrow$ and $l \in L_{k-1} = \mathrm{Max}^{\preceq_G}(L \cup P \cup \bigcup_{1 \leq i \leq k-1} \omega(p\{x \mapsto d_i\}\uparrow))$ by the second induction hypothesis, the sub-claim follows. Otherwise, if no such $l \in L_{k-1}$ exists, $\mathsf{maxSolDegs}_\omega(p\{x \mapsto d_k\}, L_{k-1}) = \mathrm{Max}^{\preceq_G}(L_{k-1} \cup \omega(p\{x \mapsto d_k\}\uparrow))$ by the first induction hypothesis, which is applicable since, by the second induction hypothesis, $L_{k-1} = \mathrm{Max}^{\preceq_G}(L \cup P \cup \bigcup_{1 \leq i \leq k-1} \omega(p\{x \mapsto d_i\}\uparrow))$, and hence L_{k-1} is pairwise incomparable w.r.t. $\preceq_G$; therefore,

$$\begin{aligned} L_k = {} & \mathsf{maxSolDegs}_\omega(p\{x \mapsto d_k\}, L_{k-1}) = \\ & \mathrm{Max}^{\preceq_G}(L_{k-1} \cup \omega(p\{x \mapsto d_k\}\uparrow)) = \\ & \mathrm{Max}^{\preceq_G}((\mathrm{Max}^{\preceq_G}(L \cup P \cup \textstyle\bigcup_{1 \leq i \leq k-1} \omega(p\{x \mapsto d_i\}\uparrow))) \cup \omega(p\{x \mapsto d_k\}\uparrow)) = \\ & \mathrm{Max}^{\preceq_G}(L \cup P \cup \textstyle\bigcup_{1 \leq i \leq k} \omega(p\{x \mapsto d_i\}\uparrow)) \ , \end{aligned}$$

which establishes the sub-claim. Thus, $L_r = \mathrm{Max}^{\preceq_G}(L \cup P \cup \bigcup_{1 \leq i \leq r} \omega(p\{x \mapsto d_i\}\uparrow)) = \mathrm{Max}^{\preceq_G}(L \cup \omega(p\uparrow)) = \mathsf{maxSolDegs}_\omega(p, L)$, since $d_1, \ldots, d_r$ is an enumeration of D_x and for each $1 \leq i \leq r$ there is a $v \in p\uparrow$ with $\pi(p\{x \mapsto d_i\}) \preceq_G \omega(v)$, which yields the claim.
In particular, if $(\Gamma_?, \Gamma^?)$ is a tight bounding pair of $(X, D^?)$-G-soft constraints for a finite set of (X, D)-G-soft constraints Γ, then

$$\begin{aligned} & \mathsf{maxSolDegs}_{(\Gamma_?, \Gamma^?)}(\lambda x \in X \,.\, ?, \emptyset) = \mathrm{Max}^{\preceq_G}(\emptyset \cup \Gamma^?((\lambda x \in X \,.\, ?)\uparrow)) = \\ & \qquad \mathrm{Max}^{\preceq_G}\{\Gamma^?(v) \mid v \in [X \to D]\} = \mathrm{Max}^{\preceq_G}\{\Gamma(v) \mid v \in [X \to D]\} = \Gamma^* \ . \end{aligned}$$

35. Given a meet monoid M and a constraint domain (X, D), an M-*soft constraint presentation* μ over (X, D), or (X, D)-M-*soft constraint presentation*, is given by a map $\mu : [X \to D] \to |M|$. Each (X, D)-M-soft constraint presentation induces an (X, D)-$cSRng\langle M\rangle$-soft constraint $\eta_M^{\mathrm{cSRng}} \circ \mu$ (viz., $(\eta_M^{\mathrm{cSRng}} \circ \mu)(v) = \{\mu(v)\}$).
Let M be a finite set of (X, D)-M-soft constraint presentations. Then

$$\textstyle\bigotimes_{cSRng\langle M\rangle}\{\eta_M^{\mathrm{cSRng}}(\mu(v)) \mid \mu \in \mathrm{M}\} = \{\prod_M\{\mu(v) \mid \mu \in \mathrm{M}\}\} \ ,$$

and thus

$$\begin{aligned} & v \leq_{\eta_M^{\mathrm{cSRng}} \circ \mathrm{M}} w \iff \{\textstyle\prod_M\{\mu(v) \mid \mu \in \mathrm{M}\}\} \preceq_{cSRng\langle M\rangle} \{\prod_M\{\mu(w) \mid \mu \in \mathrm{M}\}\} \iff \\ & \qquad \textstyle\prod_M\{\mu(v) \mid \mu \in \mathrm{M}\} \leq_M \prod_M\{\mu(w) \mid \mu \in \mathrm{M}\} \ . \end{aligned}$$

