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DISTANCES OF POLARS IN POINTED SYMMETRIC R-SPACES

PETER QUAST AND MAKIKO SUMI TANAKA

Abstract. Polars in a pointed compact symmetric space are connected components of the fixed point set of the geodesic symmetry at the origin. They carry important information about the ambient symmetric space. In this note we show that the distances to the origin of two distinct polars in a pointed indecomposable symmetric R-space are different.

1. Introduction

Symmetric spaces are connected Riemannian manifolds all of whose points admit geodesic symmetries. A *geodesic symmetry* of a Riemannian manifold through a point $o \in M$ is an isometry s_o of M that reverses the orientation all geodesics emanating in o. So it is natural to study the fixed point set $\mathrm{Fix}(s_o)$ of the geodesic symmetry s_o through a point o in a symmetric space M. While essentially trivial for non-compact symmetric spaces, $\mathrm{Fix}(s_o)$ is a very interesting object if M is compact. We will therefore assume that (M, o) is a compact pointed symmetric space. The fixed point set $\mathrm{Fix}(s_o)$ coincides with the set of midpoints of closed geodesics in M with origin o. If we choose another point $p \in M$, then we have $\mathrm{Fix}(s_p) = g(\mathrm{Fix}(s_o))$ for an isometry g of M with $p = g(o)$, because $s_p = g \circ s_o \circ g^{-1}$. The connected components of $\mathrm{Fix}(s_o)$, except $\{o\}$, are called *polars* of the compact pointed symmetric space (M, o). Polars were introduced and classified by Bang-Yen Chen and Tadashi Nagano in [2] (see also [3, 11]). Polars contain deep information about the topology of M : if the Euler characteristic $\chi(M)$ of M is positive, then $\chi(M)$ the sum of the Euler characteristics of its polars (see [13, Remark 1.10a]), and M is orientable if and only if each of its polars has even dimension (see [11, Proposition 2.10] and [12, Theorem 5.1]).

An important subclass of the class of compact symmetric spaces is formed by the *symmetric R-spaces* introduced by Masaru Takeuchi [17] and Tadashi Nagano [10] in 1965. These are compact symmetric spaces which in addition are R-spaces, that is, they also allow for a transitive action of a non-compact center-free semi-simple Lie group. Symmetric R-spaces appear in various geometric contexts, e.g. they are precisely the compact extrinsically symmetric submanifolds in Euclidean space (see [5, 6] and also [4]). We call a symmetric R-space *indecomposable*, if its is not a Riemannian product of two symmetric

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R-spaces. By a result of Masaru Takeuchi [18, Section 1] there are two kinds of indecomposable symmetric R-spaces (see also [15]):

(i) irreducible hermitian symmetric spaces of compact type;
(ii) real forms (fixed point sets of anti-holomorphic involutive isometries) of irreducible hermitian symmetric spaces of compact type.

The main result of our paper states that polars of pointed indecomposable symmetric R-spaces are characterized by their Riemannian distance (denoted by dist) to the origin.

Theorem 1. *Let M_1^+ and M_2^+ be two polars of a pointed indecomposable symmetric R-space (M, o). If $\operatorname{dist}(o, M_1^+) = \operatorname{dist}(o, M_2^+)$, then $M_1^+ = M_2^+$.*

Let f be an isometry of an indecomposable symmetric R-space M with $f(o) = o$ for some $o \in M$. Then f commutes with s_o. Thus f preserves $\operatorname{Fix}(s_o)$. Since an isometry also preserves distances, we conclude with Theorem 1 that f preserves every connected component of $\operatorname{Fix}(s_o)$.

Corollary 2. *Every isometry of an indecomposable symmetric R-space M that satisfies $f(o) = o$ for some $o \in M$ preserves every polar of (M, o).*

We will prove this theorem in Section 3, first for irreducible hermitian symmetric spaces (Paragraph 3.1, Theorem 4) and then for its real forms (Paragraph 3.2). Theorem 1 does not hold for arbitrary irreducible compact symmetric spaces as examples in Section 4 illustrate.

