Semiclassical analysis of level widths for one-dimensional potentials

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I. INTRODUCTION

The coupling of a system to environmental degrees of freedom plays an important role in many areas of physics. Already on a classical level it leads to fluctuations, as in Brownian motion, and to damping.[1] In addition, in quantum mechanics the environmental coupling induces decoherence, which, for example, is of interest in the discussion of Schrödinger cats[2] and quantum computation.[3]

An isolated quantum system has stable eigenstates and its density of states is given by a series of delta functions. Coupling to external degrees of freedom renders the original states unstable, since in general we obtain a new set of eigenstates for the entire system. When the number of degrees of freedom introduced is very large and the coupling sufficiently weak, the new spectrum, consisting of a series of closely spaced delta functions, will be signed by the original one. In particular, the reduced density of states, describing the system coupled to a large number of degrees of freedom, results from a smearing of that of the uncoupled case. The smearing of the original eigenenergies due to the coupling to an environment can be expressed as a level width which is related to the lifetime of the state.

The example of an atom coupled to the electromagnetic modes of the radiation field may help to illustrate the concept of level width. If the atom were isolated from the field, there would be no transitions from excited states to states lower in energy. This changes if we take the coupling to the radiation field into account. Then transitions between states may occur, and the finite lifetime of the excited states broadens the spectral lines associated with the transition. As a consequence of this mechanism a modification of the mode spectrum may influence the lifetime of atomic states.[4]

In the following discussion of level widths we will fix the environmental spectrum to be ohmic (for a definition see Sec. II below). On the other hand, the system considered, a particle moving in a one-dimensional power-law potential, will be quite general. While the level width of a system eigenstate will increase with increasing coupling to the environment, the properties of the eigenstate will also be of relevance. This leads to an interesting question: How do the level widths depend on the quantum number of the eigenstate? As we will see, such a rather general question has a surprisingly simple answer if one restricts one's self to the limit of large quantum numbers where semiclassical methods are applicable.

Semiclassical approaches were essential at the advent of quantum mechanics and have ever since remained a privileged tool for learning this subject, for developing our physical intuition on new problems, and for performing analytical calculations.[5,6] For a particle confined in a one-dimensional (1D) potential, the semiclassical [Wentzel–Kramers–Brillouin (WKB)] approximation yields closed expressions for the eigenenergies and eigenfunctions.[7] The applicability of the WKB approximation is restricted to large quantum numbers, where the confining potential varies smoothly on the scale of the de Broglie wavelength of the particle. In this limit, the quantum properties of the system can be obtained by means of classical trajectories.

A particularly simple case is that of a power-law potential where the eigenenergies follow a simple scaling with the quantum numbers (or the classical actions).[8,9] Restricting ourselves to power-law potentials, and in the presence of an ohmic environment, we are able to extend the scaling of Refs. 8 and 9 to level widths and demonstrate that they are simply proportional to quantum numbers. It is interesting to note that this scaling of level widths with the quantum number is even simpler than that of the eigenenergies, despite the fact that the latter are more basic quantities than the former.

The paper is organized as follows. We first present the general formalism for describing the dissipative environment and its effect on the level widths (Sec. II), and we recall the well-known case of a particle in a harmonic potential. In Sec. III, we establish the central result of this work, proving the linear dependence of level widths on quantum number under the assumptions specified above. In the concluding section we analyze the experimental implications of our findings and their possible extensions to higher dimensions. The example of a confining potential with the shape of a half-harmonic oscillator is discussed in detail in the Appendix.

II. LEVEL WIDTHS IN A DISSIPATIVE ENVIRONMENT

As our model we consider a particle of mass M moving in a one-dimensional potential $V(q)$. The spectrum of the corresponding Hamiltonian

$$H_S = \frac{p^2}{2M} + V(q) \tag{1}$$

is assumed to consist of a discrete part at low energies which may be followed by a continuous part at higher energies. It is on the discrete part of the spectrum (consisting of eigenenergies E_n, $n=0,1,2,...$) that we focus our analysis.

Since we consider the limit of large quantum numbers, we require that the number of discrete eigenstates is infinite or at least can be made sufficiently large. This includes, for example, the radial part of the Coulomb problem but excludes the Morse potential.