We write $cSRng\langle\mathrm{M}\rangle$ for the set of (X, D)-$cSRng\langle M\rangle$-soft constraints $\{\eta_M^{\mathrm{cSRng}} \circ \mu \mid \mu \in \mathrm{M}\}$.
In analogy to the notions for soft constraints, we define the *solution degree* for a $v \in [X \to D]$ by

$$\mathrm{M}(v) = \textstyle\prod_M\{\mu(v) \mid \mu \in \mathrm{M}\} \ ,$$

and the *maximum solution degrees* by

$$\mathrm{M}^* = \mathrm{Max}^{\leq_M}\{\mathrm{M}(v) \mid v \in [X \to D]\} \ .$$

36. The algorithm $\mathsf{maxSolDegs}_{(\pi,\omega)}$ for a bounding pair (π, ω) of $(X, D^?)$-G-soft constraints from §34 in fact also works under the assumptions that (X, D) is a constraint domain, M is a meet monoid, and α and ζ are $(X, D^?)$-M-soft constraint representations, such that $(\eta_M^{\mathrm{cSRng}} \circ \alpha, \eta_M^{\mathrm{cSRng}} \circ \zeta)$ is a bounding pair. We call (α, ζ) itself a *bounding pair* if $\zeta(v) \leq_M \zeta(p)$ for all $p \in [X \to D^?]$ and $v \in p{\uparrow}$, and if for each $p\{x \mapsto d\} \in [X \to D^?]$ there is a $v \in p{\uparrow}$ such that $\alpha(p\{x \mapsto d\}) \leq_M \zeta(v)$.

Assume: – (X, D) finite constraint domain
– M meet monoid
– (α, ζ) bounding pair of $(X, D^?)$-M-soft constraint presentations
In: – $p \in [X \to D^?]$ partial valuation for (X, D)
– $L \subseteq |M|$ finite and pairwise incomparable w.r.t. $\leq_M$
Return: $\mathrm{Max}^{\leq_M}(L \cup \zeta(p{\uparrow}))$

$\mathsf{maxSolDegs}_{(\alpha,\zeta)}(p, L) \equiv$
if $\forall x \in X \,.\, p(x) \neq ?$
then return $\mathrm{Max}^{\leq_M}(L \cup \{\zeta(p)\})$ **fi**
$x \leftarrow$ **choose** $\{x \in X \mid p(x) = ?\}$
$L \leftarrow \mathrm{Max}^{\leq_M}(L \cup \{\alpha(p\{x \mapsto d\}) \mid d \in D_x\})$
for $d \in D_x$
do if $\neg\exists l \in L \,.\, \zeta(p\{x \mapsto d\}) \leq_M l$
then $L \leftarrow \mathsf{maxSolDegs}_{(\alpha,\zeta)}(p\{x \mapsto d\}, L)$ **fi od**
return L

Note that $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(p, L) = L \oplus_{cSRng\langle M\rangle} \mathrm{Max}^{\leq_M} \zeta(p{\uparrow})$.
A bounding pair (α, ζ) of $(X, D^?)$-M-soft constraint presentations is *tight* for a finite set of (X, D)-M-soft constraint presentations M if $\zeta(v) = \mathrm{M}(v)$ for all $v \in [X \to D^?]$. For a tight bounding pair $(\mathrm{M}_?, \mathrm{M}^?)$ for a finite set of (X, D)-M-soft constraint presentations M we again obtain

$$\mathsf{maxSolDegs}_{(\mathrm{M}_?,\mathrm{M}^?)}(\lambda x \in X \,.\, ?, \emptyset) = \mathrm{Max}^{\leq_M}\{\mathrm{M}(v) \mid v \in [X \to D]\} = \mathrm{M}^* \,.$$

