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Abstract

We generalize the operators of classical linear time temporal logic to partial orders, such as the ones used in domain theory. This relates denotaional semantics and temporal logic. We put this into the general perspective of modal logic. $\Box$ and $\Diamond$ are viewed as standard modalities, which gives "half" of the standard axioms of LTL. Moroever, we show that the next-time operator $\bigcirc$ can be defined as a combination of two modalities. We show the role of the standard LTL axioms in narrowing the underlying partial orders to linear ones generated by the immediate-successor relation. We distinguish between modal and temporal validity of formulas and investigate their relation.

1 Introduction

In [11] a stream has been identified with the set of its finite prefixes. Based on this, we have used a special way of characterising sets of streams through sets of relevant finite "snapshots". Given a set $P \subseteq A^*$ where A is a set of atomic actions, states or data, we define

$$\mathsf{str}\, P \stackrel{\text{def}}{=} \{(\sqsubseteq Q) : Q \subseteq P \text{ directed}\} ,$$

where $\sqsubseteq$ is the prefix order and $(\sqsubseteq Q) \stackrel{\text{def}}{=} \{x \in A^* : \exists\, y \in Q : x \sqsubseteq y\}$ is the prefix closure of Q. So $\mathsf{str}\, P$ is the set of all streams "spanned" by directed subsets of P. The str operator enjoys a number of distributivity and monotonicity laws which are the basis for correct refinement of specifications into implementations. They are used in [14] in the algebraic calculation of a bounded queue module.

The str operation has been generalized in [12] to arbitrary domains. There it was conjectured that this operator has to do with the "infinitely often" operator $\Box\Diamond$ of linear temporal logic (LTL). This was made precise in [13].

So order theory admits simple characterizations of temporal operators such as "always eventually" and "always initially". In [13] we have also investigated how the other classical

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temporal operators generalize to an order-theoretic setting. It was clarified which of the classical axioms of LTL carry over to the general case; in this way also their role in the standard complete axiomatisation of LTL became clearer.

In the present paper we put this into the more general perspective of modal logic. $\Box$ and $\Diamond$ are viewed as standard modalities, which gives "half" of the standard axioms of LTL. Moreover, we show that the next-time operator $\bigcirc$ can be defined as a combination of two modalities. Extending [13] further, we distinguish between modal and temporal validity of formulas and investigate their relation. Also, the technical treatment is much simpler than in [13], leading also to a quicker proof of the above $\Box\Diamond$ conjecture.

2 Property Transformers and Modal Algebras

In this section we recall some basic notions from the theory of modal logic. A good up-to-date exposition of this topic is [16].

Modal logic can be seen as the theory of labelled transition system. It allows abbreviating quantifications about successors/predecessors of elements under a family of (labelled transition) relations. **Monomodal** systems result in the case where only a single relation is considered.

Let M, N be sets and R be a binary relation $R \subseteq M \times N$. For $P \subseteq M, Q \subseteq N$ we denote the **image** of P under R by

$$P\,R \stackrel{\text{def}}{=} \{y \in N : \exists\, x \in P : x\,R\,y\}$$

and the **inverse image** of Q under R by

$$R\,Q \stackrel{\text{def}}{=} \{x \in M : \exists\, y \in Q : x\,R\,y\}\ .$$

For singleton sets P, Q we'll omit the set braces.

In modal logic the relation R is assumed to be **homogeneous**, i.e., M and N have to be equal. The **forward** modalities $\Box_R$ and $\Diamond_R$ are universal and existential quantifiers, respectively, about the successors of an element under relation R. A formula $\Box_R F$ holds at a point $x \in M$ iff F holds at *all* points in xR. The formula $\Diamond_R F$ holds at $x \in M$ iff F holds at *some* point in xR. Dually, the **backward** modalities $⊟_R$ and $\ominus\!\!\!\Diamond_R$ quantify about R-predecessors.

Frequently it is convenient to work with algebraic counterparts of the modal formulas and quantifiers. To each formula F one associates the set $[\![F]\!] \subseteq M$ of points where F holds. We call such subsets **properties**. The properties T and F are given by

$$\begin{array}{rcl} \mathsf{T} & \stackrel{\text{def}}{=} & M\ , \\ \mathsf{F} & \stackrel{\text{def}}{=} & \emptyset\ . \end{array}$$

The propositional part of logic is reflected by

$$\begin{array}{rcl} \neg P & \stackrel{\text{def}}{=} & \mathsf{T} \backslash P\ , \\ P \wedge Q & \stackrel{\text{def}}{=} & P \cap Q\ , \\ P \vee Q & \stackrel{\text{def}}{=} & P \cup Q\ , \\ P \rightarrow Q & \stackrel{\text{def}}{=} & \neg P \vee Q\ . \end{array}$$

As is well-known, $\rightarrow$ is the algebraic counterpart of the subset relation:

Lemma 2.1 For $P, Q \subseteq M$ we have $P \subseteq Q$ iff $P \to Q = \mathsf{T}$.

Proof: (Only if) Assuming $P \subseteq Q$, which is equivalent to $\neg Q \subseteq \neg P$, we get

$$\begin{array}{ll} & \mathsf{T} \\ = & \quad \{\!\![\text{ boolean algebra }]\!\!\} \\ & \neg Q \vee Q \\ \subseteq & \quad \{\!\![\text{ monotonicity }]\!\!\} \\ & \neg P \vee Q \\ = & \quad \{\!\![\text{ definition }]\!\!\} \\ & P \to Q \ . \end{array}$$

