

dc-biased stationary transport in the absence of dissipation

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We obtain stationary transport in a Hamiltonian system with ac driving in the presence of a dc bias. A particle in a periodic potential under the influence of a time-periodic field possesses a mixed phase space with regular and chaotic components. An additional external dc bias allows to separate effectively these structures. We show the existence of a stationary current which originates from the persisting invariant manifolds (regular islands, periodic orbits, and cantori). The transient dynamics of the accelerated chaotic domain separates fast chaotic motion from ballistic type trajectories which stick to the vicinity of the invariant submanifold. Experimental studies with cold atoms in laser-induced optical lattices are ideal candidates for the observation of these unexpected findings.

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Bridging microscopics and macroscopics remains the ultimate goal in statistical mechanics, be it in or out of equilibrium. A considerable amount of work has been devoted to investigate this problem for low-dimensional stationary non-equilibrium states [1, 2, 3]. Stationary transport states under an external bias are often connected to nonzero dissipation (phase space volume contraction and violation of time reversal invariance), which, in turn, is equivalent to positive entropy production (for a recent review see Ref. [4]). Attempts to keep time reversal invariance by using Nose-Hoover thermostats [2] did not leave the grounds of dissipative systems and phase space volume contraction.

The interface between microscopic dynamics and statistical evolution and the role of nonlinearity and dissipation in nonequilibrium stationary states is an active area of research. Modern experimental studies provide an ideal testing ground to explore these problems using manipulations with cold atom ensembles in optical potentials [5]. The driven pendulum, a paradigmatic system for the study of dynamical chaos [6], is realized using a periodically modulated optical standing wave, with the possibility to control the strength of dissipation down to arbitrarily small values [5]. For example, recent cold atoms experiments [7] study the crossover from dissipative to Hamiltonian ratchets, confirming theoretical predictions [8].

In this Letter we explore the route of obtaining stationary transport using a dc bias in the absence of dissipation and corresponding phase space volume contraction. A necessary condition for such a stationary, bias driven current to occur is a *mixed* phase space [6] with coexisting regular invariant manifolds and chaotic regions for the unbiased case. A consequence of this coexistence is that directed transport may arise locally on regular invariant components of phase space. While the chaotic phase space regions transform into accelerated evolution for any arbitrary small value of the dc bias, the regu-

lar invariant submanifolds (unstable periodic orbits, regular islands and cantori with nonzero mean velocities), initially embedded in the chaotic region, persist under a finite dc bias. Moreover *unstable* periodic orbits at zero dc bias lead to the appearance of *stable* islands for some finite nonzero value of the dc bias. The presence of cantori [9] and the corresponding sticking of chaotic trajectories in the vicinity of regular manifolds is manifested by the chaotic phase space part which, though accelerated, shows large time separation from chaotic trajectories which are initially located far from the regular structures.

We consider the canonical Hamiltonian model of a particle moving in a one-dimensional space-periodic potential $U(X) = \frac{1}{2\pi} \cos(2\pi x)$ under the influence of a time-periodic space-homogenous external field $E(t) = E_{ac} \sin(\omega t)$ [6], to which we add an external dc bias E_b :

$$\dot{x} = p, \quad \dot{p} = \sin(2\pi x) + E_{ac} \sin(\omega t) - E_b \quad (1)$$

Due to time and space periodicity of the system we can map the original three-dimensional phase space (x, p, t) onto a two-dimensional cylinder, $\mathbb{T}^2 = (x \bmod 1, p)$, by using the stroboscopic Poincaré section after each period $T = 2\pi/\omega$, cf. Fig.1(a). Although the dynamics is periodic in x direction, it contains the complete information about transport in the extended system (1). The velocity $\dot{x} = p$ along a trajectory is the same for both cases.

In the case of zero dc bias, $E_b = 0$, the phase space of the system is characterized by the presence of a stochastic layer which originates from the destroyed separatrix of the undriven case $E_{ac} = 0$ [6]. The chaotic layer is confined by transporting KAM-tori, which originate from perturbed trajectories of particles with large kinetic energies. These tori are *noncontractible* (since they cannot be continuously contracted to a point on \mathbb{T}^2), and separate the cylinder.