An example of an indecomposable symmetric R-space is the special orthogonal group SO_n. We take the identity element $e \in \mathrm{SO}_n$ as our origin. The geodesic symmetry s_e is the inversion in the Lie group SO_n and $\operatorname{Fix}(s_e) = \{g \in \mathrm{SO}_n \mid g^2 = e\}$ is the set of all involutive elements of SO_n. From the classifications of polars due to Bang-Yen Chen and Tadashi Nagano (see [2, 3, 11]) we know that $\operatorname{Fix}(s_e)$ is a disjoint union of even dimensional real Grassmannians and a point, that is $\operatorname{Fix}(s_e) \setminus \{e\} \cong \bigsqcup_{r=1}^{\lfloor \frac{n}{2} \rfloor} G_{2r}(\mathbb{R}^n)$.

2. Preliminaries: Polars and root systems

In this section we recall a result of Tadashi Nagano (see [11, Proposition 6.5]) stated below in Proposition 3 relating polars and root systems, which we need in our proof of Theorem 1.

We first briefly review the definition of a root system of a symmetric space of *compact type*, that is of a compact symmetric space whose universal Riemannian cover is still compact. For details we refer to the standard literature such as [7, 9]. Let (M, o) be a pointed symmetric space of compact type. Following Nagano [11] we denote by G be the symmetry group of M, this is the closed subgroup of the full isometry group of M generated by all geodesic symmetries of M. Since G acts transitively on M we get $M = G/K$, where $K = \{g \in G \mid g(o) = o\}$ is the isotropy group of o in G. On the level of Lie algebras we get the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}.$$

Here $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{k}$ the Lie algebra of K and $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ w.r.t. a bi-invariant metric on $\mathfrak{g}$ chosen such that the canonical identification of $\mathfrak{m}$ with T_oM is a linear isometry. We now take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m} \cong T_oM$. A non-zero linear map $\alpha : \mathfrak{a} \to \mathbb{R}$ is called a *root* of M, if the dimension of the complex vector space

$$\mathfrak{g}_\alpha^{\mathbb{C}} := \left\{X \in \mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C} \mid [H, X] = i\alpha(H)X \text{ for all } H \in \mathfrak{a}\right\}$$

is positive. The set of all roots

$$\mathcal{R} := \{\alpha \in \mathfrak{a}^* \setminus \{0\} \mid \alpha \text{ is a root of } M\}$$

is known as the *root system* of M. Connected components of

$$\mathfrak{a} \setminus \Big(\bigcup_{\alpha \in \mathcal{R}} \ker(\alpha) \Big).$$

are called *Weyl chambers*. The choice of a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ corresponds to the choice of a system of *simple roots*

$$\Sigma := \left\{ \alpha \in \mathcal{R} \;\middle|\; \alpha|_{\mathfrak{a}^+} > 0;\ \frac{1}{2}\alpha \notin \mathcal{R};\ \ker(\alpha) \cap \overline{\mathfrak{a}^+} \neq \{0\} \right\},$$

which consists of $r = \dim(\mathfrak{a})$ many linearly independent elements $\alpha_1, \ldots, \alpha_r$ of $\mathcal{R}$. To Σ corresponds a so called *highest root* δ (see e.g. [7, pp. 475–476]).

Proposition 3 (Tadashi Nagano [11, Proposition 6.5])**.** *Let M^+ be a polar of an irreducible pointed symmetric space (M, o) of compact type. If $\gamma : \mathbb{R} \to M$ is a geodesic with $\gamma(0) = o$ and $\gamma(1) \in M^+$ satisfying $\operatorname{dist}(o, M^+) = L\left(\gamma|_{[0,1]}\right)$, then, after conjugation by a suitable element of K, the initial direction $\dot\gamma(0)$ of γ is under the usual identification $\mathfrak{m} \cong T_oM$ one of the following vectors:*

- $\frac{1}{2}\xi_j$, *with* $\mu_j = 1, 2$.
- $\frac{1}{2}(\xi_j + \xi_k)$, *with* $\mu_j = \mu_k = 1$ *($k = j$ is allowed),*

where $\delta = \sum_{j=1}^{r} \mu_j \alpha_j$ and $\alpha_j(\xi_k) = \pi\delta_{jk}$. Recall that $\operatorname{dist}(o, M^+)$ denotes the Riemannian distance from o to M^+ in M and $L\left(\gamma|_{[0,1]}\right)$ is the arc length of $\gamma|_{[0,1]}$.

