Anticipated synchronization in coupled inertia ratchets with time-delayed feedback: a numerical study

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Abstract

We investigate anticipated synchronization between two periodically driven deterministic, dissipative inertia ratchets that are able to exhibit directed transport with a finite velocity. The two ratchets interact through an unidirectional delay coupling: one is acting as a master system while the other one represents the slave system. Each of the two dissipative deterministic ratchets is driven externally by a common periodic force. The delay coupling involves two parameters: the coupling strength and the (positive-valued) delay time. We study the synchronization features for the unbounded, current carrying trajectories of the master and the slave, respectively, for four different strengths of the driving amplitude. These in turn characterize differing phase space dynamics of the transporting ratchet dynamics: regular, intermittent and a chaotic transport regime. We find that the slave ratchet can respond in exactly the same way as the master will respond in the future, thereby anticipating the nonlinear directed transport.

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I. INTRODUCTION

The intriguing concept of synchronization in nonlinear systems is relevant for a wide range of topics in physics, chemistry and biology. Recently, it has received much attention and first, comprehensive reviews and books have appeared [1, 2]. The case of synchronization, in particular of chaotic systems, represents a challenge, since a chaotic system is extremely sensitive to small perturbations. Nevertheless, it has been established repeatedly that the synchronization of chaotic systems is possible under certain conditions [3].

The situation when the *coupling* involves a delay in time is the focus here, and may lead to anticipated synchronization [4, 5, 6, 7, 8, 9]. In this case, one deals with two systems: a "master" and a "slave", which are coupled unidirectionally via a time-delay term, in such a way that, under some circumstances, the slave system anticipates the response of the master system. The regime of anticipated synchronization and its stability has been studied theoretically previously for a variety of systems: we mention here the case of linear set ups with delay [10], coupled chaotic maps with delays [11, 12], excitable systems [13] and nonlinear systems of practical interest, such as semiconductor lasers operating in a chaotic regime [14]. Recently, the phenomenon of anticipated synchronization has been vindicated experimentally in a semiconductor laser running in the chaotic regime [16, 17].

In different context, we witness an increasing interest during recent years in the study of intriguing transport phenomena of nonlinear systems that can extract usable work from unbiased non-equilibrium fluctuations. These, so called Brownian motor systems (or Brownian ratchets) [18] can be modelled by a Brownian particle undergoing a random walk in a periodic asymmetric potential, and being acted upon by an external time-dependent force of zero average. The recent burst of work is motivated by both, (i) the challenge to model unidirectional transport of molecular motors within the biological realm and, (ii) the potential for novel technological applications that enable an efficient scheme to shuttle, separate and pump particles on the micro- and even nanometer scale [18].

Although the vast majority of the literature in this field considers the presence of noise, there have been attempts to model the transport properties of classical deterministic inertial ratchets [19, 20, 21]. These ratchets generally possess parameter regions where the classical dynamics is chaotic; the latter in turn then decisively determines the transport properties.

In contrast to the case with two coupled oscillators possessing confined position trajectories, the position in a driven ratchet dynamics is able to undergo directed, unbounded motion with a finite transport velocity. In the following, we shall explore the case of a coupling between two deterministic, dissipative ratchets in parameter regimes where each ratchet system is individually able to exhibit either current-carrying, regular transport trajectories (which in turn possess a periodic velocity), or also a chaotic or even an intermittent, directed transport behavior [19, 20, 21]. One of the ratchet devices then acts as the "master system" while the remaining one acts as the "slave". The coupling between the two ratchets is unidirectional, meaning that the master affects the slave, but not vice versa. We shall explore the possibility of anticipated synchronization in various tailored parameter regimes of a regular, chaotic and intermittent dynamics.

