Controlling Nonlinear Stochastic Resonance by Harmonic Mixing

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Abstract

We investigate the potential for controlling the effect of nonlinear Stochastic Resonance (SR) by use of harmonic mixing signals for an overdamped Brownian dynamics in a symmetric double well potential. The periodic forcing for *harmonic mixing* consists of a first signal with a basic frequency Ω and a second, superimposed signal oscillating at twice the basic frequency 2Ω. By variation of the phase difference between these two components and the amplitude ratios of the driving the phenomenon of SR becomes *a priori* controllable. The harmonic mixing dynamically breaks the symmetry so that the time- and ensemble-average assumes a non-vanishing value. Independently of the noise level, the response can be suppressed by adjusting the phase difference. Nonlinear SR then exhibits resonances at higher harmonics with respect to the applied noise strength and relative phase. The scheme of nonlinear SR via harmonic mixing can be used to steer the nonlinear response and to sensitively measure the internal noise strength. We further demonstrate that the full Fokker-Planck dynamics can be well approximated by a two-state model.

Key words: Stochastic Resonance, harmonic mixing, two-state model, nonlinear resonances
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1 Introduction

Stochastic Resonance (SR) describes the phenomenon where an incoming, generally weak signal can become amplified upon harvesting the ambient noise in metastable, nonlinear stochastic systems [1]. This phenomenon is based on

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a stochastic synchronization between noise-induced hopping events and the periodic, externally applied signal [1,2,3,4]. SR has since been observed in an abundance of systems in physics, chemistry, engineering, biology and biomedical sciences — and the list of examples and applications is still growing. In particular SR has found widespread interest and has been applied to many differing applications within biological physics [5]. In many situations, however, the strength of the noise acting upon a system is not arbitrarily controllable; e.g. the strength of the internal noise source can be so large that SR simply will not occur, as it may happen for SR in globally coupled ion channel clusters of small size [6,7,8]. It is therefore of ultimate importance to devise control schemes to attain and manipulate SR in real systems. A concept which was proposed in prior literature [9,10] in order to enhance or suppress the spectral power is based on a modulation of the threshold in a discrete detector or in a bistable system dynamics. This in turn results in "breathing" oscillations of the barriers. By doing so, the "classical" SR effect could be both characteristically enhanced and suppressed by changing the phase difference between the threshold modulation and the input signal.

In this work we suggest a different, although related control scheme which we base on harmonic mixing input signal [11,12,13,14,15,16]. The sinusoidal input signal is superposed by a second, sinusoidal signal with twice the frequency of the former, monochromatic input signal. By controlling the phase difference between these two signal parts we obtain a powerful tool for the manipulation of SR.

2 The model

To start out, we consider the motion of a Brownian particle in a bistable and symmetric potential in the presence of noise and periodic forcing. The particle is furthermore subjected to viscous friction. With the assumption that inertia effects are negligible (overdamped dynamics), the driven Langevin dynamics reads in scaled units [1,17]:

$$\frac{d}{dt}x(t) = -\frac{d}{dx}V(x) + f(t) + \xi(t)\ , \tag{1}$$

with the static double-well potential given by $V(x) := \frac{1}{4}x^4 - \frac{1}{2}x^2$. The harmonic mixing driving signal $f(t)$ has the form,

$$f(t) = A\sin(\Omega t) + B\sin(2\Omega t + \Psi)\ , \tag{2}$$

The relative phase difference is denoted by Ψ, and it is this quantity which we shall predominantly use in the following as our control parameter for steering

SR. The coupling to the heat bath is modeled by zero-mean, Gaussian white noise $\xi(t)$ with autocorrelation function:

$$\langle \xi(t)\xi(s)\rangle = 2D\delta(t-s)\,, \tag{3}$$

where D denotes the noise strength.