Remark. If γ is a geodesic in M emanating in o whose initial direction $\dot\gamma(0)$ is a vector of the form described in Proposition 3, then $\gamma(1)$ is not necessarily a fixed point of s_o.

For later use we recall that an irreducible compact symmetric space M has type c respectively type bc, if its root system is of this kind. This means that $\mathfrak{a}$ can be identified with the standard euclidean space $\mathbb{R}^r$ by a linear isometry such that $\mathcal{R}$ becomes

$$\mathcal{R}_c = \{\pm 2\varepsilon_j \mid 1 \leq j \leq r\} \cup \{\pm\varepsilon_j \pm \varepsilon_k \mid 1 \leq j < k \leq r,\ \pm \text{independent}\}$$

respectively

$$\mathcal{R}_{bc} = \{\pm\varepsilon_j,\ \pm 2\varepsilon_j \mid 1 \leq j \leq r\} \cup \{\pm\varepsilon_j \pm \varepsilon_k \mid 1 \leq j < k \leq r,\ \pm \text{indep.}\},$$

where $\{\varepsilon_1, \ldots, \varepsilon_r\}$ is the dual basis of the standard basis $\{e_1, \ldots, e_r\}$ of $\mathbb{R}^r$, and such that Σ is identified with

$$\Sigma_c = \{\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3,\ \ldots,\ \alpha_{r-1} = \varepsilon_{r-1} - \varepsilon_r,\ \alpha_r = 2\varepsilon_r\}$$

respectively

$$\Sigma_{bc} = \{\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3,\ \ldots,\ \alpha_{r-1} = \varepsilon_{r-1} - \varepsilon_r,\ \alpha_r = \varepsilon_r\}.$$

(see e.g. [7, pp. 463, 475]). The highest root δ is in both cases

$$\delta_c = \delta_{bc} = 2\varepsilon_1$$

(see e.g. [7, p. 476]), and we get

$$\delta_c = \sum_{j=1}^{r-1} 2\alpha_j + \alpha_r \quad \text{respectively} \quad \delta_{bc} = \sum_{j=1}^{r} 2\alpha_j. \tag{1}$$

3. The proof of the main theorem

3.1. **Irreducible hermitian symmetric spaces of compact type.** We first show Theorem 1 for irreducible hermitian symmetric spaces of compact type. More precisely we prove:

Theorem 4. *Let M be an irreducible hermitian symmetric space of compact type and let M_1^+ and M_2^+ be two polars of (M, o). If $\operatorname{dist}(o, M_1^+) = \operatorname{dist}(o, M_2^+)$, then $M_1^+ = M_2^+$.*

Recall that an irreducible hermitian symmetric space M of compact type is always simply connected and has either type c or type bc (see e.g. [7, pp. 376, 518 and 532–534]). Furthermore, M is an inner symmetric space, that is, its symmetry group G is the identity component of its full isometry group (see e.g. [7, Theorem 4.5, pp. 375–376]). Thus K is connected, too.

As any polar M^+ of (M, o) is a connected component of the fixed point set of s_o, every geodesic $\gamma : \mathbb{R} \to M$ with $\gamma(0) = o$ and $\gamma(1) \in M^+$ satisfies $\gamma(2) = o$. Assuming that $\gamma|_{[0,1]}$ is shortest, we see that M^+ lies in the cut locus of (M, o). According to Takashi Sakai (see [16, p. 198]) the intersection of the tangent cut locus $C(\mathfrak{m})$ in $\mathfrak{m} \cong T_oM$ with the closure $\overline{\mathfrak{a}^+}$ of the Weyl chamber $\mathfrak{a}^+$ is

$$C(\mathfrak{m}) \cap \overline{\mathfrak{a}^+} = \{X \in \mathfrak{a} \mid \alpha(X) \geq 0 \text{ for all } \alpha \in \Sigma \text{ and } \delta(X) = \pi\}.$$

This is a simplex in the hyperplane $\{X \in \mathfrak{a} \mid \delta(X) = \pi\} \subset \mathfrak{a}$ whose r vertices $Y_j \in \mathfrak{a}$, $j = 1, \ldots, r$, are defined by

- $\alpha_k(Y_j) = 0$ for $k \neq j$,
- $\delta(Y_j) = \pi$,

where $\Sigma = \{\alpha_1, \ldots, \alpha_r\}$ is the system of simple roots corresponding to $\mathfrak{a}^+$.

