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Preprint Nr. 07/2015 — 20. Juli 2015

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<http://www.math.uni-augsburg.de/>

Impressum:

Herausgeber:

Institut für Mathematik

Universität Augsburg

86135 Augsburg

<http://www.math.uni-augsburg.de/de/forschung/preprints.html>

ViSdP:

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Abstract

We prove some geometric inequalities for p th-order chord power integrals $\mathcal{I}_p(P_d)$, $1 \leq p \leq d$, of d -parallelotopes P_d with positive volume $V_d(P_d)$. First, we derive upper and lower bounds of the ratio $\mathcal{I}_p(P_d)/V_d^2(P_d)$ which are attained by a d -cuboid C_d with the same volume resp. the same mean breadth as P_d . Second, we apply the device of Schur-convexity to obtain bounds of $\mathcal{I}_p(C_d)/V_d^2(C_d)$ which are attained by a d -cube with the same volume resp. the same mean breadth as C_d . Most of these inequalities are shown for a more general class of ovoid functionals containing, as by-product, a Pfeifer-type inequality for d -parallelotopes.

Keywords : POISSON HYPERPLANE PROCESSES, MEAN BREADTH, SCHUR-CONVEXITY, SCHUR-CRITERION, LAPLACE TRANSFORM, CARLEMAN'S INEQUALITY, PFIEFER-TYPE INEQUALITY

MSC 2010: PRIMARY 52A40 60D05 SECONDARY 52A20 52A22

1 Chord Power Integrals - General Facts and Motivation

Let K be a convex body in \mathbb{R}^d with interior points and $\mathbb{S}^{d-1} = \partial\mathbb{B}^d$ the boundary of the Euclidean unit ball $\mathbb{B}^d = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \leq 1\}$. Further, let \mathcal{H}^k denote the k -dimensional Hausdorff measure on \mathbb{R}^d for $k = 1, \dots, d$ and, thus, $V_d(K) = \mathcal{H}^d(K)$ and $\mathcal{H}^{d-1}(\partial K)$ denote the volume and surface content of K , respectively. We recall that $\kappa_d := V_d(\mathbb{B}^d) = \pi^{d/2}/\Gamma(\frac{d}{2}+1)$ and $\mathcal{H}^{d-1}(\mathbb{S}^{d-1}) = d\kappa_d$ with $\Gamma(s) := \int_0^\infty e^{-x} x^{s-1} dx$ for $s > 0$.

For any $p \geq 0$ we define the p th-order chord power integral (CPI) of K by

$$\mathcal{I}_p(K) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} \int_{K|\mathbf{u}^\perp} (\mathcal{H}^1(K \cap \ell(\mathbf{x}, \mathbf{u})))^p d\mathbf{x} \mathcal{H}^{d-1}(d\mathbf{u}) \quad (1.1)$$

(with $0^0 := 0$), where $\ell(\mathbf{x}, \mathbf{u}) := \{\mathbf{x} + \alpha \mathbf{u} : \alpha \in \mathbb{R}\}$ stands for the line in direction $\mathbf{u} \in \mathbb{S}^{d-1}$ through $\mathbf{x} \in \mathbb{R}^d$ and $K|\mathbf{u}^\perp$ is the orthogonal projection of K on \mathbf{u}^\perp ($= (d-1)$ -dimensional subspace orthogonal to \mathbf{u}). CPI's are of considerable interest in integral and stochastic geometry for a long time, see [9], [12], [13], [15], and have many applications in material sciences, physics and image analysis, see e.g. [1], [11], [3] and references therein. In textbooks of

integral and convex geometry, see e.g. [9], [12], [13] the r.h.s. of (1.1) is mostly written as integral w.r.t. the *line measure* $\mu_1^{(d)}(\cdot)$ (defined on the space $\mathbb{A}(d, 1)$ of one-dimensional affine subspaces of \mathbb{R}^d):

$$\mathcal{I}_p(K) = \frac{d \kappa_d}{2} \int_{\mathbb{A}(d,1)} (\mathcal{H}^1(K \cap L))^p \mu_1^{(d)}(dL), \quad (1.2)$$

where, for integers $p = 2, \dots, d$, the Blaschke-Petkantschin formula, see [13] (p. 363), provides the representations

$$\mathcal{I}_{k+1}(K) = \frac{(k+1) d \kappa_d}{2 \kappa_k} \int_{\mathbb{A}(d,k)} (\mathcal{H}^k(K \cap L))^2 \mu_k^{(d)}(dL) \quad (1.3)$$

for $k = 1, \dots, d-1$ with the motion-invariant *k-flat measure* $\mu_k^{(d)}(\cdot)$ (defined on the space $\mathbb{A}(d, k)$ of k -dimensional affine subspaces of \mathbb{R}^d) satisfying the normalization $\mu_k^{(d)}(\{E \in \mathbb{A}(d, k) : E \cap \mathbb{B}^d \neq \emptyset\}) = \kappa_{d-k}$. From (1.1) for $p = 0, 1$ and (1.3) for $k = d-1$ we get the following relations, see e.g. [11],

$$\mathcal{I}_0(K) = \frac{\kappa_{d-1}}{2} \mathcal{H}^{d-1}(\partial K), \quad \mathcal{I}_1(K) = \frac{d \kappa_d}{2} V_d(K), \quad \mathcal{I}_{d+1}(K) = \frac{d(d+1)}{2} V_d(K)^2.$$

Due to F. Piefke, see [11], the r.h.s. of (1.1) can be expressed for any $p > 1$ by the distribution of the interpoint distance of two randomly chosen points in K leading to

$$\mathcal{I}_p(K) = \frac{p(p-1)}{2} \int_K \int_K \frac{d\mathbf{x} d\mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|^{d+1-p}} \quad \text{for any } p > 1. \quad (1.4)$$

Note that in the special $d = 3$ the third-order CPI $\mathcal{I}_3(K)$ coincides with Newton's self-potential of the body $K \subset \mathbb{R}^d$, see e.g. [9]. In stochastic geometry there are quite a few random functionals defined on an expanding domain $\varrho K \uparrow \infty$ (as $\varrho \rightarrow \infty$) whose asymptotic variances depend on the shape of K (which is assumed to be convex containing the origin \mathbf{o} as inner point) expressed by $\mathcal{I}_p(K)$ for some $p = 1, \dots, d-1$. Let us sketch a typical example - another one is discussed in [6]. To be precise we need some further notation, for details the reader is referred to [4].

Let $\Pi_\lambda = \{P_i : i \geq 1\}$ be a stationary Poisson process on the real line with \mathbb{R}^1 intensity $\lambda := \mathbf{E}\#\{i \geq 1 : P_i \in [0, 1]\}$, and let Π_λ be independently marked with a sequence $\{U_i, i \geq 1\}$ of independent, uniformly on \mathbb{S}^{d-1} distributed random vectors and $H(P_i, U_i) := U_i^\perp + P_i U_i$ defines a random (unoriented) hyperplane in \mathbb{R}^d with orientation vector $U_i \in \mathbb{S}^{d-1}$ and signed perpendicular distance P_i from \mathbf{o} . The family $\{H(P_i, U_i) : i \geq 1\}$ represents a (motion-invariant) Poisson-hyperplane process in \mathbb{R}^d with intensity λ . Further, we consider the associated (motion-invariant) *k-flat intersection processes* $\{\bigcap_{1 \leq j \leq d-k} H(P_{i_j}, U_{i_j}) : 1 \leq i_1 < \dots < i_{d-k}\}$ for $k = 0, 1, \dots, d-1$ and introduce the mean value functionals