The corresponding Fokker-Planck equation for the probability density $P(x,t)$ [18,19] is thus given by,

$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}\left[\left(\frac{d}{dx}V(x)\right) - f(t)\right]P(x,t) + D\frac{\partial^2}{\partial x^2}P(x,t)\,. \tag{4}$$

In the absence of the second signal (i.e. $B=0$) eq. (1) forms the archetypical model for SR[1]. The dependence of SR-measures such as the spectral power amplification [20,21,22] or the signal-to-noise ratio [23], respectively, exhibits a bell-shaped behavior *vs.* the noise strength D. Moreover, due to the dynamical generalized parity symmetry [20,21,22,24] only odd higher harmonics emerge which all exhibit the effect of SR. In contrast, for asymmetric double-well potentials also the even numbered higher harmonics are generated: The generation rate of the third harmonic then depicts a characteristic noise-induced suppression [25,26]. Due to our harmonic mixing signal, and particularly due to the relative phase difference Ψ and the ratio of amplitudes A and B, we can systematically break the symmetry dynamically and thus, control the response at higher harmonics.

3 Symmetry breaking in the deterministic model

Before we elucidate the Fokker-Planck dynamics (4), the deterministic case is instructive for obtaining an understanding of the physics of the harmonic mixing driving on the dynamics of a particle in a symmetric double-well. The Langevin equation (1) thus turns into the time-dependent deterministic equation:

$$\frac{d}{dt}x(t) = -\frac{d}{dx}V(x) + f(t)\,. \tag{5}$$

In absence of any modulation, i.e. $f(t)=0$, there exist two stable attractors at $x_\pm = \pm 1$ and one unstable attractor at $x_\mathrm{u} = 0$. For sub-threshold harmonic mixing, $f(t) = A\sin(\Omega t) + B\sin(2\Omega t + \Psi)$, two oscillatory stable orbits are formed within the potential wells. The domains of attraction are separated

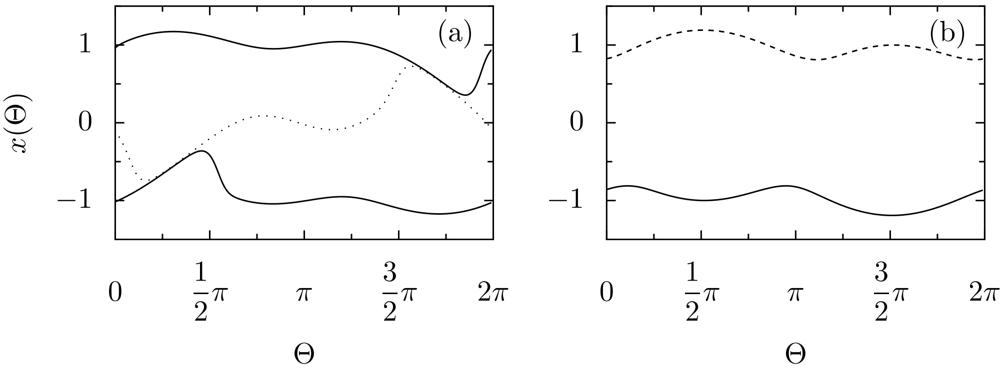

Fig. 1. The stable (solid line) and unstable (dashed line) periodic orbits for the motion of an *overdamped* particle in a quartic double-well potential driven by a harmonic mixing signal $f(t) = 0.25 \ \sin(0.1 \ t) + 0.25 \ \sin(0.2 \ t + \Psi)$ are plotted for $\Psi = 0$ in Fig. (a) with $\Theta := 0.1 \ t(\mod 2\pi)$. In contrast, there exists also a parameter regime for the phase difference Ψ where only one stable orbit exists (depicted in panel (b)). For example, the orbit is located in the left potential well for $\Psi = \pi/2$ (solid line) and in the right well for $\Psi = 3\pi/2$ (dashed line); harmonic mixing thus causes symmetry breaking.

by an unstable orbit, oscillating close to the former, unstable point near the barrier.