The next claim is the main ingredient to finish the proof of Theorem 4.

Claim. In the situation described above we have:

(i) $\|Y_j\| \neq \|Y_k\|$ for $j \neq k$.

(ii) For every polar M^+ of (M, o) there exists $j \in \{1, \dots, r\}$ such that $\mathrm{Exp}_o(Y_j) \in M^+$. Conversely, $\mathrm{Exp}_o(Y_j)$ lies in some polar of (M, o) for all $j \in \{1, \dots, r\}$. Here $\mathrm{Exp}_o : \mathfrak{m} \to M$ is the Riemannian exponential map of M at the origin o under the identification $\mathfrak{m} \cong T_oM$.

Since M is simply connected, its unit lattice $\Gamma := \{X \in \mathfrak{a} \mid \mathrm{Exp}_o(X) = o\}$ is generated over $\pi\mathbb{Z}$ by the system $\check{\mathcal{R}}$ of inverse roots of elements of $\mathcal{R}$ (see [9, pp. 25, 69, 77]), that is

$$\Gamma = \mathrm{span}_{\pi\mathbb{Z}}(\check{\mathcal{R}}) = \Big\{ \sum_{\alpha \in \mathcal{R}} \lambda_{\check{\alpha}} \cdot \check{\alpha} \mid \lambda_{\check{\alpha}} \in \pi\mathbb{Z} \Big\} \quad \text{with } \check{\alpha} = \frac{2H_\alpha}{\langle H_\alpha, H_\alpha \rangle},$$

where $H_\alpha \in \mathfrak{a}$ is defined by $\alpha = \langle H_\alpha, \cdot \rangle$ using the scalar product on $\mathfrak{m} \cong T_oM$. If $\mathcal{R}$ is of type c, then $\check{\mathcal{R}}$ is of type b, whereas if $\mathcal{R}$ is of type bc, then $\check{\mathcal{R}}$ is of type bc, too (see e.g. [9]). Identifying $\mathfrak{a}$ isometrically with $\mathbb{R}^r$ as in Section 2 and considering simple roots again, we get in both cases

$$\Gamma = \mathrm{span}_{\pi\mathbb{Z}}(e_1 - e_2,\ e_2 - e_3,\ \dots,\ e_{r-1} - e_r,\ e_r) = \mathrm{span}_{\pi\mathbb{Z}}(e_1, e_2, \dots, e_r),$$

because the systems of simple roots of root systems of type b and of type bc coincide (see [7, pp. 462, 463, 475]).

Using the notation introduced in Proposition 3 we obtain:

- If M has type c, then

$$\xi_k = \pi \sum_{j=1}^{k} e_j \text{ for } k \in \{1, \dots, r-1\} \quad \text{and} \quad \xi_r = \frac{\pi}{2} \sum_{j=1}^{r} e_j.$$

  By Proposition 3 and Equation (1) the only possible initial vectors of shortest geodesic arcs $\gamma : [0, 1] \to M$ joining o to a polar M^+ are:
  - ○ $X_k := \frac{1}{2}\xi_k = \frac{\pi}{2} \sum_{j=1}^{k} e_j$ for $k = 1, \dots, r-1$,
  - ○ $X_r := \frac{1}{2}(\xi_r + \xi_r) = \xi_r = \frac{\pi}{2} \sum_{j=1}^{r} e_j$,
  - ○ $X_{r+1} := \frac{1}{2}\xi_r = \frac{\pi}{4} \sum_{j=1}^{r} e_j$.

  One easily sees that for $l = 1, \dots, r$ we have $2X_l \in \Gamma$ and $X_l \in C(\mathfrak{m}) \cap \overline{\mathfrak{a}^+}$. Thus $\mathrm{Exp}_o(X_l)$ lies in some polar of (M, o). Whereas $2X_{r+1} \notin \Gamma$ and therefore $\mathrm{Exp}_o(X_{r+1})$ is not contained in any polar of (M, o). Since $Y_l = X_l$ for $l = 1, \dots, r$ and since these vectors all have different length if M is of type c, the claim follows.
- If M has type bc, then

$$\xi_k = \pi \sum_{j=1}^{k} e_j \text{ for } k \in \{1, \dots, r\}.$$

By Proposition 3 and Equation (1) the only possible initial vectors of shortest geodesic arcs $\gamma : [0,1] \to M$ joining o to a polar M^+ are

$$X_k := \frac{1}{2}\xi_k = \frac{\pi}{2}\sum_{j=1}^{k} e_j \quad \text{for} \quad k = 1, \dots, r.$$