(If) Assuming, conversely, $\mathsf{T} = P \to Q = \neg P \vee Q$ we get

$$\begin{array}{ll} & P \\ = & \quad \{\!\![\text{ boolean algebra }]\!\!\} \\ & P \wedge \mathsf{T} \\ = & \quad \{\!\![\text{ assumption }]\!\!\} \\ & P \wedge (\neg P \vee Q) \\ = & \quad \{\!\![\text{ boolean algebra }]\!\!\} \\ & P \wedge Q \end{array}$$

and hence $P \subseteq Q$. ■

The modal quantifiers are represented by property transformers. We'll do this only for the forward modalities, the backward ones being symmetric. To mirror $\Box_R$ we introduce for $R \subseteq M \times M$ the operator $[R] : \mathcal{P}(M) \to \mathcal{P}(M)$ by setting, for properties $P, Q \subseteq M$,

$$P \subseteq [R]Q \text{ iff } PR \subseteq Q \ .$$

By this definition, $[R]$ is the upper adjoint of a Galois connection (see e.g. [3]) between the complete lattice $(\mathcal{P}(M), \subseteq)$ and itself. Hence $[R]$ is well-defined, since the lower adjoint, the image operator for R, distributes through disjunction, i.e., preserves suprema. According to the above definition one has

$$[\![\Box_R F]\!] = [R][\![F]\!] \ .$$

By the Galois connection, $[R]$ distributes through arbitrary conjunctions,

$$[R](\bigwedge_{k \in K} Q_k) = \bigwedge_{k \in K} [R]Q_k \ ,$$

and hence is upward strict, i.e., satisfies

$$[R]\mathsf{T} = \mathsf{T} \ ,$$

and is monotonic w.r.t. $\subseteq$.

To prove the validity of some axiom schemes we need

Lemma 2.2 Assume a property transformer $G : \mathcal{P}(M) \to \mathcal{P}(M)$.

1. If G distributes through $\wedge$ we have for all $P, Q \subseteq M$
$$G(P \to Q) \subseteq G(P) \to G(Q) \ .$$

2. If G distributes through $\wedge$ and is downward strict, i.e., satisfies $G(\mathsf{F}) = \mathsf{F}$, then for all $P \subseteq M$
$$G(\neg P) \subseteq \neg G(P) \ .$$

Proof: We use Lemma 2.1.

1.
$$\begin{array}{cl}
 & G(P \to Q) \to (G(P) \to G(Q)) \\
= & \quad \{\!\![\text{ definitions }]\!\!\} \\
 & \neg G(\neg P \vee Q) \vee \neg G(P) \vee G(Q) \\
= & \quad \{\!\![\text{ de Morgan }]\!\!\} \\
 & \neg (G(\neg P \vee Q) \wedge G(P)) \vee G(Q) \\
= & \quad \{\!\![\ \wedge\text{-distributivity of } G \ ]\!\!\} \\
 & \neg G((\neg P \vee Q) \wedge P) \vee G(Q) \\
= & \quad \{\!\![\text{ boolean algebra }]\!\!\} \\
 & \neg G(Q \wedge P) \vee G(Q) \\
= & \quad \{\!\![\ \wedge\text{-distributivity of } G \ ]\!\!\} \\
 & \neg (G(Q) \wedge G(P)) \vee G(Q) \\
= & \quad \{\!\![\text{ de Morgan }]\!\!\} \\
 & \neg G(Q) \vee \neg G(P) \vee G(Q) \\
= & \quad \{\!\![\text{ boolean algebra }]\!\!\} \\
 & \mathsf{T} \ .
\end{array}$$

2.
$$\begin{array}{cl}
 & G(\neg P) \to \neg G(P) \\
= & \quad \{\!\![\text{ definition } \to \ ]\!\!\} \\
 & \neg G(\neg P) \vee \neg G(P) \\
= & \quad \{\!\![\text{ de Morgan }]\!\!\} \\
 & \neg (G(\neg P) \wedge G(P)) \\
= & \quad \{\!\![\ \wedge\text{-distributivity of } G \ ]\!\!\} \\
 & \neg G(\neg P \wedge P)) \\
= & \quad \{\!\![\text{ boolean algebra }]\!\!\} \\
 & \neg G(\mathsf{F}) \\
= & \quad \{\!\![\text{ downward strictness }]\!\!\} \\
 & \neg \mathsf{F} \\
= & \quad \{\!\![\text{ boolean algebra }]\!\!\} \\
 & \mathsf{T} \ .
\end{array}$$

$\blacksquare$

Note that this generalizes to arbitrary boolean algebras. As an application of 2. we note

$$P \wedge (Q_1 \to Q_2) \subseteq (P \wedge Q_1) \to (P \wedge Q_2) \ .$$

The algebraic counterpart of $\Diamond_R$ is the property transformer $\langle R \rangle : \mathcal{P}(M) \to \mathcal{P}(M)$, defined by

$$\langle R \rangle Q \stackrel{\text{def}}{=} \neg[R](\neg Q) \ .$$

Then, for formula F,

$$[\![\Diamond_R F]\!] = \langle R \rangle [\![F]\!] \ .$$

Moreover, idempotence of $\neg$ gives the duality properties

$$\neg[R]P = \langle R \rangle \neg P \ , \qquad \neg\langle R \rangle P = [R]\neg P \ . \tag{1}$$

One has

$$x \in \langle R \rangle Q \iff xR \cap Q \neq \emptyset \text{ iff } x \in RQ \ ,$$

i.e.,

$$\langle R \rangle Q = RQ \ . \tag{2}$$

By de Morgan's law and $\wedge$-distributivity of $[R]$ we obtain $\vee$-distributivity of $\langle R \rangle$ (which is also evident from the explicit representation (2)):

$$\langle R \rangle (\bigvee_{k \in K} Q_k) = \bigvee_{k \in K} \langle R \rangle Q_k \ .$$

Therefore $\langle R \rangle$ is also $\subseteq$-monotonic and downward strict, i.e., satisfies

$$\langle R \rangle \mathsf{F} = \mathsf{F} \ .$$

Finally, by (2) and the definition of $\langle R \rangle$ we get the domain of R as $\langle R \rangle \mathsf{T}$ and its complement as $[R]\mathsf{F}$.

3 Relation Algebra

It is useful to investigate the behaviour of the $[\cdot]$ function w.r.t. the relation algebraic operations. Let I be the **identity relation** on M, i.e., $I \stackrel{\text{def}}{=} \{(x,x) : x \in M\}$, and $1 \stackrel{\text{def}}{=} M \times M$ the **universal** relation on M. The **relational product** is denoted by juxtaposition: for relations $R, S \subseteq M \times M$ we have

$$x(RS)y \text{ iff } \exists\, z \in M : xRz \wedge zSy \ .$$

Moreover, since relations are sets of pairs, union and intersection are defined as usual.