II. TWO COUPLED INERTIAL RATCHETS WITH A TIME DELAY

To start out, let us consider a one-dimensional problem of an inertial particle driven by a periodic time-dependent external force in an asymmetric (with respect to the reflection symmetry) periodic, so called ratchet potential. In order to have no net bias at work the time average of the external force equals zero. Here, we do not take into account any sort of noise, meaning that the dynamics is deterministic. We thus deal with a rocked deterministic ratchet [19, 22] that obeys the following dimensionless inertial dynamics [20]:

$$\ddot{x} + b\dot{x} + \frac{dU(x)}{dx} = a\cos(\omega t), \tag{1}$$

where b denotes the friction coefficient, $V(x)$ is the asymmetric ratchet periodic potential, a is the amplitude of the external force and ω is the frequency of the external driving force. The dimensionless potential (which is shifted by an amount $x_0 \simeq -0.19$ in order that its minimum is located at the origin) is given by:

$$U(x) = C - U_0\left[\sin 2\pi(x-x_0) + \frac{1}{4}\sin 4\pi(x-x_0)\right] \tag{2}$$

and is depicted in Fig. 1. The constant C is chosen such that $U(0) = 0$, and is given by $C = -U_0(\sin 2\pi x_0 + 0.25\sin 4\pi x_0)$, where $U_0 = 1/4\pi^2(\sin(2\pi|x_0|) + \sin(4\pi|x_0|))$. In this case $U_0 \simeq 0.0158$ and $C \simeq 0.0173$, see also [20, 22].

This time-dependent dynamics can be embedded into a corresponding three-dimensional autonomous phase space dynamics because we are dealing with a Newtonian dynamics

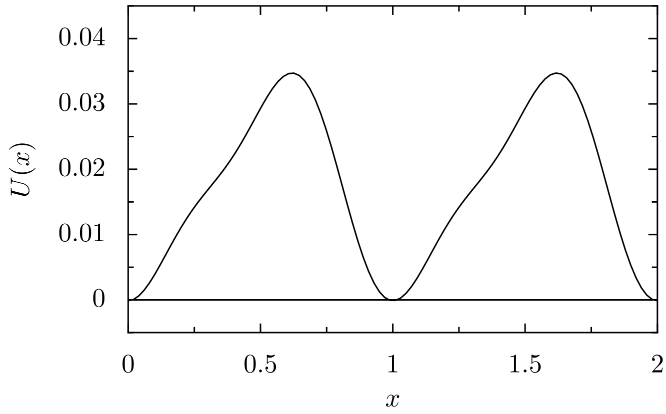

FIG. 1: The dimensionless periodic ratchet potential used in our simulations.

subjected to time dependent harmonic driving. Consequently, the equation of motion of the ratchet system can be recast as a three-dimensional dynamical system.

We next consider two ratchets, coupled in a master-slave configuration with X denoting the coordinate of the master and x that of the slave. The positive valued delay coupling K is one-way, that is, the slave is coupled to the master, but the latter is independent of the dynamics of the slave. The two ratchets obey the coupled system of equations:

$$\ddot{X}(t) + b\dot{X}(t) + U'(X(t)) = a\cos(\omega t), \tag{3}$$

$$\ddot{x}(t) + b\dot{x}(t) + U'(x(t)) = a\cos(\omega t) + K(\dot{X}(t) - \dot{x}(t-\tau)) \ , \tag{4}$$

with τ denoting the (positive-valued) time-delay. We can mathematically recast this dynamical system in terms of two coupled, autonomous three-dimensional dynamical systems, i.e.,

$$\begin{aligned}
\dot{X}(t) &= V(t) \\
\dot{V}(t) &= -bV(t) - U'(X(t)) + a\cos(\Phi(t)) \\
\dot{\Phi}(t) &= \omega \\
\dot{x}(t) &= v(t) + K(X(t) - x(t-\tau)) \\
\dot{v}(t) &= -bv(t) - U'(x(t)) + a\cos(\varphi(t)) \\
\dot{\varphi}(t) &= \omega
\end{aligned} \tag{5}$$

Given this system, we notice that the manifold $X(t) = x(t-\tau)$ presents an exact solution of the system, when the period of the external force is equal to the time delay τ. The situation when we attain anticipated synchronization yields $X(t) = x(t-\tau)$, or $X(t+\tau) = x(t)$. This implies that the slave ratchet acts at time t in exactly the same way as the master will do in the future time, $t+\tau$, thus anticipating the dynamics.