One easily sees that for $l = 1, \dots, r$ we have $2X_l \in \Gamma$ and $X_l \in C(\mathfrak{m}) \cap \overline{\mathfrak{a}^+}$. Thus $\mathrm{Exp}_o(X_l)$ lies in some polar of (M, o). Since $Y_l = X_l$ for $l = 1, \dots, r$ and these vectors all have different length if M is of type bc, the claim follows.

To finish our proof of Theorem 4, let M_1^+ and M_2^+ be two polars of (M, o). By the above claim there exist $Z_1, Z_2 \in \{X_1, \dots, X_r\}$ such that $M_1^+ = \mathrm{Exp}_o(Z_1)$ and $M_2^+ = \mathrm{Exp}_o(Z_2)$ and $\mathrm{dist}(o, M_j^+) = \|Z_j\|$ for $j = 1, 2$. If $\mathrm{dist}(o, M_1^+) = \mathrm{dist}(o, M_2^+)$, then $\|Z_1\| = \|Z_2\|$. Using the above claim again we conclude $Z_1 = Z_2$, that is $M_1^+ = M_2^+$.

Remark.
- Our above proof shows, of course, that the statement of Theorem 4 holds for any compact simply connected irreducible symmetric space of type c or bc. The non-hermitian examples of such spaces are the symplectic groups Sp_n, the quaternionic Grassmannians $G_k(\mathbb{H}^n)$ and the Caley projective plane F_4/Spin_9, (see e.g. [7, pp. 518, 532–534]), which are all symmetric R-spaces (see e.g. [1, p. 311]).
- Our proof also shows that a simply connected irreducible symmetric space M of type c or bc has exactly $\mathrm{rank}(M)$ many polars.

3.2. **Non-hermitian indecomposable symmetric R-spaces.** We now prove Theorem 1 for pointed non-hermitian indecomposable symmetric R-spaces (M, o). In [18] Masaru Takeuchi has shown that such an M is a real form of an irreducible hermitian symmetric space P of compact type (see also [15]). This means that there exists an involutive anti-holomorphic isometry τ of P with $M = \mathrm{Fix}(\tau)$. As a reflective submanifold M is geodesically convex in P by a result of the authors (see [14]). This plays an important role in the subsequent arguments.

At this point we recall that the fixed point set of an anti-holomorphic isometry of an hermitian symmetric space of compact type is always connected, because the holomorphic sectional curvatures of an hermitian symmetric space of compact type are strictly positive (see [20, Lemma 4.1] and [8, Theorem 3.8], [13, Proposition 3.2], [11, Proposition 9.1])

Since M is a totally geodesic submanifold of P, the geodesic symmetries of M are restrictions of geodesic symmetries of P. Thus there exist polars P_1^+ and P_2^+ of (P, o) such that $M_j^+ \subset P_j^+$ for $j = 1, 2$. The geodesic symmetry s_o of P through $o \in M \subset P$ is holomorphic. Hence any polar of (P, o) is again a hermitian symmetric space. From the classification of polars (see [3] and [11]) we know that every polar of P is moreover of compact type.

We notice that $\tau(o) = o$ and therefore τ and s_o commute. Thus τ preserves $\mathrm{Fix}(s_o)$. In particular, τ preserves P_1^+ and P_2^+ since τ fixes each point in M_1^+ and in M_2^+. The restrictions of τ to P_j^+ are anti-holomorphic isometries of P_j^+ and $M_j^+ = \mathrm{Fix}\big(\tau|_{P_j^+}\big)$ for $j = 1, 2$. Since M is convex in P (see [14]), $\mathrm{dist}_M(o, M_j^+) = \mathrm{dist}_P(o, M_j^+)$ for $j = 1, 2$. Here dist_M and dist_P denote the Riemannian distances in M and in P respectively. Let K_P

denote the isotropy subgroup of the symmetry group of P at o. Since every polar of (P, o) is a K_P-orbit (see [2]), we have $\mathrm{dist}_P(o, P_j^+) = \mathrm{dist}_P(o, M_j^+) = \mathrm{dist}_M(o, M_j^+)$ for $j = 1, 2$.