The eigenstates acquire a finite width if we weakly couple the particle to environmental degrees of freedom. We assume that the environment consists of a set of harmonic oscillators coupled bilinearly to the particle. This leads to the full Hamiltonian

$$H = H_S + \sum_{j=0}^{\infty} \left[\frac{p_j^2}{2m_j} + \frac{m_j \omega_j^2}{2} \left(x_j - \frac{c_j}{m_j \omega_j^2} q \right)^2 \right], \tag{2}$$

implying a coupling between system and environment through the Hamiltonian

$$H_I = -\sum_{j=0}^{\infty} c_j x_j q. \tag{3}$$

By eliminating the environmental degrees of freedom, we obtain an effective operator equation of motion,[1,10]

$$\ddot{q} + \int_0^t ds\, \gamma(t-s)\dot{q}(s) + \frac{1}{M}\frac{dV}{dq} = \frac{1}{M}\xi(t), \tag{4}$$

with damping kernel

$$\gamma(t) = \frac{2}{M}\int_0^{\infty} \frac{d\omega}{\pi} \frac{J(\omega)}{\omega} \cos(\omega t), \tag{5}$$

a spectral density of bath oscillators

$$J(\omega) = \pi \sum_{j=0}^{\infty} \frac{c_j^2}{2m_j\omega_j} \delta(\omega - \omega_j), \tag{6}$$

and a fluctuating force $\xi(t)$ which we do not need to specify further.

The special case of $J(\omega) = M\gamma\omega$ is of great importance since the damping kernel becomes local in time: $\gamma(t) = 2\gamma\delta(t)$. Noting that in Eq. (4) only half of the delta function contributes (the integral ends at $s=t$), the second term becomes $\gamma\dot{q}(t)$, describing the well-known classical damping proportional to the particle velocity. This type of damping is often referred to as ohmic because such a term also appears in equations for electrical circuits containing an ohmic resistor.

The previous approach provides a microscopic model for dissipation in quantum systems in the sense that dissipation is due to coupling to additional degrees of freedom. However, we should not conclude that in a real resistor we can identify environmental oscillators microscopically. The Hamiltonian (2) allows us to treat analytically the effect of the environment and also provides a good description of many realistic systems. It has been studied over the years[10] and more recently became known as the Caldeira–Leggett model[11] in the context of macroscopic quantum phenomena.

Assuming a weak coupling between the particle and its environment, we calculate the zero temperature width of the n-th level by means of the Fermi golden rule

$$\Gamma_n = \frac{2\pi}{\hbar} \sum_{m,j=0}^{\infty} |\langle m,1_j|H_I|n,0_j\rangle|^2 \delta(E_n - E_m - \hbar\omega_j). \tag{7}$$

This expression describes the decay of the state n to an energetically lower state m by one excitation of the j-th environmental mode that changes its occupation number from 0 to 1. Inserting the dipole matrix element

$$\langle 1_j|x_j|0_j\rangle = \left(\frac{\hbar}{2m_j\omega_j}\right)^{1/2} \tag{8}$$

of the j-th environmental oscillator we get

$$\Gamma_n = \pi \sum_{m,j=0}^{\infty} \frac{c_j^2}{m_j\omega_j} |d_{nm}|^2 \delta(E_n - E_m - \hbar\omega_j), \tag{9}$$

where $d_{nm} = \langle m|q|n\rangle$ is the dipole matrix element of the system. The properties of the environmental modes appearing in Eq. (9) can be expressed in terms of their spectral density (6), and we may write for the level width

$$\Gamma_n = \frac{2}{\hbar} \sum_{m=0}^{n-1} |d_{nm}|^2 J\left(\frac{E_n - E_m}{\hbar}\right). \tag{10}$$

The sum over the system eigenstates is restricted since an environment at zero temperature cannot excite the system into states of higher energy. The result (10) is valid for arbitrary bath density. If we used a cubic frequency dependence for $J(\omega)$, appropriate for the electromagnetic field, we would obtain the natural decay width of an excited atomic state due to spontaneous emission (apart from prefactors arising from a proper treatment of the polarization of the emitted photons).

In the sequel we will concentrate ourselves on the important case of ohmic damping where the level widths can be written as

$$\Gamma_n = \frac{2M\gamma}{\hbar^2} \sum_{m=0}^{n-1} |d_{nm}|^2 (E_n - E_m). \tag{11}$$

This expression constitutes the starting point for a calculation of level widths that will be performed in the following section. It represents, up to the factor γ, a sum over oscillator strengths

$$f_{nm} = \frac{2M}{\hbar^2} |d_{nm}|^2 (E_n - E_m). \tag{12}$$

The finiteness of the upper limit prevents application of the standard Thomas–Reiche–Kuhn sum rule[12] for oscillator strengths: $\Sigma_{m=0}^{\infty} f_{nm} = 1$.