The proof principles of **indirect inequality** and **indirect equality** for properties $P, Q \subseteq M$ are

$$\begin{array}{l} P \subseteq Q \text{ iff } \forall\, Z : Z \subseteq P \Rightarrow Z \subseteq Q \ , \\ P \subseteq Q \text{ iff } \forall\, Z : Q \subseteq Z \Rightarrow P \subseteq Z \ , \\ P = Q \text{ iff } \forall\, Z : Z \subseteq P \text{ iff } Z \subseteq Q \ . \end{array}$$

The following properties are easily checked using the Galois connection and indirect equality:

$$[I]Q = Q \ ,$$
$$[RS]Q = [R]([S]Q) \ ,$$
$$[R \cup S]Q = [R]Q \cap [S]Q \ .$$

For the first equality we have

$$
\begin{array}{ll}
& P \subseteq [I]Q \\
\Leftrightarrow & \quad \{\!\mid \text{Galois} \mid\!\} \\
& PI \subseteq Q \\
\Leftrightarrow & \quad \{\!\mid \text{neutrality} \mid\!\} \\
& P \subseteq Q \ .
\end{array}
$$

For the second one we calculate

$$
\begin{array}{ll}
& P \subseteq [RS]Q \\
\Leftrightarrow & \quad \{\!\mid \text{Galois} \mid\!\} \\
& P(RS) \subseteq Q \\
\Leftrightarrow & \quad \{\!\mid \text{relations} \mid\!\} \\
& (PR)S \subseteq Q \\
\Leftrightarrow & \quad \{\!\mid \text{Galois} \mid\!\} \\
& PR \subseteq [S]Q \\
\Leftrightarrow & \quad \{\!\mid \text{Galois} \mid\!\} \\
& P \subseteq [R]([S]Q) \ .
\end{array}
$$

For the third one we have

$$
\begin{array}{ll}
& P \subseteq [R \cup S]Q \\
\Leftrightarrow & \quad \{\!\mid \text{Galois} \mid\!\} \\
& P(R \cup S) \subseteq Q \\
\Leftrightarrow & \quad \{\!\mid \text{relations} \mid\!\} \\
& PR \cup PS \subseteq Q \\
\Leftrightarrow & \quad \{\!\mid \text{boolean algebra} \mid\!\} \\
& PR \subseteq Q \wedge PS \subseteq Q \\
\Leftrightarrow & \quad \{\!\mid \text{Galois} \mid\!\} \\
& P \subseteq [R]Q \wedge P \subseteq [S]Q \\
\Leftrightarrow & \quad \{\!\mid \text{boolean algebra} \mid\!\} \\
& P \subseteq [R]Q \cap [S]Q \ .
\end{array}
$$

This latter equality implies that $[\cdot]$ is antitonic w.r.t. $\subseteq$ in the following sense. Lift the relation $\subseteq$ to $\mathcal{P}(M) \to \mathcal{P}(M)$ by

$$F \subseteq G \text{ iff } \forall\, Q \subseteq M : FQ \subseteq GQ \ .$$

Then

$$R \subseteq S \Rightarrow [S] \subseteq [R] \ .$$

The converse of this holds also. To see this we first calculate

$$\begin{array}{ll} & xRy \\ \Leftrightarrow & \{\!\![\text{ definition of the image set }]\!\!\} \\ & y \subseteq xR \\ \Leftrightarrow & \{\!\![\text{ indirect inequality }]\!\!\} \\ & \forall\, Z : xR \subseteq Z \Rightarrow y \subseteq Z \\ \Leftrightarrow & \{\!\![\text{ definition of } [\cdot]]\!\!\} \\ & \forall\, Z : x \in [R]Z \Rightarrow y \in Z \ . \end{array}$$

So R can be reconstructed from $[R]$. Based on this one easily shows

$$[S] \subseteq [R] \Rightarrow R \subseteq S \ .$$

What about the dual $\langle R \rangle$ of $[R]$? Here, de Morgan's laws and idempotence of $\neg$ do all the work for us:

$$\begin{array}{r} \langle I \rangle Q = Q \ , \\ \langle RS \rangle Q = \langle R \rangle (\langle S \rangle Q) \ , \\ \langle R \cup S \rangle Q = \langle R \rangle Q \ \cup \ \langle S \rangle Q \ . \end{array}$$

Moreover, we get

$$R \subseteq S \ \text{ iff } \ \langle R \rangle \subseteq \langle S \rangle \ .$$

Altogether, $\langle \cdot \rangle$ is covariant w.r.t. all operations and hence easier to work with.

For the universal relation we obtain

$$\begin{array}{ll} & P \subseteq [1]Q \\ \Leftrightarrow & \{\!\![\text{ Galois }]\!\!\} \\ & P1 \subseteq Q \\ \Leftrightarrow & \{\!\![\text{ relations }]\!\!\} \\ & P = \mathsf{F} \vee Q = \mathsf{T} \end{array}$$

and hence

$$[1]Q \ = \ \mathsf{if}\ Q = \mathsf{T}\ \mathsf{then}\ \mathsf{T}\ \mathsf{else}\ \mathsf{F} \ .$$

From this we get

$$\begin{array}{ll} & \langle 1 \rangle Q \\ = & \{\!\![\text{ definition }]\!\!\} \\ & \neg[1]\neg Q \\ = & \{\!\![\text{ by the above }]\!\!\} \\ & \mathsf{if}\ \neg Q = \mathsf{T}\ \mathsf{then}\ \neg\mathsf{T}\ \mathsf{else}\ \neg\mathsf{F} \\ = & \{\!\![\text{ boolean algebra }]\!\!\} \\ & \mathsf{if}\ Q = \mathsf{F}\ \mathsf{then}\ \mathsf{F}\ \mathsf{else}\ \mathsf{T} \ . \end{array}$$

Another important relational operation is forming the **converse** $R^{\smile}$ of R:

$$R^{\smile} \stackrel{\text{def}}{=} \{(y,x) : (x,y) \in R\} \ .$$

The converse can be used to define the "backward" modal operators:

$$\begin{array}{rcl} [R]^{-} & \stackrel{\text{def}}{=} & [R^{\smile}] \ , \\ \langle R \rangle^{-} & \stackrel{\text{def}}{=} & \langle R^{\smile} \rangle \ . \end{array}$$

4 Correspondences and Partial Orders

Relation R is called

reflexive	iff $I \subseteq R$,
coreflexive	iff $R \subseteq I$,
transitive	iff $RR \subseteq R$,
dense	iff $R \subseteq RR$,
antisymmetric	iff $R \cap R^{\smile} \subseteq I$.

