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Quantum stochastic resonance in parallel

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Abstract. A study of (aperiodic) quantum stochastic resonance (QSR) in parallel is put forward. By doing so, a generally stochastic input signal is fed into an array of parallel dissipative quantum two-level systems (TLS) and its integral response is studied against increasing temperature. The response is quantified by means of an information-theoretic measure provided by the rate of mutual information per element and, in addition, by the cross-correlation between the information-carrying input signal and the output response. For ohmic-like quantum dissipation, both measures exhibit QSR for biased two-level systems. Our prime focus here, however, is on the case with zero asymmetry between the two localized stable states. We then find that the mutual information measure exhibits QSR only for sufficiently strong dissipation ($\alpha > 3/2$), as measured by the dimensionless ohmic friction strength α. Moreover, the *mutual information measure* relates QSR within quantum linear response theory to the *signal-to-noise-ratio* (SNR), being independent of the input driving frequencies in this limit. In contrast, the *cross correlation measure* connects QSR to a genuine *synchronization phenomenon*. For a single symmetric TLS, aperiodic QSR is exhibited in the cross-correlation measure for a Gaussian exponentially correlated input signal for $\alpha > 1$ already. Upon feeding the aperiodic input signal into a parallel array of unbiased TLS's, QSR successively emerges above the critical ohmic dissipation strength $\alpha > 1/2$ with increasing number n of parallel units. Thus, QSR can occur *in parallel* despite the fact that it does not occur in each individual, unbiased, TLS for $\alpha < 1$. This paradoxical phenomenon—which can be tested with an array of bistable superconducting quantum interference devices—constitutes a true quantum effect: it is due to the power-law dependence on temperature of the tunnelling rate and the stochastic linearization of quantum fluctuations with increasing number of parallel units.

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1. Introduction

The phenomenon of *stochastic resonance* (SR) constitutes a nonlinear noise-mediated cooperative phenomenon wherein feeble information of a deterministic signal can be enhanced in the presence of an optimal dose of noise. Since its inception in 1981, SR has been demonstrated in numerous systems including bistable elements such as tunnel diodes, superconducting quantum interference devices (SQUIDs), autocatalytic chemical reaction schemes, sensory neurons, or communication devices, to name only a few (the interested reader is referred to the popular reviews [1]–[3], or the comprehensive survey in [4]). Although the basic SR mechanism is by now well understood, there remain a number of challenging unsolved problems. In particular, most of the research thus far predominantly focused on classical stochastic systems. The borderline between the classical world and the quantum domain has been crossed only recently, in order to account for genuine, tunnelling-induced quantum mechanical SR-effects [5]–[9]. Moreover, these few prior studies of *quantum stochastic resonance* (QSR) have all been restricted to the *conventional* definition of SR, i.e. to stochastic resonance with a *periodic* input signal. The subject of *aperiodic* SR, i.e. stochastic resonance in the presence of a wide-band random input signal experiences a flurry of activity in the context of classical neuronal systems. The corresponding response has been quantified either by information-theoretic, or by spectral cross-correlation measures [10]–[18].

In this work, our basic challenge is to move from the classical situation and to study the quantum mechanical version of *parallel* information transfer of an aperiodic input-signal [10, 14] through a parallel array of bistable quantum systems, being typified by quantum two-level systems (TLS), see figure 1. As such, this study involves an interplay among (i) quantum dissipative dynamics, (ii) information theory aspects and (iii) nonequilibrium statistical mechanics. A certain amount of interdisciplinary knowledge is thus required which will be provided in subsequent units: after having set up our model (section 2) we derive in section 3 the result for the rate of mutual information in arrays of uncoupled TLS's. Aperiodic QSR with respect to the input–output cross-correlation measure is analyzed in section 4. An outlook together with our conclusions is presented in section 5.

2. Set up of model dynamics

In the following we review prominent results of the theory of quantum dissipation [19]–[22] as needed to set up the model dynamics. We consider an array of uncoupled quantum two-level systems which are subjected to a common, generally random classical signal, $f(t)$, of vanishing statistical average. Moreover, each individual TLS is bilinearly coupled to a separate heat bath at a common temperature. The total Hamiltonian for a single TLS element coupled to the bath reads within the *tunnelling* or localized representation [19]–[22]

$$\hat{H}(t) = -\frac{1}{2}\epsilon(t)\hat{\sigma}_z + \frac{1}{2}\hbar\Delta\hat{\sigma}_x - x_0\hat{\sigma}_z\sum_\lambda \kappa_\lambda(b_\lambda^+ + b_\lambda) + \sum_\lambda \hbar\omega_\lambda\left(b_\lambda^+ b_\lambda + \frac{1}{2}\right). \quad (1)$$

Herein $\epsilon(t) = \epsilon_0 + 2x_0 f(t)$ denotes a time-dependent energy bias between two localized states. This driven spin-boson Hamiltonian describes the reduced quantum tunnelling dynamics in an asymmetric double-well potential with minima located at $x_{min} = \pm x_0$ [19]–[22] with the corresponding time-dependent well-asymmetry denoted by $\epsilon(t)$. The boson operators b_λ^+, b_λ

