Lazy Semiring Neighbours and some Applications

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Abstract. We extend an earlier algebraic approach to Neighbourhood Logic (NL) from domain semirings to lazy semirings yielding lazy semiring neighbours. Furthermore we show three important applications for these. The first one extends NL to intervals with infinite length. The second one applies lazy semiring neighbours in an algebraic semantics of the branching time temporal logic CTL^*. The third one sets up a connection between hybrid systems and lazy semiring neighbours.

1 Introduction

Chop-based interval temporal logics, such as ITL [5] and IL [3] are useful for the specification and verification of safety properties of real-time systems. However, as it is shown in [16], these logics cannot express all desired properties, like (unbounded) liveness properties. That is why Zhou and Hansen proposed *Neighbourhood Logic* (NL) [15], a first-order interval logic which provides extra atomic formulas. In [7] an embedding and extension into the framework of semirings has been presented, giving an algebraic version of NL is given. Unfortunately neither NL nor the algebraic version can handle intervals with infinite length. To remedy this, we transfer the concept of semiring neighbours from semirings to lazy semirings [11] and present some important properties for lazy semiring neighbours.

Surprisingly, lazy semiring neighbours are not only useful for acommodating the extension of NL; they occur in different situations and structures. So, e.g., looking at the algebraic characterisation of the branching time temporal logic CTL^* of [12], the existential path quantifier E as well as the universal path quantifier A correspond to lazy semiring neighbours. Therefore, temporal logics like CTL^* are a field for applying our theory. Since we introduce more kinds of lazy semiring neighbours than occur in CTL^*, we can extend the branching time temporal logic. Another field of application is in the area of hybrid systems. Lazy semiring neighbours can be directly transferred to the algebraic model presented in [8, 9]. It turns out that some of them guarantee liveness, others guarantee non-reachability, i.e., a form of safety.

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The paper is structured into two main parts. The first one presents lazy semiring neighbours with some properties in an abstract and formal way. Therefore we recapitulate the basic algebraic foundations, like lazy semirings, in Section 2. In Section 3 we define domain and codomain for lazy semirings and give some important properties. In the next section we introduce and discuss lazy semiring neighbours and boundaries. That section contains the main contribution from a theoretical point of view.

The second part presents three different applications for the presented theory. It starts by extending Neighbourhood Logic to intervals with infinite length in Section 5. Afterwards, in Section 6, we present lazy semiring neighbours in the context of the branching time logic CTL^*. The last application is presented in Section 7 and shows how to use the theory of Section 4 in the formal description of hybrid systems.

2 Algebraic Foundations

A *lazy semiring* (L-*semiring* or *left semiring*) is a quintuple $(S, +, \cdot, 0, 1)$ where $(S, +, 0)$ is a commutative monoid and $(S, \cdot, 1)$ is a monoid such that $\cdot$ is left-distributive over $+$ and *left-strict*, i.e., $0 \cdot a = 0$. A lazy semiring structure is also at the core of process algebra frameworks. The lazy semiring is *idempotent* if $+$ is idempotent and $\cdot$ is right-isotone, i.e., $b \leq c \Rightarrow a \cdot b \leq a \cdot c$, where the *natural order* $\leq$ on S is given by $a \leq b \Leftrightarrow_{df} a + b = b$. Left-isotony of $\cdot$ follows from its left-distributivity. Moreover, 0 is the $\leq$-least element and $a + b$ is the join of a and b. Hence every idempotent L-semiring is a join semilattice. A *semiring* (for clarity sometimes also called *full semiring*) is a lazy semiring in which $\cdot$ is also right-distributive and right-strict. An L-semiring is *Boolean* if it is idempotent and its underlying semilattice is a Boolean algebra. Every Boolean L-semiring has a greatest element $\top$.

A *lazy quantale* is an idempotent L-semiring that is also a complete lattice under the natural order with $\cdot$ being universally disjunctive in its left argument. A *quantale* is a lazy quantale in which $\cdot$ is universally disjunctive also in its right argument. Following [1], one might also call a quantale a *standard Kleene algebra*. A lazy quantale is *Boolean* if it is right-distributive and a Boolean L-semiring.

An important lazy semiring (that is even a Boolean quantale) is REL, the algebra of binary relations over a set under relational composition.

To model assertions in semrings we use the idea of tests as introduced into Kleene algebras by Kozen [10]. In REL a set of elements can be modelled as a subset of the identity relation; meet and join of such partial identities conincide with their conposition and union. Generalising this, one defines a *test* in a (left) quantale to be an element $p \leq 1$ that has a complement q relative to 1, i.e., $p + q = 1$ and $p \cdot q = 0 = q \cdot p$. The set of all tests of a quantale S is denoted by $\mathsf{test}(S)$. It is not hard to show that $\mathsf{test}(S)$ is closed under $+$ and $\cdot$ and has 0 and 1 as its least and greatest elements. Moreover, the complement $\neg p$ of a test is uniquely determined by the definition. Hence $\mathsf{test}(S)$ forms a Boolean algebra. If S itself is Boolean then $\mathsf{test}(S)$ coincides with the set of all elements below 1. We will consistently write $a, b, c \ldots$ for arbitrary semiring elements and $p, q, r, \ldots$ for tests.

With the above definition of tests we deviate slightly from [10], in that we do not allow an arbitrary Boolean algebra of subidentities as $\mathsf{test}(S)$ but only the maximal complemented one. The reason is that the axiomatisation of domain to be presented below forces this maximality anyway (see [2]).

In the remainder we give another important example of an L-semiring (especially with regard to temporal logics like CTL^* and hybrid systems). It is based on trajectories (cf. e.g. [13]) that reflect the values of the variables over time and was introduced in [9]. We give a slightly extended version of our model in [8].

Let V be a set of *values* and D a set of *durations* (e.g. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\ldots$). We assume a cancellative addition $+$ on D and an element $0 \in D$ such that $(D, +, 0)$ is a commutative monoid and the relation $x \leq y \Leftrightarrow_{df} \exists z \,.\, x + z = y$ is a linear order on D. Then 0 is the least element and $+$ is isotone w.r.t. $\leq$. Moreover, 0 is indivisible, i.e., $x + y = 0 \Leftrightarrow x = y = 0$. D may include the special value ∞. It is required to be an annihilator w.r.t. $+$ and hence the greatest element of D (and cancellativity of $+$ is restricted to elements in $D - \{\infty\}$). For $d \in D$ we define the interval $\mathsf{intv}\, d$ of admissible times as

$$\mathsf{intv}\, d =_{df} \begin{cases} [0, d] & \text{if } d \neq \infty \\ [0, d[& \text{otherwise} . \end{cases}$$

A *trajectory* t is a pair (d, g), where $d \in D$ and $g : \mathsf{intv}\, d \to V$. Then d is the *duration* of the trajectory. This view models *oblivious* systems in which the evolution of a trajectory is independent of the history before the starting time.