6. Constraint Hierarchies

37. A *constraint hierarchy* [2] $H = (C_k)_{1\leq k\leq n}$ over a constraint domain (X, D), or (X, D)-*constraint hierarchy*, is given by a family of sets of (X, D)-constraints. The constraints in *level* $1 \leq k \leq n$ are considered as *strictly more important* than the constraints in level $k + 1$. An (X, D)-constraint hierarchy is *finite* if $\bigcup_{1\leq k\leq n} C_k$ is finite.
Let $H = (C_k)_{1\leq k\leq n}$ be a finite (X, D)-constraint hierarchy, let $W = (M_i)_{1\leq k\leq n}$ be a corresponding family of meet monoids, and let for each $1 \leq k \leq n$ and each $c \in C_k$, $\mu(c)$ be an (X, D)-M_k-soft constraint presentation. We call $\mathrm{H} = (\mathrm{M}_k)_{1\leq k\leq n}$ with $\mathrm{M}_k = \{\mu(c) \mid c \in C_k\}$ for $1 \leq k \leq n$ a (X, D)-W-*soft constraint hierarchy presentation*. For a $v \in [X \to D]$ the *solution degree* for $(\mathrm{M}_k)_{1\leq k\leq n}$ of v is defined to be $(\mathrm{M}_k(v))_{1\leq k\leq n}$. Define a binary relation $<_\mathrm{H} \subseteq [X \to D] \times [X \to D]$ by

$$v <_\mathrm{H} w \iff \exists 1 \leq k \leq n \,.\, (\forall 1 \leq i \leq k-1 \,.\, \mathrm{M}_i(v) = \mathrm{M}_i(w)) \wedge \mathrm{M}_k(v) <_{M_k} \mathrm{M}_k(w) \,,$$

saying that the valuation w is *strictly better* than the valuation v, and denote its reflexive closure on $[X \to D]$ by $\leq_\mathrm{H}$, which is the lexicographic order on the set $\{(\mathrm{M}_k(v))_{1\leq k\leq n} \mid v \in [X \to D]\}$. In particular,

$$v \leq_\mathrm{H} w \iff (\mathrm{M}_k(v))_{1\leq k\leq n} \leq_{M_1 \ltimes \ldots \ltimes M_n} (\mathrm{M}_k(w))_{1\leq k\leq n}$$

if, on the one hand, every M_k is a bounded meet monoid for all $2 \leq k \leq n$, and, on the other hand, $\mathrm{M}_k(v), \mathrm{M}_k(w) \notin \mathcal{C}(M_k)$ for all $1 \leq k \leq n$, or, equivalently, if $\mu(c)(v), \mu(c)(w) \notin \mathcal{C}(M_k)$ for each $c \in C_k$, $1 \leq k \leq n$. The first requirement, that each M_k is bounded, can be achieved by moving from M_k to its lifted variant $(M_k)_\perp$.

38. Consider a single level k of a finite (X, D)-constraint hierarchy $H = (C_k)_{1 \leq k \leq n}$, and let $C = C_k$. The *locally-predicate-better* (*LPB*) *comparator* for C corresponds to requiring

$$v <_C^{\mathrm{LPB}} w \iff \{c \in C \mid w \not\models c\} \subsetneq \{c \in C \mid v \not\models c\} .$$

This can be expressed by choosing the meet monoid $M = (\mathcal{P}_{\mathrm{fin}}(C), \cup, \emptyset, \supseteq)$ and the set of (X, D)-M-soft constraint representations $\mathrm{M} = \{\mu(c) \mid c \in C\}$ with $\mu(c)(v) = \{c\}$ if $v \not\models c$ and $\mu(c)(v) = \emptyset$ otherwise, for each $c \in C$. However, all elements of M are idempotent, and thus the collapsing elements of M are $\mathcal{P}_{\mathrm{fin}}(C) \setminus \{\emptyset\}$. Hence, M is not suitable for a lexicographic product.
Choosing instead the meet monoid $N = (\mathcal{M}_{\mathrm{fin}}(C), \uplus, \Lbag\Rbag, \sqsupseteq)$ which has no collapsing elements and the set of (X, D)-N-soft constraint representations $\mathrm{N} = \{\nu(c) \mid c \in C\}$ with $\nu(c)(v) = \Lbag c \Rbag$ if $v \not\models c$ and $\nu(c)(v) = \Lbag\Rbag$ otherwise, for each $c \in C$, deviates this situation, since we have

$$\mathrm{M}(v) \leq_M \mathrm{M}(w) \iff \mathrm{N}(v) \leq_N \mathrm{N}(w)$$

for all $v, w \in [X \to D]$.