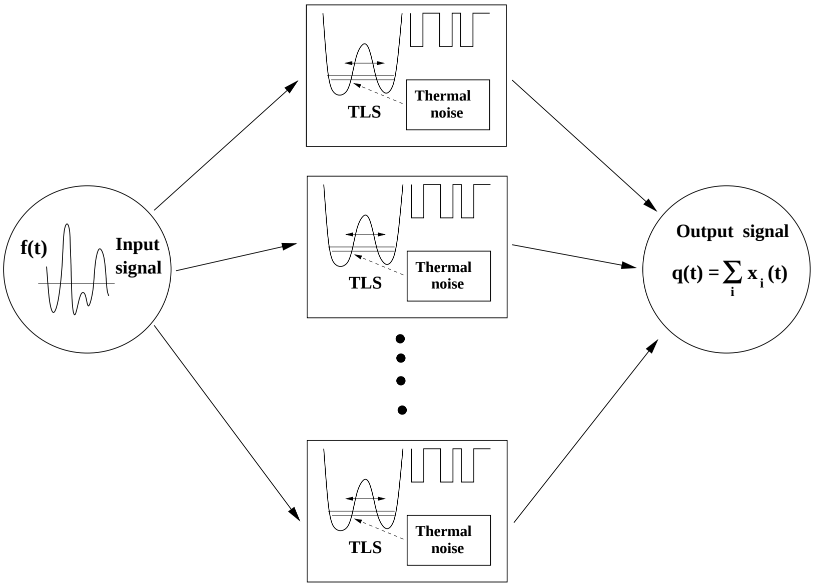

Figure 1. Quantum stochastic resonance in parallel: a generally stochastic input information signal is fed into a parallel array of uncoupled bistable, dissipative quantum systems, being modelled by two-level systems. The output information $q(t)$ corresponds to the combined sum of individual TLS responses. Note that the two level systems become mutually dependent via their common input signal $f(t)$.

correspond to normal mode oscillators of the thermal bath with frequencies ω_λ. The operators $\hat{\sigma}_z, \hat{\sigma}_x$ denote the usual Pauli matrices. The tunnelling dynamics itself can be characterized by the time-dependent position operator $\hat{x}(t) = x_0\hat{\sigma}_z(t)$. Furthermore, $\hbar\Delta$ in (1) is the tunnelling matrix element between the two lowest-lying energy levels. The effect of the thermal bath is captured by an *operator* random force $\hat{\xi}(t) = \sum_\lambda \kappa_\lambda(b_\lambda^+ \mathrm{e}^{\mathrm{i}\omega_\lambda t} + b_\lambda \mathrm{e}^{-\mathrm{i}\omega_\lambda t})$. Due to the inherent Gaussian statistics of the harmonic bath, its statistical properties are determined by the complex-valued autocorrelation function [19]–[22]

$$\langle \hat{\xi}(t)\hat{\xi}(0)\rangle_\beta = \frac{\hbar}{\pi}\int_0^\infty J(\omega)[\coth(\beta\hbar\omega/2)\cos(\omega t) - \mathrm{i}\sin(\omega t)]\mathrm{d}\omega. \quad (2)$$

Here, the spectral density $J(\omega) = (\pi/\hbar)\sum_\lambda \kappa_\lambda^2\delta(\omega-\omega_\lambda)$ of the thermal bath has been introduced, $\langle\ldots\rangle_\beta$ denotes the thermal average wherein $\beta = 1/k_BT$ denotes the inverse temperature. We assume that $J(\omega)$ acquires an ohmic form, i.e. $J(\omega) = (2\pi\hbar/4x_0^2)\alpha\omega\mathrm{e}^{-\omega/\omega_c}$. The dissipation parameter α quantifies the dimensionless viscous friction strength and ω_c characterizes the physically relevant exponential cut-off of the spectral density. The driving force $f(t)$ plays the role of an information-carrying *input signal*. For instance, in the case of SQUIDs the input signal corresponds to an applied magnetic flux variation whilst the *output* relates to the total magnetic flux [23].

This two-level approximation for the tunnelling dynamics is well justified at low temperatures $k_BT \ll \hbar\omega_g$ and for a time-dependent bias $|\epsilon_0 + 2x_0f(t)| \ll \hbar\omega_g$, where $\hbar\omega_g$ measures the energy splitting between the lowest tunnel doublet and the closest higher-lying excited state in the full bistable double well.

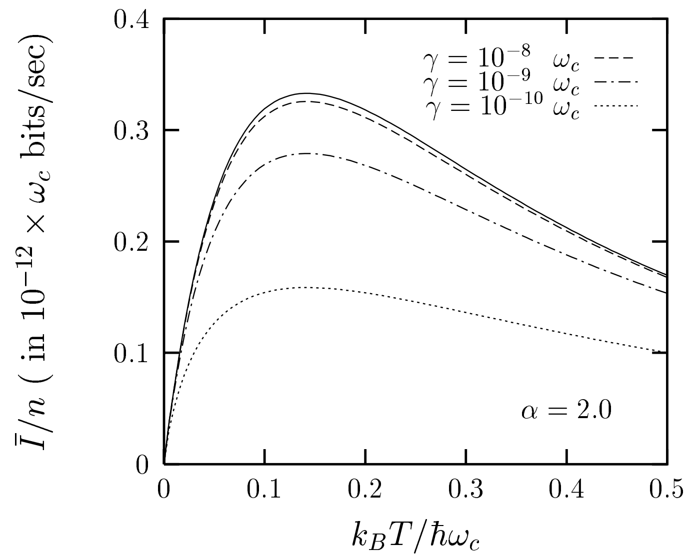

Figure 2. Average rate of mutual information $\bar{I}/n$ plotted against the scaled temperature in an array of $n = 10^3$ symmetric TLS's. The different curves correspond to differing signal bandwidths γ of a Gaussian exponentially correlated input signal $f(t)$ (Ornstein–Uhlenbeck process). The dimensionless parameter values used are: friction strength $\alpha = 2.0$, tunnelling coupling $\Delta = 10^{-4}\omega_c$, and strength of input signal variance $x_0A_0 = 10^{-2}\hbar\omega_c$. The solid curve compares these findings against the channel informational capacity per element, C_n/n; see text.