The set of all trajectories is denoted by TRA. Composition of trajectories (d_1, g_1) and (d_2, g_2) is defined by

$$(d_1, g_1) \cdot (d_2, g_2) =_{df} \begin{cases} (d_1 + d_2, g) & \text{if } d_1 \neq \infty \wedge g_1(d_1) = g_2(0) \\ (d_1, g_1) & \text{if } d_1 = \infty \\ \text{undefined} & \text{otherwise} \end{cases}$$

with $g(x) = g_1(x)$ for all $x \in [0, d_1]$ and $g(x + d_1) = g_2(x)$ for all $x \in \mathsf{intv}\, d_2$.

For a value $v \in V$, let $\underline{v} =_{df} (0, g)$ with $g(0) = v$ be the corresponding zero-length trajectory.

A *process* is a set of trajectories. The *infinite and finite parts* of a process A are the processes $\mathsf{inf}\, A =_{df} \{(d, g) \in A \mid d = \infty\}$ and $\mathsf{fin}\, A =_{df} A - \mathsf{inf}\, A$. Composition is lifted to processes as follows:

$$A \cdot B =_{df} \mathsf{inf}\, A \cup \{a \cdot b \mid a \in \mathsf{fin}\, A, b \in B\} .$$

With $I =_{df} \{\underline{v} \mid v \in V\}$ being the set of all zero-length trajectories, the structure

$$\text{PRO} =_{df} (\mathcal{P}(\text{TRA}), \cup, \cdot, \emptyset, I) ,$$

forms a lazy Boolean quantale which can be extended to a test quantale by $\mathsf{test}(\text{PRO}) =_{df} \mathcal{P}(\{\underline{v} \mid v \in V\})$.

Note that $A \in \mathrm{PRO}$ consists of infinite trajectories only, i.e., $A = \mathsf{inf}\, A$, iff $A \cdot B = A$ for all $B \in \mathrm{PRO}$. We call such a process *infinite*, too. Contrarily, A consists of finite trajectories only, i.e., $A = \mathsf{fin}\, A$, iff $A \cdot \emptyset = \emptyset$. We call such a process *finite*, too.

Finally, we note that for a discrete infinite set D of durations, e.g. $D = \mathbb{N}$, trajectories are isomorphic to nonempty finite or infinite words over the value set V. If V consists of states of computations, then the elements of PRO can be viewed as sets of computation streams; therefore we also write $\mathrm{STR}(V)$ instead of PRO in this case.

We now generalise the notions of infinite and finite parts of processes from PRO to an arbitrary L-semiring$S$. An element $a \in S$ is called *infinite* if it is a left zero, i.e., $a \cdot b = a$ for all $b \in S$, which is equivalent to $a \cdot 0 = a$. By this property, $a \cdot 0$ may be considered as the *infinite part* of a, i.e., the part consisting just of infinite computations (if any). We assume that there exists a largest infinite element N, i.e.,

$$a \leq \mathsf{N} \Leftrightarrow_{df} a \cdot 0 = a \ .$$

Dually, we call an element a *finite* if its infinite part is trivial, i.e., if $a \cdot 0 = 0$. We also assume that there is a largest finite element F, i.e.,

$$a \leq \mathsf{F} \Leftrightarrow_{df} a \cdot 0 = 0 \ .$$

In Boolean quantales N and F always exist[1] and satisfy $\mathsf{N} = \top \cdot 0$ and $\mathsf{F} = \overline{\mathsf{N}}$, where $\bar{\ }$ denotes complementation. Moreover, every element can be split into its finite and infinite parts: $a = \mathsf{fin}\, a + \mathsf{inf}\, a$, where $\mathsf{fin}\, a =_{df} a \sqcap \mathsf{F}$ and $\mathsf{inf}\, a =_{df} a \sqcap \mathsf{N}$. In particular, $\top = \mathsf{N} + \mathsf{F}$.

Other examples of lazy (test) semirings will be given in Sections 5–7, where applications for lazy semiring neighbours are presented.

3 Domain and Codomain in L-Semirings

Domain and codomain are intended to abstractly characterise, in the form of tests, the sets of initial and final states of a set of computations. In contrast to the domain and codomain operators of full semirings and Kleene algebras [2] the operators for L-semirings are not symmetric. Therefore we recapitulate their definitions [11] and establish some properties which we need afterwards.

Definition 3.1 A *lazy semiring with domain* ($\ulcorner$-L-*semiring*) is a structure $(S, \ulcorner)$, where S is an idempotent lazy test semiring and the *domain operation* $\ulcorner : S \to \mathsf{test}(S)$ satisfies for all $a, b \in S$ and $p \in \mathsf{test}(S)$

$$a \leq {\ulcorner}a \cdot a \quad \text{(d1)}, \qquad {\ulcorner}(p \cdot a) \leq p \quad \text{(d2)}, \qquad {\ulcorner}(a \cdot {\ulcorner}b) \leq {\ulcorner}(a \cdot b) \quad \text{(d3)}.$$

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[1] In general N and F need not exist. In [11] lazy semirings where these elements exist are called *separated*.

The axioms are the same as in [2]. Since the domain describes all possible starting states of an element, it is easy to see that "laziness" of the underlying semiring doesn't matter. Most properties of [2, 11] can still be proved in L-semirings with domain. We only give some properties which we need in the following sections. First, the conjunction of (d1) and (d2) is equivalent to each of

$$ {}^{\ulcorner}a \leq p \Leftrightarrow a \leq p \cdot a \quad \text{(llp)}, \qquad\qquad {}^{\ulcorner}a \leq p \Leftrightarrow \neg p \cdot a \leq 0 \quad \text{(gla)}. $$

(llp) says that ${}^{\ulcorner}a$ is the least left preserver of a; (gla) that $\neg{}^{\ulcorner}a$ is the greatest left annihilator of a. By Boolean algebra, (gla) is equivalent to

$$ p \cdot {}^{\ulcorner}a \leq 0 \Leftrightarrow p \cdot a \leq 0 \; . \tag{1} $$

Lemma 3.2 *[11] Let S be a ${}^{\ulcorner}$-L-semiring.*

(a) ${}^{\ulcorner}$ *is isotone.*

(b) ${}^{\ulcorner}$ *is uiversally disjunctive;*
in particular ${}^{\ulcorner}0 = 0$ *and* ${}^{\ulcorner}(a+b) = {}^{\ulcorner}a + {}^{\ulcorner}b$.

(c) ${}^{\ulcorner}a \leq 0 \Leftrightarrow a \leq 0$. (Full Strictness)

(d) ${}^{\ulcorner}p = p$. (Stability)

(e) ${}^{\ulcorner}(p \cdot a) = p \cdot {}^{\ulcorner}a$. (Import/Export)

(f) ${}^{\ulcorner}(a \cdot b) \leq {}^{\ulcorner}a$.