$$\widehat{\zeta}_{k,d}(\lambda, \varrho K) := \frac{1}{V_d(\varrho K)} \sum_{1 \leq i_1 < \dots < i_{d-k}} \mathcal{H}^k \left(\bigcap_{1 \leq j \leq d-k} H(P_{i_j}, U_{i_j}) \cap \varrho K \right) \quad (1.5)$$

having the expectations $\mathbb{E}\widehat{\zeta}_{k,d}(\lambda, \varrho K) = \frac{\kappa_d}{\kappa_k} \binom{d}{k} \left(\frac{\lambda \kappa_{d-1}}{d \kappa_d}\right)^{d-k}$ and the asymptotic variances $\sigma_{k,d}^2(\lambda, K)$ are given by the limit

$$\lim_{\varrho \rightarrow \infty} \varrho \text{Var}(\widehat{\zeta}_{k,d}(\lambda, \varrho K)) = \frac{2 \kappa_{d-1}^2}{d \kappa_k^2} \binom{d-1}{k}^2 \left(\frac{\lambda \kappa_{d-1}}{d \kappa_d}\right)^{2(d-k)-1} \frac{\mathcal{I}_d(K)}{V_d(K)^2} \quad (1.6)$$

for $k = 0, 1, \dots, d-1$. Note that $\sqrt{\varrho}(\widehat{\zeta}_{k,d}(\lambda, \varrho K) - \mathbb{E}\widehat{\zeta}_{k,d}(\lambda, \varrho K))$ is asymptotically normally distributed with variance $\sigma_{k,d}^2(\lambda, K)$, see [4]. The dependence of $\sigma_{k,d}^2(\lambda, K)$ on the shape of K (not only on $V_d(K)$) is caused by the *long-range correlations* within the random union set $\bigcup_{i \geq 1} H(P_i, U_i)$, see similar results for Poisson cylinder processes in [6].

Statisticians aim at creating experimental designs such that estimators of model parameters have minimal variances. In our model this means to minimize the ratio $\mathcal{I}_d(K)/V_d(K)^2$ in (1.6) if another ovoid functional of K , e.g. the *mean breadth* $b_d(K)$ of K , is fixed. Here the mean breadth of K is defined by

$$b_d(K) := \frac{1}{d \kappa_d} \int_{\mathbb{S}^{d-1}} (h(K, u) + h(K, -u)) \mathcal{H}^{d-1}(d\mathbf{u}) = \frac{2 \kappa_{d-1}}{d \kappa_d} V_1(K) \quad , \text{ see [13] (p. 601) ,}$$

where $h(K, u)$ denotes the *support function* of K in direction $\mathbf{u} \in \mathbb{S}^{d-1}$ and $V_1(K)$ is the first intrinsic volume of K , see [13] (p. 600).

In the planar case best lower bounds of $\mathcal{I}_2(K)/V_2(K)^2$ have been proved for particular classes of convex discs in [5] when the perimeter $\mathcal{H}^1(\partial K) = \pi b_2(K)$ is given. In convex geometry, see [2], [12] or [13], one is mostly interested to maximize $\mathcal{I}_k(K)$ for $k = 1, \dots, d$ when $V_d(K)$ is fixed. Among all convex bodies the ball with radius $(V_d(K)/\kappa_d)^{1/d}$ is the unique maximizer due to *Carleman's inequality*, see [13] (p. 364),

$$\mathcal{I}_k(K) \leq (\geq) \frac{2^{k-1} d \kappa_d \kappa_{d-1+k}}{\kappa_k} \left(\frac{V_d(K)}{\kappa_d}\right)^{(d-1+k)/d} \quad \text{for } 1 \leq k \leq d+1 \quad (k \geq d+1). \quad (1.7)$$

Upper and apparently best possible lower bounds of $\mathcal{I}_p(\mathbb{E}(\mathbf{a}))$, $1 \leq p \leq d$, for d -dimensional ellipsoids $\mathbb{E}(\mathbf{a})$ with positive semi-axes $\mathbf{a} = (a_1, \dots, a_d)$ have been obtained in [6].

2 Preliminaries and a Basic Lemma

In order to generalize the class of (motion-invariant) ovoid functionals (1.4) we consider integrals of the form

$$\int_K \int_K f(\|\mathbf{x} - \mathbf{y}\|^2) d\mathbf{x} d\mathbf{y} = \int_{K \oplus (-K)} V_d(K \cap (K + \mathbf{z})) f(\|\mathbf{z}\|^2) d\mathbf{z} \quad (2.1)$$

for any convex body $K \subset \mathbb{R}^d$ and any Borel-measurable function $f|(0, \infty) \rightarrow \mathbb{R}^1$ satisfying

$$\int_0^\tau x^{d-1} |f(x^2)| dx < \infty \quad \text{for all } \tau > 0. \quad (2.2)$$

Since the difference body $K \oplus (-K) := \{x - y : x, y \in K\}$ is contained in a ball centred at \mathbf{o} with radius $\text{diam}(K) := \sup_{x, y \in K} \|x - y\|$ the condition (2.2) guarantees the existence of the above integrals over all convex bodies K . Hence, if additionally $V_d(K) > 0$, the functional

$$Q_d(f, K) := \frac{1}{V_d(K)^2} \int_K \int_K f(\|\mathbf{x} - \mathbf{y}\|^2) d\mathbf{x} d\mathbf{y} = \mathbb{E}f(\|X_K - Y_K\|^2) \quad (2.3)$$

is well-defined, where X_K, Y_K are independent random vectors uniformly distributed on K .

The rest of this paper is organized as follows: In the below Sections 3 and 4 we derive a lower (upper) bound of $Q_d(f, K)$ when K belongs to the class of d -parallelotopes with fixed mean breadth (volume) and f is convex (concave) or continuous and non-decreasing (non-increasing). For this, we need the below Lemma 1 which seems to be of interest for its own rights. In the final Section 5 we prove sharp bounds of $Q_d(f, K)$ for d -cuboids $K = \times_{i=1}^d [0, a_i]$ by applying the concept of Schur-convexity.

Lemma 1 *If the function $f|(0, \infty) \rightarrow \mathbb{R}^1$ is convex (concave), then*

$$f(x + c) + f(x - c) \geq (\leq) 2 f(x) \quad \text{for all } c \in (-x, x), x > 0. \quad (2.4)$$

If $f|(0, \infty) \rightarrow \mathbb{R}^1$ is continuous and non-increasing (non-decreasing) and $g|[0, 1] \rightarrow [0, \infty)$ is non-increasing then the parameter integrals

$$J(f, g; a, b, c) := \int_0^1 g(x) f(a^2 + (bx + c)^2) dx$$

satisfy the inequality

$$J(f, g; a, b, c) + J(f, g; a, b, -c) \leq (\geq) 2 J(f, g; a, b, 0) \quad \text{for all } a, b, c \in \mathbb{R}^1. \quad (2.5)$$

For $a = 0$ we suppose in addition that $\int_0^\tau |f(x^2)| dx < \infty$ for all $\tau > 0$.

Proof of Lemma 1. It suffices to prove (2.4) for $c \geq 0$. Due to the assumed convexity (concavity) of f on $(0, \infty)$ we have $f(x + c) - f(x) \geq (\leq) f(x) - f(x - c)$ for $0 \leq c < x$ which immediately yields the asserted inequality.