Upon increasing the amplitudes A and B, the oscillations of the stable and instable orbits become larger; consequently, the corresponding orbits approach each other. At even stronger driving the situation changes drastically and the variation of the phase difference Ψ possesses salient effects: the symmetry breaking by the harmonic mixing signal becomes evident: In Fig. 1 the amplitudes A and B both equal the barrier height 1/4 and the basic frequency is $\Omega = 0.1$. For phases Ψ around 0 or π two stable and one unstable orbits are present, whereas for $\Psi = \pi/2$ or $\Psi = 3\pi/2$ there is only one attractor located in either the left or in the right potential well of the static potential, respectively. The reason for this symmetry breaking is the interplay between the asymmetry of the harmonic mixing signal and the non-linearity of the quartic double-well potential.

For large signal amplitudes and arbitrary phase differences, there occurs only one stable periodic orbit which spreads over both potential wells (not depicted).

4 Two-State model

In view of our findings for the deterministic dynamics, we expect, that the noisy system exhibits SR similarly to that occurring in asymmetric potentials driven by sinusoidal signals [1,17]. In order to check the former statement we have numerically solved the continuous Fokker-Planck model and developed an approximate treatment for (1) in terms of a two-state model.

For small driving frequencies, i.e. for frequencies which are much smaller than the noise induced hopping rate, the adiabatic potential modulation can be invoked. Applying Kramers rate formula [27,28,29,30] for the transition rates among potential wells we find to leading order in the driving amplitudes [31] the results:

$$k_\pm(t) = k_0 \; \exp\left\{\pm\frac{A}{D}\sin(\Omega t) \pm \frac{B}{D}\sin(2\Omega t + \Psi)\right\} , \tag{6}$$

wherein k_0 is the Kramers rate of the unperturbed symmetric system, i.e. $k_0 := 1/(\pi\sqrt{2})\exp\{-1/(4D)\}$. The occupation probabilities $p_\pm(t)$ for the two states $x_\pm = \pm 1$ obey the following master equation [23,32]:

$$\frac{d}{dt}p_\pm(t) = k_\pm(t)p_\mp(t) - k_\mp(t)p_\pm(t)\,. \tag{7}$$

Due to normalization of probabilities, i.e. $p_+(t) + p_-(t) = 1$, the differential equation for the mean value ($\langle x(t)\rangle = p_+ - p_-$) reads

$$\frac{d}{dt}\langle x(t)\rangle = -\left[k_+(t) + k_-(t)\right]\langle x(t)\rangle + k_+(t) - k_-(t)\,. \tag{8}$$

Assuming $A \approx B$ and $A/D \ll 1$ the asymptotic, periodic long time solution of eq. (8) can be expanded beyond linear response into a series with respect to the ratios A/D and B/D. Next, in order to identify higher harmonics, we expand $\langle x(t)\rangle$ into a Fourier series:

$$\langle x(t)\rangle = \gamma_0 + \sum_{n=1}^{\infty}\gamma_n \sin(n\Omega t + \phi_n)\,, \tag{9}$$

with corresponding Fourier coefficients γ_n and phase lags ϕ_n. The spectral amplification factors η_n, which are defined as ratio of the output power stored at the corresponding higher harmonic driving frequency to the input power, are given by

$$\eta_n = \frac{\gamma_n^2}{A^2 + B^2},\;\; n = 1, 2, \ldots\,. \tag{10}$$

4.1 The time- and ensemble-averaged mean value

We start our discussion with the zeroth order Fourier coefficient γ_0, namely the time- and noise averaged mean value. This nonlinear response reads in