With SR generically operating in the *overdamped* regime, we consider the TLS quantum dynamics in the *incoherent* regime where the population dynamics of the localized states obey a nonstationary Markovian dynamics. This description holds true for ohmic friction at arbitrary temperature if the tunnelling coupling is small, i.e. $\Delta \ll \omega_c$, and the coupling to the heat bath is sufficiently strong, $\alpha > 1/2$ [22]. The approximation in addition covers the regime at smaller dissipation strengths $\alpha < 1/2$, if only the temperature is sufficiently high, i.e. $k_BT \gg \hbar\Delta$ [19, 21, 22]. As a consequence, the localized populations $P_\pm(t) = (1 \pm \langle\sigma_z(t)\rangle_\beta)/2$ obey the balance equations [6], [22], [24]–[26]

$$\frac{\mathrm{d}P_+(t)}{\mathrm{d}t} = -W_+(t)P_+(t) + W_-(t)P_-(t),$$
$$\frac{\mathrm{d}P_-(t)}{\mathrm{d}t} = -W_-(t)P_-(t) + W_+(t)P_+(t), \qquad (3)$$

with the time-dependent relaxation rates governed by the golden rule result

$$W_\pm(t) = \frac{1}{2}\Delta^2 \int_0^\infty \mathrm{d}\tau \exp[-Q'(\tau)] \cos\left[Q''(\tau) \pm \frac{1}{\hbar}\int_{t-\tau}^{t} \epsilon(t')\mathrm{d}t'\right]. \qquad (4)$$

The functions $Q'(t)$ and $Q''(t)$ in (4) denote the real and imaginary parts of the bath correlation function, respectively, i.e. [21]

$$Q'(t) + \mathrm{i}Q''(t) = \frac{4x_0^2}{\hbar^2}\int_0^t \mathrm{d}t_1 \int_0^{t_1} \langle\hat{\xi}(t_2)\hat{\xi}(0)\rangle_\beta \mathrm{d}t_2 + \mathrm{i}\lambda t,$$

where $\hbar\lambda = 4x_0^2 \int_0^\infty \mathrm{d}\omega J(\omega)/\pi\omega$ denotes the bath reorganization energy [25]. For the situation considered herein, the function $Q(t)$ can be evaluated in closed analytical form to yield [21, 27]

$$Q''(t) = 2\alpha \arctan(\omega_c t),$$
$$Q'(t) = 2\alpha \ln\left\{\sqrt{1+\omega_c^2 t^2}\frac{\Gamma^2(1+\kappa)}{|\Gamma(1+\kappa+\mathrm{i}\omega_\beta t)|^2}\right\}. \tag{5}$$

In (5), $\Gamma(z)$ denotes the complex gamma-function, $\omega_\beta = k_B T/\hbar$, and $\kappa = \omega_\beta/\omega_c$. Note that in the limit of *adiabatic* driving varying on a time-scale τ_f such that both $\omega_c\tau_f, \alpha k_B T\tau_f/\hbar \gg 1$, the time-dependent transition rates $W_\pm(t)$ follow the *instantaneous* value of the bias $\epsilon(t)$. In this case, the relaxation rates $W_\pm(t)$ obey the detailed balance condition in the form $W_+(t) = \mathrm{e}^{-\epsilon(t)/k_B T} W_-(t)$. Moreover, at extremely low temperatures, $\pi k_B T \ll \hbar\omega_c$, and a small bias, $\epsilon_0 \ll \hbar\omega_c$, one arrives from (4), (5) at the well known [19]–[21], [28] analytical approximation for the static relaxation rates $W_\pm(\epsilon_0)$,

$$W_\pm(\epsilon_0) = \frac{\Delta^2}{4\omega_c\,\Gamma(2\alpha)}\left(\frac{2\pi k_B T}{\hbar\omega_c}\right)^{2\alpha-1}\left|\Gamma\left(\alpha + \mathrm{i}\frac{\epsilon_0}{2\pi k_B T}\right)\right|^2 \exp(\mp\epsilon_0/2k_B T), \tag{6}$$

where $\Gamma(z)$ is the complex gamma-function. This result is applicable for any value of the viscous friction strength α.

It is worth noting that the considered incoherent limit for the tunnel dynamics of driven, dissipative TLS allows for an effective *quasiclassical* interpretation in terms of a classical random telegraph process. Put differently, the position operator $\hat{x}(t)$ assumes effectively a classical two-state process $\hat{x}(t) \to x(t) = \pm x_0$. Its transition rates, however, are governed by the *quantum expressions* in (4); see also appendix 1. As such, the model presents the quantum analogue of the classical aperiodic SR-investigation in [14].