For proving the second inequality (2.5), let $b > 0$ and $c > 0$ without loss of generality. At first, let additionally $a^2 > 0$. By obvious rearrangements and the partial integration formula for Riemann - Stieltjes integrals we rewrite $J(f, g; a, b, c)$ as follows:

$$\begin{aligned}
J(f, g; a, b, \pm c) &= \int_0^1 g(x) \, dx \left(\int_0^x f(a^2 + (by \pm c)^2) \, dy \right) \\
&= g(1) \int_0^1 f(a^2 + (by \pm c)^2) \, dy + \int_0^1 \int_0^x f(a^2 + (by \pm c)^2) \, dy \, d(-g(x)) \\
&= \frac{g(1)}{b} \int_{\pm c}^{b \pm c} f(a^2 + z^2) \, dz + \frac{1}{b} \int_0^1 \int_{\pm c}^{bx \pm c} f(a^2 + z^2) \, dz \, d(-g(x)).
\end{aligned}$$

This gives

$$\begin{aligned}
\frac{\partial J(f, g; a, b, \pm c)}{\partial c} &= \frac{g(1)}{b} (\pm f(a^2 + (b \pm c)^2) \mp f(a^2 + c^2)) \\
&\quad + \frac{1}{b} \int_0^1 (\pm f(a^2 + (bx \pm c)^2) \mp f(a^2 + c^2)) \, d(-g(x)),
\end{aligned}$$

whence we obtain that

$$\begin{aligned}
\frac{\partial J(f, g; a, b, c)}{\partial c} + \frac{\partial J(f, g; a, b, -c)}{\partial c} &= \frac{g(1)}{b} (f(a^2 + (b+c)^2) - f(a^2 + (b-c)^2)) \\
&\quad + \frac{1}{b} \int_0^1 [f(a^2 + (bx+c)^2) - f(a^2 + (bx-c)^2)] \, d(-g(x)) \leq (\geq) 0.
\end{aligned}$$

The latter is justified by $(bx+c)^2 \geq (bx-c)^2$ and the monotonicity of f on $(0, \infty)$ and of g on $[0, 1]$. Hence, the even function $c \mapsto J(f, g; a, b, c) + J(f, g; a, b, -c)$ is non-increasing (non-decreasing) for $c \geq 0$ attaining its maximum (minimum) at $c = 0$. The inequalities (2.5) remain valid for $a = 0$ by passing to the limit $a \rightarrow 0$ provided that $\int_0^\tau |f(x^2)| \, dx < \infty$. \square

To avoid ambiguity let us recall that a d -*parallelootope* is a convex body spanned by linearly independent vectors $\mathbf{a}_i = (a_i^{(1)}, \dots, a_i^{(d)})$, $i = 1, \dots, d$, in \mathbb{R}^d , i.e. $P_d(\mathbf{a}_1, \dots, \mathbf{a}_d) := \{\sum_{i=1}^d \lambda_i \mathbf{a}_i : 0 \leq \lambda_1, \dots, \lambda_d \leq 1\}$. For brevity, we write P_d instead of $P_d(\mathbf{a}_1, \dots, \mathbf{a}_d)$ (if no confusion is possible). In what follows we often compare functionals of d -parallelotopes with corresponding functionals d -*cuboids* $C_d(a_1, \dots, a_d) := \times_{i=1}^d [0, a_i]$ having edge lengths $a_1, \dots, a_d > 0$.

From analytic geometry it is well-known that the d -volume $V_d(P_d(\mathbf{a}_1, \dots, \mathbf{a}_d))$ coincides with the absolute value of the determinant

$$\det \left((a_j^{(i)})_{i,j=1}^d \right) = \det [\mathbf{a}_1^\top, \dots, \mathbf{a}_d^\top].$$

Since any two distinct points $\mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{y} = (y_1, \dots, y_d) \in P_d$ can be expressed as linear combination $\mathbf{x} = \lambda_1 \mathbf{a}_1 + \dots + \lambda_d \mathbf{a}_d$ resp. $\mathbf{y} = \mu_1 \mathbf{a}_1 + \dots + \mu_d \mathbf{a}_d$ with unique $\lambda_1, \mu_1, \dots, \lambda_d, \mu_d \in [0, 1]$, we may apply the integral transformation formula with the Jacobian determinants

$$\left| \det \left(\left(\frac{\partial x_i}{\partial \lambda_j} \right)_{i,j=1}^d \right) \right| = \left| \det \left(\left(\frac{\partial y_i}{\partial \mu_j} \right)_{i,j=1}^d \right) \right| = \left| \det \left((a_j^{(i)})_{i,j=1}^d \right) \right| = V_d(P_d),$$

leading to the following representation of (2.3) for $K = P_d(\mathbf{a}_1, \dots, \mathbf{a}_d)$:

$$Q_d(f, P_d) = \int_0^1 \int_0^1 \cdots \int_0^1 \int_0^1 f\left(\left\|\sum_{i=1}^d (\lambda_i - \mu_i) \mathbf{a}_i\right\|^2\right) d\lambda_d d\mu_d \cdots d\lambda_1 d\mu_1. \quad (2.6)$$

Notice the remarkable fact that the mean breadth $b_d(P_d(\mathbf{a}_1, \dots, \mathbf{a}_d))$ only depends on the sum of the edge lengths $\|\mathbf{a}_1\|, \dots, \|\mathbf{a}_d\|$, but not on the angles between the edges, see e.g. [9] (p. 227) for $d = 3$. More precisely, it holds $V_1(P_d(\mathbf{a}_1, \dots, \mathbf{a}_d)) = \|\mathbf{a}_1\| + \cdots + \|\mathbf{a}_d\|$ as can be seen from Steiner's formula, see [13] (p. 600), so that

$$b_d(P_d(\mathbf{a}_1, \dots, \mathbf{a}_d)) = \frac{2\kappa_{d-1}}{d\kappa_d} \left(\sum_{i=1}^d \|\mathbf{a}_i\| \right) = b_d(C_d(\|\mathbf{a}_1\|, \dots, \|\mathbf{a}_d\|)). \quad (2.7)$$

3 Lower bounds of $Q_d(f, P_d)$ for convex f

First we rewrite the $2d$ -fold integral (2.6) as a sum of 2^d d -fold integrals which allow to estimate $Q_d(f, P_d)$ from below. By the following straightforward rearrangements

$$\begin{aligned} Q_d(f, P_d) &= \int_0^1 \int_{-\mu_1}^{1-\mu_1} \cdots \int_0^1 \int_{-\mu_d}^{1-\mu_d} f\left(\left\|\lambda_1 \mathbf{a}_1 + \cdots + \lambda_d \mathbf{a}_d\right\|^2\right) d\lambda_d d\mu_d \cdots d\lambda_1 d\mu_1 \\ &= \int_0^1 \int_0^{\mu_1} \cdots \int_0^1 \int_0^{\mu_d} \sum_{\nu_1, \dots, \nu_d \in \{0,1\}} f\left(\left\|\sum_{i=1}^d (-1)^{\nu_i} \lambda_i \mathbf{a}_i\right\|^2\right) d\lambda_d d\mu_d \cdots d\lambda_1 d\mu_1 \\ &= \int_0^1 \int_{\lambda_1}^1 \cdots \int_0^1 \int_{\lambda_d}^1 \sum_{\nu_1, \dots, \nu_d \in \{0,1\}} f\left(\left\|\sum_{i=1}^d (-1)^{\nu_i} \lambda_i \mathbf{a}_i\right\|^2\right) d\mu_d d\lambda_d \cdots d\mu_1 d\lambda_1 \end{aligned}$$

we arrive at

$$Q_d(f, P_d) = \int_0^1 \cdots \int_0^1 \sum_{\nu_1, \dots, \nu_d \in \{0,1\}} f\left(\left\|\sum_{i=1}^d (-1)^{\nu_i} \lambda_i \mathbf{a}_i\right\|^2\right) \prod_{i=1}^d (1 - \lambda_i) d\lambda_d \cdots d\lambda_1. \quad (3.1)$$