The first four of these properties of relations are reflected in their associated modal operators as follows (see e.g. [16]): relation R is

$$\begin{array}{ll} \text{reflexive} & \text{iff } \forall\, P \subseteq M : [R]P \subseteq P, \\ \text{coreflexive} & \text{iff } \forall\, P \subseteq M : P \subseteq [R]P, \\ \text{transitive} & \text{iff } \forall\, P \subseteq M : [R]P \subseteq [R]([R]P), \\ \text{dense} & \text{iff } \forall\, P \subseteq M : [R]([R]P) \subseteq [R]P. \end{array}$$

The proofs are immediate from the relation algebraic characterizations and the above properties of $[\cdot]$. Moreover, using coreflexivity, we obtain that R is

$$\text{antisymmetric iff } \forall\, P \subseteq M : P \subseteq [R \cap R^{\smile}]P. \tag{3}$$

A binary relation is a **preorder** iff it is reflexive and transitive. It is a **partial order** iff it is an antisymmetric preorder.

We have just seen modal characterizations of reflexivity and transitivity. Together we get

$$R \text{ is a preorder iff } \forall\, P \subseteq M : [R]P \subseteq P \cap [R]([R]P) \ .$$

For partial orders we get in addition (3).

5 Modal Validity

A property $P \subseteq M$ is **modally valid**, denoted $\models_m P$, iff P holds for all elements of M, i.e., iff $P = M = \mathsf{T}$. Following LTL we define the abbreviation

$$P \Rightarrow Q \stackrel{\text{def}}{=} [R](P \rightarrow Q) \ .$$

From these definitions we obtain the following properties of validity:

Lemma 5.1 1. $\models_m P \rightarrow Q$ iff $P \subseteq Q$.

2. $\models_m P$ implies $\models_m [R]P$.
   In particular, $\models_m P \rightarrow Q$ implies $\models_m P \Rightarrow Q$.

3. $\models_m [R]P$ implies $\models_m P$ iff R is surjective.

4. For reflexive R we have $\models_m [R]P \rightarrow P$.

5. $\models_m [R](P \rightarrow Q) \rightarrow ([R]P \rightarrow [R]Q)$.

Proof: 1. is immediate from Lemma 2.1.

2. is just upward strictness of $[R]$.

3. We note first that by the Galois connection

$$[R]P = \mathsf{T} \text{ iff } \mathsf{T} \subseteq [R]P \text{ iff } \mathsf{T}R \subseteq P \quad (4)$$

Now

$$\begin{array}{ll} & \forall\, P : \models_m [R]P \text{ implies } \models_m P \\ \Leftrightarrow & \quad \{\!\![\text{ by (4) }]\!\!\} \\ & \forall\, P : \mathsf{T}R \subseteq P \text{ implies } \mathsf{T} \subseteq P \\ \Leftrightarrow & \quad \{\!\![\text{ by indirect inequality }]\!\!\} \\ & \mathsf{T} \subseteq \mathsf{T}R \\ \Leftrightarrow & \quad \{\!\![\text{ relations }]\!\!\} \\ & R \text{ surjective }. \end{array}$$

4. is a direct consequence of 2. and the modal correspondences.

5. is immediate from Lemma 2.2.1.

■

In particular,

$$[R](P \rightarrow Q) \subseteq [R]P \rightarrow [R]Q \; .$$

A somewhat surprising property is

$$\langle R\rangle(P \rightarrow Q) = [R]P \rightarrow \langle R\rangle Q \; ,$$

shown by

$$\begin{array}{ll} & \langle R\rangle(P \rightarrow Q) \\ = & \quad \{\!\![\text{ definition } \rightarrow]\!\!\} \\ & \langle R\rangle(\neg P \vee Q) \\ = & \quad \{\!\![\text{ distributivity }]\!\!\} \\ & \langle R\rangle\neg P \vee \langle R\rangle Q \\ = & \quad \{\!\![\text{ duality }]\!\!\} \\ & \neg[R]P \vee \langle R\rangle Q \\ = & \quad \{\!\![\text{ definition } \rightarrow]\!\!\} \\ & [R]P \rightarrow \langle R\rangle Q \; . \end{array}$$

6 A Next-Time Operator

The next-time operator we define is a combination of the $[\cdot]$ and $\langle\cdot\rangle$ operations. It asserts that a sucessor element exists and that a certain property holds for all successors. So we set

$$((R))P \stackrel{\text{def}}{=} \langle R\rangle \mathsf{T} \wedge [R]P \ .$$

By $\wedge$-distributivity of $[R]$ we get the same distributivity for $((R))$:

$$((R))(\bigwedge_{k\in K} Q_k) = \bigwedge_{k\in K} ((R))Q_k \ .$$

Hence $((R))$ is upward strict, i.e., satisfies

$$((R))\mathsf{T} = \langle R\rangle \mathsf{T} \ ,$$

and is monotonic w.r.t. $\subseteq$. Moreover, $((R))$ is downward strict:

$$\begin{array}{ll} & ((R))\mathsf{F} \\ = & \quad \{\![\text{ definition }]\!\} \\ & \langle R\rangle \mathsf{T} \wedge [R]\mathsf{F} \\ = & \quad \{\![\text{ duality (1) }]\!\} \\ & \neg[R]\mathsf{F} \wedge [R]\mathsf{F} \\ = & \quad \{\![\text{ boolean algebra }]\!\} \\ & \mathsf{F} \ . \end{array}$$

From this we get one half of each of two standard axioms:

Lemma 6.1 1. $\models_m ((R))(P \to Q) \to (((R))P \to ((R))Q)$.