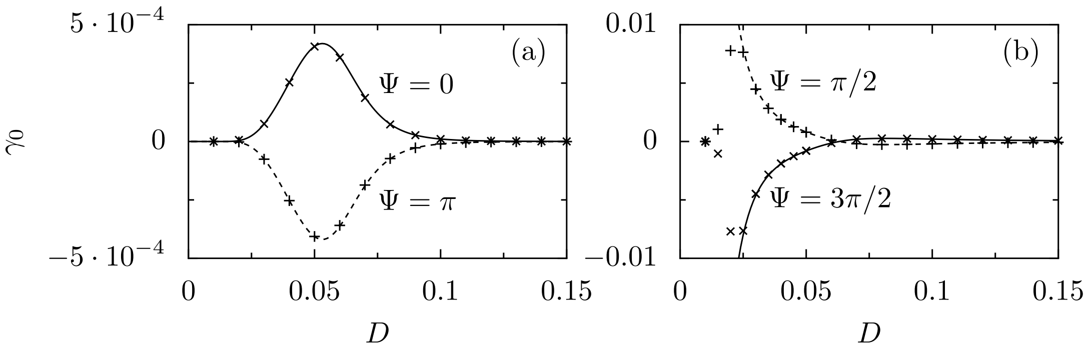

Fig. 2. Time- and ensemble-averaged mean value γ_0 of the nonlinear response of the two-state system driven by a harmonic mixing signal with amplitudes $A = B = 0.01$, fundamental frequency $\Omega = 0.01$ and phase differences $\Psi = 0, \pi/2, \pi$ and $3\pi/2$. The lines correspond to the analytic solution, i.e. eq. (11), while the two symbols ("+" and "$\times$") belong to the corresponding numerical solution of eq. (8). This driving induced zero-frequency response γ_0 exhibits versus noise strength D a bell-shaped behavior, similar to the behavior of Stochastic Resonance. Interestingly, for specific noise levels and chosen relative phases Ψ the symmetry can be restored, cf. panel 2(b) and Fig. 4(a).

leading order:

$$\gamma_0 = \frac{A^2 B}{D^3}\,\frac{1}{8}\,\frac{1}{(4k_0^2+\Omega^2)(k_0^2+\Omega^2)}\times \left\{3k_0\Omega^3\cos\Psi + (8k_0^4 + 4k_0^2\Omega^2 - \Omega^4)\sin\Psi\right\}. \tag{11}$$

Please note that generally γ_0 differs from zero. This is so, because the unbiased, but asymmetric input signal, possessing particularly nonvanishing time-averaged odd numbered higher moments $n \geq 3$ dynamically breaks the symmetry of the system [9,10,33,34,35,36,37]. For illustration, we depict this driving induced, nonvanishing mean γ_0 for $f(t) = 0.01\sin(0.01t) + 0.01\sin(0.02t + \Psi)$ and different relative phases Ψ in Fig. 2. For a phase difference $\Psi = 0$, the accumulation in the state "+" increases initially, reaches a maximum, and then decreases as the noise strength D is increased further, cf. Fig. 2(a). At an optimum noise level D the accumulation in one state is extremal. Similar to the phenomenon of Stochastic Resonance, this effect manifests itself by a synchronization of noise-activated hopping events between the two metastable states and the driving force $f(t)$.

Upon changing the relative phase difference Ψ we thus can control the dynamical asymmetry of the harmonic mixing driving signal and therefore the time averaged mean value γ_0. As a consequence for $\Psi = 0$ and π the accumulation in the states "+ and $-$" undergo an SR - like behavior. For other phase differences Ψ, the time averaged mean value vanishes at certain noise strengths; thus, the symmetry in the system can, dynamically, be restored accidentally at selected parameter choices, cf. Fig. 2(b). According to the expansion in A/D and B/D, the analytic solution worsens for small noise strengths D, this feature is apparent in Fig. 2(b).

4.2 Spectral amplification factors

The spectral amplification factors (10) at the first and second harmonic of the system output are evaluated to leading, non-vanishing order as:

$$\eta_1 = \frac{A^2}{D^2}\,\frac{1}{A^2+B^2}\,\frac{4k_0^2}{4k_0^2+\Omega^2}\,, \quad \eta_2 = \frac{B^2}{D^2}\,\frac{1}{A^2+B^2}\,\frac{k_0^2}{k_0^2+\Omega^2}. \tag{12}$$

We observe that, within this two-state approximation scheme, γ_1 depends in lowest order only on A/D (linear response limit). Likewise, the spectral amplification at 2Ω is determined in linear response by the second harmonic component of the harmonic mixing signal, yielding the spectral amplification of the second harmonic η_2. The two components of the driving do not interact with each other in this lowest order, particularly because of the suppression of even-numbered higher harmonic generation in symmetric systems driven by sinusoidal signals. Therefore, SR manifests itself at both frequencies with the well-known bell-shaped amplification behavior, cf. Fig. 3(a) and (b).