The stochastic layer is not uniform and contains regular invariant manifolds. There exists a whole hierarchy of

embedded regular islands, which are filled by *contractible* tori, cf. Fig.1(a). The centers of the islands contain elliptic periodic orbits $\hat{\mathbf{X}}_{T_p}(t) = \{x_{T_p}(t), p_{T_p}(t)\}$ with the period $T_p = kT$, $k = 1, 2, \dots$ and integer shifting distance L , such that

$$x_{T_p}(t + T_p) = x_{T_p}(t) + L, \quad p_{T_p}(t + T_p) = p_{T_p}(t) \quad (2)$$

The transport properties of such an orbit (and of all trajectories inside the corresponding island) are characterized by a winding number $\nu = L/T_p$. If $\nu \neq 0$ then that PO and the corresponding island are transporting. In addition, within the chaotic sea there is an infinite number of unstable periodic orbits (UPO) with different periods and winding numbers. UPOs are unstable with respect to small perturbations and they are not isolated from the chaotic layer. Finally *cantori* [9] exist which generate semipenetrable barriers in phase space.

Eq.(1) is invariant under time reversal

$$S : t \rightarrow -t + T/2, \quad x \rightarrow x, \quad p \rightarrow -p, \quad (3)$$

which changes the sign of the current $J = \nu$. Thus, transporting invariant POs and islands appear as symmetry-related pairs in phase space with opposite winding numbers.

Contrary to the common expectation that all trajectories acquire unbounded acceleration for nonzero dc bias $E_b > 0$, recent numerical evidence tells that this is not true for all trajectories [10]. In Fig.1(b) we plot the escape time T_{esc} to be accelerated below the threshold $p_t = -10$ as a function of initial conditions $\{x_0, p_0\}$ for nonzero E_b . The window $\{x_0, p_0\}$ coincides with Fig.1(a). The escape time is reached by a trajectory if $p(T_{esc}) = -10$, and we integrate up to $t = 200$. While most of the chaotic layer in Fig.1(a) is quickly accelerated, regular islands persist and their trajectories are in fact not accelerated at all. Moreover we observe phase space regions with delayed acceleration. These trajectories stick for large time to the persisting regular islands.

Noncontractible KAM-tori do not survive in the biased Hamiltonian system. If there exists at least one accelerating trajectory, then KAM-tori do not persist [11]. Such accelerating trajectories always exist for $E_b \neq 0$, both with $E_{ac} = 0$ and $E_{ac} \neq 0$.

Let us give analytical proof that POs at $E_b = 0$ persist for nonzero E_b . In order to formulate the proof we define the stability property of a PO $\hat{\mathbf{X}}_{T_p}(t)$. Towards this goal we linearize the phase space flow around it, and map it onto itself by integrating over one period T_p . The resulting 2×2 symplectic Floquet matrix \mathbf{M} has eigenvalues (Floquet multipliers) λ_1 and λ_2 with $\lambda_1 \lambda_2 = 1$ [12]. For a stable PO, both multipliers are located on the unit circle, while for an UPO, both multipliers are located either both on the negative or positive real axis.

Denote a solution of (1) with initial conditions $\mathbf{X}_0 = \{x_0, p_0\}$ at $t = 0$ by $\hat{\mathbf{X}}(t, \mathbf{X}_0, E_b) =$