We note, that the phase difference $\Phi(t) - \varphi(t)$ of the driving forces remains fixed during its time evolution. Thus, the only way to achieve the synchronization – with anticipation time τ – is to choose the initial phase $\varphi(0)$ of the slave to match precisely

$$\Phi(0) = \varphi(0) + \omega\tau. \tag{6}$$

This dynamical system is similar to the excitable system studied in [13], where the Adler equation [23] is considered for a particle in a tilted symmetric (non-ratchet) periodic potential. In our case the number of degrees of freedom is increased since we are addressing the inertial dynamics. We remark, however, that in the limit of two over-damped ratchets, our system dynamics becomes similar to the one studied previously in [13], except that we deal here with a common periodic forcing instead of a common random external forcing used in [13].

III. ANTICIPATED SYNCHRONIZATION: NUMERICAL RESULTS

In the following, we numerically analyze thoroughly the dynamics of the two coupled ratchets (5). Let us consider the case where both ratchets, master and slave, are identical, that is, they have the same parameters a, b, and ω; the parameters that enter in the ratchet potential are also identical. In Fig. 2 we depict transporting trajectories for the master and the slave ratchet, when $a = 0.08$, $b = 0.1$ and $\omega = 0.67$. The delay coupling involves two parameters: the coupling strength $K = 0.6$ and the delay time $\tau = 1.3$. We notice that the master and the time-shifted slave essentially coincide; that is, the slave is anticipating the response of the master system, see in the blow up in Fig. 2.

The parameter space of the full nonlinear dynamical system is far too large for a systematic numerical analysis. In this paper we will choose the dynamics of the master to be in one of the following representative regimes: regular, chaotic and intermittent transport dynamics. Then, the relevant parameter space for synchronization (τ, K) will be scanned

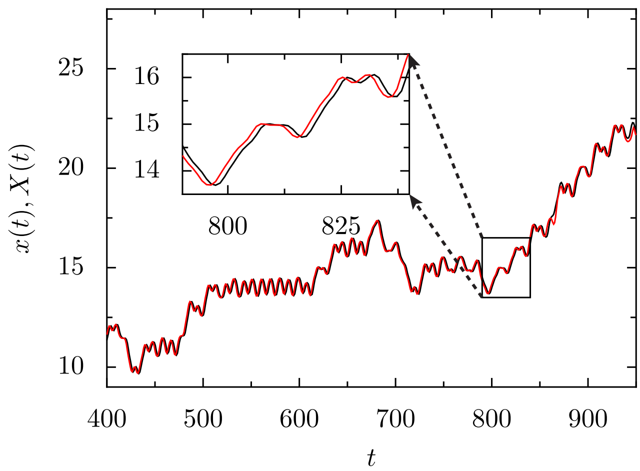

FIG. 2: (Color online) Trajectories for the master (red line) and slave (black line) ratchets in a typical synchronization scenario. The magnified part of the trajectory reveals the anticipation effect. The parameters are: $a = 0.08$, $b = 0.1$, $\omega = 0.67$, $K = 0.6$, $\tau = 1.3$.

numerically. We also seek a quantity that can characterize the quality of synchronization. A first natural candidate would be the position correlation function. We found, however, that this measure does not provide an intuitive answer whether the master and the slave are synchronized. Instead, we will use the fraction of the time during which the two ratchets are synchronized within some prescribed accuracy. That is, we consider that the master and the slave attain a regime of anticipated synchronization when the difference between their trajectories is smaller than some small given value ϵ, that is, $|x(t) - X(t + \tau)| \leq \epsilon$. We always have set this value to read $\epsilon = 0.01$.