3. Mutual information

We next consider the transfer of information from a random aperiodic classical input signal $f(t)$ through a parallel array consisting of quantum two-level systems as depicted in figure 1. In doing so, we consider the observable for the sum of individual TLS responses, i.e.

$$\hat{q}(t) = \sum_{i=1}^{n} \hat{x}_i(t). \tag{7}$$

Within the considered quasiclassical approximation, any quantum coherence can safely be neglected (incoherent quantum dynamics), and consequently the output and its sum become classical objects, i.e.,

$$\hat{q}(t) \to q(t) = \sum_{i=1}^{n} x_i(t).$$

Our focus here concerns the rate of mutual information between the summed output $q(t)$ and the aperiodic input signal $f(t)$. The average amount of mutual information per unit time [29] (or the transinformation rate) between two continuous-time random processes $f(t)$ and $q(t)$, with $t \in [0,\ \mathcal{T}]$, is defined by the double functional integral [30]:

$$\bar{I}(q:f) \equiv \lim_{\mathcal{T}\to\infty}\frac{1}{\mathcal{T}}\int\int Df(t)Dq(t)P[f(t),q(t)]\log_a\frac{P[f(t),q(t)]}{P[f(t)]P[q(t)]}. \tag{8}$$

$P[f(t), q(t)]$ denotes the joint probability density functional for the random processes $q(t)$ and $f(t)$; $P[f(t)]$ is the *a priori* given probability density functional of the signal, and $P[q(t)] = \int Df(t)P[f(t), q(t)]$ is the probability density functional of the output. Moreover, depending on the basis a of the logarithm in (8) the transinformation rate $\bar{I}$ is measured in binary units, *bits/s*, i.e. $a = 2$, natural units *nats/s* ($a = e$), or *digits/s (a=10)*.

In the absence of any external driving, the integral output signal $q(t)$ can be described as a sum of identical independent random telegraph noises with signal-dependent transition rates (4). Note that due to the *common* signal $f(t)$, an array of initially uncoupled TLS's becomes statistically dependent through the common input information $f(t)$. In view of the subadditivity of the mutual information one finds that

$$\bar{I}(q:f) \leq n\bar{I}_0(x:f), \tag{9}$$

where $\bar{I}_0(x:f)$ is the rate of mutual information for a single TLS element. Thus, the average amount of mutual information per element cannot exceed the one for a single element $\bar{I}_0$. The focus of this work is on the situation with many elements. Then, in absence of the information signal $f(t)$ we can invoke the the central limit theorem to treat the output signal $q(t)$ approximately as a Gaussian process. The information-carrying signal $f(t)$ is assumed to be approximately a Gaussian process as well. Then, by addressing mainly the case of *weak* signals, $f(t)$, it follows that the integral output $q(t)$ is approximated by Gaussian statistics as well. Consequently, the rate of mutual information between $q(t)$ and $f(t)$ is governed by a nontrivial result due to Pinsker [31] for the transinformation rate between two stationary Gaussian processes [32], reading

$$\bar{I}(q:f) = -\frac{1}{2\pi}\int_0^\infty \log_a\left(1 - |\rho(\omega)|^2\right)\mathrm{d}\omega, \tag{10}$$

where

$$\rho(\omega) = \frac{S_{qf}(\omega)}{\sqrt{S_{qq}(\omega)}\sqrt{S_{ff}(\omega)}} \tag{11}$$

denotes the so-termed coherence function and $S_{qf}(\omega)$ and $S_{qq}(\omega)$ are the cross-spectral power density and the output spectral power density, respectively. For weak adiabatic driving $f(t)$, one obtains—in close analogy to the classical case [14]—from quantum linear response theory, cf appendix, the result

$$\begin{aligned} S_{qq}(\omega) &= n^2|\tilde{\chi}(\omega)|^2 S_{ff}(\omega) + nS_{xx}^{(0)}(\omega), \\ S_{fq}(\omega) &= n\tilde{\chi}(\omega)S_{ff}(\omega), \end{aligned} \tag{12}$$

where $\tilde{\chi}(\omega)$ is the linear susceptibility of a single TLS. Equation (12) allows one to recast (10) into the more familiar form [33]

$$\bar{I}(q:f) = \frac{1}{2\pi}\int_0^\infty \log_a\left(1 + n\frac{|\tilde{\chi}(\omega)|^2 S_{ff}(\omega)}{S_{xx}^{(0)}(\omega)}\right)\mathrm{d}\omega. \tag{13}$$

Upon close inspection, (13) just coincides with the celebrated Shannon's formula for the rate of transinformation across a Gaussian, memory-free channel [29]. Here, Shannon's formula is applied to the *filtered* Gaussian signal $s(t) = n\int_{-\infty}^t \chi(t-\tau)f(\tau)\mathrm{d}\tau$. By use of the rms amplitude A_0, i.e. $A_0^2 = \int_{-\infty}^\infty S_{ff}(\omega)\mathrm{d}\omega/2\pi$, one can recast (13) by use of (A.7) into the appealing form

$$\bar{I}(q:f) = \frac{1}{2\pi}\int_0^\infty \log_a\left(1 + \frac{n}{\pi A_0^2}SNR(\omega)S_{ff}(\omega)\right)\mathrm{d}\omega, \tag{14}$$

where $SNR(\omega)$ is the signal-to-noise ratio (SNR) for a single TLS driven by a weak periodic input perturbation at angular frequency ω. In the considered situation of weak adiabatic signals $SNR(\omega)$ does not depend on ω, cf (A.8). Therefore, by use of the inequality, $\log_a(1+x) \leq x/\ln a$, we obtain

$$\bar{I}(q:f) \leq C_n := n\ SNR/(2\pi \ln a). \tag{15}$$

As a result, we find that the maximal achievable amount of information being transmitted by the parallel array is approximately determined by the conventional signal-to-noise ratio, *independently* of the spectrum of the information carrying input signal.