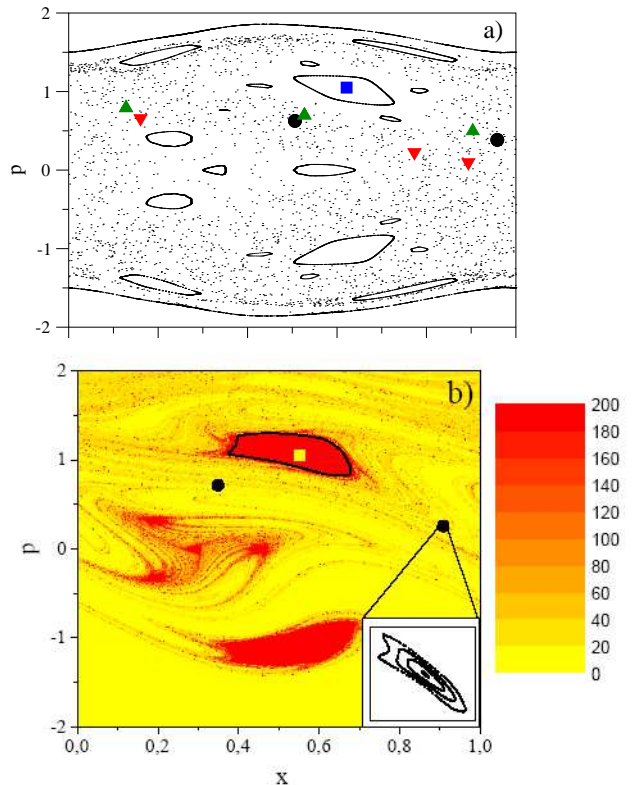


FIG. 1: (a) Poincaré section for the system Eq.(1), $\omega = 2\pi, T = 1, E_{ac} = 5.8$, for (a) $E_b = 0$ and (b) $E_b = 0.183$. Several POs are shown: one stable ($k = 1, L = 1$)(squares), and several unstable POs ($k = 2, L = 1$)(circles), ($k = 3, L = 2$)(upward triangles), and ($k = 3, L = 1$)(downward triangles). The island group near $p = 0$ has zero winding number $\nu = 0$. (b) Time T_{esc} to be accelerated below the threshold $p_t = -10$ as a function of the initial conditions in phase space. Trajectories from the islands are not accelerated at all. Inset in Fig.1(b): Poincare section zoom of phase space structure showing the transformation of an UPO into a stable PO with a surrounding invariant regular island.

$\{x(t, x_0, p_0, E_b), p(t, x_0, p_0, E_b)\}$. If \mathbf{X}_0 is close enough to $\hat{\mathbf{X}}_{T_p}(0)$ and E_b is sufficiently small, then the solution $\hat{\mathbf{X}}(t, \mathbf{X}_0, E_b)$ on the interval $[0, T_p]$ is close to the unperturbed PO. $\hat{\mathbf{X}}$ is T_p -periodic if and only if the vector function

$$\mathbf{F}(\mathbf{X}_0, E_b) = \begin{pmatrix} x(T_p, x_0, p_0, E_b) - x_0 - L \\ p(T_p, x_0, p_0, E_b) - p_0 \end{pmatrix} = \mathbf{0}. \quad (4)$$

Since $\hat{\mathbf{X}}(t, x_{T_p}(0), p_{T_p}(0), 0) \equiv \hat{\mathbf{X}}_{T_p}(t)$, Eq. (4) is satisfied by $\hat{\mathbf{X}}_{T_p}(0)$ at $E_b = 0$. By the Implicit Function Theorem [13], if the Jacobian of the vector function $\mathbf{F}(\mathbf{X}_0, E_b)$ with respect to \mathbf{X}_0 is non-zero at $\hat{\mathbf{X}}_{T_p}(0)$, $E_b = 0$, then the solution $\hat{\mathbf{X}}_{T_p}(0)$ can be continued to $E_b \neq 0$, $\mathbf{X}_0(E_b) = \{x_0(E_b), p_0(E_b)\}$ which corresponds to a T_p -periodic orbit of the perturbed system. The Jacobian of the function (4) at $\hat{\mathbf{X}}_{T_p}(0)$ is $\det \partial \mathbf{F} / \partial \mathbf{X} = \det[\mathbf{M} - \mathbf{I}]$, where \mathbf{I} is the unity matrix. The Jacobian is nonzero if

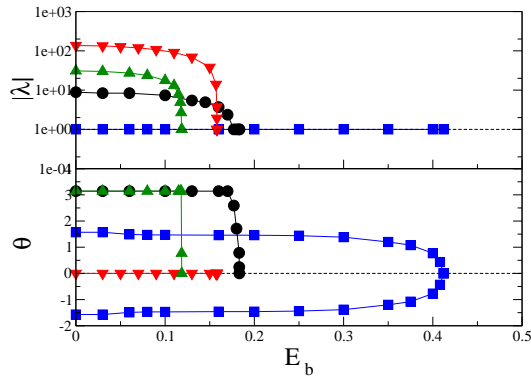


FIG. 2: Variation of Floquet multipliers, $\lambda_j = |\lambda_j|e^{i\theta_j}$, of PO's vs bias E_b for Eq.(1) with parameters and symbols as in Fig.(1). We show both multipliers for stable PO (squares) and $|\lambda_1| > 1$ for unstable POs.