39. A *weighting* for a set C of (X, D)-constraints is given by a function $g : C \times [X \to D] \to \mathbb{R}_{\geq 0}$ with $g(c, v) = 0$ if $v \models c$ for $v \in [X \to D]$ and $c \in C$.
We consider the following *level weightings* for a level k of a finite (X, D)-constraint hierarchy $H = (C_k)_{1 \leq k \leq n}$, letting $C = C_k$:
– *Weighted sum*: $W(v) = \sum_{c \in C} g(c, v)$.
– *Least squares*: $W(v) = \sqrt{\sum_{c \in C} g(c, v)^2}$.
– *Worst case*: $W(v) = \max\{g(c, v) \mid c \in C\}$.
Each of these level weightings W induces a relation $\leq_C^W \subseteq [X \to D] \times [X \to D]$ on valuations defined by

$$v \leq_C^W w \iff W(w) \leq W(v) ,$$

where $\leq$ is the usual order over the real numbers.
Let $R = (\mathbb{R}_{\geq 0}, \otimes, 0, \geq)$ be a meet monoid (where $\leq$ is the usual order over the real numbers) and let $g : C \times [X \to D] \to \mathbb{R}_{\geq 0}$ be a weighting for C. Consider $W_R(v) = \bigotimes_{c \in C} g(c, v)$. Then we have the following correspondences:
– Weighted sum: $r \otimes s = r + s$.
– Least squares: $r \otimes s = \sqrt{r^2 + s^2}$.
– Worst case: $r \otimes s = \max\{r, s\}$.
For any $p > 0$, $r \otimes s = (r^p + s^p)^{1/p}$ is strictly monotonic in both arguments and thus the corresponding R has no collapsing elements, which covers the cases of weighted sum and least squares.
In the case of worst case, $\mathcal{C}(R) = \mathbb{R}_{\geq 0} \setminus \{0\}$, since $\otimes$ is idempotent. Assume that C has three different constraints c_1, c_2, and c_3; that there are valuations v_1 violating only c_1, v_2 violating only c_2, v_{13} violating exactly c_1 and c_3, and v_{23} violating exactly c_2 and c_3; that $g(c, v) = 0$ if, and only if, $c \in \{c_1, c_2, c_3\}$ is satisfied by $v \in \{v_1, v_2, v_{13}, v_{23}\}$; and that the weightings are independent of the valuation, i.e., $g(c_1, v_1) = g(c_1, v_{13})$ and $g(c_2, v_2) = g(c_2, v_{23})$ and $g(c_3, v_{13}) = g(c_3, v_{23})$. Also assume that the level weightings for the valuations v_1, v_2, v_{13}, and v_{23} for the worst case are related by

$$W_R(v_1) = g(c_1, v_1) > g(c_2, v_2) = W_R(v_2) ,$$
$$W_R(v_{13}) = \max\{g(c_1, v_{13}), g(c_3, v_{13})\} = \max\{g(c_2, v_{23}), g(c_3, v_{23})\} = W_R(v_{23}) .$$

Any set of (X, D)-M-soft constraint representations $\mathrm{M} = \{\mu(c) \mid c \in C\}$ reflecting the ordering induced by W_R on valuations, i.e., $\mathrm{M}(v) \leq_M \mathrm{M}(w) \iff W_R(w) \leq W_R(v)$, would thus have $\mu(c_3)$ as collapsing element in M.

7. Constraint Relationships

40. A *constraint relationship* over a constraint domain (X, D), or (X, D)-*constraint relationship*, is given by a dag C, where $|C|$ is a set of (X, D)-constraints. We think of a constraint $c' \in |C|$ as *more important* than another constraint $c \in |C|$ if $c \to_C c'$. An (X, D)-constraint relationship C is *finite* if $|C|$ is finite.

41. Let C be an (X, D)-constraint relationship and let M be a meet monoid with a partial order homomorphism $\varphi : PO\langle C\rangle \to PO(M)$. For each $c \in |C|$, define the (X, D)-M-soft constraint representation $c_{M,\varphi} : [X \to D] \to |M|$ by

$$c_{M,\varphi}(v) = \begin{cases} \varphi(c) & \text{if } v \not\models c \\ \varepsilon_M & \text{otherwise} \end{cases} .$$

We write $C_{M,\varphi}$ for the set of (X, D)-M-soft constraint presentations $\{c_{M,\varphi} \mid c \in |C|\}$.