For sufficiently simple confining potentials $V(q)$ the evaluation of the level widths can easily be done. For instance, a harmonic potential with frequency ω_0 has dipole matrix elements that only couple nearest neighbor states,

$$d_{nm} = \sqrt{\frac{\hbar}{2M\omega_0}} (\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}). \tag{13}$$

This leads to the well-known result for the level widths of a damped harmonic oscillator[13]

$$\Gamma_n = n\gamma. \tag{14}$$

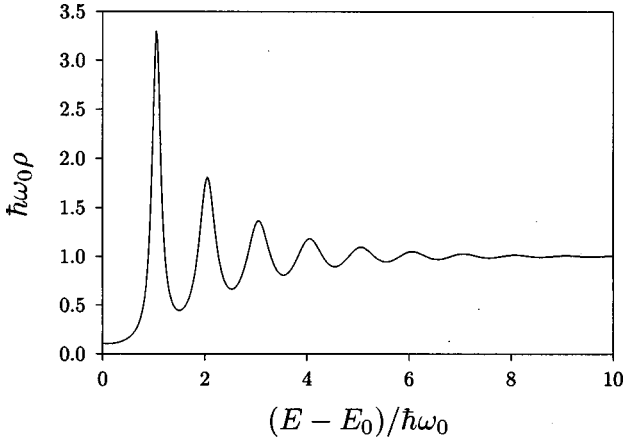

Fig. 1. Density of states for a harmonic oscillator of frequency ω_0 coupled to an ohmic environment with $\gamma/2\omega_0=0.1$. The delta function corresponding to the stable ground state at energy $E_0=\hbar\omega_0/2$ is not shown.

In this case one can even go beyond the level widths and calculate the reduced density of states $\rho(E)$ in closed form.[14] As an illustration we show in Fig. 1 the result for ohmic damping with $\gamma/2\omega_0=0.1$. A delta function corresponding to the stable ground state has been omitted. The levels broaden with increasing energies, according to Eq. (14). For the relatively large coupling chosen, the regime where the eigenstates of the system are completely smeared out is reached for rather small quantum numbers.

The strength of coupling of a system to its environment can vary widely according to the physical problem under consideration. The theoretical analysis that we pursue in the following addresses the situation where the coupling is much weaker than that in Fig. 1. Thus we expect a large series of broadened eigenstates before the level widths become of the order of the mean level spacing.

III. LEVEL WIDTHS FOR POWER-LAW POTENTIALS

In this section we establish the main result of our work, the proportionality of level widths to quantum numbers for the model described in Sec. II and power-law confining potentials of the form $V(q)=A|q|^\alpha$. The amplitude A and the exponent α should have the same sign to allow for bound states. The accessible classical region might be limited by an infinite potential wall, and we assume that such a wall is present at $q=0$ whenever $\alpha<0$. The case $\alpha=0$ will be excluded because it requires two walls and thus reduces to the exactly solvable case of a particle in a box (see also $\alpha=0$ in Table I). Furthermore, we will restrict the exponent to $\alpha>-2$. At $\alpha=-2$ the action becomes independent of energy as will become clear from Eq. (21) below. Therefore we must exclude this pathological case. We emphasize that an attractive $1/q^2$ potential for small q can never appear in a radial equation of motion in $d>1$ after elimination of the angular degrees of freedom.

Since our semiclassical approach requires sufficiently large energies, the discussion applies also to confinements which effectively behave like a power-law potential at higher energies, regardless of the shape at the bottom of the potential. For example, the quartic double-well potential is included in our discussion since it has the form Aq^4 for large energies.

Table I. Eigenenergies, dipole matrix elements, and level widths in the semiclassical limit for 1D box ($\alpha=0$ with two walls), half oscillator ($\alpha=2$), and radial part of the Coulomb problem ($\alpha=-1$).