2. $\models_m ((R))\neg P \to \neg((R))P$.

Proof: This is immediate from Lemma 2.2. ■

7 Temporal Operators

We will now concentrate on special relations R that are supposed to model temporal succession of certain events. To find suitable requirements on $R \subseteq M \times M$, we introduce a central notion for studying timed systems: the **interval** between $x, y \in M$ is

$$[x, y] \stackrel{\text{def}}{=} xR \cap Ry \ .$$

A reasonable assumption is that $[x, y] \neq \emptyset$ iff xRy. However, only in the case of a transitive R we can infer $x \leq y$ from $[x, y] \neq \emptyset$. On the other hand, only for reflexive R we can infer $x, y \in [x, y]$ from $x \leq y$. This suggests that having a reflexive and transitive R, i.e., a preorder, is a minimum requirement for working reasonably with intervals. Antisymmetry, on the other hand is not strictly necessary, although it seems a reasonable assumption for time points.

For the special case where R is a preorder $\leq$ we simply write $\Box$ and $\Diamond$ instead of $[\leq]$ and $\langle\leq\rangle$. We will only define the "forward" temporal operators, the "backward" ones being symmetric. The basic operator is the until-operator $\mathcal{U}$. In LTL the formula $P \,\mathcal{U}\, Q$ is satisfied in a state if there is a future state where Q holds and $P \vee Q$ holds for all intermediate states. We now generalize this idea by mirroring the notions "future" and "intermediate" order-theoretically: $P \,\mathcal{U}\, Q$ holds for a point x iff there is a later point y for which Q holds and $P \vee Q$ holds for all intermediate points. "Later" of course means $x \leq y$ and the intermediate points are the $z \in [x, y]$. This is precisely reflected in the definition below:

$$x \in P \,\mathcal{U}\, Q \quad \text{iff} \quad \exists\, y \in Q : x \leq y \wedge [x, y] \subseteq P \vee Q\ .$$

From this it is immediate that $\mathcal{U}$ is monotonic in both arguments. Note, however, that $\wedge$-distributivity in the left argument and $\vee$-distributivity in the right one fail for arbitrary pre-orders.

If $\leq$ is not linear then $\mathcal{U}$ is "angelic" in the sense that a suitable successor y on *one* continuation of the computation path leading to x is sufficient to ensure $x \in P \,\mathcal{U}\, Q$. We have chosen this particular definition, since it most directly corresponds to the informal requirement used in LTL.

We have

$$\Diamond Q = \mathsf{T} \,\mathcal{U}\, Q = \leq Q\ . \qquad (5)$$

Moreover,

$$\Box P \wedge \Diamond Q \subseteq P \,\mathcal{U}\, Q\ , \qquad (6)$$
$$P \,\mathcal{U}\, \mathsf{F} = \mathsf{F}\ . \qquad (7)$$

The definition of the while operator is copied verbatim from LTL:

$$P \,\mathcal{W}\, Q \stackrel{\text{def}}{=} \Box P \vee P \,\mathcal{U}\, Q\ .$$

From this we get

$$\Box P = P \,\mathcal{W}\, \mathsf{F}\ .$$

We have the following further properties of modal validity:

Lemma 7.1 1. $\models_m \Box P \rightarrow P \,\mathcal{W}\, Q$.

2. $\models_m P \,\mathcal{U}\, Q \leftrightarrow P \,\mathcal{W}\, Q \wedge \Diamond Q$.

3. $\models_m Q \rightarrow P \,\mathcal{U}\, Q$.

Proof: 1. is straightforward from the definition.

2. We have

$$\begin{array}{ll} & P \,\mathcal{W}\, Q \wedge \Diamond Q \\ = & \quad \{\!\!\{\ \text{definition } \mathcal{W}\ \}\!\!\} \\ & (\Box P \vee P \,\mathcal{U}\, Q) \wedge \Diamond Q \\ = & \quad \{\!\!\{\ \text{distributivity}\ \}\!\!\} \end{array}$$

$$
\begin{array}{ll}
 & (\Box P \wedge \Diamond Q) \vee (P \,\mathcal{U}\, Q \wedge \Diamond Q) \\
= & \quad \{\!\![\text{ by (5) }]\!\!\} \\
 & (\Box P \wedge \Diamond Q) \vee (P \,\mathcal{U}\, Q \wedge \mathsf{T} \,\mathcal{U}\, Q) \\
= & \quad \{\!\![\text{ monotonicity of } \mathcal{U} \text{ and boolean algebra }]\!\!\} \\
 & (\Box P \wedge \Diamond Q) \vee P \,\mathcal{U}\, Q \\
= & \quad \{\!\![\text{ by (6) and boolean algebra }]\!\!\} \\
 & P \,\mathcal{U}\, Q \, .
\end{array}
$$

3. is immediate from reflexivity of $\leq$.

■

Note that verification of 3. is the only place where reflexivity of $\leq$ is used. Transitivity and antisymmetry aren't used at all.

Let now $<$ be the strict part of $\leq$, i.e., assume

$$x < y \text{ iff } x \leq y \wedge \neg\, y \leq x \, .$$

Then the set of **maximal elements** of $N \subseteq M$ is

$$\mathsf{max}\, N \stackrel{\text{def}}{=} N \backslash (< N) \, ,$$

whereas the set of **minimal elements** is

$$\mathsf{min}\, N \stackrel{\text{def}}{=} N \backslash (N <) \, .$$

The LTL next-time operator can be viewed as the instantiation of our generalized one by the immediate successor relation on M, which is defined by

$$x \,\mathsf{suc}\, y \text{ iff } y \in \mathsf{min}\,(x <) \, .$$

Now we set

$$\bigcirc P \stackrel{\text{def}}{=} (\!(\,\mathsf{suc}\,)\!) P \, .$$

8 Ideals as a Temporal Base

We now study particular partial orders as used in denotational semantics as a way of talking about temporal succession. We recall that usually algebraic cpos are used as semantic domains. In such a domain each element is the supremum of a directed set of compact elements and can be identified with the ideal of all compact elements below it.