EXAMPLE. Let C be a finite constraint relationship over (X, D).

(1) We first consider the single-predecessor lifiting introduced in §16.

Let $M_C = \mathit{mMon}(\mathit{jMon}\langle PO\langle C\rangle\rangle) = \mathit{mMon}\langle PO\langle C\rangle^{-1}\rangle$ and define $m_C : PO\langle C\rangle \to PO(M_C)$ by $m_C(c) = \eta^{\mathrm{mMon}}_{PO\langle C\rangle^{-1}}(c) = \wr c \int$; in particular $\varepsilon_{M_C} = \wr\int$. Then for $v, w \in [X \to D]$

$$\begin{aligned}
&v \leq_{cSRng\langle C_{M_C,m_C}\rangle} w \iff \\
&\quad \textstyle\prod_{M_C}\{c_{M_C,m_C}(v) \mid c \in |C|\} \leq_{M_C} \prod_{M_C}\{c_{M_C,m_C}(w) \mid c \in |C|\} \iff \\
&\quad \wr c \mid v \not\models c,\ c \in |C| \int \Subset^{PO\langle C\rangle^{-1}} \wr c \mid w \not\models c,\ c \in |C| \int \iff \\
&\quad \wr c \mid w \not\models c,\ c \in |C| \int \Subset_{PO\langle C\rangle} \wr c \mid v \not\models c,\ c \in |C| \int .
\end{aligned}$$

Thus, w is considered a better solution if each constraint that is not satisfied by w can be paired off with a constraint that is not satisfied by v and which is more important.

(2) We now consider the transitive-predecessors lifiting introduced in §17.

Let $U_C = \mathit{mMon}(\mathit{jMon}(\mathit{uSL}\langle PO\langle C\rangle\rangle))$ and define $u_C : PO\langle C\rangle \to PO(U_C)$ by $u_C(c) = \{c\}$; in particular, $\varepsilon_{U_C} = \emptyset$. Then for $v, w \in [X \to D]$

$$\begin{aligned}
&v \leq_{cSRng\langle C_{U_C,u_C}\rangle} w \iff \\
&\quad \textstyle\prod_{U_C}\{c_{U_C,u_C}(v) \mid c \in |C|\} \leq_{U_C} \prod_{U_C}\{c_{U_C,u_C}(w) \mid c \in |C|\} \iff \\
&\quad \mathrm{Max}^{\leq_C}\{c \mid v \not\models c,\ c \in |C|\} \supseteq_{PO\langle C\rangle} \mathrm{Max}^{\leq_C}\{c \mid w \not\models c,\ c \in |C|\} \iff \\
&\quad \mathrm{Max}^{\leq_C}\{c \mid w \not\models c,\ c \in |C|\} \subseteq_{PO\langle C\rangle} \mathrm{Max}^{\leq_C}\{c \mid v \not\models c,\ c \in |C|\} \iff \\
&\quad \forall c \in \{c_w \in |C| \mid w \not\models c_w\} \,.\, \exists c' \in \{c_v \in |C| \mid v \not\models c_v\} \,.\, c \leq_{PO\langle C\rangle} c' .
\end{aligned}$$

Thus, w is considered a better solution if each constraint that is not satisfied by w can be covered by a constraint that is not satisfied by v and which is more important. □

42. The *scope* of a constraint c over a constraint domain (X, D) is given by the set of variables it depends on, i.e.,

$$\mathrm{sc}(c) = \{x \in X \mid \exists v \in [X \to D], d_1 \neq d_2 \in D_x \,.\, c(v\{x \mapsto d_1\}) \neq c(v\{x \mapsto d_2\})\} .$$

For a partial valuation $p \in [X \to D^?]$, we write $p \not\models c$ if $\mathrm{sc}(c) \subseteq \mathrm{def}(p)$ and $v \not\models c$ for some $v \in p\!\uparrow$ (which is well-defined, since then c only depends on variables that are in the domain of definition of p).
Let C be a finite constraint relationship over (X, D), let M be a meet monoid, and let $\varphi : PO\langle C\rangle \to PO(M)$. Define $\alpha_{M,\varphi}, \zeta_{M,\varphi} : [X \to D^?] \to |M|$ by

$$\alpha_{M,\varphi}(p) = \textstyle\prod_M\{\varphi(c) \mid c \in |C|,\ \mathrm{sc}(c) \subseteq \mathrm{def}(p),\ p \not\models c\} \cdot_M \prod_M\{\varphi(c) \mid \mathrm{sc}(c) \not\subseteq \mathrm{def}(p)\}\ ,$$
$$\zeta_{M,\varphi}(p) = \textstyle\prod_M\{\varphi(c) \mid c \in |C|,\ \mathrm{sc}(c) \subseteq \mathrm{def}(p),\ p \not\models c\}\ .$$