α	E_n	$d_{n,n-l}$	$\Gamma_n/\gamma n$
0	$\sim n^2$	$\sim l^{-2}$	$\frac{7}{\pi^2}\zeta(3)=0.85\ldots$
2	$\sim n$	$\sim n^{1/2}l^{-2}$	$\frac{8}{\pi^2}=0.81\ldots$
-1	$\sim n^{-2}$	$\sim n^2 l^{-5/3}$	$\frac{6^{1/3}}{\pi^2}[\Gamma(2/3)]^2\zeta(7/3)=0.47\ldots$

According to Eq. (11) the n-dependence of the level widths is determined by the dipole matrix element d_{nm} and the energy difference E_n-E_m. We start the analysis of these quantities by recalling some relations concerning energy quantization in the semiclassical limit. The central quantity is the action

$$S(E)=M\oint dq\dot{q}, \tag{15}$$

taken over one cycle of the classical periodic motion. The period T itself may be obtained by taking the derivative of S with respect to energy, $T=dS/dE$. Within semiclassical quantization, the eigenenergies are determined by

$$S(E_n)=2\pi\hbar(n+\mu), \tag{16}$$

where μ is a constant depending on the details of the potential. We are interested in large quantum numbers n and m where $n,\ m\gg l=n-m$. Then we obtain for the energy difference

$$E_n-E_m\approx\frac{\partial E}{\partial n}l=\frac{2\pi\hbar}{T}l. \tag{17}$$

The dependence of the energy E on quantum number n can be obtained by using the scaling properties of the classical energy conservation condition

$$E=\frac{M}{2}\dot{q}^2+Aq^\alpha. \tag{18}$$

To this end we introduce dimensionless coordinate and time

$$q'=\left(\frac{|A|}{E}\right)^{1/\alpha}q,\quad t'=\frac{|A|^{1/\alpha}E^{(\alpha-2)/2\alpha}}{M^{1/2}}t, \tag{19}$$

which simplifies Eq. (19) to the dimensionless form

$$1=\tfrac{1}{2}\dot{q}'^2+\mathrm{sign}(A)q'^\alpha. \tag{20}$$

Here, $\dot{q}'=dq'/dt'$. The quantization condition (16) in scaled variables reads

$$S(E)=\frac{M^{1/2}E^{(2+\alpha)/2\alpha}}{|A|^{1/\alpha}}\oint dq'\dot{q}'=2\pi\hbar n, \tag{21}$$

where again the integral runs over one period. On the right-hand side we have omitted the constant μ introduced in (16)

which becomes irrelevant for large n. With $S'=\oint dq'\dot{q}'$ we then find for the energy eigenvalues

$$E=\left[\left(\frac{2\pi\hbar}{S'}\right)^{2\alpha}\frac{|A|^2}{M^\alpha}\right]^{1/(2+\alpha)}n^{2\alpha/(2+\alpha)}, \tag{22}$$

in agreement with previous results for power-law potentials[8,9] where S' has been evaluated explicitly. Equation (22) is still correct for sufficiently large n if the potential behaves like a power-law only asymptotically. Then, in general, S' can no longer be evaluated analytically.

In the semiclassical limit the dipole matrix elements can be related to the Fourier components of the classical motion of the particle:[15]

$$\begin{aligned} d_{n,n-l}&=-\frac{1}{\pi l}\int_{q_1}^{q_2}dq\,\sin\left(2\pi l\frac{t(q)}{T}\right)\\ &=\frac{1}{T(E)}\int_0^{T(E)}dt\,q(t,E)\cos\left(2\pi l\frac{t}{T(E)}\right). \end{aligned} \tag{23}$$

Here the second line has been obtained by means of the substitution $dq=\dot{q}\,dt$ and partial integration.

Equation (23), first found by Debye in 1927, extends Bohr's correspondence principle, which states that in the limit $\hbar\to 0$ the frequencies of an atomic transition should agree with electrodynamics, to intensities which are related to the square of dipole matrix elements, Eq. (12). Equation (23) is the leading-order approximation in $\hbar$. Higher order corrections have been derived in Ref. 16 but we will not make use of them here.

In the presence of hard walls the WKB approximation is still applicable and an extra phase in the semiclassical wave function takes into account the infinite potential.[5] Therefore, the semiclassical approximation to the dipole matrix element is the same as for smooth potentials.