We now turn to the dual case of the domain operation. In the case where we have (as in full semirings) right-distributivity and right-strictness, a codomain operation ${}^{\urcorner}$ is easily defined as a domain operation in the opposite L-semiring (i.e., the one that swaps the order of composition). But due to the absence of right-distributivity and right-strictness we need an additional axiom.

Definition 3.3 A *lazy semiring with codomain* (${}^{\urcorner}$-L-*semiring*) is a structure $(S, {}^{\urcorner})$, where S is an idempotent lazy test semiring and the *codomain operation* ${}^{\urcorner} : S \to \mathsf{test}(S)$ satisfies for all $a, b \in S$ and $p \in \mathsf{test}(S)$

$$ \begin{aligned} a &\leq a \cdot a^{\urcorner} \quad \text{(cd1)}, & (a \cdot p)^{\urcorner} &\leq p \quad \text{(cd2)}, \\ (a^{\urcorner} \cdot b)^{\urcorner} &\leq (a \cdot b)^{\urcorner} \quad \text{(cd3)}, & (a+b)^{\urcorner} &\geq a^{\urcorner} + b^{\urcorner} \quad \text{(cd4)}. \end{aligned} $$

(cd4) guarantees isotony of the codomain operator. As for domain, the conjunction of (cd1) and (cd2) is equivalent to

$$ a^{\urcorner} \leq p \Leftrightarrow a \leq a \cdot p \; , \tag{lrp} $$

i.e., $a^{\urcorner}$ is the least right preserver of a. However, due to lack of right-strictness $\neg a^{\urcorner}$ need not be the greatest right annihilator; we only have the weaker equivalence

$$ a^{\urcorner} \leq p \Leftrightarrow a \cdot \neg p \leq a \cdot 0 \; . \tag{wgra} $$

Lemma 3.4 *Let S be a ${}^{\urcorner}$-L-semiring.*

(a) ${}^{\urcorner}$ *is isotone.*

(b) ${}^{\urcorner}$ *is universally disjunctive;*
in particular $0^{\urcorner} = 0$ *and* $(a+b)^{\urcorner} = a^{\urcorner} + b^{\urcorner}$.

(c) $a^{\urcorner} \leq 0 \Leftrightarrow a \leq \mathsf{N}$.

(d) $p^{\urcorner} = p$. (Stability)
(e) $(a \cdot p)^{\urcorner} = a^{\urcorner} \cdot p$. (Import/Export)
(f) $(a \cdot b)^{\urcorner} \le b^{\urcorner}$.

Lemma 3.2(c) and Lemma 3.4(c) show the asymmetry of domain and codomain.

As in [11], a *modal lazy semiring* (ML-semiring) is an L-semiring with domain and codomain. The following lemma has some important consequences for the next sections, and illustrates again the asymmetry of L-semirings.

Lemma 3.5 *In an* ML*-semiring with a greatest element* $\top$*, we have*

(a) $\neg p \cdot a \le 0 \Leftrightarrow \ulcorner a \le p \Leftrightarrow a \le p \cdot a \Leftrightarrow a \le p \cdot \top$.
(b) $a \cdot \neg p \le a \cdot 0 \Leftrightarrow a^{\urcorner} \le p \Leftrightarrow a \le a \cdot p \Leftrightarrow a \le \top \cdot p$.
(c) $a \le \mathsf{F} \Leftrightarrow (a \le a \cdot p \Leftrightarrow a \cdot \neg p \le 0) \Leftrightarrow (a \le \top \cdot p \Leftrightarrow a \cdot \neg p \le 0)$.
Therefore, in general, $a \le a \cdot p \not\Leftarrow a \cdot \neg p \le 0$ *and* $a \le \top \cdot p \not\Leftarrow a \cdot \neg p \le 0$.

Proof.

(a) The first equivalence is (gla), the second (llp). $a \le p \cdot a \Rightarrow a \le p \cdot \top$ holds by isotony of $\cdot$ and $a \le p \cdot \top \Rightarrow \ulcorner a \le p$ by isotony of domain and $\ulcorner(p \cdot \top) \overset{3.2(e)}{=} p \cdot \ulcorner\top = p$, since $\ulcorner\top \ge \ulcorner 1 = 1$ by Lemma 3.2(d).
(b) Symmetrically to (a).
(c) $a \le \mathsf{F} \Rightarrow (a \le a \cdot p \Leftrightarrow a \cdot \neg p \le 0)$ holds by (b) and $a \cdot 0 \le 0 \Leftrightarrow a \le \mathsf{F}$.
The converse implication is shown by setting $p = 1$, Boolean algebra and definition of F: $a \le a \Rightarrow a \cdot \neg 1 \le 0 \Leftrightarrow a \cdot 0 \le 0 \Leftrightarrow a \le \mathsf{F}$.
The second equivalence follows from $a \le a \cdot p \Leftrightarrow a \le \top \cdot p$ (see (b)). $\square$

(c) says that we do not have a law for codomain that is symmetric to (a).

Further properties of (co)domain and ML-semirings can be found in [2,11].

4 Neighbours — Definitions and Basic Properties

In [7] semiring neighbours and semiring boundaries are motivated by Neighbourhood Logic [15,16]. The definitions there require full semirings as the underlying algebraic structure. In this section we use the same axiomatisation as in [7] to define neighbours and boundaries in L-semirings. Since the domain and codomain operators are not symmetric we also discuss some properties and consequences of the lack of right-distributivity and right-strictness. Note that in [7] the semiring neighbours and boundaries work on predomain and precodomain, i.e., assumed only (d1)–(d2) and (cd1)–(cd2), resp. Here we assume (d3)/(cd3) as well.

In the remainder some proofs are done only for one of a series of similar cases.

Definition 4.1 Let S be an ML-semiring and $a, b \in S$. Then
(a) a is a *left neighbour* of b (or $a \le {\Diamond\!\!\!\!\!\!\scriptstyle n\;\,}_l b$ for short) iff $a^{\urcorner} \le \ulcorner b$,
(b) a is a *right neighbour* of b (or $a \le {\Diamond\!\!\!\!\!\!\scriptstyle n\;\,}_r b$ for short) iff $\ulcorner a \le b^{\urcorner}$,
(c) a is a *left boundary* of b (or $a \le {\Diamond\!\!\!\!\!\!\scriptstyle b\;\,}_l b$ for short) iff $\ulcorner a \le \ulcorner b$,
(d) a is a *right boundary* of b (or $a \le {\Diamond\!\!\!\!\!\!\scriptstyle b\;\,}_r b$ for short) iff $a^{\urcorner} \le b^{\urcorner}$.