By means of the identity

$$\|z_1 \mathbf{a}_1 + \cdots + z_d \mathbf{a}_d\|^2 = \sum_{i=1}^d z_i^2 \|\mathbf{a}_i\|^2 + 2 \sum_{1 \leq i < j \leq d} z_i z_j \langle \mathbf{a}_i, \mathbf{a}_j \rangle$$

for $z_1, \dots, z_d \in [-1, 1]$, where $\langle \mathbf{a}_i, \mathbf{a}_j \rangle$ denotes the scalar product of \mathbf{a}_i and \mathbf{a}_j , we deduce from (3.1) for pairwise orthogonal vectors \mathbf{a}_i that

$$Q_d(f, C_d(\|\mathbf{a}_1\|, \dots, \|\mathbf{a}_d\|)) = 2^d \int_0^1 \cdots \int_0^1 f\left(\sum_{i=1}^d \lambda_i^2 \|\mathbf{a}_i\|^2\right) \prod_{i=1}^d (1 - \lambda_i) d\lambda_d \cdots d\lambda_1. \quad (3.2)$$

Next, under the assumption that $x \mapsto f(x)$ is convex for $x > 0$, we get a lower bound of the d -fold integral on the r.h.s of (3.1). (Note that, if $x \mapsto f(x)$ is concave for $x > 0$, then $-f(x)$ is convex for $x > 0$ leading to an upper bound.) For this purpose we apply the elementary inequality (2.4) for $x = \|\sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \mathbf{a}_i\|^2 + \lambda_d^2 \|\mathbf{a}_d\|^2$ and $c = 2 \langle \lambda_d \mathbf{a}_d, \sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \mathbf{a}_i \rangle$ (satisfying $-x \leq c \leq x$) implying that

$$\begin{aligned} & \sum_{\nu_d \in \{0,1\}} f(\|(-1)^{\nu_1} \lambda_1 \mathbf{a}_1 + \cdots + (-1)^{\nu_d} \lambda_d \mathbf{a}_d\|^2) \\ = & \sum_{\nu_d \in \{0,1\}} f\left(\left\|\sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \mathbf{a}_i\right\|^2 + \lambda_d^2 \|\mathbf{a}_d\|^2 + 2(-1)^{\nu_d} \lambda_d \sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \langle \mathbf{a}_i, \mathbf{a}_d \rangle\right) \\ \geq & 2f\left(\left\|\sum_{i=1}^{d-1} (-1)^{\nu_i} \lambda_i \mathbf{a}_i\right\|^2 + \lambda_d^2 \|\mathbf{a}_d\|^2\right). \end{aligned}$$

Proceeding in this way leads to

$$\begin{aligned} & \sum_{\nu_k \in \{0,1\}} f(\|(-1)^{\nu_1} \lambda_1 \mathbf{a}_1 + \cdots + (-1)^{\nu_k} \lambda_k \mathbf{a}_k\|^2 + \lambda_{k+1}^2 \|\mathbf{a}_{k+1}\|^2 + \cdots + \lambda_d^2 \|\mathbf{a}_d\|^2) \\ \geq & 2f\left(\left\|\sum_{i=1}^{k-1} (-1)^{\nu_i} \lambda_i \mathbf{a}_i\right\|^2 + \lambda_k^2 \|\mathbf{a}_k\|^2 + \cdots + \lambda_d^2 \|\mathbf{a}_d\|^2\right) \end{aligned}$$

for $k = d-1, \dots, 2$. Summarizing all these inequalities yields

$$\sum_{\nu_1, \dots, \nu_d \in \{0,1\}} f(\|(-1)^{\nu_1} \lambda_1 \mathbf{a}_1 + \cdots + (-1)^{\nu_d} \lambda_d \mathbf{a}_d\|) \geq 2^d f(\lambda_1^2 \|\mathbf{a}_1\|^2 + \cdots + \lambda_d^2 \|\mathbf{a}_d\|^2)$$

whence it follows together with (3.2) the assertion of

Theorem 1 *If the function $f|(0, \infty) \rightarrow \mathbb{R}^1$ is convex (concave) satisfying (2.2) then*

$$Q_d(f, P_d(\mathbf{a}_1, \dots, \mathbf{a}_d)) \geq (\leq) Q_d(f, C_d(\|\mathbf{a}_1\|, \dots, \|\mathbf{a}_d\|)). \quad (3.3)$$

4 Lower Bounds of $Q_d(f, P_d)$ for non-decreasing f

The volume $V_d(P_d)$ as well as the integral defined in (2.1) are invariant under rigid motions of P_d . In particular, we have $Q_d(f, P_d) = Q_d(f, P_d \mathbf{O})$ for any orthogonal $d \times d$ -matrix \mathbf{O} . We define such an orthogonal matrix by the equations $\mathbf{a}_j \mathbf{O} = \mathbf{b}_j = (b_j^{(1)}, \dots, b_j^{(j)}, 0, \dots, 0)$ and put

$a_j := |b_j^{(j)}| > 0$ for $j = 1, \dots, d$ with $a_1 = \|\mathbf{a}_1\|$, where the components $b_j^{(i)}, 1 \leq i \leq j \leq d$ can be calculated step by step from the equations

$$\langle \mathbf{a}_i, \mathbf{a}_j \rangle = \langle \mathbf{b}_i, \mathbf{b}_j \rangle = \sum_{k=1}^i b_i^{(k)} b_j^{(k)} \quad \text{for } 1 \leq i \leq j \leq d$$

which are equivalent to the recursive relations

$$b_j^{(i)} = \frac{1}{b_i^{(i)}} \left(\langle \mathbf{a}_i, \mathbf{a}_j \rangle - \sum_{k=1}^{i-1} b_i^{(k)} b_j^{(k)} \right) \quad \text{for } i = 1, \dots, j \text{ and } j = 1, \dots, d. \quad (4.1)$$

It is immediately clear that $V_d(P_d) = |\det [\mathbf{b}_1^\top, \dots, \mathbf{b}_d^\top]| = \prod_{j=1}^d a_j = V_d(C_d(a_1, \dots, a_d))$.

$$\begin{aligned} & \| (-1)^{\nu_1} \lambda_1 \mathbf{a}_1 + \dots + (-1)^{\nu_d} \lambda_d \mathbf{a}_d \|^2 = \sum_{j,k=1}^d (-1)^{\nu_j + \nu_k} \lambda_j \lambda_k \langle \mathbf{a}_j, \mathbf{a}_k \rangle \\ &= \sum_{j,k=1}^d (-1)^{\nu_j + \nu_k} \lambda_j \lambda_k \langle \mathbf{b}_j, \mathbf{b}_k \rangle = \sum_{j=1}^d \lambda_j^2 \sum_{i=1}^j (b_j^{(i)})^2 + 2 \sum_{1 \leq j < k \leq d} (-1)^{\nu_j + \nu_k} \lambda_j \lambda_k \sum_{i=1}^j b_j^{(i)} b_k^{(i)} \\ &= \sum_{i=1}^d \sum_{j=i}^d (b_j^{(i)} \lambda_j^2)^2 + 2 \sum_{i=1}^d \sum_{i \leq j < k \leq d} (-1)^{\nu_j + \nu_k} b_j^{(i)} b_k^{(i)} \lambda_j \lambda_k = \sum_{i=1}^d \left(\sum_{j=i}^d (-1)^{\nu_j} b_j^{(i)} \lambda_j \right)^2 \end{aligned}$$