After scaling, the dipole matrix element (23) reads

$$d_{n,n-l}=\left(\frac{E}{|A|}\right)^{1/\alpha}d'_l\,, \tag{24}$$

with

$$d'_l=-\frac{1}{\pi l}\int_{q'_1}^{q'_2}dq'\,\sin\left(2\pi l\frac{t'(q')}{T'}\right), \tag{25}$$

where T' denotes the period T scaled according to Eq. (19). The entire dependence of the dipole matrix element on the quantum number n is now contained in the energy factor $E^{1/\alpha}$.

In view of Eqs. (22) and (24), the level widths can be expressed as

$$\Gamma_n=\gamma\frac{8\pi^2}{S'^2}\frac{2+\alpha}{2\alpha}\left[\sum_{l=1}^{n}l{d'_l}^2\right]n. \tag{26}$$

Apart from the factor n this result still depends on the state number n via the upper limit of the sum. As a last step, we therefore have to consider the convergence properties of this sum.

For $\alpha>0$ and in the absence of an infinite potential wall, the velocity of the particle as a function of time is continuous and consequently the scaled dipole moment (25) will decay at least as l^{-2}. This still holds for a wall with finite potential on one side since in that case the reflection will lead to a triangular cusp singularity in the trajectory and the dipole moment will decay as l^{-2}.

The case of negative exponent α with an infinite potential wall at $q=0$ where the potential diverges is more interesting. Close to $q=0$ we may neglect the constant on the left-hand side of (20). Assuming that the reflection happens at $t'=0$ we find for the trajectory close to the reflection point $q'\sim|t'|^{2/(2-\alpha)}$. For sufficiently large l, this singular part yields the asymptotic behavior $d'_l\sim l^{-(4-\alpha)/(2-\alpha)}$ for the scaled dipole matrix element. For $\alpha>-2$ it follows that d'_l decays always faster than $l^{3/2}$.

As a consequence, the argument of the sum in Eq. (26) decays faster than $1/l^2$ for all potentials under consideration. Neglecting terms of order $1/n$, as is consistent with our previous approximations, we may extend the upper summation limit to infinity and arrive at our final result

$$\Gamma_n=\gamma\frac{8\pi^2}{S'^2}\frac{2+\alpha}{2\alpha}\left[\sum_{l=1}^{\infty}l{d'_l}^2\right]n. \tag{27}$$

For sufficiently large energies, the level widths are therefore proportional to the state number n. We point out that the proportionality constant depends on α and γ but not on M and A.

Apart from the harmonic oscillator discussed at the end of Sec. II there exist a few more systems for which the level widths can be evaluated exactly. Table I summarizes the results for the box, the radial part of the Coulomb problem (with the dipole matrix element given in Ref. 17), and the half-harmonic oscillator for which we sketch the calculation in the Appendix. All quantities are given for large quantum number n. In addition, for the dipole matrix element the limit of large l with $n\gg l$ is taken, as was the case in the general derivation given above. For these three examples, the table shows that the linear dependence of the level widths on quantum number results in a nontrivial way from the n-dependences of the eigenenergies and the dipole matrix elements.

While these properties are special to the case of ohmic damping, an extension to other bath densities along the lines presented here is straightforward. Often one assumes a power-law behavior for the spectral density of bath oscillators $J(\omega)\sim\omega^\beta$. The behavior at large frequencies may lead to divergencies and one is often forced to introduce a cutoff which might be sharp or exponential in nature. If the cutoff frequency is much larger than all other energies of interest, one finds along the lines indicated above for the level widths

$$\Gamma_n\sim n^{1-(\beta-1)(2-\alpha)/(2+\alpha)}. \tag{28}$$

This clearly shows that the universality found above is special to the case of ohmic damping ($\beta=1$). For larger exponent β the level widths may even decrease with increasing quantum number n as is well known from the stability of Rydberg atoms.

IV. CONCLUSIONS

Making use of a semiclassical expression for the dipole matrix element in terms of the Fourier transform of the classical paths, we have shown that the level widths of a particle in a power-law potential coupled to an ohmic environment are proportional to the number n of the state. This result is

valid for sufficiently large n and can therefore be extended to potentials possessing only an asymptotic power-law behavior. The cases with one or two hard walls are also shown to obey such a scaling. The proportionality of level widths with the state number known for the harmonic oscillator is thus generalized to a large class of one-dimensional potentials. The prefactor of such a linear law depends on the specific potential.