To gain further insight, we next model the stationary Gaussian input signal $f(t)$ by an exponentially correlated process (a so-called Ornstein–Uhlenbeck process) with zero average and autocorrelation

$$\langle f(t+\tau)f(t)\rangle_f = A_0^2 \mathrm{e}^{-\gamma|\tau|}. \tag{16}$$

With the decay rate γ obeying both, $\gamma \ll \omega_c, \alpha k_B T/\hbar$, we stay within the regime of adiabatic driving. The signal's power spectrum is clearly of Lorentzian shape, i.e.,

$$S_{ff}(\omega) = 2A_0^2\gamma/(\gamma^2+\omega^2). \tag{17}$$

Upon combining (17) and (A.8) in (14) we arrive at the main result

$$\bar{I}(q:f) = \frac{\gamma}{2\ln a}\left[\sqrt{1+\frac{2n}{\pi}\frac{SNR}{\gamma}}-1\right] = \begin{cases} (n\gamma SNR)^{1/2}/(\sqrt{2\pi}\ln a) : \gamma \ll n\ SNR \\ n\ SNR/(2\pi\ln a) : \gamma \gg n\ SNR \end{cases}. \tag{18}$$

Note that for input signals of small bandwidth, $\gamma \ll nSNR$, the transinformation rate becomes proportional to both the *square root* of SNR and to the *square root* of the signal bandwidth γ. The maximal (versus temperature) transinformation rate consequently coincides with the maximum of the *signal-to-noise ratio* measure. The position of this maximum T^I_{max} depends neither on the signal bandwidth γ, nor on the number n of elements in the parallel array. We also observe that no saturation in the temperature dependence of the information flow—the so-called 'stochastic resonance without tuning' [10]—occurs at $n \to \infty$. Moreover, with increasing bandwidth γ the rate of mutual information (18) increases *monotonically* and achieves the upper boundary C_n at $\gamma \gg n\ SNR$. Thus, the quantity C_n provides the informational capacity [29] of the *whole* array.

As a consequence of this analysis, the conditions for occurrence of aperiodic QSR—being quantified by the mutual information transmission—are essentially identical to those for conventional QSR, being quantified by the SNR-measure [1]–[4]. By use of the rate expression in (6) and the result for SNR in (A.8), its temperature dependence is determined by

$$SNR \propto T^{2\alpha-3}/\cosh(\epsilon_0/2k_BT). \tag{19}$$

Thus, the mutual information per unit time does *not* exhibit QSR in unbiased systems (i.e. $\epsilon_0 = 0$) if $\alpha < 3/2$. In this regime, QSR requires a finite bias $\epsilon_0 \neq 0$. However, QSR does also occur for unbiased systems *if* $\alpha > 3/2$. In this case, it is necessary to go beyond the low temperature approximation in (6) by using the full result for the incoherent quantum rates in (4) and (5). This is in agreement with conventional QSR, as shown previously in [35]. The results for the transinformation rate per element are depicted in figure 2 for a large ensemble $(n = 10^3)$ of unbiased parallel TLS as a function of differing bandwidths γ for a viscous friction strength of $\alpha = 2$. The solid line shows the result for the averaged informational capacity C_n/n : this limit is approached rather quickly $(\gamma/\omega_c > 10^{-8})$ as the bandwidth parameter increases. The maximal value assumed by $\bar{I}/n$ increases monotonically with increasing γ, cf (18), and saturates at the value of $SNR/(2\pi \ln 2)$ for a single bistable element.

Most importantly, the transinformation rate for aperiodic, parallel QSR connects this phenomenon with conventional QSR for a single unit as characterized by the SNR measure. Because the maximum position at $T^{\bar{I}}_{max}$ does *not* depend on the bandwidth parameter γ, the transinformation measure *does not characterize parallel (aperiodic) QSR as a synchronization phenomenon*.

4. Aperiodic QSR as synchronization phenomenon

In search for a quantification of aperiodic QSR as a synchronization phenomenon we consider the cross-correlation coefficient [10, 14, 17, 18]

$$\rho = \frac{\int_0^\infty \mathrm{Re} S_{qf}(\omega)\mathrm{d}\omega}{\sqrt{\int_0^\infty S_{qq}(\omega)\mathrm{d}\omega}\sqrt{\int_0^\infty S_{ff}(\omega)\mathrm{d}\omega}}. \tag{20}$$

It worth recalling that conventional aperiodic classical SR has originally been introduced for the Fitzhugh–Nagumo model of the neuronal dynamics [10]. For this model, it was shown that the cross-correlation coefficient ρ and the rate of mutual information $\bar{I}$ provide equivalent measures. As we show below, however, for the case of aperiodic QSR in parallel these two measures no longer provide the same information, but behave instead rather distinctly. Within quantum linear response theory, the application of equations (12), (17), (A.5), and (20) yields for the cross-correlation coefficient the result

$$\rho_n = \frac{W(\epsilon_0)}{\gamma + W(\epsilon_0)} \frac{\sqrt{n}c(T)}{\sqrt{1 + nc^2(T)\frac{W(\epsilon_0)}{\gamma + W(\epsilon_0)}}}, \tag{21}$$

where $c(T) = x_0 A_0/k_B T \cosh(\epsilon_0/2k_B T)$ and $W(\epsilon_0) := W_+(\epsilon_0) + W_-(\epsilon_0)$.