We will see below that the notation using $\leq$ is justified. By *lazy semiring neighbours* we mean both, left/right neighbours and boundaries. Most of the properties given in [7] use Lemma 3.5(a) in their proofs and a symmetric version of it for codomain which holds in full semirings. Unfortunately, by Lemma 3.5(b) and 3.5(c), we do not have this symmetry. Hence we have to check all properties in the setting of L-semirings again. Definition 4.1 works for all ML-semirings. However, most of the interesting properties postulate a greatest element $\top$. Therefore we assume the existence of such an element in the remainder.

Lemma 4.2 *Neighbours and boundaries can be expressed explicitly as*

$$\Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_l b \;=\; \top\cdot\ulcorner b\;, \quad \Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_r b \;=\; b\urcorner\cdot\top\;, \quad \Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_l b \;=\; \ulcorner b\cdot\top\;, \quad \Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_r b \;=\; \top\cdot b\urcorner\;.$$

Proof. We use the principle of indirect (in)equality.
By definition and Lemma 3.5(b) we get

$$a \leq \Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_l b \;\Leftrightarrow\; a\urcorner \leq \ulcorner b \;\Leftrightarrow\; a \leq \top\cdot\ulcorner b\;. \qquad \square$$

For nested neighbours we have the following cancellation properties.

Lemma 4.3
(a) $\Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_l \Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_r b \;=\; \Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_r b$ *and* $\Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_r \Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_l b \;=\; \Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_l b$,
(b) $\Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_l \Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_r b \;=\; \Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_r b$ *and* $\Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_r \Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_l b \;=\; \Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_l b$,
(c) $\Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_l \Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_l b \;=\; \Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_l b$ *and* $\Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_r \Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_r b \;=\; \Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_r b$,
(d) $\Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_l \Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_l b \;=\; \Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_l b$ *and* $\Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_r \Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_r b \;=\; \Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_r b$.

Proof. The proof of [7] can immediately be adopted, since it only uses the explicit representations of neighbours and boundaries, which are identical for L-semirings and full semirings. E.g., by definition (twice), $\ulcorner p\cdot\top \;=\; p$ and definition again,

$$\Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_l \Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_r b \;=\; \Diamond\!\!\!\!\!\!{\scriptstyle n}\;\;_l (b\urcorner\cdot\top) \;=\; \top\cdot\ulcorner(b\urcorner\cdot\top) \;=\; \top\cdot b\urcorner \;=\; \Diamond\!\!\!\!\!\!{\scriptstyle b}\;\;_r b\;. \qquad \square$$

Now we draw some conclusions when S is Boolean.

Lemma 4.4 *For a Boolean* ML*-semiring S, we have*
(a) $\neg\ulcorner a \leq \ulcorner\overline{a}$ *and* $\neg a\urcorner \leq \overline{a}\urcorner$.
(b) $\overline{p\cdot\top} \;=\; \neg p\cdot\top$
(c) *If S is right-distributive,* $\overline{\top\cdot p} \;=\; \mathsf{F}\cdot\neg p$

Proof.

(a) By Boolean algebra and additivity of domain we have

$$1 \;=\; \ulcorner\top \;=\; \ulcorner(a+\overline{a}) \;=\; \ulcorner a + \ulcorner\overline{a}$$

and the first claim follows by shunting. The second inequality can be shown symmetrically.

(b) By Boolean algebra we only have to show that $\neg p\cdot\top + p\cdot\top \;=\; \top$ and $\neg p\cdot\top \sqcap p\cdot\top \;=\; 0$. The first equation follows by left-distributivity, the second one by Boolean algebra and the law [11]

$$p\cdot a \sqcap q\cdot a \;=\; p\cdot q\cdot a\;. \qquad (2)$$

(c) By left and right distributivity, Boolean algebra and N being a left zero,

$$\begin{aligned}\mathsf{F}\cdot\neg p+\top\cdot p \;&=\; \mathsf{F}\cdot\neg p+(\mathsf{F}+\mathsf{N})\cdot p \;=\; \mathsf{F}\cdot\neg p+\mathsf{F}\cdot p+\mathsf{N}\cdot p\\ &=\; \mathsf{F}\cdot(\neg p+p)+\mathsf{N} \;=\; \mathsf{F}+\mathsf{N} \;=\; \top\ .\end{aligned}$$

Next, again by distributivity,

$$\begin{aligned}\mathsf{F}\cdot\neg p\sqcap\top\cdot p \;&=\; \mathsf{F}\cdot\neg p\sqcap(\mathsf{F}+\mathsf{N})\cdot p \;=\; \mathsf{F}\cdot\neg p\sqcap(\mathsf{F}\cdot p+\mathsf{N}\cdot p)\\ &=\; (\mathsf{F}\cdot\neg p\sqcap\mathsf{F}\cdot p)+(\mathsf{F}\cdot\neg p\sqcap\mathsf{N}\cdot p)\ .\end{aligned}$$

The first summand is 0, since the law symmetric to (2) holds for finite a and hence for F. The second summand is, by $p, \neg p \leq 1$ and isotony, below $\mathsf{F}\sqcap\mathsf{N}=0$ and thus 0, too. $\square$

Similarly to [7], we now define perfect neighbours and boundaries.

Definition 4.5 Let S be a Boolean ML-semiring and $a, b \in S$.
(a) a is a *perfect left neighbour* of b (or $a \leq \boxed{n}_l b$ for short) iff $a^{\urcorner}\cdot{}^{\ulcorner}\overline{b} \leq 0$,
(b) a is a *perfect right neighbour* of b (or $a \leq \boxed{n}_r b$ for short) iff $\overline{b}^{\urcorner}\cdot{}^{\ulcorner}a \leq 0$,
(c) a is a *perfect left boundary* of b (or $a \leq \boxed{b}_l b$ for short) iff ${}^{\ulcorner}a\cdot{}^{\ulcorner}\overline{b} \leq 0$,
(d) a is a *perfect right boundary* of b (or $a \leq \boxed{b}_r b$ for short) iff $a^{\urcorner}\cdot\overline{b}^{\urcorner} \leq 0$.

From this definition, we get the following exchange rule for perfect neighbours.

$$a \leq \boxed{n}_l b \;\Leftrightarrow\; \overline{b} \leq \boxed{n}_r \overline{a}\ . \tag{3}$$

Lemma 4.6 *Perfect neighbours and perfect boundaries have the following explicit forms:*

$$\boxed{n}_l b \;=\; \top\cdot\neg{}^{\ulcorner}\overline{b}\ , \quad \boxed{n}_r b \;=\; \neg\overline{b}^{\urcorner}\cdot\top\ , \quad \boxed{b}_l b \;=\; \neg{}^{\ulcorner}\overline{b}\cdot\top\ , \quad \boxed{b}_r b \;=\; \top\cdot\neg\overline{b}^{\urcorner}\ .$$

Proof. By definition, shunting and Lemma 3.5(b)

$$a \leq \boxed{n}_l b \;\Leftrightarrow\; a^{\urcorner}\cdot{}^{\ulcorner}\overline{b} \leq 0 \;\Leftrightarrow\; a^{\urcorner} \leq \neg{}^{\ulcorner}\overline{b} \;\Leftrightarrow\; a \leq \top\cdot\neg{}^{\ulcorner}\overline{b}\ .$$ $\square$