LEMMA. *$(\alpha_{M,\varphi}, \zeta_{M,\varphi})$ is a tight bounding pair of $(X, D^?)$-M-soft constraint presentations for $C_{M,\varphi}$.*

Proof. For a $p \in [X \to D^?]$ let $V(p) = \{c \in |C| \mid \mathrm{sc}(c) \subseteq \mathrm{def}(p),\ p \not\models c\}$ and $W(p) = \{c \in |C| \mid \mathrm{sc}(c) \not\subseteq \mathrm{def}(p)\}$. Then $\alpha_{M,\varphi}(p) = \prod_M \varphi(V(p)) \cdot_M \prod_M \varphi(W(p))$ and $\zeta_{M,\varphi}(p) = \prod_M \varphi(V(p))$ for all $p \in [X \to D^?]$.
Let $p \in [X \to D^?]$ and $v \in p\!\uparrow$ be given. Then $V(p) \subseteq V(v)$, and thus $\zeta_{M,\varphi}(v) \leq_M \zeta_{M,\varphi}(p)$: For a $c \in V(p)$, i.e., $\mathrm{sc}(c) \subseteq \mathrm{def}(p)$ and $p \not\models c$, also $c \in V(v)$, since $v \in p\!\uparrow$ and thus $v \not\models c$.
Now let $p' = p\{x \mapsto d\} \in [X \to D^?]$ and let $v \in p\!\uparrow$ be arbitrary. Then $V(v) \subseteq V(p') \cup W(p')$, and thus $\alpha_{M,\varphi}(p') \leq_M \zeta_{M,\varphi}(v)$: Let $c \in V(v)$, i.e., $v \not\models c$. If $\mathrm{sc}(c) \subseteq \mathrm{def}(p')$, then $p' \not\models c$, and hence $c \in V(p')$; otherwise $c \in W(p')$.
Finally, $\zeta_{M,\varphi}(v) = C_{M,\varphi}(v)$ and thus $(\alpha_{M,\varphi}, \zeta_{M,\varphi})$ is tight for $C_{M,\varphi}$. ☐

EXAMPLE. Consider the constraint domain (X, D) given by

$$\mathrm{X} = \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}\ ,$$
$$\mathrm{D_x} = \mathrm{D_y} = \mathrm{D_z} = \{1, 2, 3\}$$

as well as the constraint relationship $\mathrm{C} = (\{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}, \{(\mathrm{c}_2, \mathrm{c}_1), (\mathrm{c}_3, \mathrm{c}_1)\})$, i.e., $\mathrm{c}_2 \to_\mathrm{C} \mathrm{c}_1$ and $\mathrm{c}_3 \to_\mathrm{C} \mathrm{c}_1$, with

$$\mathrm{c}_1 : \mathrm{x} + 1 = \mathrm{y}\ ,$$
$$\mathrm{c}_2 : \mathrm{z} = \mathrm{y} + 2\ ,$$
$$\mathrm{c}_3 : \mathrm{x} + \mathrm{y} \leq 3\ .$$

Let $M_\mathrm{C} = mMon\langle PO\langle\mathrm{C}\rangle^{-1}\rangle$ and $m_\mathrm{C} = \eta^{\mathrm{mMon}}_{PO\langle\mathrm{C}\rangle^{-1}}$. Then

$$\alpha_{M_\mathrm{C},m_\mathrm{C}}(p) = \wr c \in |\mathrm{C}| \mid \mathrm{sc}(c) \subseteq \mathrm{def}(p),\ p \not\models c \int \uplus \wr c \in |\mathrm{C}| \mid \mathrm{sc}(c) \not\subseteq \mathrm{def}(p) \int\ ,$$
$$\zeta_{M_\mathrm{C},m_\mathrm{C}}(p) = \wr c \in |\mathrm{C}| \mid \mathrm{sc}(c) \subseteq \mathrm{def}(p),\ p \not\models c \int\ ,$$