We view the compact elements as stations in computations that approximate a (possibly non-compact) element of the domain. The approximation order is used as the temporal order, since it reflects progress in information. Our generalized temporal setting serves to characterize sets of stations of computation by temporal formulae.

8.1 Basic Definitions

Assume a partially ordered set $(M, \leq)$ and $N \subseteq M$. A subset $N \subseteq M$ is **directed** if every finite subset of N has an upper bound in N. Equivalently, N is directed if $N \neq \emptyset$ and any two elements in N have a common upper bound in N. For $P \subseteq M$ we set

$$\mathsf{dir}\, P = \{D \subseteq P : D \text{ directed}\} .$$

An **ideal** of $(M, \leq)$ is a directed and downward closed subset, i.e., a $Q \in \mathsf{dir}\, M$ with $(\leq Q) \subseteq Q$. A **principal ideal** is an ideal of the form $(\leq x)$ for some $x \in M$. By $I(M)$ we denote the set of all ideals of M.

An element $x \in M$ is **compact** iff for every $D \in \mathsf{dir}\, M$ with $x \leq \sqcup D$ we have also $x \leq z$ for some $z \in D$. Equivalently, x is compact iff for every $I \in I(M)$ with $x \leq \sqcup I$ we have $x \in I$. $(M, \leq)$ is **algebraic** iff every element of M is the supremum of a directed set of compact elements. A non-compact element of M is then called a **limit point** or an **infinite element**.

The partial order $(M, \leq)$ is called a **complete** or a **cpo** iff every set $D \in \mathsf{dir}\, M$ has a least upper bound $\sqcup D$. With these notions one has (see e.g. [1, 3])

Theorem 8.1 Let $(M, \leq)$ be an ordered set.

1. The set $(I(M), \subseteq)$ ordered by set inclusion is an algebraic cpo, the compact elements being the principal ideals $(\leq x)$ for $x \in M$. The mapping $\iota : x \mapsto (\leq x)$ is an embedding of M into $I(M)$. In particular, $x \leq y$ iff $(\leq x) \subseteq (\leq y)$.

2. For every monotonic mapping $h : M \to P$ into a dir-complete set $(P, \leq)$ there is a unique continuous mapping $\overline{h} : I(M) \to P$ extending h, i.e., with $\overline{h}(\leq x) = h(x)$. $\overline{h}$ is given by $\overline{h}(I) = \sqcup\, h(I)$ for $I \in I(M)$; hence $\overline{h}(\leq D) = \sqcup\, h(D)$ for $D \in \mathsf{dir}\, M$.

The order $(I(M), \subseteq)$ is called the **ideal completion** of $(M, \leq)$.

Let us give another characterization of infinite ideals. Since an ideal with maximal element x is by directedness the principal ideal $(\leq x)$, an infinite (i.e., non-compact) ideal cannot have a maximal element. It is easy to show that we also have the reverse implication. So, an ideal J is infinite iff $\mathsf{max}\, J = \emptyset$.

8.2 Streams as Ideals

We now briefly show the above notions at work in the particular partial order of streams. Assume an alphabet A. As usual, A^* is the set of all finite words over A. By ε we denote the empty word, whereas concatenation is denoted by $\bullet$. A subset of A^* is called a **formal language**.

A word u is a **prefix** of a word v, written $u \sqsubseteq v$, iff there is a word w such that $u \bullet w = v$. The partial order $\sqsubseteq$ is even well-founded. Moreover, ε is the least element in this order. The corresponding strict-order is denoted by $\sqsubset$.

An ideal of $(A^*, \sqsubseteq)$ is then a prefix-closed language. Note that every ideal contains ε.

In $(A^*, \sqsubseteq)$, directedness has a special property:

Lemma 8.2 $D \subseteq A^*$ is directed w.r.t. $\sqsubseteq$ iff D is totally ordered by $\sqsubseteq$.

Thus, by prefix-closedness, an ideal is a set of words of increasing length "growing at the right end". This set may be finite or infinite. A simple example is, for $a \in A$, the infinite ideal

$$a^* = \{\varepsilon,\ a,\ a \bullet a,\ a \bullet a \bullet a,\ a \bullet a \bullet a \bullet a,\ \ldots\} .$$

For the special case of words under the prefix ordering, we therefore call the elements of $I(A^*)$ **streams** over A. The compact elements of $I(A^*)$ correspond to the elements of A^*, whereas the non-compact elements are precisely the (cardinally) infinite ideals. Hence, for countable A, the set $(I(A), \subseteq)$ has a countable basis of compact elements and therefore is countably algebraic.

The non-compact elements correspond to infinite sequences over A and hence we set

$$A^\omega \stackrel{\text{def}}{=} \{J \in I(A) : \mathsf{max}\, J = \mathsf{F}\} .$$

9 A Generalized Temporal Setting and Temporal Validity

Recall the usual setting of LTL (see. e.g. [9, 10]): the underlying structure is $\Sigma^\omega \times \mathbb{N}$ where Σ is a set of states and Σ^ω is the set of all infinite sequences of states. A pair (σ, n) denotes a particular point in the computation history given by $\sigma \in \Sigma^\omega$.

To move towards an order-theoretic setting we want to get rid of the explicit recourse to $\mathbb{N}$ here. This can be achieved by replacing the time point n by the initial segment $\sigma_0, \ldots, \sigma_{n-1}$; the time n can be retrieved as the length of that initial segment. Now we employ our view of streams, in particular of elements of Σ^ω, as ideals within A^*. Then a finite prefix of some stream in Σ^ω simply is an element of that stream. Hence an equivalent representation of the LTL structure is the set

$$\{(\sigma, s) : \sigma \in \Sigma^\omega, s \in \Sigma^* : s \in \sigma\} .$$

This form now generalizes directly to arbitrary partial orders.

Assume a partial order $(M, \leq)$. The elements of M are thought of as finite approximations of elements of some domain isomorphic to the ideal completion $(I(M), \subseteq)$ of $(M, \leq)$. A **stage (of computation)** is an element of the set

$$\text{STAGE} \stackrel{\text{def}}{=} \{(J, x) : J \in I(M) \wedge x \in J\} .$$

This corresponds to the usual definition of state in modal logic (see e.g. [2, 16]).