Lemma 4.7 *Each perfect neighbour (boundary) is a neighbour (boundary):*

$$\boxed{n}_l b \leq \Diamond\hspace{-0.6em}{\scriptstyle n}\hspace{0.25em}_l b\ , \qquad \boxed{n}_r b \leq \Diamond\hspace{-0.6em}{\scriptstyle n}\hspace{0.25em}_r b\ , \qquad \boxed{b}_l b \leq \Diamond\hspace{-0.6em}{\scriptstyle b}\hspace{0.25em}_l b\ , \qquad \boxed{b}_r b \leq \Diamond\hspace{-0.6em}{\scriptstyle b}\hspace{0.25em}_r b\ .$$

Proof. The claim follows by definition, shunting, Lemma 4.4(a), Boolean algebra and definition again:

$$a \leq \boxed{n}_l b \;\Leftrightarrow\; a^{\urcorner}\cdot{}^{\ulcorner}\overline{b} \leq 0 \;\Leftrightarrow\; a^{\urcorner} \leq \neg{}^{\ulcorner}\overline{b} \;\Rightarrow\; a^{\urcorner} \leq {}^{\ulcorner}b \;\Leftrightarrow\; a \leq \Diamond\hspace{-0.6em}{\scriptstyle n}\hspace{0.25em}_l b\ .$$ $\square$

Similarly to Lemma 4.3, we have cancellative laws for all box-operators. By $\Box\Box a \;=\; \overline{\Diamond\Diamond\overline{a}}$ for all kinds of perfect lazy semiring neighbours, we have

Corollary 4.8
(a) $\boxed{n}_l\,\boxed{n}_r b \;=\; \boxed{b}_r b$ *and* $\boxed{n}_r\,\boxed{n}_l b \;=\; \boxed{b}_l b$,
(b) $\boxed{b}_l\,\boxed{n}_r b \;=\; \boxed{n}_r b$ *and* $\boxed{b}_r\,\boxed{n}_l b \;=\; \boxed{n}_l b$,
(c) $\boxed{b}_l\,\boxed{b}_l b \;=\; \boxed{b}_l b$ *and* $\boxed{b}_r\,\boxed{b}_r b \;=\; \boxed{b}_r b$,
(d) $\boxed{n}_l\,\boxed{b}_l b \;=\; \boxed{n}_l b$ *and* $\boxed{n}_r\,\boxed{b}_r b \;=\; \boxed{n}_r b$.

There are also cancellation rules for mixed diamond/box expressions, e.g.,

$$\langle b\rangle_l \boxed{b}_l b = \boxed{b}_l b \quad \textit{and} \quad \boxed{b}_l \langle b\rangle_l b = \langle b\rangle_l b \ . \tag{4}$$

By straightforward calculations we get the de Morgan duals of right neighbours and left boundaries, respectively.

$$\begin{array}{lcl} \overline{\langle n\rangle_r \overline{b}} = \boxed{n}_r b & \text{and} & \overline{\boxed{n}_r \overline{b}} = \langle n\rangle_r b \ , \\ \overline{\langle b\rangle_l \overline{b}} = \boxed{b}_l b & \text{and} & \overline{\boxed{b}_l \overline{b}} = \langle b\rangle_l b \ . \end{array} \tag{5}$$

Furthermore, we have the following Galois connections.

Lemma 4.9 *We have* $\langle n\rangle_r a \le b \Leftrightarrow a \le \boxed{n}_l b$ *and* $\langle b\rangle_l a \le b \Leftrightarrow a \le \boxed{b}_r b$.

Proof. By de Morgan duality, Boolean algebra and the exchange rule (3)

$$\langle n\rangle_r a \le b \Leftrightarrow \overline{\boxed{n}_r \overline{a}} \le b \Leftrightarrow \overline{b} \le \boxed{n}_r \overline{a} \Leftrightarrow a \le \boxed{n}_l b \ . \qquad \square$$

Since Galois connections are useful as theorem generators and dualities as theorem transformers we get many properties of (perfect) neighbours and (perfect) boundaries for free. For example we have

Corollary 4.10

(a) $\langle n\rangle_r$, $\langle b\rangle_l$ *and* $\boxed{n}_l$, $\boxed{b}_r$ *are isotone.*
(b) $\langle n\rangle_r$, $\langle b\rangle_l$ *are disjunctive and* $\boxed{n}_l$, $\boxed{b}_r$ *are conjunctive.*
(c) *We also have cancellative laws:*
$\langle n\rangle_r \boxed{n}_l a \le a \le \boxed{n}_l \langle n\rangle_r a$ *and* $\langle b\rangle_l \boxed{b}_r a \le a \le \boxed{b}_r \langle b\rangle_l a$.

But, because of Lemma 4.4(c), we do not have the full semiring de Morgan dualities of left neighbours and right boundaries, respectively. We only obtain

Lemma 4.11 *Let* S *be right-distributive.*

(a) $\overline{\langle n\rangle_l \overline{b}} \le \boxed{n}_l b$ *and* $\overline{\boxed{n}_l \overline{b}} \le \langle n\rangle_l b$,
(b) $\overline{\langle b\rangle_r \overline{b}} \le \boxed{b}_r b$ *and* $\overline{\boxed{b}_r \overline{b}} \le \langle b\rangle_r y$.

Proof. (a) By Lemma 4.2, 4.4(c), isotony and Lemma 4.6,
$\overline{\langle n\rangle_l \overline{b}} = \overline{\top \cdot \ulcorner\overline{b}} = \mathsf{F} \cdot \neg\ulcorner\overline{b} \le \top \cdot \neg\ulcorner\overline{b} = \boxed{n}_l b.$
The equation $\overline{\boxed{n}_l \overline{b}} \le \langle n\rangle_l b$ then follows by shunting. $\square$

The converse inequations do not hold. For example, setting $b = \top$ implies $\overline{\langle n\rangle_l \overline{\top}} = \overline{\top \cdot \ulcorner 0} = \overline{\top \cdot 0} = \overline{\mathsf{N}} = \mathsf{F}$ and $\boxed{n}_l \top = \top \cdot \neg\ulcorner 0 = \top$. But in general, $\top \le \mathsf{F}$ is false (if there is at least one infinite element $a \neq 0$). Also, the Galois connections of [7] are not valid for left neighbours and right boundaries, but one implication can still be proved.

Lemma 4.12 *Let* S *be right-distributive, then*

$$\langle n\rangle_l a \le b \Rightarrow a \le \boxed{n}_r b \ , \qquad \langle b\rangle_r a \le b \Rightarrow a \le \boxed{b}_r b \ .$$

Proof. By Lemma 4.11(a), Boolean algebra and the exchange rule (3)

$$\langle n\rangle_l a \le b \Rightarrow \overline{[n]_l \overline{a}} \le b \Leftrightarrow \overline{b} \le [n]_l \overline{a} \Leftrightarrow a \le [n]_r b \ . \qquad \square$$

By lack of Galois connections, we do not have a full analogue to Corollary 4.10.