Let $x \mapsto f(x)$ be non-decreasing (non-increasing) and continuous for $x > 0$ and $\int_0^\tau x^{d-1} |f(x)| dx < \infty$ for all $\tau > 0$. Under this assumption we derive a lower (upper) bound of the d -fold integral

$$Q_d(f, P_d(\mathbf{a}_1, \dots, \mathbf{a}_d)) = \int_0^1 \dots \int_0^1 \sum_{\nu_1, \dots, \nu_d \in \{0,1\}} f \left(\left\| \sum_{i=1}^d (-1)^{\nu_i} \lambda_i \mathbf{b}_i \right\|^2 \right) \prod_{i=1}^d (1 - \lambda_i) d\lambda_d \dots d\lambda_1.$$

Using the inequality (2.5) of Lemma 1 with $g(x) = 1 - x$ for $a^2 = \sum_{i=2}^d \left(\sum_{j=i}^d (-1)^{\nu_j} b_j^{(i)} \lambda_j \right)^2$, $b = b_1^{(1)}$ (and $a_1 = |b_1^{(1)}|$) and $c = (-1)^{\nu_1} \sum_{j=2}^d (-1)^{\nu_j} b_j^{(i)} \lambda_j$ we find that

$$\begin{aligned} & \sum_{\nu_1 \in \{0,1\}} \int_0^1 f \left(\sum_{i=1}^d \left(\sum_{j=i}^d (-1)^{\nu_j} b_j^{(i)} \lambda_j \right)^2 \right) (1 - \lambda_1) d\lambda_1 \\ & \geq (\leq) 2 \int_0^1 f \left(a_1^2 \lambda_1^2 + \sum_{i=2}^d \left(\sum_{j=i}^d (-1)^{\nu_j} b_j^{(i)} \lambda_j \right)^2 \right) (1 - \lambda_1) d\lambda_1 \end{aligned}$$

Analogously, we get successively for $k = 2, \dots, d$ that

$$\begin{aligned} & \sum_{\nu_k \in \{0,1\}} \int_0^1 f \left(a_1^2 \lambda_1^2 + \dots + a_{k-1}^2 \lambda_{k-1}^2 + \sum_{i=k}^d \left(\sum_{j=i}^d (-1)^{\nu_j} b_j^{(i)} \lambda_j \right)^2 \right) (1 - \lambda_k) d\lambda_k \\ & \geq (\leq) 2 \int_0^1 f \left(a_1^2 \lambda_1^2 + \dots + a_k^2 \lambda_k^2 + \sum_{i=k+1}^d \left(\sum_{j=i}^d (-1)^{\nu_j} b_j^{(i)} \lambda_j \right)^2 \right) (1 - \lambda_k) d\lambda_k. \end{aligned}$$

In this way we obtain

Theorem 2 *If the function $f|(0, \infty) \rightarrow \mathbb{R}^1$ is continuous and non-decreasing (non-increasing) satisfying (2.2) then*

$$Q_d(f, P_d(\mathbf{a}_1, \dots, \mathbf{a}_d)) \geq (\leq) Q_d(f, C_d(a_1, \dots, a_d)), \quad (4.2)$$

where the edge lengths $a_j = |b_j^{(j)}|$, $j = 1, \dots, d$, are defined by (4.1).

In Section 5 we establish lower resp. upper bounds of the p th-order CPI of d -cuboids C_d in terms of $b_d(C_d)$ resp. $V_d(C_d)$ and the p th-order CPI of the unit cube $[0, 1]^d$.

5 Lower and upper bounds of $\mathcal{I}_p(C_d)$ for $1 < p \leq d$

From (1.4), (2.3) and (3.2) it is easily seen that in case of the d -cuboid $C_d = \times_{i=1}^d [0, a_i]$ the ratio $\mathcal{I}_p(C_d)/V^2(C_d)$ is equal to $(d-q)(d-q+1)2^{d-1} \mathcal{J}_q(a_1, \dots, a_d)$ with the d -fold parameter integral

$$\mathcal{J}_q(a_1, \dots, a_d) := \int_0^1 \cdots \int_0^1 \frac{(1-x_1) \cdots (1-x_d)}{(a_1^2 x_1^2 + \cdots + a_d^2 x_d^2)^{q/2}} dx_d \cdots dx_1 \quad (5.1)$$

for $q = d + 1 - p \in [0, d)$, i.e., $1 < p \leq d + 1$. We mostly write shorthand $\mathcal{J}_q(\mathbf{a})$ with $\mathbf{a} = (a_1, \dots, a_d)$ instead of $\mathcal{J}_q(a_1, \dots, a_d)$.

Definition (see [16]) *For $-\infty \leq a < b \leq \infty$ and $d \geq 2$, a symmetric function $F|(a, b)^d \rightarrow \mathbb{R}^1$ is said to be Schur-convex (Schur-concave) if for every doubly stochastic matrix $\mathbf{S} = (s_{ij})_{i,j=1}^d$ (i.e., $s_{ij} \geq 0$ such that $s_{i1} + \cdots + s_{id} = s_{1j} + \cdots + s_{dj} = 1$ for $1 \leq i, j \leq d$,*

$$F(\mathbf{x}\mathbf{S}) \leq (\geq) F(\mathbf{x}) \text{ for all } \mathbf{x} = (x_1, \dots, x_d) \in (a, b)^d. \quad (5.2)$$

Obviously, F is Schur-concave if and only if $-F$ is Schur-convex.

The following condition which goes back to I. Schur provides a useful criterion to prove Schur-convexity.

Lemma 2 (see [16]) *A symmetric function $F(\mathbf{x}) = F(x_1, \dots, x_d)$ with continuous partial derivatives on $(a, b)^d$ is Schur-convex (Schur-concave) if and only if*

$$(x_1 - x_2) \left(\frac{\partial F(\mathbf{x})}{\partial x_1} - \frac{\partial F(\mathbf{x})}{\partial x_2} \right) \geq (\leq) 0 \text{ for all } \mathbf{x} = (x_1, \dots, x_d) \in (a, b)^d. \quad (5.3)$$

For alternative definitions, historical background and further details related with Schur-convexity the reader is referred to the monographs [7], [8], and [14].

Theorem 3 For $1 \leq q < d$ the mapping $(a_1, \dots, a_d) \mapsto \mathcal{J}_q(a_1, \dots, a_d)$ is Schur-convex on $(0, \infty)^d$. Furthermore, the mapping

$$\mathbf{b} = (b_1, \dots, b_d) \mapsto \mathcal{J}(f; \mathbf{b}) := \int_0^1 \cdots \int_0^1 f\left(\sum_{i=1}^d x_i^2 e^{2b_i}\right) \prod_{i=1}^d (1-x_i) dx_d \cdots dx_1 \quad (5.4)$$

is Schur-convex (Schur-concave) on \mathbb{R}^d if the function $f|_{(0, \infty)} \rightarrow \mathbb{R}^1$ is continuous and non-decreasing (non-increasing) satisfying (2.2).