For the generation of the third higher harmonic, however, the two parts of the harmonic mixing signal do interact, and, in lowest, leading order, η_3 is given by the expression:

$$\begin{aligned}
\eta_3 = \frac{1}{D^6}\,\frac{1}{A^2+B^2}\,&\left[\left(k_0^2+\Omega^2\right)\left(4\,k_0^2+\Omega^2\right)^2\left(4\,k_0^2+9\,\Omega^2\right)\right]^{-1}\times\\
&\Bigg\{A^6\,\frac{1}{144}\,k_0^2\,\left(\Omega^2+16\,k_0^2\right)\left(k_0^2+\Omega^2\right)\left(4\,k_0^2+\Omega^2\right)\\
&-A^4B^2\,k_0^2\,\Bigg[\frac{1}{12}\left(\Omega^6+64\,k_0^6+36\,k_0^4\Omega^2-9\,k_0^2\Omega^4\right)\cos^2\Psi\\
&\qquad+\frac{1}{2}\,k_0\,\Omega^3\,\left(\Omega^2-2\,k_0^2\right)\,\sin\Psi\,\cos\Psi\\
&\qquad-\frac{1}{24}\,\left(64\,k_0^6+36\,k_0^4\Omega^2-9\,k_0^2\Omega^4+\Omega^6\right)\Bigg]\\
&+A^2B^4\,\frac{1}{16}\,k_0^2\,\left(\Omega^4-7\,k_0^2\,\Omega^2+16\,k_0^4\right)\left(4\,k_0^2+\Omega^2\right)\Bigg\}\,.
\end{aligned} \tag{13}$$

Just as for the case with an asymmetric double well potential [26,38,39], the spectral amplification at the third harmonic exhibits in our case a noise-

