

# Forcing inertial Brownian motors: efficiency and negative differential mobility

Marcin Kostur<sup>a</sup> Lukasz Machura<sup>a,b</sup> Peter Hänggi<sup>a</sup>  
 Jurek Luczka<sup>b</sup> Peter Talkner<sup>a</sup>

<sup>a</sup>*Institute of Physics, University of Augsburg, Universitätsstrasse 1, D-86135 Augsburg, Germany*

<sup>b</sup>*Institute of Physics, University of Silesia, P-40-007 Katowice, Poland*

---

## Abstract

The noise-assisted, directed transport in a one-dimensional dissipative, inertial Brownian motor of the rocking type that is exposed to an external bias is investigated. We demonstrate that the velocity-load characteristics is distinctly non-monotonic, possessing regimes with a *negative differential mobility*. In addition, we evaluate several possible efficiency quantifiers which are compared among each other. These quantifiers characterize the mutual interplay between the viscous drag and the external load differently, weighing the inherent rectification features from different physical perspectives.

*Key words:* inertial Brownian motor, efficiency, negative differential mobility

*PACS:* 05.60.Cd, 05.40.-a, 05.45.-a

---

## 1 Introduction

Brownian motors are small physical micro- or even nano-machines that operate far from thermal equilibrium by extracting the energy from both, thermal and nonequilibrium fluctuations in order to generate work against external loads [1,2,3,4,5,6]. They present the physical analogue of bio-molecular motors that also work out of equilibrium to direct intracellular transport and to control motion and sensation in cells [7]. The most popular models assume an overdamped Brownian dynamics [8,9,10,11,12,13]. In many situations, such as in biological applications, such a simplification can be well justified from physical grounds. There exist several situations, however, where

---

*Email address:* Marcin.Kostur@physik.uni-augsburg.de (Lukasz Machura).

the inertial effects are prominent [14,15]; being intrinsically the case for quantum Brownian motors [16,17]. In this paper we will deal with inertial Brownian motors [18,19,20,21,22,23,24,25]. The underlying deterministic dynamics can be chaotic [18,19,23,26,27] and thus it is distinctly more complex than its overdamped counterpart [8,28]. Despite an abundance of research works dealing with numerous variants of Brownian motors and Brownian ratchets [1,2,3,4,5,6], there remain still intriguing features awaiting to be discovered. This present study is to the best of our knowledge the first work that considers the behavior of the noise-activated, *directed current* of an inertial Brownian motor versus an *external bias*; thus yielding the *velocity-load behavior* when inertial effects dominate. Here, we will demonstrate that a rocked, inertial Brownian motor degree of freedom, if put to work against a load, can exhibit *negative differential mobility*. This striking phenomenon has been observed within a quantum mechanical setting for electron transfer phenomena [29] or for ac-dc-driven tunnelling transport [30], in the dynamics of cooperative Brownian motors [31,32,33], Brownian transport with complex topology (entropic ratchets) [34,35,36,37,38,39] and in some stylized, multistate models with state-dependent noise [40,41], to name but a few.

Furthermore, we also investigate the efficiency for this forced Brownian inertial transport; in this case, it is possible to devise and to compare several qualifiers characterizing the efficiency of energy conversion and of rectification.

## 2 Biased, rocked inertial Brownian motor

Upon introducing an appropriate scaling of time and length (the details are elaborated in Ref. [24]) the dynamics of a massive Brownian particle can be written in dimensionless form; i.e.,

$$\ddot{x} + \gamma\dot{x} = -V'(x) + F + a \cos(\omega t) + \sqrt{2\gamma D} \xi(t), \quad (1)$$

where  $\gamma$  denotes the friction coefficient,  $V(x) = V(x+1)$  is a spatially periodic and asymmetric ratchet potential (i.e. no reflection symmetry holds) with both, the period and the barrier height set equal to one. The quantity  $F$  denotes the external, constant load force. Additionally, the particle is driven by an unbiased, time-periodic force of amplitude  $a$  and angular frequency  $\omega$ . The interaction with the thermal bath is modeled by white Gaussian noise  $\xi(t)$  with auto-correlation function  $\langle \xi(t)\xi(s) \rangle = \delta(t-s)$ , satisfying Einstein's fluctuation-dissipation relation.  $D$  stands for the re-scaled noise intensity and as such it is proportional to the physical temperature.