The applicability of our results to physically realizable situations is limited by the following restrictions: (i) the motion of the particle whose state may decay has to be one-dimensional; (ii) large quantum numbers are involved; (iii) the coupling giving rise to the decay has to be dipolar and the environment of ohmic density of bath modes; (iv) the coupling has to be sufficiently weak in order to have well defined states at large n. These conditions could be met on the one hand in experiments on mesoscopic electronic devices, which enable one to build quantum confined systems of reduced spatial dimensionality. Other candidates are trapped atoms: They can, e.g., be highly excited into elongated (quasi 1D) Stark-type states and their coupling to the environment can be measured with high precision. Hence, experimental scenarios to test the simple general scaling of the widths with quantum number appear possible.

One condition that would be interesting to be relaxed among the above requirements is the one concerning the dimensionality of the system. In three dimensions there exists a semiclassical treatment of radiative lifetimes in hydrogenlike atoms[18] which is relevant for spectroscopy of Rydberg atoms. Apart from this special case, we are not aware of any systematic study of level widths for a whole class of potentials in the limit of large quantum numbers.

Multidimensional systems allow us to take into account the rich behavior that the underlying classical mechanics yields. Semiclassics is appropriate to treat such cases (in the limit of large energies) and allows us to address the differences arising from the chaotic or integrable nature of the classical motion. Therefore, an extension of the approach presented here would contribute to the understanding of the interplay between quantum chaos and dissipation.

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APPENDIX: HALF-OSCILLATOR

We discuss in this Appendix the case of a half-harmonic oscillator, where the confining potential has the form of a harmonic oscillator for $q>0$ and an infinite wall at $q=0$ (the particle is then confined to $q\geqslant 0$). The interest of treating this nontrivial example separately stems from the fact that both approaches, the direct calculation and the semiclassical approximation, are feasible and can be compared. In contrast to the case of the harmonic oscillator, the dipole matrix elements of the half-oscillator couple not only nearest neighbor states. The eigenstates of the half-oscillator are given by the odd eigenstates of the harmonic oscillator with the prefactor adjusted to account for the restricted interval of normalization. The dipole matrix element d_{nm} may then be evaluated by expressing the Hermite polynomial with the higher quantum number $n>m$ by means of the Rodrigues formula,

$$d_{nm}=-\left(\frac{\hbar}{M\omega\pi}\right)^{1/2}\frac{1}{2^{n+m}[(2n+1)!(2m+1)!]^{1/2}}\times\int_0^\infty d\xi\,\xi H_{2m+1}(\xi)\frac{d^{2n+1}}{d\xi^{2n+1}}\exp(-\xi^2). \tag{A1}$$

After repeated partial integration and use of

$$\frac{d^k}{d\xi^k}(\xi H_{2m+1}(\xi))\bigg|_{\xi=0}=(-1)^{m+1-k/2}\frac{2^{k-1}k(2m+1)!}{(m+1-k/2)!}, \tag{A2}$$

which holds for k even and yields zero for k odd, one arrives at

$$d_{nm}=2\left(\frac{\hbar}{M\pi\omega}\right)^{1/2}\left(\frac{(2m+1)!}{2^{2(m+n+1)}(2n+1)!}\right)^{1/2}\times(-1)^{n-m+1}\sum_{l=1}^{m+1}2^{2l}l\frac{[2(n-l)]!}{(m-l+1)!(n-l)!}. \tag{A3}$$

By induction one can show that

$$d_{nm}=\left(\frac{\hbar}{M\pi\omega}\right)^{1/2}\frac{(-1)^{n-m+1}}{2^{n+m-1}}\times\frac{[(2n+1)!(2m+1)!]^{1/2}}{n!m!}\frac{1}{4(n-m)^2-1}. \tag{A4}$$

The semiclassical evaluation of the dipole matrix element using Eq. (23) is considerably easier and leads to

$$d_{n,n-l}=-\frac{4}{\pi}\left[\frac{\hbar}{M\omega}\left(n-\frac{1}{4}\right)\right]^{1/2}\frac{1}{4l^2-1}. \tag{A5}$$

For $n\gg l$ this agrees with the exact result up to an irrelevant sign. For the level widths we then obtain

$$\Gamma_n=\frac{8}{\pi^2}\gamma n\simeq 0.81\gamma n \tag{A6}$$

and thus the proportionality to the quantum number.