Proof of Theorem 3. First we apply Schur's criterion (5.3) to show that the symmetric function $\mathcal{J}_q(\mathbf{a})$ is Schur-convex. This means that, for $a_1 \geq a_2 > 0$ and any fixed $a_3, \dots, a_d > 0$, we have to verify the inequality

$$\frac{\partial \mathcal{J}_q(\mathbf{a})}{\partial a_1} \geq \frac{\partial \mathcal{J}_q(\mathbf{a})}{\partial a_2}. \quad (5.5)$$

After differentiation and partial integration w.r.t. x_1 we arrive at

$$\begin{aligned} \frac{\partial \mathcal{J}_q(\mathbf{a})}{\partial a_1} &= -q \int_0^1 \int_0^1 \cdots \int_0^1 \frac{a_1 x_1^2 (1-x_1)(1-x_2) \cdots (1-x_d)}{(a_1^2 x_1^2 + a_2^2 x_2^2 + \cdots + a_d^2 x_d^2)^{q/2+1}} dx_d \cdots dx_2 dx_1 \\ &= \frac{1}{a_1} \int_0^1 \int_0^1 \cdots \int_0^1 x_1 \prod_{i=1}^d (1-x_i) dx_1 \left(\sum_{i=1}^d a_i^2 x_i^2 \right)^{-q/2} dx_2 \cdots dx_d \\ &= -\frac{1}{a_1} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{(1-2x_1)(1-x_2)(1-x_3) \cdots (1-x_d)}{(a_1^2 x_1^2 + a_2^2 x_2^2 + \cdots + a_d^2 x_d^2)^{q/2}} dx_d \cdots dx_2 dx_1 \end{aligned}$$

and, likewise, we get that

$$\frac{\partial \mathcal{J}_q(\mathbf{a})}{\partial a_2} = -\frac{1}{a_2} \int_0^1 \int_0^1 \cdots \int_0^1 \frac{(1-x_1)(1-2x_2)(1-x_3) \cdots (1-x_d)}{(a_1^2 x_1^2 + a_2^2 x_2^2 + \cdots + a_d^2 x_d^2)^{q/2}} dx_d \cdots dx_2 dx_1.$$

Unfortunately, to the best of the authors knowledge, it seems that there is no direct way to prove the relation (5.5). For this reason we rewrite the derivatives $\frac{\partial \mathcal{J}_q}{\partial a_1}$ and $\frac{\partial \mathcal{J}_q}{\partial a_2}$ by means of Laplace transforms. Setting $r := a_1/a_2 \geq 1$ and $r_i := a_i/a_2$ for $i = 2, \dots, d$ and using the identity

$$\frac{\Gamma(q/2)}{s^{q/2}} = \int_0^\infty e^{-st} t^{q/2-1} dt = 2 \int_0^\infty e^{-st^2} t^{q-1} dt$$

for $s = a_1^2 x_1^2 + a_2^2 x_2^2 + \cdots + a_d^2 x_d^2$ with the Laplace transforms

$$u(t) = \int_0^1 e^{-t^2 x^2} (1-x) dx \quad , \quad v(t) = \int_0^1 e^{-t^2 x^2} (1-2x) dx = u(t) - \frac{1-e^{-t^2}}{2t^2} \quad , \quad t \geq 0$$

we obtain that

$$\Gamma\left(\frac{q}{2}\right) \frac{\partial \mathcal{J}_q(\mathbf{a})}{\partial a_1} = -\frac{2}{a_1} \int_0^\infty v(a_1 t) \prod_{i=2}^d u(a_i t) t^{q-1} dt = -\frac{2}{a_1 a_2^q} \int_0^\infty v(rt) u(t) \prod_{i=3}^d u(r_i t) t^{q-1} dt$$

and

$$\Gamma\left(\frac{q}{2}\right) \frac{\partial \mathcal{J}_q(\mathbf{a})}{\partial a_2} = -\frac{2}{a_2} \int_0^\infty v(a_2 t) \prod_{\substack{i=1 \\ i \neq 2}}^d u(a_i t) t^{q-1} dt = -\frac{2}{a_2 a_1^q} \int_0^\infty v\left(\frac{t}{r}\right) u(t) \prod_{i=3}^d u\left(\frac{r_i}{r} t\right) t^{q-1} dt.$$

Hence, (5.5) can be equivalently expressed by

$$\int_0^\infty v(r t) u(t) \prod_{i=3}^d u(r_i t) t^{q-1} dt \leq r^{1-q} \int_0^\infty v\left(\frac{t}{r}\right) u(t) \prod_{i=3}^d u\left(\frac{r_i}{r} t\right) t^{q-1} dt. \quad (5.6)$$

The function $u(t)$ can be calculated by partial integration as follows

$$t u(t) = \int_0^1 \frac{1-x}{2tx} dx (1 - e^{-t^2 x^2}) = - \int_0^1 \frac{1 - e^{-t^2 x^2}}{2t} dx \left(\frac{1-x}{x}\right) = \int_0^1 \frac{1 - e^{-t^2 x^2}}{2t x^2} dx$$

so that

$$u(t) = \frac{1}{2t} \int_0^t \frac{1 - e^{-x^2}}{x^2} dx \quad \text{and} \quad v(t) = \frac{1}{2t} \int_0^t \left(\frac{1 - e^{-x^2}}{x^2} - \frac{1 - e^{-t^2}}{t^2} \right) dx \geq 0.$$

The latter holds since the mapping $t \mapsto (1 - e^{-t^2})/t^2$ is strictly decreasing for $t > 0$. Obviously, the Laplace transform $u(t)$ is strictly decreasing whereas the function $\hat{u}(t) := t u(t)$ is strictly increasing for $t > 0$. Since the derivative $(t v(t))' = [1 - (1 + t^2) e^{-t^2}] t^{-2}$ is strict positive for $t > 0$ the function $\hat{v}(t) := t v(t)$ turns out strictly increasing.

In view of $r \geq 1$ and the monotonicity of $u(t)$ we have $u(r_i t) \leq u\left(\frac{r_i}{r} t\right)$ for all $t > 0$, $r_i > 0$ and $i = 3, \dots, d$. Thus, for proving (5.6) it suffices to show that

$$\int_0^\infty v(r t) u(t) t^{q-1} dt \leq r^{1-q} \int_0^\infty v\left(\frac{t}{r}\right) u(t) t^{q-1} dt$$

which is just the desired inequality for $d = 2$. By substituting $t = s/r$ on the l.h.s. and $t = s r$ on the r.h.s. of the latter inequality we get that

$$\int_0^\infty v(s) u\left(\frac{s}{r}\right) s^{q-1} ds \leq r^{q+1} \int_0^\infty v(s) u(s r) s^{q-1} ds$$

which in turn is equivalent to

$$\int_0^\infty v(s) \hat{u}\left(\frac{s}{r}\right) s^{q-2} ds \leq r^{q-1} \int_0^\infty v(s) \hat{u}(s r) s^{q-2} ds. \quad (5.7)$$

Since $\hat{u}(s/r) \leq \hat{u}(s r)$, the monotonicity of $\hat{u}(t)$ reveals that (5.7) and therefore (5.6) hold at least for $q \geq 1$. In other words, Schur's criterion (5.5) is satisfied for $q \geq 1$.

In the second part we prove that the function $\mathbf{b} \mapsto \mathcal{J}(f; \mathbf{b})$ is Schur-convex on \mathbb{R}^d if $f|_{(0, \infty)} \rightarrow \mathbb{R}^1$ is continuous and non-decreasing. Since $\mathcal{J}(f; b_1, \dots, b_d)$ is symmetric and has continuous

partial derivatives (as seen from the below formula (5.9)) we may apply Lemma 2 in the case of Schur-convexity which means to verify that

$$\frac{\partial \mathcal{J}(f; \mathbf{b})}{\partial b_1} \geq \frac{\partial \mathcal{J}(f; \mathbf{b})}{\partial b_2}. \quad (5.8)$$

for $-\infty < b_2 \leq b_1 < \infty$ and any fixed $b_3, \dots, b_d \in \mathbb{R}^1$, For brevity put $A_1 = e^{2b_1}$, $A_2 = e^{2b_2}$ with $A_1 \geq A_2 > 0$ and $B = e^{2b_3} x_3^2 + \dots + e^{2b_d} x_d^2 \geq 0$. To avoid the differentiation of the function f we apply the partial integration formula for Riemann-Stieltjes integrals yielding

$$\begin{aligned} \int_0^1 (1-x_1) f(A_1 x_1^2 + A_2 x_2^2 + B) dx_1 &= \int_0^1 (1-x_1) dx_1 \left(\int_0^{x_1} f(A_1 y^2 + A_2 x_2^2 + B) dy \right) \\ &= \int_0^1 \int_0^{x_1} f(A_1 x_1^2 + A_2 x_2^2 + B) dy dx_1 = e^{-b_1} \int_0^1 \int_0^{e^{b_1} x_1} f(y^2 + A_2 x_2^2 + B) dy dx_1. \end{aligned}$$