4.1. Aperiodic QSR for a single element

Note that (21) is valid also for the case $n = 1, \rho_1 := \rho$, i.e. for QSR in a single element. The corresponding result can be simplified upon noting that $c(T) \ll 1$, yielding

$$\rho \approx \frac{1}{k_B T} \frac{x_0 A_0}{\cosh(\epsilon_0/2k_B T)} \frac{W(\epsilon_0)}{\gamma + W(\epsilon_0)} . \tag{22}$$

With the focus being on unbiased TLS's the analysis of (22) shows that aperiodic QSR for the cross-correlation measure already occurs for $\alpha > 1$. Therefore, with $1 < \alpha < 3/2$ the input–output cross-correlations can be optimized by applying an appropriate dose of thermal noise whilst for the mutual information measure QSR only occurs for $\alpha > 3/2$. The maximal value for ρ is assumed at a temperature

$$T^{\rho}_{max} = \frac{\hbar\omega_c}{2\pi k_B}[2(\alpha - 1)]^{1/(2\alpha-1)}(\gamma/w_0)^{1/(2\alpha-1)}, \tag{23}$$

where $w_0 = \Delta^2\Gamma^2(\alpha)/2\omega_c\Gamma(2\alpha)$. The substitution of (23) into (6) yields the relation

$$W_\pm(T^{\rho}_{max}) = (\alpha - 1)\gamma, \ \alpha > 1. \tag{24}$$

This result inherits the condition for an *approximate* matching between the time-scales of (incoherent) tunnelling events and the autocorrelation time of the input signal at maximal cross

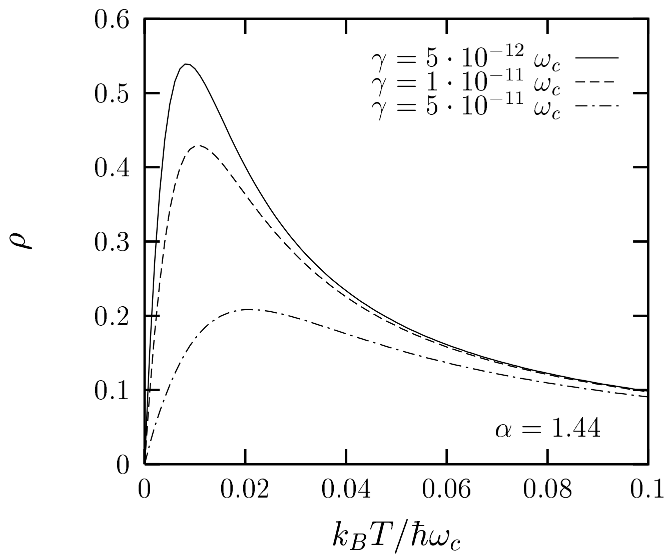

Figure 3. Aperiodic quantum stochastic resonance as a synchronization phenomenon: the cross correlation measure ρ for a single unbiased TLS unit is depicted versus the scaled temperature for differing bandwidth parameters γ of an Ornstein–Uhlenbeck input signal at an ohmic friction strength of $\alpha = 1.44$; its maximal value now exhibits a distinct dependence on the chosen value of inverse noise correlation time γ. The remaining parameter values are: $\Delta = 10^{-4}\omega_c$, $x_0 A_0 = 10^{-2}\hbar\omega_c$.

correlation. Thus, we indeed find that the cross-correlation coefficient ρ characterizes aperiodic QSR as a genuine *synchronization phenomenon*!

The corresponding bell-shaped aperiodic QSR behaviour is depicted in figure 3 for a dissipative strength of $\alpha = 1.44$; this specific value is of relevance for the observed experimental SQUID dynamics as investigated in [36] in absence of driving. Naturally, it is expected that this novel aperiodic QSR phenomenon can be verified experimentally as well. Note that for this value of ohmic dissipative strength no maximum for the mutual information rate occurs. Moreover, the cross correlation measure for synchronization is an increasing function versus decreasing bandwidth strength γ; see figure 3. This latter result is in accordance with conventional (periodic) SR where the maximum of spectral amplification increases with decreasing driving frequency for a periodic input signal [37].

4.2. Parallel aperiodic QSR

The case of a large ensemble of parallel units, cf figure 1, with $n \gg 1$ is even more striking. Then, upon combining (6) with (21) we find that QSR in the cross correlation measure emerges already for $\alpha > 1/2$. Put differently, a large ensemble of identical independent, unbiased TLS's is able to exhibit QSR whilst a single element does not. This paradoxical result is depicted in figure 4 for the case with $\alpha = 0.9$. The bottom curve in the figure depicts the result for a single *symmetric* TLS, where in agreement with the previous analysis no QSR occurs. The QSR phenomenon successively occurs with increasing number of parallel units. This surprising