[1]See, e.g., Ulrich Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 1993); Thomas Dittrich, Peter Hänggi, Gert-Ludwig Ingold, Bernhard Kramer, Gerd Schön, and Wilhelm Zwerger, *Quantum Transport and Dissipation* (Wiley-VCH, Weinheim, 1998), Chap. 4.

[2]M. Brune, E. Hagley, J. Dreyer, X. Maître, A. Maali, C. Wunderlich, J. M. Raimond, and S. Haroche, ''Observing the progressive decoherence of the 'Meter' in a quantum measurement,'' Phys. Rev. Lett. **77**, 4887–4890 (1996).

[3]David P. DiVincenzo, ''Quantum computation,'' Science **270**, 255–261 (1995).

[4]See, e.g., Serge Haroche and Daniel Kleppner, ''Cavity quantum electrodynamics,'' Phys. Today **42**, 24–30 (January 1989).

[5]Martin C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, Berlin, 1990).

[6]Matthias Brack and Rajat K. Bhaduri, *Semiclassical Physics*, Frontiers in Physics (Addison-Wesley, Reading, 1997), Vol. 96.

[7]For recent work on accurate WKB wave functions for 1D potentials see: H. Friedrich and J. Trost, Phys. Rev. Lett. **76**, 4869–4873 (1996); Phys. Rev. A **54**, 1136–1145 (1996).

[8]Uday P. Sukhatme, ''WKB energy levels for a class of one-dimensional potentials,'' Am. J. Phys. **41**, 1015–1016 (1973).

[9]J. F. Carĩnena, C. Farina, and Cássio Sigaud, ''Scale invariance and the Bohr-Wilson-Sommerfeld (BWS) quantization for power law one-dimensional potential wells,'' Am. J. Phys. **61**, 712–717 (1993).

[10]As an early reference we mention V. B. Magalinskiĭ, ''Dynamical model

in the theory of the Brownian motion,'' Zh. Eksp. Teor. Fiz. **36**, 1942–1944 (1959) [Sov. Phys. JETP **9**, 1381–1382 (1959)]. Further references can be found for example in Ref. 1.
[11]A. O. Caldeira and A. J. Leggett, ''Quantum tunneling in a dissipative system,'' Ann. Phys. (N.Y.) **149**, 374–456 (1983).
[12]C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics* (Wiley, New York, 1978), Vol. 2, p. 1318.
[13]See, e.g., Claude Cohen-Tannoudji, Jacques Dupont-Roc, and Gilbert Grynberg, *Atom-Photon Interactions* (Wiley, New York, 1992).
[14]A. Hanke and W. Zwerger, ''Density of states of a damped quantum oscillator,'' Phys. Rev. E **52**, 6875–6878 (1995).
[15]See, e.g., L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, Sec. 48 (Pergamon, New York, 1959).
[16]Robert Karrlein and Hermann Grabert, ''Semiclassical theory of vibrational energy relaxation,'' J. Chem. Phys. **108**, 4972–4983 (1998).
[17]S. M. Susskind and R. V. Jensen, ''Numerical calculations of the ionization of one-dimensional hydrogen atoms using hydrogenic and Sturmian basis functions,'' Phys. Rev. A **38**, 711–728 (1988).
[18]Hermann Marxer and Larry Spruch, ''Semiclassical estimation of the radiative mean lifetimes of hydrogen like states,'' Phys. Rev. A **43**, 1268–1274 (1991).

UNIVERSAL GRAVITATION

Stanley and Emin and their followers trekked for several months to the east African coast, reaching the sea at a small German post in today's Tanzania.

A German battery fired an artillery salute in their honor, and officials gave the two of them a banquet at the local officers' mess. A naval band played; Stanley, Emin, and a German major gave speeches. ''The wines were choice and well-selected and iced,'' writes Stanley. Then the near-sighted Emin, who had been moving up and down the banquet table, chatting with the guests and drinking champagne, stepped through a second-floor window that he apparently thought opened on a veranda. It didn't. He fell to the street and was knocked unconscious. He had to remain in a local German hospital for two months, and Stanley was unable to bring him back to Europe in triumph. Most embarrassing of all for Stanley was that Emin Pasha, once he recovered, went to work neither for his British rescuers nor for Leopold, but for the Germans.

Adam Hochschild, *King Leopold's Ghost—A Story of Greed, Terror, and Heroism in Colonial Africa* (Houghton Mifflin, Boston, 1998), p. 100.