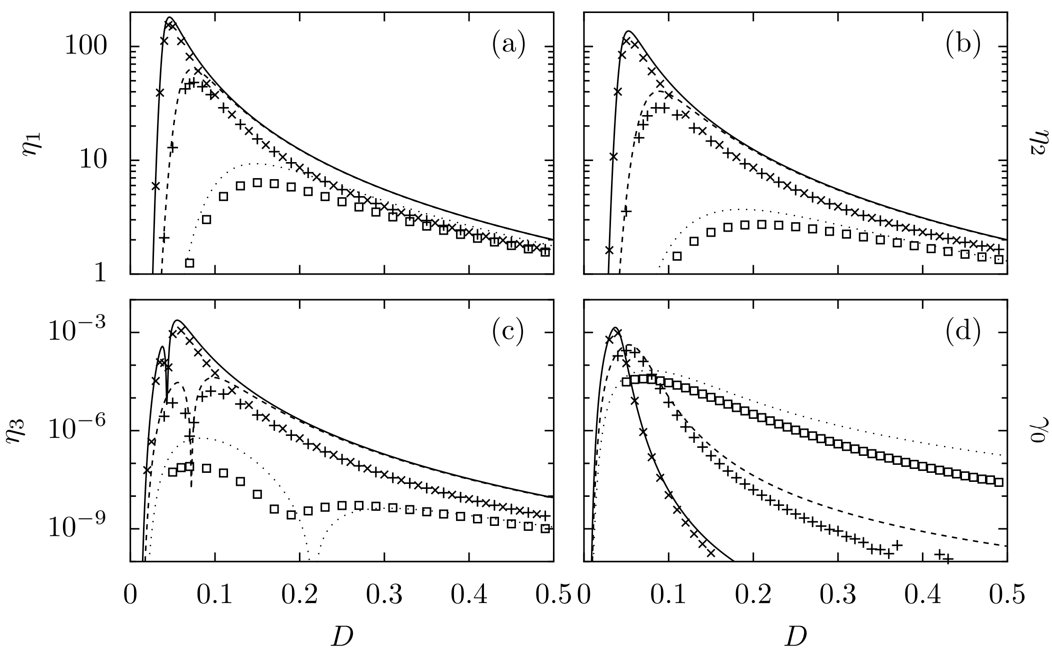

Fig. 3. The dependence of the spectral power amplification factors (a)-(c) and the time averaged mean value (d) versus the noise strength D is depicted for the driving amplitudes $A = B = 0.01$ at vanishing relative phase $\Psi = 0$ and at the fundamental driving frequency $\Omega = 0.001$: analytic estimate (solid line), corresponding numerical Fokker-Planck solution (crosses "$\times$"). The same for the driving fundamental at $\Omega = 0.01$: analytic estimate (dashed line), numerical solution ("+" signs). Likewise, the same for the high frequency drive at $\Omega = 0.1$: analytic estimate (dotted line), numerical Fokker-Planck solution (squares). Note that at large driving frequencies there is a good agreement between analytic results (lines) and the numerical results (symbols) for the Fokker-Planck equation (4).

induced suppression. This characteristic suppression at a tailored noise strength depends on the driving frequency Ω and is accompanied with a corresponding π-phase jump (not depicted). In Fig. 3(c) we depict this behavior for amplitudes $A = B = 0.01$, and a vanishing relative phase difference $\Psi = 0$ and for different fundamental frequencies. Agreement with the two-state theory is best at moderate fundamental driving frequencies; this corroborates with the fact that the linear response analysis and its corrections to higher orders indeed work best at moderate-to-large frequencies and increasingly fails at very small frequencies [40,41].

4.3 Comparison with the Fokker-Planck treatment

Additionally, we have numerically integrated the Fokker-Planck equation (4) and evaluated the time-periodic, asymptotic mean value $\langle x(t) \rangle$ together with an expansion according to eq. (9) into a Fourier series. The results are depicted in Fig. 3. There is good agreement between the analytic solution of the two-state approximation and the numerical solution of the continuous-state problem. Although the two-state approximation, i.e. the Kramers-rate approximation fails for large driving frequencies and large noise strengths, respectively, there is nevertheless still qualitative good agreement, cf. Fig. 3.

5 Controlling nonlinear SR with noise and relative phase Ψ

Within the range of small harmonic mixing driving amplitudes, where the agreement of the two-state and the continuous system is very good, the time-averaged and noise averaged mean value γ_0 and the spectral amplification factor of the third harmonic η_3 depict a striking dependency on the relative phase Ψ; in contrast the amplification factors of the first and second harmonic generations are in lowest order independent on the phase difference. This is because these former quantifiers depend nonlinearly on the driving amplitudes (nonlinear response regime). In Fig. 4 this dependence of the time averaged mean value γ_0 (a) and the third spectral amplification factor η_3 (b) are plotted versus the noise strength and the relative phase difference by means of contour-line plots. Because a shift of π will not change the spectral amplification factors and only inverts the sign of the time averaged mean value γ_0, it is sufficient to vary Ψ in the range from 0 to π.

As noted above, the mean value vanishes for certain, tailored noise strengths D and relative phases Ψ, cf. Fig. 2(b). The resulting zero-lines converge for large noise strengths to multiples of π, cf. Fig. 4(a). Interestingly enough, for every phase difference there exists only one value of noise strength for which γ_0 vanishes and, thus, symmetry restoring occurs accidentally. This feature can be used to determine and characterize sensitively the operating internal noise level in metastable systems. Additionally, by changing the relative phase difference the time averaged mean value and, consequently, the output power of the dynamically induced bias value of the response signal can be controlled. A maximum enhancement of γ_0 is obtained for relative phases Ψ around $\pi/2$ and $3\pi/2$, respectively.

By variation of the phase difference Ψ, the noise strength D at which suppression takes place could be controlled as well, cf. Fig. 4(b). Yet another feature to be obtained upon controlling the relative phase difference Ψ is a large enhancement of η_3 up to a factor of ten.