In the following, we calculate the trajectories for both ratchet dynamics, compute the amount of time that they stay synchronized (according to the above criterion), and then determine the ratio p between the synchronized and total timespan of the considered trajectory. This measure of synchronization p therefore varies between zero and one.

¿From the literature it is known [4, 5, 9] that the delay coupling scheme used here requires some constraints on the positive anticipation time τ and the coupling strength K for the anticipated synchronization to occur. Thus, we investigate the stability regions, in the parameter space (K, τ), for coupled chaotic ratchets as described by eq. (5). In Fig. 3 we show the synchronization properties for the case $b = 0.1$ and $\omega = 0.67$, for four values of the amplitude a. These values, indicated in the figure, correspond to regular, intermittent and a chaotic regime of the master dynamics [20]. The gray-scale depicts the value of the

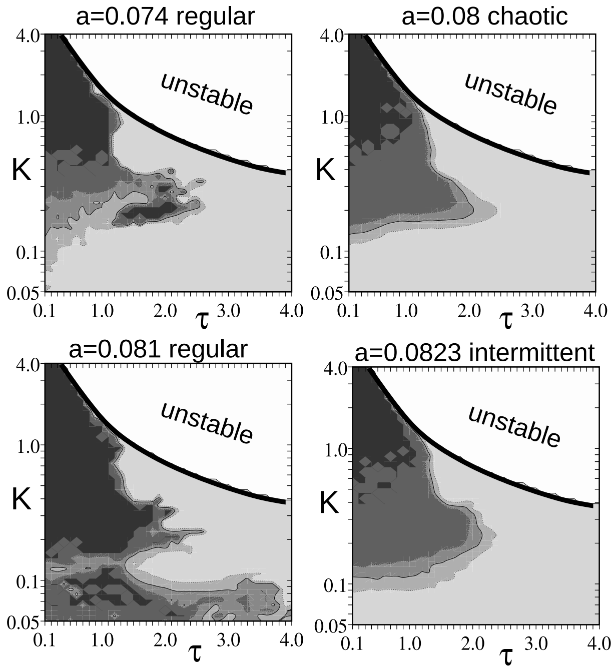

FIG. 3: Parameter space indicating the stability region for the anticipated synchronization to take place. We depict four cases: one with intermittent, directed transport dynamics at a driving amplitude of $a = 0.0823$, one with a chaotic transport behavior at $a = 0.08$ and two regular situations with $a = 0.074$ and $a = 0.081$, respectively. The remaining parameters are as follows: $b = 0.1$, $\omega = 0.67$. Within the white area the slave exhibits exponentially growing oscillations. The gray-scale represents the quality parameter p (defined in the text). Four gray-tones from light to dark gray correspond to the corresponding ranges of p: $0 \ldots 0.25$, $0.25 \ldots 0.5$, $0.5 \ldots 0.75$ and $0.75 \ldots 1$, respectively. Each point in this plot was obtained by averaging the ratio p over 40 trajectories of the system (5) with random initial conditions. Each trajectory has a length of 20000 time units. Every plot consumed ca. 200 hours of CPU on a typical workstation. The bold line exhibits the necessary stability condition $K\tau = \pi/2$ resulting from the linear part of the delay equation, see also the text.

fraction of synchronization time p, as defined above. The darkest region corresponds to a well synchronized behavior with values of p being close to 1; the light gray represents an unsynchronized master and slave behavior with a value $p \simeq 0$.