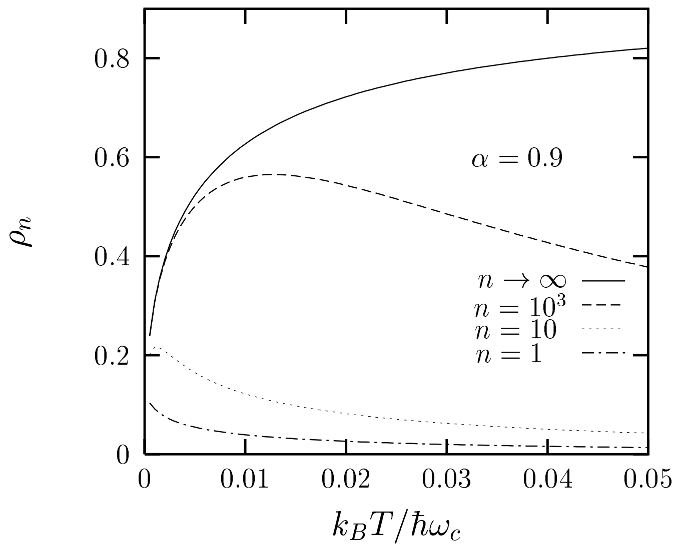

Figure 4. Parallel aperiodic quantum stochastic resonance: the cross-correlation measure between a stochastic Ornstein–Uhlenbeck input signal process and the integral output in arrays containing a differing number of elements n is depicted versus the scaled temperature. The corresponding parallel arrays are composed of symmetric dissipative TLS's at the ohmic frictional strength $\alpha = 0.9$, and $\Delta = 10^{-4}\omega_c$, $\gamma = 10^{-9}\omega_c$, $x_0 A_0 = 10^{-3}\hbar\omega_c$. While no QSR occurs for a single TLS unit, an increasing number of elements n in the parallel array provides the stochastic resonance effect.

phenomenon is rooted in the diminishing role of internal, individual fluctuations of $x_i(t)$ in a large ensemble of parallel elements, cf (12). The phenomenon is due to a combination of this fact together with the *power law* dependence on temperature of the incoherent *quantum* rates in (6); as such the effect is of genuine quantum origin.

Next we consider the limit $n \to \infty$ in (21), i.e.,

$$\rho_{n\to\infty} \approx \sqrt{\frac{W(\epsilon_0)}{\gamma + W(\epsilon_0)}} \,. \tag{25}$$

Using the result for the quantum rate in (6) we find that in the considered limit the cross-correlation ρ increases *monotonically* with increasing temperature for $\alpha > 1/2$, until ρ reaches its maximal value $\rho \approx 1$ at $W(\epsilon_0) \gg \gamma$. This behaviour of growing cross-correlation with increasing temperature, which saturates at large noise dose, has been termed in the literature SR without tuning [10]; it can be explained in terms of a 'stochastic linearization' [14, 17] as $n \to \infty$.

5. Summary and conclusions

The main primer of this work has been the investigation of quantum stochastic resonance through an array of independent, parallel quantum two level systems. Before concluding it may be useful to recapitulate again our main ideas, involved assumptions and main findings. Our idea has been to investigate the transduction of information for a generally aperiodic (stochastic) input signal through an array of parallel bistable quantum systems—all being in contact with a thermal, identical environment—which we modelled in terms of ohmic-like, dissipative two level systems. Then, we proceeded by applying the rate of transinformation by Shannon's formula (13). The main assumptions used in this work, being valid in many practical situations, are: (i) use of an incoherent quantum dynamics for individual TLS systems, (ii) weak adiabatic signals $f(t)$ with Gaussian statistics, and, for the case of QSR in parallel, (iii) a large number n of elements in the array.

Explicit findings have been obtained for stochastic signals from an exponentially correlated Gaussian process, such as the insightful result for the rate of mutual information in (18). This very result demonstrates unambiguously that the rate of mutual information is determined by conventional SNR for a single element with external cosinusoidal driving. This statement is also valid for classical systems. Henceforth, we have established a universal connection between SR in parallel and the SNR characterization of conventional SR. Because the maximum position of the rate of transinformation does not depend on the characteristic time scale of the input signal, this measure does *not* quantify QSR as a synchronization effect. Moreover, the averaged amount of transinformation per one element per unit time, $\bar{I}/n$, is generally less than that of the single element, $\bar{I}_0$. The main reason for this behaviour is related to the fact that the dynamical behaviour of an array of independent (in the absence of the input signal) TLS's become statistically *dependent* when a common signal is present; the theoretical maximum of $\bar{I}_{max}/n \approx \bar{I}_0$ is assumed only when the elements in the array become *completely* uncorrelated. Therefore, the introduction of additional mutual coupling among the TLS's will only result in a further *deterioration* of mutual information between input and output.

In contrast, the cross correlation measure ρ_n indeed characterizes QSR as a synchronization phenomenon. For weak adiabatic Ornstein–Uhlenbeck signals, it was demonstrated that the input–output cross-correlation can be optimized by a corresponding dose of thermal noise in a single symmetric TLS if $\alpha > 1$. The study of the cross-correlation for QSR in a parallel array revealed a new paradoxical phenomenon: the appearance of QSR in ensembles of *independent* elements which by themselves all do not display QSR, cf figure 4. This surprising behaviour is the result of a synergetic interplay between classical stochastic linearization [17, 14, 16] and the inherent power-law dependence on temperature of the quantum rates. In generalizing the experimental setup used in [38, 39] for detecting SR in a single SQUID element, this result can possibly be examined by use of a parallel array of SQUIDs of the type put forward recently by Wernsdorfer *et al* [40].