$\lambda_{1,2} \neq 1$ for **M**. Thus, almost all POs of the unperturbed system persist for nonzero dc bias since generically their Floquet multipliers are not equal to unity. POs can be continued up to a value of E_b where their Floquet multipliers collide at +1. Linearly stable POs in Hamiltonian systems are always enclosed by quasiperiodic tori, which form a regular island. Thus, the transporting contractible islands also persist for nonzero dc bias. Note that the symmetry (3) persists for nonzero E_b . Consequently any invariant transporting manifolds which persist for nonzero E_b come in pairs and stationary currents occur in both directions [8].

We expect that for large enough dc bias $E_b \gg 1$, a transporting PO will disappear. This occurs when its Floquet multipliers collide at +1, $\lambda_1(E_b^c) = \lambda_2(E_b^c) = 1$. This is a saddle-center bifurcation [14], when two POs of the same period coalesce, one of them being initially stable and the other one unstable. Since stable POs are enclosed by regular islands, this can happen only when the corresponding island surrounding the stable PO shrinks to zero.

In Fig.1(a) we depict the phase space structure of Eq.(1) for $\omega = 2\pi, E_{ac} = 5.8$ and $E_b = 0$. Using the Newton method, cf. e.g. [14], we find stable and unstable POs (various symbols in Fig.1), and continue them to nonzero values of E_b . Stable POs exist up to $E_b \approx 0.412$. The evolution of the corresponding Floquet multipliers $\lambda = |\lambda| \exp(i\theta)$ is shown in Fig.2.

Transporting UPOs become more stable with increasing dc bias, since they have to acquire $\lambda_{1,2} = 1$ at a critical dc bias value before disappearing. Moreover, UPOs with negative real Floquet multipliers have to become stable POs before a critical bifurcation happens, since the only way to move the multipliers from the negative real axis to +1 is to transport them to -1 and subsequently to shift them along the unit circle. During that last stage an originally unstable PO transforms into a stable PO, and a new transporting regular island emerges

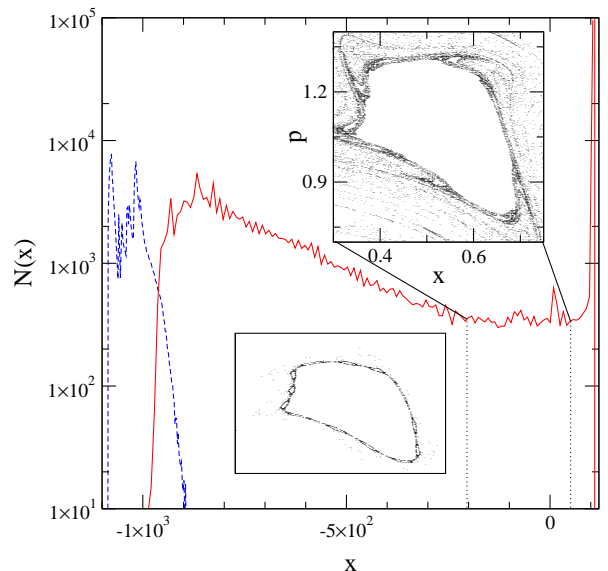


FIG. 3: Solid line: spatial distribution $N(x)$ after $t = 100$ for $\omega = 2\pi, E_{ac} = 5.8$ and $E_b = 0.183$ and when $2.5 \cdot 10^4$ trajectories started out from the uniform distribution in the rectangle ($x \in [0.3, 0.75], p \in [0.7, 1.4]$), which enclose the transporting island with $v = 1$. The upper inset shows initial conditions which correspond to the tail of $N(x)$, $x \in [-200, 50]$. Lower inset depicts the Poincare section of a single trajectory that started out in the "sticky" vicinity of the island. Dashed line: distribution $N(x)$ with initial conditions shifted by $\Delta p_0 = -2.2$ and additional shift in x (see text).

in phase space. Thus, along with an overall shrinking of the size of transporting regular islands with increasing E_b , new islands are born as well. In the inset in Fig1(b) a zoom of the Poincare section shows such a case. For $E_b = 0$ we found an UPO with $T_p = 2$ and winding number $v = 1/2$ and $\lambda_1 \approx -8.365, \lambda_2 = 1/\lambda_1 \approx -0.119$. This orbit becomes stable at $E_b \approx 0.178$ (when the multipliers collide at -1 and enter the unit circle) and disappears at $E_b \approx 0.185$, when $\lambda_1 = \lambda_2 = 1$. In between these two bias values the now stable PO is surrounded by a regular island of finite size.