For the ratchet potential  $V(x)$  we choose a linear superposition of three spatial harmonics [24]; i.e.,

$$V(x) = V_0[\sin(2\pi x) + c_1 \sin(4\pi x) + c_2 \sin(6\pi x)], \quad (2)$$

where  $V_0$  normalizes the barrier height to unity and the parameters  $c_1$  and  $c_2$  determine the specific ratchet profile. Below, we analyze in detail the case when  $c_1 = 0.245$  and  $c_2 = 0.04$ , yielding  $V_0 \simeq 0.461\dots$

### 3 Rectification efficiency in presence of friction and load

The efficiency of a machine is defined as the ratio of the power  $P = F\langle v \rangle$  done against an external force  $F$  and the input power  $P_{in}$ , i.e.  $\eta = P/P_{in}$ . The same definition of *efficiency of energy conversion* was used for Brownian motors [3,4,5,42,43]:

$$\eta_E = \frac{F\langle v \rangle}{P_{in}}. \quad (3)$$

A grave disadvantage of such a characterization is that it yields a vanishing measure (i.e.  $\eta_E = 0$ ) in the absence of a load force  $F$ . In many cases, however, like e.g. for protein transport within a cell, the Brownian motor operates at a zero bias regime ( $F = 0$ ) and its objective is to carry a cargo across a viscous environment. Clearly, the minimal energy input required to move a particle in presence of friction  $\gamma$  over a given distance depends on the velocity, tending to zero when we move it very slowly. If one is interested in delivering the cargo in a finite time one should require that the transport is accomplished at an average motor velocity  $\langle v \rangle$ . In this case, the necessary energy input is finite. Thus, we replace the load force in the expression (3) by the viscous force  $\gamma\langle v \rangle$  to obtain the called *Stokes efficiency* [44]; i.e.,

$$\eta_S = \frac{\gamma\langle v \rangle^2}{P_{in}}. \quad (4)$$

Upon combining the two above given notions we recover the *rectification efficiency* originally proposed by Suzuki and Munakata [45,46] or its equivalent version presented by Derenyi *et al.* [47]

$$\eta_R = \frac{F\langle v \rangle + \gamma\langle v \rangle^2}{P_{in}}. \quad (5)$$

It is made up of the sum of the efficiencies  $\eta_S$  and  $\eta_E$ . Therefore, it accounts for both, the work that the Brownian motor performs against the external bias  $F$  as well as the work that is necessary to move the object a given distance in a viscous environment at the average velocity  $\langle v \rangle$ .

The average input power for a tilted rocking ratchet is given by [24,25,48]:

$$P_{in} = F\langle v \rangle + \gamma[\langle v^2 \rangle - D_0]. \quad (6)$$

This expression follows from an energy balance of the underlying equation of motion (1) [24].

## 4 Numerical analysis

Focussing on the directed current, we investigate the asymptotic, time-periodic regime after effects of the initial conditions and transient processes have died out. Then, the statistical quantifiers of interest can be determined in terms of the statistical average over the different realizations of the process (1) and over the driving period  $T$ .

Clearly, there exist no analytical methods of analyzing eq. (1) in presence of inertia. Therefore, we performed extensive, precise numerical studies by employing the Stochastic Runge-Kutta (SRK) algorithm of order 2 with a time step  $h = 10^{-3}$ . For the initial conditions we used a uniform distribution of the initial position  $x(t = t_0)$  at time  $t_0$  on an interval lying between two neighboring maxima of the ratchet potential given in (2). The initial starting velocities  $v(t = t_0)$  were randomly chosen from an uniform distribution over the interval  $[-0.2, 0.2]$ . All quantities were averaged over 100 different trajectories, each of which evolved over  $10^5$  driving-periods  $T$ . For the investigation of the efficiency quantifiers defined in section 3 above, we restrict the discussion here to a set of optimal driving parameters, reading,  $a = 3.7$ ,  $D_0 = 0.001$ ,  $\omega = 4.9$  and  $\gamma = 0.9$  (see for the details in Ref. [24,25]).

### 4.1 Current-load behavior

In Fig. 1 we depict the load-velocity characteristics of the non-equilibrium Brownian motor dynamics (1). Contrary to the familiar, usually monotonic dependence found for overdamped ratchet dynamics [1,28,49], the velocity-load-behavior becomes now considerably more complex, exhibiting distinct non-monotonic characteristics. Around the forces  $F \simeq -1.4$  and  $F \simeq 0$  an increase of the bias  $F$  results in a corresponding decrease of the average velocity. This behavior is termed *negative differential mobility*. The effect is extremely pronounced in the neighborhood of  $F = 0$ .