In all four panels in Fig. 3 we can observe regions of synchronization. Although, their shapes differ from panel to panel they contain common features. First of all, we note that at sufficiently small K, and in particular when $K = 0$, the slave and the master are not coupled and evolve independently of each other. Secondly, the increase of the coupling strength K causes the onset of synchronization for small and moderate values of the delay time τ. Synchronization is lost, however, for too large values of the delay time τ regardless of the coupling K. The third common feature is the loss of the stability of the slave, if both K and τ are too large. The origin of the instability derives from the linear delay equation $\dot{x}(t) = -Kx(t-\tau)$, resulting from eq. (5) upon neglecting the velocity $v(t)$, see also Ref. [10]. For $K \cdot \tau > \pi/2$ the trajectories of this linear equation grow exponentially. This criterion agrees perfectly well with our numerical findings. The unstable region can be neighboring to the synchronized or to unsynchronized one, depending on the delay τ. Hence, two scenarios have been observed: increasing K at larger τ first de-synchronizes the system until the slave finally becomes unstable (see e.g. $a = 0.08$, $\tau = 1.7$ and $K = 0.2 \ldots 4$); at smaller values of τ the synchronized state becomes unstable with growing K (see e.g. $a = 0.08$, $\tau = 1.0$ and $K = 0.2 \ldots 4$).

One can distinguish two regions of synchronization in these four plots in Fig. 3. The first one is present in all cases for coupling strengths $K > 0.1$. It has a similar shape although the underlying dynamics may differ dramatically. The second region for $K < 0.1$ exists in the case of a regular dynamics with $a = 0.081$. Remarkably, in this regime the system can reach synchronization for delay times τ much larger than in the other cases, and it takes place even at a small coupling strength $K \simeq 0.06$.

Clearly, the parameter p does not provide the full information on the dynamics. If we know that e.g. 50% of time a slave synchronizes with a master, we still do not know much about the nature of these events. Thus, we shall next investigate in more detail for each of the above four characteristic situations the representative time series.

At the driving strength $a = 0.074$, the master possesses a stable regular trajectory, being characterized by a period-two orbit in the corresponding Poincaré section, see e.g. Fig. 2 in [20]). Starting with independent random initial conditions within the same period of the potential for the master and the slave, the master reaches the stable orbit after a transient time. This is depicted in Fig. 4. Already before the master has reached its stable orbit the slave starts to synchronize at $t = 600$. Only at $t = 750$ both the master and the slave reach the regular orbit and the synchronization becomes even better, note the drop of the value $\ln|x(t) - X(t+\tau)|$. In this case $p = 1$, the slave will never de-synchronize from the master. Yet another scenario is possible: depending on the chosen initial conditions, the slave and