For any PO the momentum $p(t)$ is a periodic function and can be represented in the form $p(t) = v + p^0(t)$, such that $p^0(t + T_p) = p^0(t)$, $\langle p^0(t) \rangle_{T_p} = 0$. Then we obtain the energy balance equation:

$$vE_b = \langle p^0(t)E_{ac}(t) \rangle_{T_p}. \quad (5)$$

An external ac-field mimics an additional effective friction in the nonequilibrium Nosé-Hoover oscillator [15]: it pumps energy into the system in order to compensate the work against bias (for POs with positive v) and acts as an energy sink to compensate the acceleration in the case of current along a bias (for POs with negative v). But in our case, at variance with the Nosé-Hoover system, the generating equations are Hamiltonian and dissipationless.

All trajectories except the regular islands and the POs will be accelerated without bound for nonzero E_b . Nevertheless, the mixed phase space structure for $E_b = 0$ leads to a strongly nonuniform acceleration in time for trajectories with different initial conditions. That can be already observed in Fig.1(b). Indeed, for $E_b = 0$ chaotic trajectories in the vicinity of regular islands may stick to the island boundary for long times and produce a unidirectional flight with the velocity v due to the *stickiness* effect [16]. The origin of such a behavior is the presence of partial barriers in phase space formed by cantori [9]. This feature is also observed on finite times for nonzero E_b . Trajectories may stick for long times to regular island boundaries without acceleration (see Poincaré section in lower inset in Fig.3). Only after some rather large escape times T_{esc} such a trajectory will eventually leave the island vicinity and suddenly accelerate. In Fig.3 (solid line) we compute the distribution $N(x)$ of displacements of trajectories with initial conditions uniformly covering a phase space part which includes a transporting island. We observe that the transient time T_{esc} can be extremely long for initial conditions in the island's vicinities (see also Fig.1(b)). In Fig.3 the right peak corresponds to initial conditions inside the island (no acceleration for all times). The left front peak corresponds to initial conditions that are accelerated right from the beginning. In the absence of trajectories which stick to regular islands with a broad distribution of escape times, the two peaks would be separated by a depleted region, i.e. a gap. Instead we observe a smeared out distribution which corresponds to trajectories with long transient sticking to the island boundary. Indeed, initial conditions which correspond to the gap region stick to the island (upper inset in Fig.3). A shift of the initial conditions by $\Delta p_0 = -2.2$ down into the homogenous part of the phase space in Fig.1(b) yields a distribution $N(x)$ which contains a single peak due to immediate acceleration (Fig.3, dashed line). Note that we shifted $N(x)$ here by $\Delta x_0 = 220$ which occurs for particles with $V(x) = 0$ and $E_{ac} = 0$ if the initial momentum is shifted by $\Delta p_0 = -2.2$.

In conclusion, we show that stationary, dissipationless transport is sustained by a nonzero dc bias. The dc bias provides a unique possibility to separate the different parts of a mixed phase space, since it accelerates the chaotic part away and leaves the regular invariant manifolds basically untouched. Moreover, even within the accelerated part, proper distribution functions after finite acceleration times reveal the intriguing properties of surviving cantori and sticking. We give an analytical proof for the persistence of periodic orbits and thus for regular islands. We found evidence for the persistence of cantori as well. Our results are not limited to the case in Eq.(1) and are valid as well for the general case of a potential $U(t, x) = U(t + T, x) = U(t, x + L)$. The presence

of weak dissipation will not change the situation drastically. Since it can be treated as a smooth perturbation, POs will persist for nonzero damping. Stable POs become attractors (limit cycles), and unstable POs saddles [12]. A side result is then that an infinite number of stable POs for zero damping will lead to an infinite number of attractors for infinitesimally weak damping [17].

A possible experimental realization can be obtained with cold atoms in a standing wave potential [5], where an additional dc-bias can be realized by a gravitational force. In such a case, all of the above results could be verified. The persistence of transporting regular islands and POs can be used for the preparation of monochromatic matter waves.

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