We note that by working with generalized stages $(J, [x, y])$ for $x, y \in J$ with $x \leq y$ one obtains an analogous generalization of interval temporal logic (see e.g. [15]). This is beyond the scope of the present paper, though.

It is convenient to extend the order $\leq$ to a relation between stages: we set

$$(J_1, x_1) \leq (J_2, x_2) \text{ iff } J_1 = J_2 \wedge x_1 \leq x_2 .$$

One checks immediately that this defines again a partial order. This relation allows a much simpler definition of the relevant notions than in [13], since it keeps the ideal part fixed and hence implies a relativization of the order to stages within the same ideal.

A **property** now is a set of stages, i.e., an element of $\mathcal{P}(\mathrm{STAGE})$. For the partial order $(\mathrm{STAGE}, \leq)$ all our definitions of the generalized temporal operators apply.

Following classical temporal logic, we define a different notion of validity for this setting. We assume that the underlying partial order $(M, \leq)$ has a least element $\perp$. Note that then every ideal contains $\perp$.

We call a property $P \subseteq \mathrm{STAGE}$ **temporally valid** on an ideal J, in signs $J \models_t P$, if P holds in the initial stage of J, i.e. if $(J, \perp) \in P$. We have

$$J \models_t \Box P \text{ iff } \mathit{lift}(J) \subseteq P \tag{8}$$

where

$$\mathit{lift}(J) \stackrel{\text{def}}{=} \{(J, x) : x \in J\} ,$$

since $\mathit{lift}(J) = (J, \perp) \leq$.

General **temporal validity** is defined by

$$\models_t P \text{ iff } \forall\, J \in I(M) : J \models_t P .$$

Of course we have

$$\models_m P \text{ implies } \models_t P .$$

A useful property is

$$\models_t P \Rightarrow Q \text{ iff } P \subseteq Q ,$$

shown by

$$\begin{array}{ll}
 & \models_t P \Rightarrow Q \\
\Leftrightarrow & \quad \{\!\!\{ \text{ definitions } \}\!\!\} \\
 & \forall\, J : J \models_t \Box(P \rightarrow Q) \\
\Leftrightarrow & \quad \{\!\!\{ \text{ by (8) } \}\!\!\} \\
 & \forall\, J : \mathit{lift}(J) \subseteq P \rightarrow Q \\
\Leftrightarrow & \quad \{\!\!\{ \text{ definitions } \}\!\!\} \\
 & \forall\, J : \forall\, x \in J : (J, x) \notin P \vee (J, x) \in Q \\
\Leftrightarrow & \quad \{\!\!\{ \text{ boolean algebra } \}\!\!\} \\
 & P \subseteq Q .
\end{array}$$

10 Temporal Validity and the Nexttime Operator

The nexttime operator $\bigcirc$ is a lot more tricky w.r.t. validity than the other modal and temporal operators.

First we note that from the definition of $\bigcirc$ it is clear that

$$J \models_t \Box \bigcirc \mathsf{T} \text{ implies } \max J = \emptyset .$$

This means that J is non-compact in $(I(M), \subseteq)$.

We have already seen that one half each of two standard axioms of LTL are even modally valid. The converses are qualified:

- $\forall\, P : J \models_t \neg \bigcirc P \;\Rightarrow\; \bigcirc \neg P$ iff
  $\forall\, x \in J : |J \cap x\,\mathsf{suc}\,| = 1.$
- $\forall\, P : J \models_t (\bigcirc P \;\rightarrow\; \bigcirc Q) \;\Rightarrow\; \bigcirc(P \;\rightarrow\; Q)$ iff
  $\forall\, x \in J : |J \cap x\,\mathsf{suc}\,| = 1.$

So these converses hold only for linear orders with a total successor relation.

For the induction axiom we get

- $\forall\, P : J \models_t (P \;\Rightarrow\; \bigcirc P) \;\rightarrow\; (P \;\Rightarrow\; \Box P)$ iff
  $\forall\, x \in J : J \cap (x \leq) = J \;\cap\; x\; \mathsf{suc}^*.$

This means that the axiom holds only in partial orders in which the upper cones are **reachable**, i.e., generated by suc.

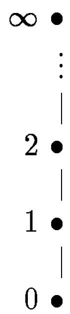
The weirdest beast, however, is the while axiom. We have

- $\forall\, P : J \models_t (Q \;\vee\; (P \;\wedge\; \bigcirc(P \;\mathcal{W}\; Q))) \;\Leftarrow\; P \;\mathcal{W}\; Q$ iff
  $\forall\, x \in J :\; J \cap x\,\mathsf{suc} \;\neq\; \emptyset \;\wedge\; J \;\cap\; (x \leq) \;\;=\;\; J \cap x\; \mathsf{suc}^*.$

So this holds only for partial orders which have no maximal elements and in which the order is generated by suc. For the converse we get

- $\forall\, P : J \models_t (Q \;\vee\; (P \;\wedge\; \bigcirc(P \;\mathcal{W}\; Q))) \;\Rightarrow\; P \;\mathcal{W}\; Q$ iff
  $\forall\, x \in J :\; J \cap (x \leq) \;\;=\;\; \{x\} \;\cup \bigcup_{y \in J \,\cap\, x\,\mathsf{suc}} J \cap (y \leq).$

Note that this condition is not equivalent to reachability. It is satisfied e.g. in the partial order



where the limit point ∞ is not reachable.

11 Behaviours and Refinement

11.1 Basic Definitions

We apply ideals to describe runs of systems. To model non-determinacy, we define a **behaviour** to be a set of ideals. An example of a partial order $(M, \leq)$ more general than the one of finite words under the prefix order is the set of all finite sets of partially ordered events under the initial-segment order.

It should be noted that using *sets* of ideals as behaviours allows only "trace-like" semantics in which there is no distinction between internal and external non-determinacy. Still our generalization of the temporal operators applies.

As our refinement relation between behaviours we choose inclusion, i.e., behaviour $\mathcal{T}$ **refines** behaviour $\mathcal{S}$ if $\mathcal{T} \subseteq \mathcal{S}$. To allow correct local refinements one therefore has to ensure monotonicity of all operations w.r.t. inclusion.