such that, for example,

$$\alpha_{M_\mathrm{C},m_\mathrm{C}}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto 1, \mathrm{z} \mapsto ?\}) = \wr \mathrm{c}_1, \mathrm{c}_2 \int\ , \qquad \zeta_{M_\mathrm{C},m_\mathrm{C}}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto 1, \mathrm{z} \mapsto ?\}) = \wr \mathrm{c}_1 \int\ ,$$
$$\alpha_{M_\mathrm{C},m_\mathrm{C}}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto 2, \mathrm{z} \mapsto ?\}) = \wr \mathrm{c}_2 \int\ , \qquad \zeta_{M_\mathrm{C},m_\mathrm{C}}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto 2, \mathrm{z} \mapsto ?\}) = \wr\int\ ,$$
$$\alpha_{M_\mathrm{C},m_\mathrm{C}}(\{\mathrm{x} \mapsto 2, \mathrm{y} \mapsto 3, \mathrm{z} \mapsto ?\}) = \wr \mathrm{c}_2, \mathrm{c}_3 \int\ , \qquad \zeta_{M_\mathrm{C},m_\mathrm{C}}(\{\mathrm{x} \mapsto 2, \mathrm{y} \mapsto 3, \mathrm{z} \mapsto ?\}) = \wr \mathrm{c}_3 \int\ ;$$

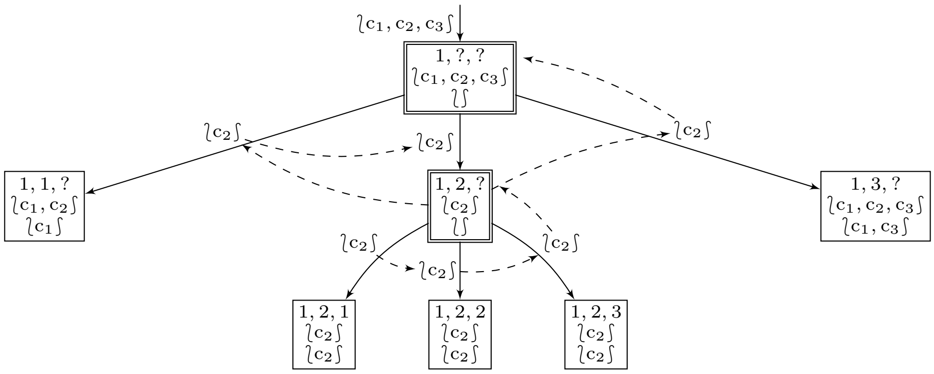
in particular

$$\zeta_{M_\mathrm{C},m_\mathrm{C}}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto 1, \mathrm{z} \mapsto ?\}) = \wr \mathrm{c}_1 \int <_{M_\mathrm{C}} \wr \mathrm{c}_2 \int = \alpha_{M_\mathrm{C},m_\mathrm{C}}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto 2, \mathrm{z} \mapsto ?\})\ .$$

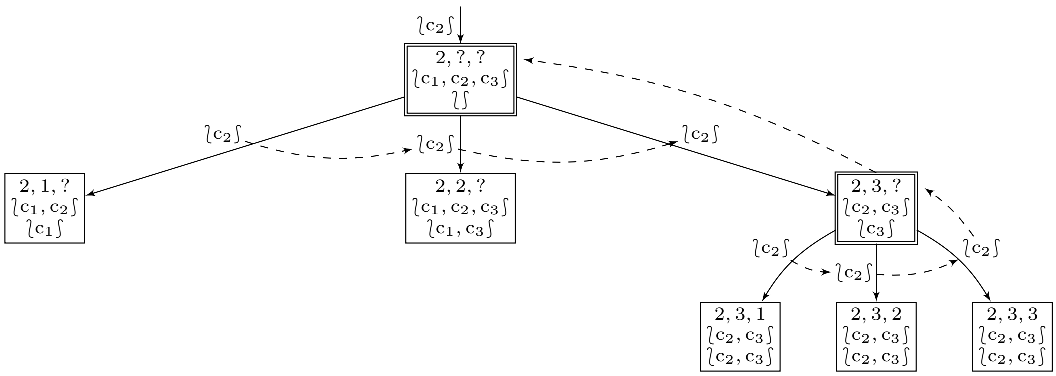
We abbreviate $\alpha_{M_\mathrm{C},m_\mathrm{C}}$ by α and $\zeta_{M_\mathrm{C},m_\mathrm{C}}$ by ζ. We follow an execution of $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto ?, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \emptyset)$, choosing the variables x, y, and z in this order and running through $\{1, 2, 3\}$ in the natural order; we select x and y first, since $\mathrm{sc}(\mathrm{c}_1) = \{\mathrm{x}, \mathrm{y}\}$ and c_1 is the top element in C.
The first step in evaluating $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\lambda x \in \mathrm{X} \,.\, ?, \emptyset)$ is to evaluate $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \wr \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3 \int)$, since