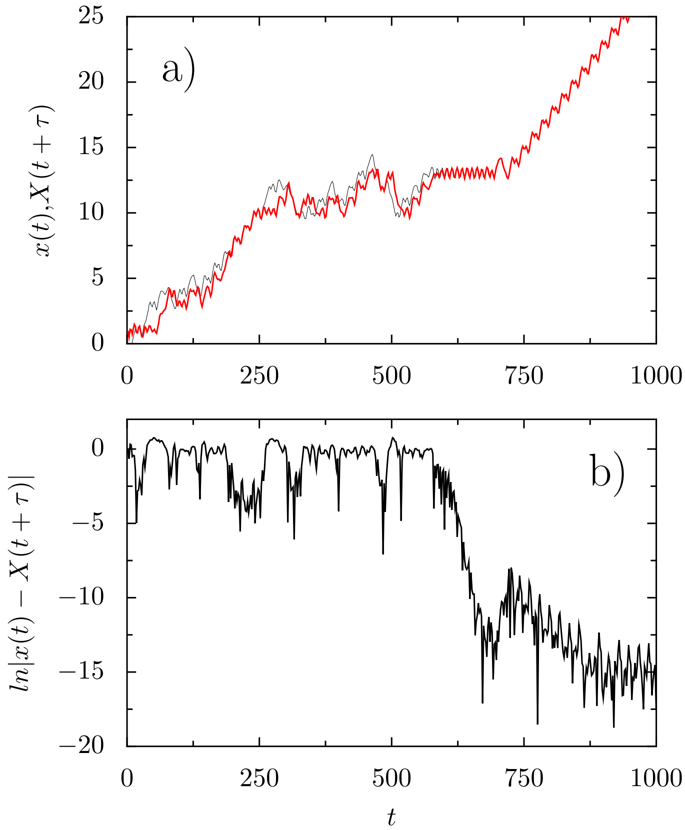

FIG. 4: (Color online) Regular regime at the driving amplitude $a = 0.074$. (a): Trajectories for the master (black line) and slave (red line) ratchet dynamics. (b): The logarithm of the absolute value of the difference between the future position of the master and the present one of the slave. In the depicted scenario the slave already synchronizes to the transient of the master; later both systems together reach the growing, regular (period-2) orbit, see text. The parameters are: $a = 0.074$, $b = 0.1$, $\omega = 0.67$, $K = 0.2$, $\tau = 1.8$.

the master may reach their regular orbits while keeping a spatial separation equal to one period of the potential (which in our case is 1.0). Apparently, the attractor is strong enough to dominate the coupling term $K \simeq 1.0$ and the system, although not synchronized, evolves periodically (modulo the spatial period of the potential). This mechanism is responsible for the irregular shape of the dark area in Fig. 3.

At $a = 0.081$, as in the previous case, the attractor of the master exhibits a stable regular trajectory possessing a periodic orbit in the Poincaré section (with period four [20]). There exists, however, a dramatic difference in the synchronization behavior. In Fig. 3 one can detect the appearance of an additional, well synchronized region for small coupling strengths

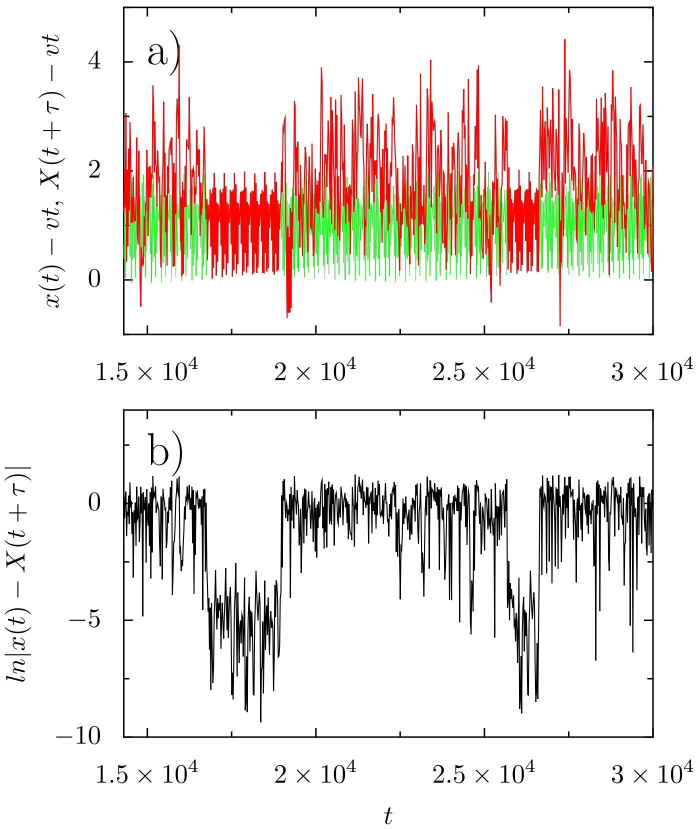

FIG. 5: (Color online) Regular regime for a driving amplitude $a = 0.081$. (a): Trajectories for the master (black line) and slave (red line) ratchets (b): The logarithm of the absolute value of the difference between the respective positions. In this case the master has already reached the regular orbit. The slave occasionally synchronizes and desynchronizes again. The parameters are: $a = 0.081$, $b = 0.1$, $\omega = 0.67$, $K = 0.0981$, $\tau = 3.4$. The symbol v stands for the average velocity of the ratchet dynamics. The trajectories are depicted in the frame moving with velocity $v \simeq -0.025$ in order to pronounce the oscillations.