Lemma 4.13
(a) $\langle n\rangle_l$, $\langle b\rangle_r$, $[n]_r$ *and* $[b]_l$ *are isotone.*
(b) *If* S *is right-distributive, then*
$\langle n\rangle_l$, $\langle b\rangle_r$ *are disjunctive and* $[n]_r$, $[b]_l$ *are conjunctive.*

Proof.
(a) The claim follows directly by the explicit representation of (perfect) neighbours and boundaries (Lemma 4.2 and Lemma 4.6).
(b) By Lemma 4.2, additivity of domain and right-distributivity we get
$\langle n\rangle_l(a+b) = \top \cdot {}^{\ulcorner}(a+b) = \top \cdot ({}^{\ulcorner}a + {}^{\ulcorner}b) = \top \cdot {}^{\ulcorner}a + \top \cdot {}^{\ulcorner}b = \langle n\rangle_l a + \langle n\rangle_l b \ . \quad \square$

Until now, we have shown that most of the properties of [7] hold in L-semirings, too. At some points, we need additional assumptions like right-distributivity. Many more properties, like $b^{\urcorner} \le \langle n\rangle_r b$, can be shown. Most proofs use the explicit forms for lazy semiring neighbours or the Galois connections (Lemma 4.9) and Lemma 4.12. However, since L-semirings reflect some aspects of infinity, we get some useful properties, which are different from all properties given in [7]. Some are summarised in the following lemma.

Lemma 4.14
(a) $\langle n\rangle_l \mathsf{F} = \langle n\rangle_r \mathsf{F} = \langle b\rangle_l \mathsf{F} = \langle b\rangle_r \mathsf{F} = \top$.
(b) $b \le \mathsf{N} \Leftrightarrow \langle n\rangle_r b \le 0 \Leftrightarrow \langle b\rangle_r b \le \mathsf{N}$.
(c) $[n]_l \mathsf{N} = [b]_r \mathsf{N} = \mathsf{N}$ *and* $[n]_r \mathsf{N} = [b]_l \mathsf{N} = 0$.
(d) $\overline{b} \le \mathsf{N} \Leftrightarrow \mathsf{F} \le b \Leftrightarrow [n]_r b = \top \Leftrightarrow [b]_r b = \top$.

Proof. First we note that by straightforward calculations using Lemma 3.2 and 3.4, we get

$$\top \cdot p \le \top \cdot q \Leftrightarrow p \le q \Leftrightarrow p \cdot \top \le q \cdot \top \ . \tag{6}$$

(a) Directly by Lemma 4.2 and ${}^{\ulcorner}\mathsf{F} = \mathsf{F}^{\urcorner} = 1$, since $1 \le \mathsf{F}$:
$$\langle n\rangle_l \mathsf{F} = \top \cdot {}^{\ulcorner}\mathsf{F} = \top \cdot 1 = \top \ .$$
(b) By Lemma 3.4, (6), left-strictness and definition of $\langle n\rangle_l$
$$b \le \mathsf{N} \Leftrightarrow b^{\urcorner} \le 0 \Leftrightarrow b^{\urcorner} \cdot \top \le 0 \cdot \top \Leftrightarrow \langle n\rangle_r b \le 0 \ .$$
(c) By Lemma 4.6 and ${}^{\ulcorner}\mathsf{F} = 1$ we get
$$[n]_l \mathsf{N} = \top \cdot \neg {}^{\ulcorner}\overline{\mathsf{N}} = \top \cdot \neg {}^{\ulcorner}\mathsf{F} = \top \cdot 0 = \mathsf{N}.$$
(d) Similar to (b). $\square$

Note that (a) implies $\langle n\rangle_l \top = \langle n\rangle_r \top = \langle b\rangle_l \top = \langle b\rangle_r \top = \top$ using isotony. (c) shows again that the inequations of Lemma 4.11 cannot be strengthened to equations.

Since the above theory concerning lazy semiring neighbours is based on lazy semirings, it is obvious that one can use it also in the framework of lazy Kleene algebra and lazy omega algebra [11]. The former one provides, next to the L-semiring operators, an operator for finite iteration. The latter one has an additional operator for infinite iteration.

5 Neighbourhood Logic with Infinite Durations

Using the theory of the previous section, we can now formulate a generalisation of NL, which includes infinite elements (intervals with infinite duration). Those intervals are not included in the original Neighbourhood Logic of [15, 16], i.e., if we compose two intervals $[a, b]$ and $[b, c]$ (where intervals are defined, as usual, as $[a, b] =_{df} \{x \mid a \leq x \leq b, a \leq b\}$), it is assumed that the points of $[b, c]$ are reached after finite duration $b-a$. However, for many applications, e.g. for hybrid systems, as we will see in Section 7, a time point ∞ of infinity is reasonable. But then the composition of the intervals $[a, \infty[$ and $[b, c]$ never reaches the second interval. This gives rise to an L-semiring.

Neighbourhood Logic and its Embedding. In this paragraph the Neighbourhood Logic [15, 16] and its embedding [7] are briefly recapitulated.

Chop-based interval temporal logics, such as ITL [5] and IL [3] are useful for the specification and verification of safety properties of real-time systems. In these logics, one can easily express a lot of properties such as "if ϕ holds for an interval, then there is a subinterval where ψ holds". As shown in [16], these logics cannot express all desired properties. E.g., (unbounded) liveness properties such as "eventually there is an interval where ϕ holds" are not expressible in these logics. As it is shown in [16] the reason is that the modality *chop* $\frown$ is a *contracting* modality, in the sense that the truth value of $\phi \frown \psi$ on $[a, b]$ only depends on subintervals of $[a, b]$:

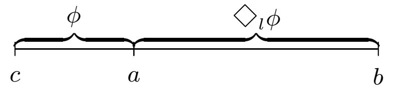
$$\begin{gathered}\phi \frown \psi \text{ holds on } [a, b] \text{ iff} \\ \text{there exists } c \in [a, b] \text{ such that } \phi \text{ holds on } [a, c] \text{ and } \psi \text{ holds on } [c, b].\end{gathered}$$

Hence Zhou and Hansen proposed a first-order interval logic called *Neighbourhood Logic* (NL) in 1996 [15]. In this logic they introduce *left* and *right neighbourhoods* as new primitive intervals to define other unary and binary modalities of intervals in a first-order logic. The two proposed simple expanding modalities $\Diamond_l \phi$ and $\Diamond_r \phi$ are defined as follows:

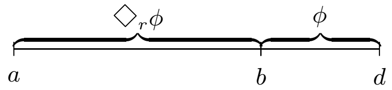
$$\Diamond_l \phi \text{ holds on } [a, b] \text{ iff there exists } \delta \geq 0 \text{ such that } \phi \text{ holds on } [a-\delta, a], \quad (7)$$

$$\Diamond_r \phi \text{ holds on } [a, b] \text{ iff there exists } \delta \geq 0 \text{ such that } \phi \text{ holds on } [b, b+\delta], \quad (8)$$

where ϕ is a *formula*[2] of NL. These modalities can be illustrated by



where $c = a - \delta$



where $d = b + \delta$

With $\Diamond_r (\Diamond_l)$ one can reach the *left* (*right*) *neighbourhood* of the beginning (ending) point of an interval. In contrast to the chop operator, the neighbourhood modalities are *expanding* modalities, i.e., $\Diamond_l$ and $\Diamond_r$ depend not only on subintervals of an interval $[a, b]$, but also on intervals "outside". In [15] it is shown

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[2] The exact definition of the syntax of formulas can be found e.g. in [15].

that the modalities of [6] and [14] as well as the chop operator can be expressed by the neighbourhood modalities.

In [7] we present an embedding and extension of NL into the framework of full semirings. There, (perfect) neighbours and boundaries are defined on full semirings in the same way as we have done this for L-semirings in Section 4. Consider the structure

$$\mathrm{INT} =_{df} (\mathcal{P}(\mathrm{Int}), \cup, ;, \emptyset, \mathbb{1}) ,$$

where $\mathbb{1} =_{df} \{[a,a]\}$ denotes the set of all intervals consisting of one single point and Int is the set of all intervals $[a,b]$ with $a, b \in \mathbb{T}\mathrm{ime}$ and $\mathbb{T}\mathrm{ime}$ is a totally ordered poset, e.g. $\mathbb{R}$. Further we assume that there is an operation $-$ on $\mathbb{T}\mathrm{ime}$, which gives us the duration of an interval $[a,b]$ by $b-a$. By this operation $\mathbb{1}$ consists of all 0-length intervals.

For the moment we exclude intervals with infinite duration. The symbol ; denotes the pointwise lifted composition of intervals which is defined by

$$[a,b] \,;\, [c,d] =_{df} \begin{cases} [a,d] & \text{if } b = c \\ \text{undefined otherwise} . \end{cases}$$

It can easily be checked that INT forms a full semiring. In [7] we have shown that

$$\begin{array}{l} \Diamond_l \phi \text{ holds on } [a,b] \;\Leftrightarrow\; \{[a,b]\} \le {\Diamond\mkern-14mu{\scriptstyle n}\,}_r \mathbb{I}_\phi \ , \\ \Diamond_r \phi \text{ holds on } [a,b] \;\Leftrightarrow\; \{[a,b]\} \le {\Diamond\mkern-14mu{\scriptstyle n}\,}_l \mathbb{I}_\phi \ , \end{array}$$

where $\mathbb{I}_\phi =_{df} \{i \mid i \in \mathrm{Int}, \phi \text{ holds on } i\}$. This embedding gives us the possibility to use the structure of a semiring to describe NL. Many simplifications of NL and properties concerning the algebraic structure are given in [7].

Adding Infinite Durations. Now, we assume a point of infinity $\infty \in \mathbb{T}\mathrm{ime}$, e.g. $\mathbb{T}\mathrm{ime} = \mathbb{R} \cup \{\infty\}$. If there is such an element, it has to be the greatest element. Consider the slightly changed structure

$$\mathrm{INT}^i =_{df} (\mathcal{P}(\mathrm{Int}), \cup, ;, \emptyset, \mathbb{1}) ,$$

where ; is now the pointwise lifted composition defined as

$$[a,b] \,;\, [c,d] =_{df} \begin{cases} [a,d] & \text{if } b = c, b \ne \infty \\ [a,b] & \text{if } b = \infty \\ \text{undefined otherwise} . \end{cases}$$

Again, it is easy to check that INT^i forms an L-semiring, which even becomes an ML-semiring by setting, for $A \in \mathcal{P}(\mathrm{Int})$,

$$\ulcorner A =_{df} \{[a,a] \mid [a,b] \in A\} \quad \text{and} \quad A^{\urcorner} =_{df} \{[b,b] \mid [a,b] \in A, b \ne \infty\} .$$

Note that INT^i is right-distributive, so that all Lemmas and Corollaries of Section 4 hold in this model.

Thereby we have defined a new version NL^i of NL which handles intervals with infinite durations. NL^i also subsumes the theory presented in [17]. In particular it builds a bridge between NL and a duration calculus for infinite intervals.

6 Lazy Semiring Neighbours and CTL^*

Lazy semiring neighbours do not only appear in NL^i. Also in other areas of computer science they play an important role. We now present an application of neighbours and boundaries in the field of CTL^*.

The branching time temporal logic CTL^* (see e.g. [4]) is a well-known tool for analysing and describing parallel as well as reactive and hybrid systems. In CTL^* one distinguishes state formulas and path formulas, the former ones denoting sets of states, the latter ones sets of computation traces.

The language Ψ of *CTL^* formulas* over a set Φ of atomic propositions is defined by the grammar

$$\Psi \ ::= \ \bot \mid \Phi \mid \Psi \to \Psi \mid \mathsf{X}\,\Psi \mid \Psi\,\mathsf{U}\,\Psi \mid \mathsf{E}\Psi \ ,$$

where X and U are the next-time and until operators and E is the existential quantifier on paths. As usual,

$$\begin{array}{rlcrl} \neg\varphi & =_{df} \ \varphi \to \bot \ , & \qquad & \varphi \wedge \psi & =_{df} \ \neg(\varphi \to \neg\psi) \ , \\ \varphi \vee \psi & =_{df} \ \neg\varphi \to \psi \ , & & \mathsf{A}\varphi & =_{df} \ \neg\,\mathsf{E}\neg\varphi \ . \end{array}$$

In [12] a connection between CTL^* and Boolean modal quantales is presented. Since these are right-distributive, all the lemmas of the previous sections are again available. If A is a set of states one could, e.g., use the algebra $\mathrm{STR}(A)$ of finite and infinite streams of A-states as a basis. For an arbitrary Boolean modal quantale S, the concrete standard semantics for CTL^* is generalised to a function $\llbracket\_\rrbracket : \Psi \to S$ as follows, where $\llbracket\varphi\rrbracket$ abstractly represents the set of paths satisfying formula φ. One fixes an element n (n standing for "next") as representing the transition system underlying the logic and sets

$$\begin{aligned} \llbracket\bot\rrbracket &= 0 \ , \\ \llbracket p\rrbracket &= p \cdot \top \ , \\ \llbracket\varphi \to \psi\rrbracket &= \overline{\llbracket\varphi\rrbracket} + \llbracket\psi\rrbracket \ , \\ \llbracket\mathsf{X}\,\varphi\rrbracket &= \mathsf{n} \cdot \llbracket\varphi\rrbracket \ , \\ \llbracket\varphi\,\mathsf{U}\,\psi\rrbracket &= \bigsqcup_{j\geq 0} (\mathsf{n}^j \cdot \llbracket\psi\rrbracket \sqcap \prod_{k<j} \mathsf{n}^k \cdot \llbracket\varphi\rrbracket) \ , \\ \llbracket\mathsf{E}\varphi\rrbracket &= {}^{\ulcorner}\llbracket\varphi\rrbracket \cdot \top \ . \end{aligned}$$