After differentiating w.r.t. b_1 and partial integration w.r.t. x_1 we get the relations

$$\begin{aligned} &\frac{\partial}{\partial b_1} \left(e^{-b_1} \int_0^1 \int_0^{e^{b_1} x_1} f(y^2 + A_2 x_2^2 + B) dy dx_1 \right) \\ &= -e^{-b_1} \int_0^1 \int_0^{e^{b_1} x_1} f(y^2 + A_2 x_2^2 + B) dy dx_1 + e^{-b_1} \int_0^1 e^{b_1} x_1 f(A_1 x_1^2 + A_2 x_2^2 + B) dx_1 \\ &= -e^{-b_1} \int_0^{e^{b_1}} f(y^2 + A_2 x_2^2 + B) dy + 2 \int_0^1 x_1 f(A_1 x_1^2 + A_2 x_2^2 + B) dx_1 \\ &= \int_0^1 (2x_1 - 1) f(A_1 x_1^2 + A_2 x_2^2 + B) dx_1. \end{aligned}$$

This leads to the partial derivatives

$$\frac{\partial \mathcal{J}(f; \mathbf{b})}{\partial b_1} = \int_0^1 \dots \int_0^1 \int_0^1 (2x_1 - 1) \prod_{i=2}^d (1-x_i) f(A_1 x_1^2 + A_2 x_2^2 + B) dx_1 dx_2 \dots dx_d. \quad (5.9)$$

and likewise

$$\frac{\partial \mathcal{J}(f; \mathbf{b})}{\partial b_2} = \int_0^1 \dots \int_0^1 \int_0^1 (2x_2 - 1) \prod_{\substack{i=1 \\ i \neq 2}}^d (1-x_i) f(A_1 x_1^2 + A_2 x_2^2 + B) dx_1 dx_2 \dots dx_d.$$

Hence,

$$\frac{\partial \mathcal{J}(f; \mathbf{b})}{\partial b_1} - \frac{\partial \mathcal{J}(f; \mathbf{b})}{\partial b_2} = \int_0^1 \dots \int_0^1 \int_0^1 (x_1 - x_2) \prod_{i=3}^d (1-x_i) f(A_1 x_1^2 + A_2 x_2^2 + B) dx_1 dx_2 \dots dx_d.$$

In order to prove that the d -fold integral on the r.h.s. takes non-negative values it suffices to show that

$$h(f; A_1, A_2) := \int_0^1 \int_0^1 (x_1 - x_2) f(A_1 x_1^2 + A_2 x_2^2 + B) dx_1 dx_2 \geq 0 \quad \text{iff} \quad A_1 \geq A_2.$$

For this we rewrite $h(f; A_1, A_2)$ as follows:

$$\begin{aligned} h(f; A_1, A_2) &= \int_0^1 \int_0^{x_1} (x_1 - x_2) f(A_1 x_1^2 + A_2 x_2^2 + B) dx_2 dx_1 \\ &+ \int_0^1 \int_0^{x_2} (x_1 - x_2) f(A_1 x_1^2 + A_2 x_2^2 + B) dx_1 dx_2 \\ &= \int_0^1 \int_0^1 (x_1 - x_1 y) x_1 f(A_1 x_1^2 + A_2 x_1^2 y^2 + B) dy dx_1 \\ &+ \int_0^1 \int_0^1 (x_2 x - x_2) x_2 f(A_1 x_2^2 x^2 + A_2 x_2^2 + B) dx dx_2 \\ &= \int_0^1 \int_0^1 x^2 (1 - y) (f(A_1 x^2 + A_2 x^2 y^2 + B) - f(A_1 x^2 y^2 + A_2 x^2 + B)) dy dx. \end{aligned}$$

Obviously, $f(A_1 x^2 + A_2 x^2 y^2 + B) \geq f(A_1 x^2 y^2 + A_2 x^2 + B)$ for all $x, y \in [0, 1]$ iff $A_1 \geq A_2$ which confirms (5.10) and hence (5.8) for a non-decreasing function f . The reverse inequality (5.8) for a non-increasing function f follows by applying the above arguments to $-f$. Thus, Theorem 3 is completely proved. \square

Corollary 1 For $1 \leq q < d$ the parameter integral (5.1) allows the inclusion

$$\frac{\mathcal{J}_q(1, \dots, 1)}{(\text{AM}(\mathbf{a}))^q} \leq \mathcal{J}_q(\mathbf{a}) \leq \frac{\mathcal{J}_q(1, \dots, 1)}{(\text{GM}(\mathbf{a}))^q} \quad \text{for all } \mathbf{a} = (a_1, \dots, a_d) \in (0, \infty)^d, \quad (5.10)$$

where $\text{AM}(\mathbf{a}) := (a_1 + \dots + a_d)/d$ and $\text{GM}(\mathbf{a}) := (a_1 \dots a_d)^{1/d}$.

In particular, $\inf\{\mathcal{J}_q(r_1, \dots, r_d) : r_1, \dots, r_d \geq 0, r_1 + \dots + r_d = 1\} = \mathcal{J}_q(1/d, \dots, 1/d)$.

Proof of Corollary 1. Since $\mathcal{J}_q(t\mathbf{a}) = t^{-q} \mathcal{J}_q(\mathbf{a})$ for $t > 0$ we have

$$\mathcal{J}_q(a_1, \dots, a_d) = \frac{\mathcal{J}_q(r_1, \dots, r_d)}{(a_1 + \dots + a_d)^q} \quad \text{with} \quad r_i = a_i / (a_1 + \dots + a_d), \quad i = 1, \dots, d.$$

Choosing a doubly stochastic matrix \mathbf{S}^* with identical entries equal to $s_{ij}^* = 1/d$ the Schur-convexity of $a \mapsto \mathcal{J}_q(\mathbf{a})$ implies that $\mathcal{J}_q(\mathbf{r}) \geq \mathcal{J}_q(\mathbf{r} \mathbf{S}^*) = \mathcal{J}_q(1/d, \dots, 1/d) = d^q \mathcal{J}_q(1, \dots, 1)$ for all $\mathbf{r} = (r_1, \dots, r_d)$ satisfying $r_1, \dots, r_d \geq 0$ and $r_1 + \dots + r_d = 1$. Combining this with the foregoing equality yields the lower bound of (5.10). The upper bound of (5.10) follows from the second assertion of Theorem 3 for the strictly decreasing function $f(x) = x^{-q/2}$, and $b_i = \log a_i$ for $i = 1, \dots, d$ and $\bar{b} := \text{AM}(\mathbf{b}) = (b_1 + \dots + b_d)/d = \log(\text{GM}(\mathbf{a}))$. $\mathcal{J}(f; \mathbf{b}) \leq \mathcal{J}(f; \mathbf{b} \mathbf{S}^*) = \mathcal{J}_q(\exp\{\bar{b}\}, \dots, \exp\{\bar{b}\}) = \exp\{-q\bar{b}\} \mathcal{J}_q(1, \dots, 1) = (\text{GM}(\mathbf{a}))^{-q} \mathcal{J}_q(1, \dots, 1)$. \square

Next, we formulate a Pfiefer-type inequality for d -parallelotopes. Pfiefer's original result says that, for given $V_d(K) > 0$ and strictly decreasing f on $(0, \infty)$ satisfying (2.2), the functional (2.3) yields the maximum for balls with radius $V_d(K)^{1/d}$, see [10] or [13] (p. 363).