Let us elucidate the underlying working mechanism in greater detail: At a zero load the corresponding deterministic dynamics possesses one stable attractor of period one (in velocity space, see in [25]) which translocates the particle

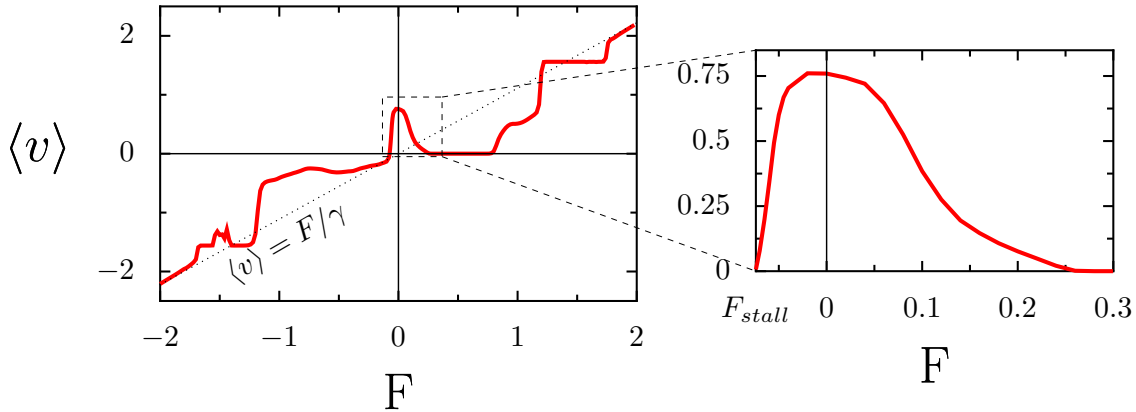


Fig. 1. Average velocity of the inertial Brownian motor (1) as a function of an external, constant force  $F$ . The system parameters are:  $a = 3.7$ ,  $\omega = 4.9$ ,  $\gamma = 0.9$  and  $D = 0.001$ . The dotted line denotes the average velocity of a particle moving in the absence of a periodic potential, being the limiting case for the Brownian motor dynamics at  $F \rightarrow \infty$ . One can notice a few regimes where the differential mobility ( $\partial\langle v \rangle / \partial F$ ) assumes a negative value. The most pronounced such behavior occurs for small positive values of the bias  $F$  (depicted in the inset). For bias forces  $F \in (F_{stall}, 0)$ ,  $F_{stall} \simeq -0.074$ , the Brownian motor performs against the external load.

from one to the next potential well during one period  $T$  of driving. The particle moves with a high Stokes efficiency as a consequence of small fluctuations of the velocity from its average value. A residence within this regime, however, requires that all system parameters are precisely tuned. A consecutive increase of the external load  $F$ , regardless of its sign, drives the system away from this most efficient regime and the average velocity starts dropping to small values. This is a result of the complex inertial dynamics where a forcing of the particle into the direction of its motion diminishes, rather than increases the average velocity. In contrast, at very large magnitudes of the load force  $F$ , the velocity assumes its asymptotic value, reading  $\langle v \rangle = F/\gamma$ .

#### 4.2 Efficiency for forced, rocking Brownian motors

As we remarked already above, near the bias  $F \simeq 0$ , the Brownian motor operates optimally. With Fig. 2 (a), we depict the behavior of the Stokes efficiency within an interval of bias forces  $F \in (F_{stall}, 0)$  where the motor does work against the external force. The Stokes efficiency assumes a value around 0.75 at  $F = 0$ , and monotonically decreases, reaching zero at the stall force  $F_{stall}$ , where the average velocity vanishes. In between,  $F \in (F_{stall}, 0)$ , the ratchet device pumps particles uphill, cf. Fig. 1. The behavior of the rectification efficiency in this regime closely matches the behavior of the Stokes efficiency. Indeed, within this forcing regime the efficiency of energy transduction  $\eta_E$  assumes much smaller values, see Fig. 2 (b). Within this forcing regime the

bell-shaped character of  $\eta_E$  is an immediate consequence of its definition in eq. (3): It acquires vanishing values of both, at the stall force, where the velocity becomes zero and at  $F = 0$ , where the output power vanishes. In this regime  $P_{in}$  varies only slightly.