Using these definitions, it is straightforward to check that $\llbracket\varphi \vee \psi\rrbracket = \llbracket\varphi\rrbracket + \llbracket\psi\rrbracket$, $\llbracket\varphi \wedge \psi\rrbracket = \llbracket\varphi\rrbracket \sqcap \llbracket\psi\rrbracket$ and $\llbracket\neg\varphi\rrbracket = \overline{\llbracket\varphi\rrbracket}$.

By simple calculations we get the following result.

Lemma 6.1 *[12] Let φ be a state formula of CTL^*. Then*

$$\llbracket\mathsf{A}\varphi\rrbracket \ = \ \neg{}^{\ulcorner}(\overline{\llbracket\varphi\rrbracket}) \cdot \top \ .$$

Hence we see that $\llbracket\mathsf{E}\varphi\rrbracket$ corresponds to a left boundary and $\llbracket\mathsf{A}\varphi\rrbracket$ to a perfect left boundary, i.e.,

$$\llbracket\mathsf{E}\varphi\rrbracket \ = \ \Diamond\!\!\!\!\!\;\,b_{\,l}\,\llbracket\varphi\rrbracket \quad \text{and} \quad \llbracket\mathsf{A}\varphi\rrbracket \ = \ \boxed{b}_{\,l}\,\llbracket\varphi\rrbracket \ .$$

With these equations we have connected lazy neighbours with CTL^*. From Lemma 4.3, Corollary 4.8 and equations (4) we obtain immediately

$$\begin{aligned} [\![\mathsf{EE}\varphi]\!] &= [\![\mathsf{E}\varphi]\!]\ , & \qquad [\![\mathsf{AA}\varphi]\!] &= [\![\mathsf{A}\varphi]\!]\ , \\ [\![\mathsf{EA}\varphi]\!] &= [\![\mathsf{A}\varphi]\!]\ , & \qquad [\![\mathsf{AE}\varphi]\!] &= [\![\mathsf{E}\varphi]\!]\ . \end{aligned}$$

The other two boundaries as well as all variants of (perfect) neighbours do not occur in CTL^* itself.

A connection to hybrid systems will be set up in the next section.

7 Lazy Semiring Neighbours and Hybrid Systems

Hybrid systems are dynamical heterogeneous systems characterised by the interaction of discrete and continuous dynamics. In [9] we use the L-semiring PRO of processes from Section 2 for the description of hybrid systems.

Hybrid systems and NL. In PRO the left/right neighbours describe a kind of composability, i.e., for processes A, B,

$$A \le {\Diamond\!\!\!\!\!{\scriptstyle n}\;}_l B \quad \text{iff} \quad \forall\, a \in A : \exists\, b \in B : a \cdot b \text{ is defined,} \tag{9}$$

$$A \le {\Diamond\!\!\!\!\!{\scriptstyle n}\;}_r B \quad \text{iff} \quad \forall\, a \in A : \exists\, b \in \mathsf{fin}\,(B) : b \cdot a \text{ is defined.} \tag{10}$$

These equivalences are closely related to (7) and (8), respectively. ${\Diamond\!\!\!\!\!{\scriptstyle n}\;}_r$ and ${\Diamond\!\!\!\!\!{\scriptstyle n}\;}_l$ each guarantee existence of a composable element. Especially, ${\Diamond\!\!\!\!\!{\scriptstyle n}\;}_r \neq 0$ guarantees that there exists a process, and therefore a trajectory, that can continue the current process (trajectory). Therefore it is a form of liveness assertion. In particular, the process ${\Diamond\!\!\!\!\!{\scriptstyle n}\;}_r B$ contains all trajectories that are composable with the "running" one. If ${\Diamond\!\!\!\!\!{\scriptstyle n}\;}_r B = \emptyset$, we know that the system will terminate if all trajectories of the running process have finite durations. Note that in the above characterisation of ${\Diamond\!\!\!\!\!{\scriptstyle n}\;}_l$ the composition $a \cdot b$ is defined if either $f(d_1) = g(0)$ (assuming $a = (d_1, f)$ and $b = (d_2, g)$) or a has infinite duration, i.e., $d = \infty$. As we will see in the next paragraph, left and right boundaries of lazy semirings are closely connected to temporal logics for hybrid systems. But, by Lemma 4.3, they are of course also useful as operators that simplify nestings of semiring neighbours.

The situation for right/left perfect neighbours is more complicated. As shown in [7], $\boxed{n}_r B$ is the set of those trajectories which can be reached only from B, not from $\overline{B}$. Hence it describes a situation of guaranteed non-reachability from $\overline{B}$. The situation with $\boxed{n}_l$ is similar for finite processes, because of the symmetry between left and right perfect neighbours.

Hybrid systems and CTL^*. Above we have shown how lazy semiring neighbours are characterised in PRO. Since they are also closely connected to CTL^* (cf. Section 6) we have built a bridge between CTL^* and hybrid systems in an algebraic manner. As shown above, we have on the one hand $[\![\mathsf{E}\varphi]\!] = {\Diamond\!\!\!\!\!{\scriptstyle b}\;}_l [\![\varphi]\!]$ and on the other hand that ${\Diamond\!\!\!\!\!{\scriptstyle b}\;}_l$ is connected to hybrid systems. Therefore we now have an interpretation of the existence operator E of CTL^* in hybrid systems. That the existence fits well into our model, can be seen in equations (9) and (10), where the existence quantifier occurs. Of course all other kinds of left and right (perfect) neighbours and boundaries have their own interpretation in PRO and in (the extended) CTL^*, respectively. A detailed discussion of all these interpretations is part of our future work (cf. Section 8).

8 Conclusion and Outlook

In the paper we have presented a second extension of Neighbourhood Logic. Now this logic is able to handle intervals which either have finite or infinite length. Therefore we established semiring neighbours over lazy semirings. During the development of lazy semiring neighbours it turned out that they are not only useful and necessary for NL but also in other areas of computer science. We have sketched a connection to temporal logics and to hybrid systems.

Since we have only given a short overview over the connections between lazy semiring neighbours, CTL^* and hybrid systems, one of our aims for further work is a longer treatment of CTL^* interpreted for hybrid system. Furthermore it will be interesting to see if there are even more applications for semiring neighbours.

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