K, which spreads to relatively large values of τ. Also the temporal dynamics exhibits another behavior. Let us inspect more closely the system dynamics for the parameters $K = 0.0981$ and $\tau = 3.4$. In Fig. 5 we present a small portion of the trajectory at large times. First, one can observe that the master system has already reached the period-4 orbit. The slave, however, in contrast to the behavior in the previous case, synchronizes and de-synchronizes in an intermittent manner. Another observation is that the "distance" parameter $\ln|X(t+\tau) - x(t)|$ is around -5 when synchronization takes place, compared to a typical values -15 characterizing synchronization in other parameter regimes. We also checked the behavior of the system at $K = 0.25$ and $\tau = 1.6$, i.e. where synchronization is observed for all parameter values shown in Fig. 3. In that case the scenario resembles the one at $a = 0.074$, the slave and the master reach the regular orbit and stay there forever.

A typical chaotic transporting trajectory (possessing a small, positive-valued transport-velocity) at the amplitude strength of $a = 0.08$ is depicted with Fig. 6. Similarly to the case shown in Fig. 5 we notice bursts of de-synchronization (or synchronization, respectively, depending on the chosen parameters). The "distance" parameter exhibits a random walk pattern, of the type discussed in chapter 13 of Ref. [1].

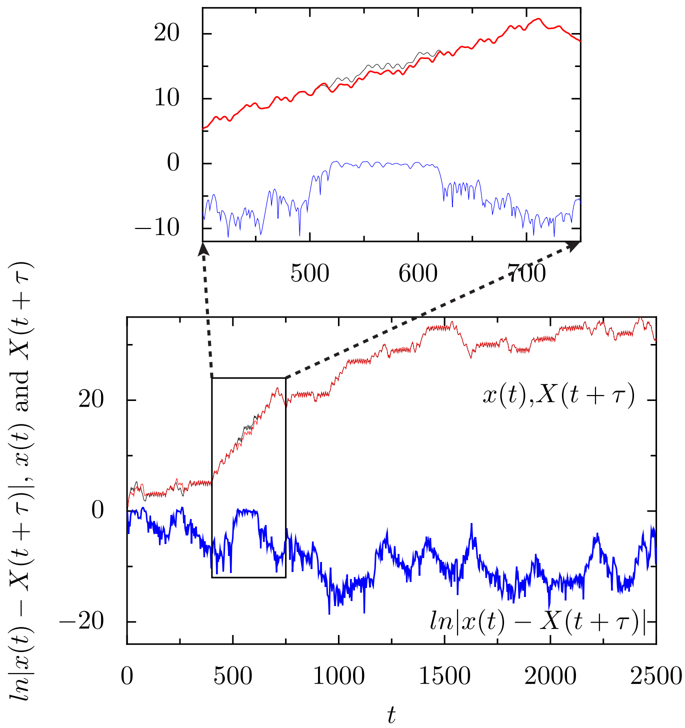

FIG. 6: (Color online) Directed transport in the chaotic regime with a driving strength of $a = 0.08$. Depicted are the trajectories for the master and slave (upper curves, black and red lines, respectively) and the logarithm of the absolute value of the difference between the two positions (lower curves). The slave occasionally synchronizes with, and subsequently de-synchronizes again from the master. The distance $\ln |X(t+\tau) - x(t)|$ exhibits a random walk like pattern. The magnification in the upper panel depicts a short desynchronization event at $t = 500$. The parameters are: $a = 0.08$, $b = 0.1$, $\omega = 0.67$, $K = 0.305$, $\tau = 1.6$.