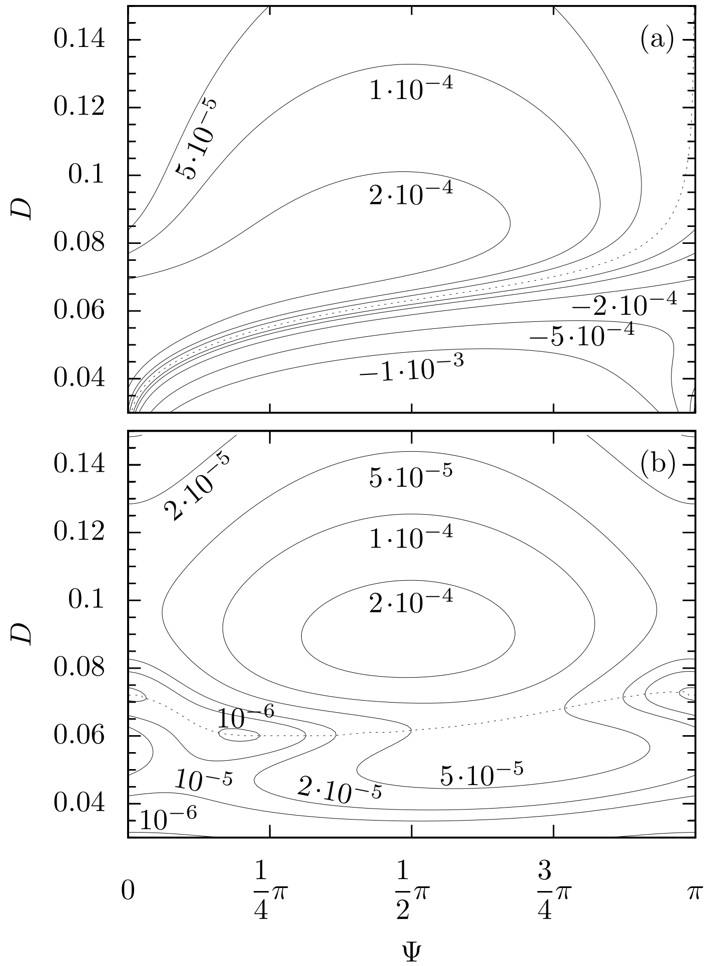

Fig. 4. The contour plot of the time averaged mean value γ_0 (a) and of the spectral amplification factor of the third harmonic η_3 (b) are depicted for varying phase difference Ψ and noise strength D according eq. (11) and eq. (13) ($A = B = 0.01$, $\Omega = 0.01$). The two dashed lines indicate the zero contour-line, meaning the symmetry restoring condition in (a) and the corresponding line in panel (b) the regime of noise-induced suppression of η_3.

6 Summary

We have investigated the influence of a harmonic mixing signal on the phenomenon of nonlinear Stochastic Resonance [1,32] for a Brownian dynamics in a double well. In the deterministic limit of harmonic mixing driving we can distinguish three situations: for small driving amplitudes the particle oscillates in one of the wells, depending on the initial starting value. In the range of large amplitudes the oscillation extend over both wells. For moderate driving amplitudes, however, a symmetry breaking occurs: Independent on the initial starting values the motion dwells only one specific well. By varying the relative phase difference between the two components of the mixing signal we can selectively control the dynamics in one of the two wells.

For the phenomenon of nonlinear Stochastic Resonance we monitor the nonlinear response due to harmonic mixing versus the noise strength D. Despite the somewhat coarse nature of the applied two-state approximation, it nevertheless provides very good agreement for the dynamics of the full Fokker-Planck dynamics; it is only for very small frequencies and/or large noise strength where the approximation starts to fail. The analytic estimate predicts a dynamical symmetry breaking which can be selectively controlled by the relative phase between the two driving modes and the noise strength D.

The spectral amplification measures of the higher harmonics exhibit the characteristic features of nonlinear SR in systems possessing an asymmetry. At selected noise strengths and relative phase differences the time averaged mean value accidentally vanishes thereby restoring the symmetry via the combined action of noise and driving. The dynamically induced bias value and the spectral amplification factor of the third harmonic generation depend sensitively on the relative phase difference of the two sinusoidal input signals. This can be used from a technological viewpoint to selectively control the enhancement and the suppression, respectively, of the nonlinear system response up to factor of ten. Moreover, the dynamically induced restoration of symmetry can be harvested to measure very sensitively the internal noise strength in a symmetric system.

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