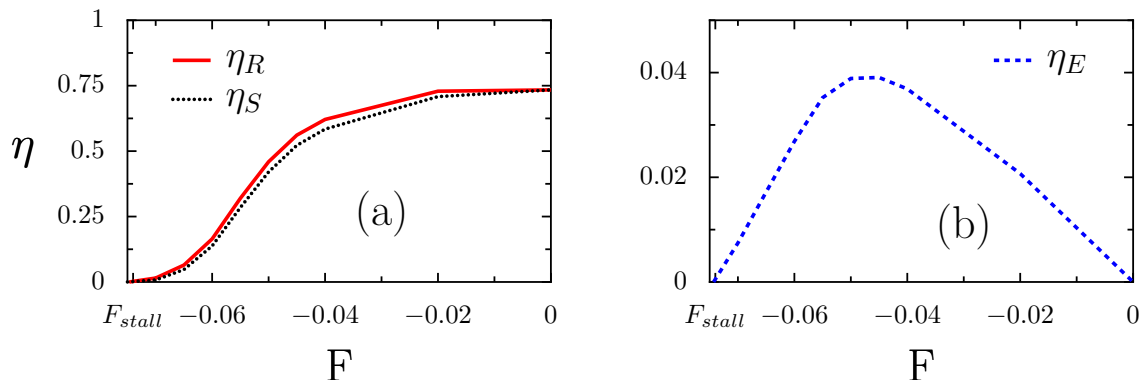


Fig. 2. Behavior of different efficiency measures within the regime of "uphill motion". Depicted are the efficiency of rectification  $\eta_R$ , the closely related Stokes efficiency  $\eta_S$ , in panel (a), and efficiency of energy conversion  $\eta_E$ , panel (b), versus the external load  $F$ , varying between the stall force  $F_{stall}$  and the vanishing bias  $F = 0$ . The Stokes efficiency assumes much larger values than the corresponding energetic one; it is therefore dominating the viscous, noise-assisted transport.

## 5 Summary

Biased, inertial rocking Brownian motors can exhibit an intriguing velocity-load characteristics. We discovered that the average velocity assumes a non-monotonic behavior as a function of the external load; i.e., in certain regimes of external forcing the differential mobility is *negative-valued*. Near small negative load forces the rocked, inertial Brownian motor is able to perform "uphill"-motion against the external force. Within this regime the bell-shaped energetic efficiency  $\eta_E$  is distinctly smaller than the corresponding efficiency of rectification and also smaller than the related "Stokes" efficiency. These latter two efficiencies clearly dominate over conversion of energy within this very regime, where particles move against an externally applied load.

## Acknowledgements

The authors gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft via grant HA 1517/13-4, the Graduiertenkolleg 283 (LM, PH, PT), the collaborative research grant SFB 486 (PH), the DAAD-KBN (German-Polish project *Stochastic Complexity*) (JL,PH), and the ESF, Program *Stochas-*

*tic Dynamics: fundamentals and applications; STOCHDYN (JL,PH), and the Bavarian collaboration program: Bavaria-Rio de Janeiro (PH).*

## References

- [1] P. Hänggi, R. Bartussek, Lect. Notes Phys. 476 (1996) 294.
- [2] R. D. Astumian, P. Hänggi, Phys. Today 55 (2002) 33.
- [3] P. Reimann, Phys. Rep. 361 (2002) 57.
- [4] P. Reimann, P. Hänggi, Appl. Phys. A 75 (2002) 169.
- [5] H. Linke, Appl. Phys. A 75 (2002) 167.
- [6] P. Hänggi, F. Marchesoni, F. Nori, Ann. Phys. (Leipzig) 14 (2005) 51.
- [7] F. Jülicher, A. Ajdari, J. Prost, Rev. Mod. Phys. 69 (1997) 1269.
- [8] R. Bartussek, P. Hänggi, J. G. Kissner, Europhys. Lett 28 (1994) 459.
- [9] J. Łuczka, R. Bartussek, P. Hänggi, Europhys. Lett 31 (1995) 431.
- [10] P. Hänggi, R. Bartussek, P. Talkner, J. Łuczka, Europhys. Lett 35 (1996) 315.
- [11] M. Kostur, J. Łuczka, Phys. Rev. E. 63 (2001) 021101.
- [12] S. Savelev, F. Marchesoni, P. Hänggi, F. Nori, Europhys. Lett 67 (2004) 179.
- [13] S. Savelev, F. Marchesoni, P. Hänggi, F. Nori, Phys. Rev. E. 70 (2004) 066109.
- [14] E. Pollak, J. Bader, B. J. Berne, P. Talkner, Phys. Rev. Lett. 70 (1993) 3299.
- [15] M. Borromeo, F. Marchesoni, Phys. Rev. Lett. 84 (2000) 203.
- [16] P. Reimann, M. Grifoni, P. Hänggi, Phys. Rev. Lett. 79 (1997) 10.
- [17] I. Goychuk, M. Grifoni, P. Hänggi, Phys. Rev. Lett. 81 (1998) 649; Phys. Rev. Lett. 81 (1998) 2837 (addendum).
- [18] P. Jung, J. G. Kissner, P. Hänggi, Phys. Rev. Lett. 76 (1996) 3436.
- [19] J. L. Mateos, Phys. Rev. Lett. 84 (2000) 258.
- [20] B. Lindner, L. Schimansky-Geier, P. Reimann, P. Hänggi, M. Nagaoka, Phys. Rev. E. 59 (1999) 1417.
- [21] T. Sintes, K. Sumithra, Physica A 312 (2002) 86.
- [22] W. S. Son, I. Kim, Y. J. Park, C. M. Kim, Phys. Rev. E. 68 (2003) 067201.
- [23] S. Sengupta, R. Guantes, S. Miret-Artes, P. Hänggi, Physica A 338 (2004) 406.