Now $\mathrm{dist}_M(o, M_1) = \mathrm{dist}_M(o, M_2)$ implies $\mathrm{dist}_P(o, P_1^+) = \mathrm{dist}_P(o, P_2^+)$. By Theorem 4 we get $P_1^+ = P_2^+$. Since $M_j^+ = \mathrm{Fix}(\tau|_{P_j^+}) = P_j^+ \cap M$ is connected for $j = 1, 2$ (see also [19, Lemma 4.2]), we conclude $M_1^+ = M_2^+$.

4. Non-examples

Theorem 1 obviously does not extend to decomposable symmetric R-spaces. In this section we give two examples that illustrate that Theorem 1 fails if we modify the assumption that M is an indecomposable symmetric R-space. Example 1 shows that Theorem 1 can be wrong for a symmetric space that is non-trivially covered by an indecomposable symmetric R-space. Example 2 shows that Theorem 1 neither extends to simply connected irreducible symmetric spaces of compact type.

Example 1. Let $\hat{M} = \mathrm{Sp}_4$ and $\check{M} = \mathrm{Sp}_4/\mathbb{Z}_2$. Then $\check{M}$ is doubly covered by $\hat{M}$ and it is not a symmetric R-space, whereas $\hat{M}$ is a symmetric R-space (see e.g. [1, p. 311]). Polars of $(\hat{M}, o)$ are $\hat{M}_1^+ = G_1(\mathbb{H}^4)$, $\hat{M}_2^+ = G_2(\mathbb{H}^4)$, $\hat{M}_3^+ = G_3(\mathbb{H}^4)$, and $\hat{M}_4^+$ being a singleton (see [3, Appendix]). Initial vectors X_j of shortest geodesic arcs $\gamma_j : [0, 1] \to \hat{M}$ joining o to $\hat{M}_j^+$ are $X_j := \frac{\pi}{2} \sum_{k=1}^{j} e_k$ for $j = 1, \ldots, 4$ (see Paragraph 3.1).

Let $\pi : \hat{M} \to \check{M}$ denote the projection and $\check{o} = \pi(o)$. Then the polars of $(\check{M}, \check{o})$ are $\check{M}_1^+ = \pi(\hat{M}_1^+) = \pi(\hat{M}_3^+) = G_1(\mathbb{H}^4)$, $\check{M}_2^+ = \pi(\hat{M}_2^+) = G_2(\mathbb{H}^4)/\mathbb{Z}_2$ and $\check{M}_3^+ = (\mathrm{Sp}_4/\mathrm{U}_4)/\mathbb{Z}_2$ (see [3, Appendix]). Initial vectors $\check{X}_j$ of shortest geodesic arcs $\check{\gamma}_j : [0, 1] \to \check{M}$ joining $\check{o}$ to $\check{M}_j^+$ for $j = 1, 2, 3$ are $\check{X}_1 := \frac{\pi}{2} e_1$, $\check{X}_2 := \frac{\pi}{2}(e_1 + e_2)$ and $\check{X}_3 := \frac{\pi}{4}(e_1 + e_2 + e_3 + e_4)$. Hence $\mathrm{dist}(\check{o}, \check{M}_1^+) = \mathrm{dist}(\check{o}, \check{M}_3^+)$ in $\check{M} = \mathrm{Sp}_4/\mathbb{Z}_2$ but $\check{M}_1^+$ and $\check{M}_3^+$ are non-isomorphic.

Example 2. The center of the compact Lie group Spin_{4n}, $n \geq 2$, is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and therefore contains three elements of order 2. (see e.g. [7, Table IV, p. 516]). If we take the identity e as our origin, the pointed symmetric space (Spin_{4n}, e) has three poles (singleton polars). If $n \neq 2$, then two of these poles have the same distance from e. If $n = 2$, then all three poles have the same distance from e (they are mapped to each other by the triality automorphism of Spin_8).

The situation is similar for the Grassmannian $\tilde{G}_{2n}(\mathbb{R}^{4n})$, $n \geq 2$, of oriented $2n$-dimensional real subspaces of $\mathbb{R}^{4n}$, the only other simply connected irreducible symmetric space of compact type that shares the type d_n with Spin_{4n}.

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