The directed, transporting ratchet dynamics of the driven ratchet at the driving strength $a = 0.0823$ (see Fig. 2 in [20]) is intermittent. The region of synchronization (Fig. 3) does not significantly differ from the case at $a = 0.08$. The trajectory (see Fig. 7), however, exhibits typical features of the intermittency: the regular behavior is intermittently interrupted by finite "bursts" in which the orbit behaves in a chaotic manner [20]. Similarly to the chaotic case the slave occasionally synchronizes with the master, and subsequently de-synchronizes again from the master. Those (de)synchronization events are not directly connected with the "bursts" of the master dynamics! However, we have observed some regularities. In Fig. 7 at $t = 7200$ the master changes its dynamics from a neighborhood of a period-two orbit to the period-four orbit. Simultaneously, the value of $\ln|X(t+\tau) - x(t)|$ rises by ten orders of magnitude. For this particular event, however, the slave does not loose its synchronization with the master.

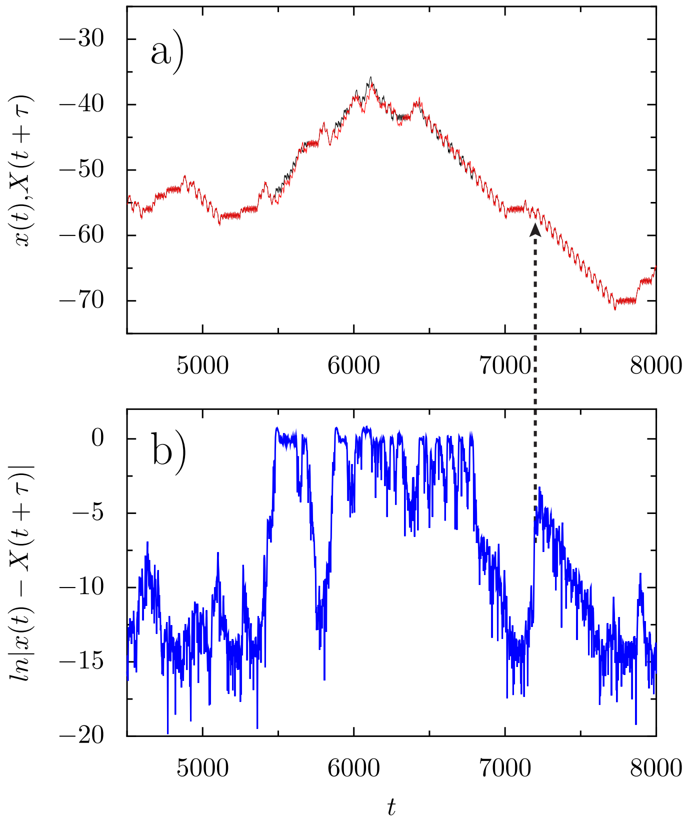

FIG. 7: (Color online) Intermittent transport regime for the driving strength $a = 0.0823$. (a): Trajectories for the master (black line) and the slave (red line). (b): The logarithm of the absolute value of the difference between the two positions. The slave occasionally de-synchronizes from the master. The parameters are: $a = 0.0823$, $b = 0.1$, $\omega = 0.67$, $K = 0.305$, $\tau = 1.6$.

IV. RESUME

In summary, we numerically studied two deterministic ratchets coupled unidirectionally via a time delay. We established the conditions under which one can obtain anticipated synchronization for the two coupled transporting ratchet trajectories and postulated a necessary stability criterion for the motion of the slave which is perfectly confirmed by our numerical results. A further necessary condition for the occurrence of synchronization is a strict relation between the phases of the driving forces of the master and the slave.

Within the stable parameter region $K\tau < \pi/2$, we quantified the degree of synchronization by means of its relative frequency p. For p values close to unity the slave ratchet anticipates the dynamics of the master in an almost perfect way irrespectively of whether it moves regularly or performs an intermittent or fully chaotic motion. These results allow one to predict the directed transport features of particles on a ratchet potential using a copy of the same system that acts as a slave.

Acknowledgments

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