- [24] L. Machura, M. Kostur, P. Talkner, J. Luczka, F. Marchesoni, P. Hänggi, *Phys. Rev. E.* 70 (2004) 061105.
- [25] L. Machura, M. Kostur, F. Marchesoni, P. Talkner, P. Hänggi, J. Luczka, *J. Phys.: Condens. Matter* 17 (2005) S3741–S3752.
- [26] C. M. Arizmendi, F. Family, A. L. Salas-Brito, *Phys. Rev. E.* 63 (2001) 061104.
- [27] M. Barbi, M. Salerno, *Phys. Rev. E.* 62 (2000) 1988.
- [28] R. Bartussek, R. Reimann, P. Hänggi, *Phys. Rev. Lett.* 76 (1996) 1166.
- [29] F. Nava, C. Canali, F. Catellani, G. Gavioli, G. Ottaviani, *J. Phys. C: Solid State Phys.* 9 (1976) 1685.
- [30] L. Hartmann, M. Grifoni, P. Hänggi, *Europhys. Lett* 38 (1997) 497.
- [31] C. V. den Broeck, I. Bena, P. Reimann, J. Lehmann, *Ann. Phys. (Leipzig)* 9 (2000) 713.
- [32] C. V. den Broeck, B. Cleuren, R. Kawai, M. Kambon, *International Journal of Modern Physics C* 13 (2002) 1195.
- [33] R. Eichhorn, P. Reimann, *Acta. Phys. Pol. B* 35 (2004) 1407.
- [34] S. R. White, M. Barma, *J. Phys. A* 17 (1984) 2995.
- [35] V. Balakrishnan, C. V. den Broeck, *Physica A* 217 (1995) 1.
- [36] G. A. Cecchi, M. O. Magnasco, *Phys. Rev. Lett.* 76 (1996) 1968.
- [37] G. W. Slater, H. L. Guo, G. I. Nixon, *Phys. Rev. Lett.* 78 (1997) 1170.
- [38] R. Eichhorn, P. Reimann, P. Hänggi, *Phys. Rev. Lett.* 88 (2002) 190601.
- [39] R. Eichhorn, P. Reimann, P. Hänggi, *Phys. Rev. E.* 66 (2002) 066132.
- [40] B. Cleuren, C. V. den Broeck, *Phys. Rev. E.* 65 (2002) 030101.
- [41] A. Haljas, R. Mankin, A. Sauga, E. Reiter, *Phys. Rev. E.* 70 (2004) 041107.
- [42] H. X. Zhou, Y. D. Chen, *Phys. Rev. Lett.* 77 (1996) 194.
- [43] K. Sekimoto, *J. Phys. Soc. Jpn.* 66 (1997) 1234.
- [44] H. Wang, G. Oster, *Europhys. Lett* 57 (2002) 134.
- [45] D. Suzuki, T. Munakata, *Phys. Rev. E.* 68 (2003) 021906.
- [46] D. Suzuki, T. Munakata, *J. Phys. Soc. Jpn.* 74 (2005) 550.
- [47] I. Derenyi, M. Bier, R. D. Astumian, *Phys. Rev. Lett.* 83 (1999) 903.
- [48] H. Linke, M. T. Downton, M. J. Zuckermann, *Chaos* 15 (2005) 026111.
- [49] I. Zapata, R. Bartussek, F. Sols, P. Hänggi, *Phys. Rev. Lett.* 77 (1996) 2292.