Corollary 2 *If $f|(0, \infty) \rightarrow \mathbb{R}^1$ is continuous and non-increasing satisfying (2.2), then*

$$Q_d(f, P_d(\mathbf{a}_1, \dots, \mathbf{a}_d)) \leq Q_d(f, V_d(P_d(\mathbf{a}_1, \dots, \mathbf{a}_d))^{1/d} [0, 1]^d). \quad (5.11)$$

In other words, among all d -parallelotopes P_d with given volume $V_d(P_d) > 0$, precisely the cubes provide the maximum of the functional $Q_d(f, P_d)$.

Proof of Corollary 2. In view of (4.2) and $V_d(P_d(\mathbf{a}_1, \dots, \mathbf{a}_d)) = V_d(C_d(a_1, \dots, a_d))$, where the edge lengths $a_j = |b_j^{(j)}|$, $j = 1, \dots, d$, are defined by (4.1), it suffices to show that $Q_d(f, C_d(a_1, \dots, a_d)) \leq Q_d(f, (a_1 \cdot \dots \cdot a_d)^{1/d} [0, 1]^d)$. Since $f|(0, \infty) \rightarrow \mathbb{R}^1$ is continuous and non-increasing we may apply Theorem 3 to the Schur-concave mapping $(\log a_1, \dots, \log a_d) = \mathbf{b} \mapsto \mathcal{J}(f; \mathbf{b})$ and take \mathbf{S}^* as in the proof of Corollary 1. Thus, we get the desired inequality

$$Q_d(f, C_d(a_1, \dots, a_d)) = 2^d \mathcal{J}(f; \mathbf{b}) \leq 2^d \mathcal{J}(f; \mathbf{b} \mathbf{S}^*) = Q_d(f, (a_1 \cdot \dots \cdot a_d)^{1/d} [0, 1]^d).$$

□

Corollary 3 *Let $P_d = P_d(\mathbf{a}_1, \dots, \mathbf{a}_d)$ be a d -parallelotope spanned by linearly independent vectors $\mathbf{a}_1, \dots, \mathbf{a}_d \in \mathbb{R}^d$. Then the inclusion*

$$\left(\frac{2\kappa_{d-1}}{\kappa_d} \right)^{d+1-p} \frac{\mathcal{I}_p([0, 1]^d)}{b_d(P_d)^{d+1-p}} = \frac{d^{d+1-p} \mathcal{I}_p([0, 1]^d)}{(\|\mathbf{a}_1\| + \dots + \|\mathbf{a}_d\|)^{d+1-p}} \leq \frac{\mathcal{I}_p(P_d)}{V_d(P_d)^2} \leq \frac{\mathcal{I}_p([0, 1]^d)}{V_d(P_d)^{(d+1-p)/d}}$$

holds for $1 \leq p \leq d$. This means that for given mean breadth $b_d(P_d)$ (resp. volume $V_d(P_d)$) the ratio $\mathcal{I}_p(P_d)/V_d(P_d)^2$ attains its minimum (resp. maximum) for cubes with edge length $(\|\mathbf{a}_1\| + \dots + \|\mathbf{a}_d\|)/d$ (resp. $V_d(P_d)^{1/d}$). Moreover, P_d satisfies the inequalities

$$\mathcal{I}_p(P_d) \begin{cases} \leq V_d(P_d)^{(d+p-1)/d} \mathcal{I}_p([0, 1]^d) & \text{for } 1 \leq p \leq d+1, \\ \geq V_d(P_d)^{(d+p-1)/d} \mathcal{I}_p([0, 1]^d) & \text{for } p \geq d+1. \end{cases} \quad (5.12)$$

with equality for a cube with edge length $V_d(P_d)^{1/d}$.

Proof of Corollary 3. The equality of the lower bounds in the asserted inclusion follows from (2.7). For $p = 1$, the r.h.s. of the inclusion is trivial since $\mathcal{I}_1(K) = \frac{1}{2} d \kappa_d V_d(K)$ for any convex

body K , whereas the l.h.s. is just the volume inequality $V_d(P_d) \leq ((\|\mathbf{a}_1\| + \dots + \|\mathbf{a}_d\|)/d)^d$ which follows directly by comparing the lower and upper bound for $p = d$. For $1 < p \leq d$, the desired lower bound of $\mathcal{I}_p(P_d)/V_d(P_d)^2$ is obtained by combining the inequality (3.3) applied to the convex function $f(x) = x^{-(d+1-p)/2}$ (which satisfies (2.2)) with the lower bound of (5.10) and the fact that $\mathcal{I}_p(C_d) = V_d(C_d)^2 p(p-1)2^{d-1} \mathcal{J}_{d+1-p}(a_1, \dots, a_d)$ for $C_d = \times_{i=1}^d [0, a_i]$. Similarly, the upper bound of $\mathcal{I}_p(P_d)/V_d(P_d)^2$ follows by combining (4.2) applied to the non-increasing function $f(x) = x^{-(d+1-p)/2}$ with the upper bound of (5.10). Finally, the bounds in (5.12) are obtained from (5.11) applied to the non-increasing functions $f(x) = x^{-(d+1-p)/2}$ for $1 \leq p \leq d+1$ and $f(x) = -x^{(p-d-1)/2}$ for $p \geq d+1$. \square

Remark Both inequalities of (5.12) are stronger than those of (1.7) for $K = P_d$ since $\mathcal{I}_k([0, 1]^d) \leq (\geq) \mathcal{I}_k(\mathbb{B}^d)/\kappa_d^{(d+k-1)/d}$ for $1 \leq k \leq d+1$ ($k \geq d+1$) which also follows from (1.7). Note that the lower bound of $\mathcal{I}_p(P_d)/V_d(P_d)^2$ in Corollary 3 is a still unproved for $d < p < d+1$. The crucial point is to show the first assertion of Theorem 3 for $0 < q < 1$.

To conclude with we give the explicit values for the second-order CPI of squares $[0, a]^2$ and the third-order CPI of cubes $[0, a]^3$ with edge-length $a > 0$. Using (1.4) and (5.1) we obtain after rather lengthy calculations that

$$\begin{aligned} \mathcal{I}_2([0, a]^2) &= \int_{[0, a]^2} \int_{[0, a]^2} \frac{d\mathbf{x} d\mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|} = 4a^3 \mathcal{J}_1(1, 1) \approx 0.97881799 a^3, \\ \mathcal{I}_3([0, a]^3) &= 3 \int_{[0, a]^3} \int_{[0, a]^3} \frac{d\mathbf{x} d\mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|} = 24a^5 \mathcal{J}_1(1, 1, 1) \approx 5.64693794 a^5, \end{aligned}$$

where

$$\begin{aligned} \mathcal{J}_1(1, 1) &= \int_0^1 \int_0^1 \frac{(1-x_1)(1-x_2)dx_2 dx_1}{\sqrt{x_1^2 + x_2^2}} = \log(1 + \sqrt{2}) - \frac{\sqrt{2}-1}{3} \approx 0.2447045, \\ \mathcal{J}_1(1, 1, 1) &= \int_0^1 \int_0^1 \int_0^1 \frac{(1-x_1)(1-x_2)(1-x_3)dx_3 dx_2 dx_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} = \arcsin\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right) - \frac{\pi}{2} \\ &+ \frac{\log((2+\sqrt{3})(1+\sqrt{2}))}{4} + \frac{1+\sqrt{2}-2\sqrt{3}}{20} \approx 0.2352891. \end{aligned}$$

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