The infinite ideals of a behaviour are selected by

$$\mathsf{inf}\,\mathcal{B} \stackrel{\text{def}}{=} \{J \in \mathcal{B} : \max J = \emptyset\} \ .$$

11.2 Describing Behaviours by Snapshots

Generalizing from the particular domain of streams we want to characterize the ideals in a behaviour by certain sets of "relevant" or "admissible" approximations. For such a set $P \subseteq M$ we define by

$$\mathsf{ide}\,P \stackrel{\text{def}}{=} \{(\leq D) : D \in \mathsf{dir}\,P\}$$

the set of all ideals "spanned" by directed subsets of P. For the case of finite and infinite sequences over some alphabet a related notion occurs in [4]. Note that ide is monotonic w.r.t. inclusion. Further properties of ide are

Lemma 11.1 1. For $J \in I(M)$ and $Q \subseteq M$ we have

$$J \in \mathsf{ide}\,Q \quad \text{iff} \quad J \subseteq (\leq (J \cap Q)) \quad \text{iff} \quad J = (\leq (J \cap Q)) \ .$$

2. Consider $N, P \subseteq M$. Then $\mathsf{ide}\,(N \cup P) = \mathsf{ide}\,N \cup \mathsf{ide}\,P$.

For the proof see [12]. It should be noted, however, that ide only distributes through finite unions and hence is not "continuous". For an instance of this see Example 11.3.

In connection with safety issues one is interested in elements for which *all* finite approximations are admissible. The part of a snapshot set $P \subseteq M$ that is closed under finite approximations is

$$\mathsf{alw}\,P \stackrel{\text{def}}{=} \{x \in M : (\leq x) \subseteq P\} \ .$$

Note that alw does not distribute through union.

11.3 Streams and Snapshots

The set of streams spanned by a subset $P \subseteq A^*$ is

$$\mathsf{str}\,P \stackrel{\text{def}}{=} \mathsf{ide}\,P \ .$$

Note that it would not be adequate to work with the set $\mathsf{str}\,(\sqsubseteq P)$ instead of $\mathsf{str}\,P$. The reason is that by prefix-closure infinite substreams may "sneak" into a cone although it results from a language of mutually $\sqsubseteq$-incomparable words which represent systems with finite behaviour only.

Example 11.2 The language $0^* \bullet 1$ represents a behaviour with arbitrarily long but finite sequences of 0s terminated by the "explicit endmarker" 1. However, its prefix closure $(0^* \bullet 1)^{\sqsubseteq}$ contains the infinite ideal 0^* representing an infinite stream of 0s. ■

Using König's Lemma one can even show that for finite A *every* infinite cone contains an infinite stream. The general definition of ide omits these undesired streams.

We want to show now that str (and hence ide) does not distribute through general union:

Example 11.3 Take $U = 0^*$. Then $U = \bigcup_{i \in \mathbb{N}} 0^i$. However, $\mathsf{str}\, U = \{0^*\} \cup \{(0^i)^{\sqsubseteq} : i \in \mathbb{N}\}$, whereas $\bigcup_{i \in \mathbb{N}} \mathsf{str}\, 0^i = \{(0^i)^{\sqsubseteq} : i \in \mathbb{N}\}$. ■

Using the operations $\_^*$ and str we can also define the infinite repetition of words. If $U \subseteq A^*$ satisfies the Fano condition (or, equivalently, is prefix-free), i.e., the words in U are mutually incomparable w.r.t. $\sqsubseteq$, then we can define the behaviour U^ω, i.e., the set of streams which result from infinite repetition of words in U, as

$$U^\omega = \mathsf{inf}\, \mathsf{str}\, U^* \ .$$

Hence our order-theoretic definition of infinite repetition fits in well with concepts known from the theory of languages with infinite words (see e.g. [17]). This, the original characterization of the ide operation and the relation with [4] nurture

Feeling 1: ide corresponds to $\Box\Diamond$ in LTL.

Moreover, the informal characterization of the alw operation nurtures

Feeling 2: alw corresponds to $\boxed{\mathrm{i}}$ in interval temporal logic [15]).

We can now convert these feelings into theorems.

11.4 From Feelings to Theorems

Consider again an arbitrary partial order $(M, \leq)$. For a snapshot set $N \subseteq M$ we define the property

$$\underline{N} \stackrel{\text{def}}{=} \{(J, x) : J \in I(M) \wedge x \in L\} \ .$$

Feeling 1 is confirmed by

$$J \models_t \Box\Diamond\underline{N}$$

$\Leftrightarrow$ $\{\!|$ by (8) and definition of $\Box$ $|\!\}$

$$\mathit{lift}(J) \subseteq\ \leq \underline{N}$$

$\Leftrightarrow$ $\{\!|$ definition of $\underline{N}$ $|\!\}$

$$J \subseteq\ \leq (J \cap N)$$

$\Leftrightarrow$ $\{\!|$ by Lemma 11.1 $|\!\}$

$$I \in \mathsf{ide}\, N \ .$$

Feeling 2 is confirmed analogously with the "backward" operator $\Box^-$.

12 Conclusion

All these properties together clearly indicate that the classical LTL axioms are complete precisely for the set of linear partial orders that are freely generated by the successor relation, i.e., for structures in which the domain is isomorphic to the set of countable sequences over some basic set.

What we have achieved is a simple generalization of LTL. It is quite different, however, from branching time logics such as CTL/CTL* which allow quantification over the paths in the underlying graph.

It is more widely applicable than the embeddings of LTL into the sequential calculus [8] or relational algebra [7], since it is not based on a sequential composition operator. So it is directly usable for systems such as CSP or CCS which are not centered around a composition operator but have domain and hence partial order semantics.

The framework should also easily generalize to categories as domains [18].

There remain plenty of open questions:

- Can various properties of subsets of an ordered set such as linearity, reachability and infinity, be separated more cleanly by temporal formulas?
- What is a "good" class of domains for which a complete axiomatization analogous to the standard LTL one can be given?
- What is the precise relation to CTL/CTL*, TLA (see e.g. [5]) and other temporal logics as well as to general modal logic (see e.g. [2, 16])?

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