In conclusion, we can assert that the measure of mutual information overtakes within the theme of *aperiodic* (quantum) stochastic resonance the role of SNR, whilst the cross-correlation coefficient overtakes the role of the spectral amplification measure [37]. Both the cross-correlation coefficient and the spectral amplification characterize QSR as a genuine noise-optimized, averaged synchronization measure. Moreover, our novel findings for aperiodic QSR in single elements and in parallel arrays are expected to be become experimentally observable in mesoscopic bistable quantum systems such as tunnelling of magnetic flux in rf-driven SQUIDs [36, 38, 39], tunnelling of impurities in mesoscopic bismuth wires [41], or in proton-transferring molecular complexes, as well as in parallel arrangements of such systems. Likewise, the results herein may also be of importance when nature optimizes electron transfer reactions due to nonequilibrium noise influences in biological complexes.

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Appendix 1: Quantum linear response theory

In this appendix we provide the readers with results of quantum linear response theory (LRT) as they are of relevance for this study on aperiodic QSR. Within the framework of LRT, the deviation of the thermal average $\langle \delta\hat{x}(t)\rangle_\beta = \langle \hat{x}(t)\rangle_\beta - \bar{x}$, from the *equilibrium* value $\bar{x}$ due to the external perturbation $f(t)$ is

$$\langle \delta\hat{x}(t)\rangle_\beta = \int_{-\infty}^{t} \chi(t-t')f(t')\mathrm{d}t', \tag{A.1}$$

where $\chi(t)$ denotes the response function. The linear susceptibility is defined as the one-sided Fourier transform $\tilde{\chi}(\omega) = \int_0^\infty \mathrm{e}^{\mathrm{i}\omega t}\chi(t)\mathrm{d}t$. Furthermore, the spectral power of fluctuations reads $S_{xx}(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega\tau}\bar{C}_{xx}(\tau)\mathrm{d}\tau$, with

$$\bar{C}_{xx}(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_0^T \langle \delta\hat{x}(t)\delta\hat{x}(t+\tau) + \delta\hat{x}(t+\tau)\delta\hat{x}(t)\rangle_\beta \mathrm{d}t \tag{A.2}$$

being the time-averaged, symmetrized autocorrelation function of the TLS fluctuations. Note that within LRT the spectral power $S_{xx}(\omega)$ can be decomposed as

$$S_{xx}(\omega) = |\tilde{\chi}(\omega)|^2 S_{ff}(\omega) + S_{xx}^{(0)}(\omega). \tag{A.3}$$

Here, $S_{xx}^{(0)}(\omega)$ stands for the spectral power of spontaneous fluctuations of the TLS in the absence of driving, and $S_{ff}(\omega)$ denotes the spectral power of the signal defined analogously to $S_{xx}(\omega)$. Moreover, $S_{xx}^{(0)}(\omega)$ is related to the linear susceptibility $\tilde{\chi}(\omega)$ by the well-known *fluctuation-dissipation theorem* (FDT) [42]

$$S_{xx}^{(0)}(\omega) = \hbar \coth\left(\frac{\hbar\omega}{2k_BT}\right) \mathrm{Im}\tilde{\chi}(\omega). \tag{A.4}$$

An evaluation of either $S_{xx}^{(0)}(\omega)$, or $\tilde{\chi}(\omega)$ for the spin-boson model (1) presents a nontrivial task which can be solved only approximately. To this end, let us consider the TLS dynamics subjected to weak harmonic driving of the form

$$f(t) = A_0\cos(\Omega t). \tag{A.5}$$

Then, an analysis of the asymptotic ($t\to\infty$) solution of the equation (3) yields [6, 22]

$$\tilde{\chi}(\Omega) = \frac{1}{k_BT}\frac{x_0^2}{\cosh^2(\epsilon/2k_BT)}\frac{W(\epsilon_0)}{W(\epsilon_0)-\mathrm{i}\Omega} \tag{A.6}$$

with the relaxation rate $W(\epsilon_0) := W_+(\epsilon_0) + W_-(\epsilon_0)$ given by (4) with $\epsilon(t) \equiv \epsilon_0$. The expression (A.5) is valid for $x_0A_0, \hbar\Omega \ll \hbar\omega_c, \alpha k_BT$ [6]. Moreover, we assume the condition $W(\epsilon_0) \ll k_BT$ to hold, being obeyed for all practical purposes. In this case, the *quantum* FDT (A.3) can safely be substituted by its *classical* analogue, yielding the unperturbed spectral density of the TLS

$$S_{xx}^{(0)}(\omega) = \frac{x_0^2}{\cosh^2(\epsilon_0/2k_BT)}\frac{2W(\epsilon_0)}{W^2(\epsilon_0)+\omega^2}\,. \tag{A.7}$$

The spectral power density (A7) contemplates the random transitions between levels of the TLS with the *switching rates* $W_\pm(\epsilon_0)$ determined by the relaxation of *mean* populations. It thus reflects the quasiclassical Onsager regression hypothesis which underpins the quasiclassical interpretation of the incoherent Markovian TLS dynamics as classical random telegraph process.

Furthermore, the signal-to-noise ratio, SNR, is the ratio of the spectral amplitude of signal, $\pi A_0^2|\tilde{\chi}(\Omega)|^2$, to the spectral power density of fluctuations (A.6) at the same frequency Ω, i.e. [4],

$$SNR(\Omega) = \frac{\pi A_0^2|\tilde{\chi}(\Omega)|^2}{S_{xx}^{(0)}(\Omega)}. \tag{A.8}$$

Upon combining (A.5) and (A.6) one obtains the result

$$SNR = \frac{\pi A_0^2 x_0^2}{2(k_B T)^2} \frac{W(\epsilon_0)}{\cosh^2(\epsilon_0/2k_B T)} . \tag{A.9}$$

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