$$\alpha((\lambda x \in \mathrm{X} \,.\, ?)\{x \mapsto d\}) = \wr \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3 \int\ ,$$

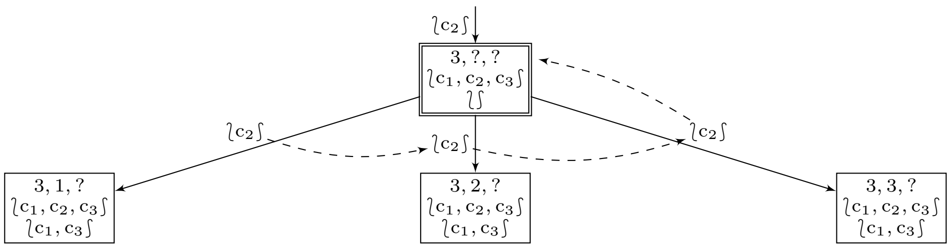
for all $d \in \{1,2,3\}$ and $\zeta((\lambda x\,.\,\mathrm{X})\{x \mapsto 1\}) = \lbag\rbag$; which leads to the following graph:



The annotations on the solid edges represent the current values of the lower bound set L, where we have omitted the set braces. Each node gives the respective partial valuation p for x, y, and z at the top, $\alpha(p)$ in the middle, and $\zeta(p)$ at the bottom. Doubly outlined nodes represent calls to $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(p, L)$; singly outlined nodes represent the successful test whether $\zeta(p)$ already is dominated by a lower bound in L. Finally, the dashed edges show the flow of the lower bounds.
Thus, $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto 1, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \emptyset) = \{\lbag c_2 \rbag\}$, and $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto 2, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \{\lbag c_2 \rbag\})$ is executed:



Hence, $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto 2, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \{\lbag c_2 \rbag\}) = \{\lbag c_2 \rbag\}$ and $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto 3, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \{\lbag c_2 \rbag\})$ is executed:



Therefore, $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto 3, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \{\lbag c_2 \rbag\}) = \{\lbag c_2 \rbag\}$, and we have as the final result that $\mathsf{maxSolDegs}_{(\alpha,\zeta)}(\{\mathrm{x} \mapsto ?, \mathrm{y} \mapsto ?, \mathrm{z} \mapsto ?\}, \emptyset) = \{\lbag c_2 \rbag\}$. □

References

[1] Stefano Bistarelli, Ugo Montanari, Francesca Rossi, Thomas Schiex, Gérard Verfaillie, and Hélène Fargier. Semiring-Based CSPs and Valued CSPs: Frameworks, Properties, and Comparison. *Constraints*, 4(3):199–240, 1999.

[2] Alan Borning, Bjorn Freeman-Benson, and Molly Wilson. Constraint Hierarchies. *LISP Symb. Comp.*, 5:223–270, 1992.

[3] Fabio Gadducci, Matthias M. Hölzl, Giacoma V. Monreale, and Martin Wirsing. Soft Constraints for Lexicographic Orders. In Félix Castro, Alexander Gelbukh, and Miguel González, editors, *Proc. 12th Mexican Int. Conf. Artificial Intelligence (MICAI'13)*, Lect Notes Comp. Sci. 8265, pages 68–79. Springer, 2013.

[4] Matthias Hölzl, Max Meier, and Martin Wirsing. Which Soft Constraints do you Prefer? In *Proc. 7th Int. Wsh. Rewriting Logic and its Applications (WRLA'08)*, volume 238 of *Electronic Notes in Theoretical Computer Science*, pages 189–205, 2009.

[5] P. Meseguer, Francesca Rossi, and T. Schiex. Soft Constraints. In Francesca Rossi, P. van Beek, and Toby Walsh, editors, *Handbook of Constraint Programming*, chapter 9. Elsevier, 2006.