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Preprint Nr. 03/2016 — 15. Juni 2016

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Impressum:

Herausgeber:

Institut für Mathematik
Universität Augsburg
86135 Augsburg
`http://www.math.uni-augsburg.de/de/forschung/preprints.html`

ViSdP:

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Mixing Properties of Stationary Poisson Cylinder Models

Christian Bräu and Lothar Heinrich [1]

Abstract

We study a particular class of stationary random closed sets in $\mathbb{R}^d$ called Poisson k-cylinder models (short: P-k-CM's) for $k = 1, \ldots, d-1$. We show that all P-k-CM's are weakly mixing and possess long-range correlations. Further, we derive necessary and sufficient conditions in terms of the directional distribution of the cylinders under which the corresponding P-k-CM is mixing. Regarding the P-$(d-1)$-CM as union of "thick hyperplanes" which generates a stationary process of polytopes we prove that the distribution of the polytope containing the origin does not depend on the thickness of the hyperplanes.

Keywords : random closed set, hitting functional, random k-cylinder, independently marked Poisson process, tail σ-algebra, typical cell, zero cell

AMS 2010 MSC : Primary: 60D05 , 37A25; Secondary: 60G55 , 60G60

1 Introduction and Preliminaries

A stationary *Poisson k-cylinder model* (short: P-k-CM) in the d-dimensional Euclidean space $\mathbb{R}^d$ (for $d \geq 2$ and some $k \in \{1, \ldots, d-1\}$) is defined as union of randomly dilated k-flats whose individual spatial extensions, positions and directions are determined by a stationary independently marked Poisson process on $\mathbb{R}^{d-k}$. In this way a *random closed set* (short: RACS) in $\mathbb{R}^d$ with positive volume fraction (if the cylinder base in $\mathbb{R}^{d-k}$ has positive volume) is obtained which allows explicite formulas for a number of characteristics, e.g. n-point probabilities for any $n \in \mathbb{N} = \{1, 2, \ldots\}$, see [20]. Although Poisson cylinder models have been considered already at the very beginning of the systematic study of RACSs, see [16] for $k = d-1$, [17] for any $k \in \{0, 1, \ldots, d-1\}$, and [5] for stereological relationships, their importance as well-tractable model in stochastic geometry with interesting properties (partly in contrast to the

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frequently used Boolean model) was recognized just recently, see [20] and [8], [10] for central limit theorems of the volume and surface content in expanding windows.

To be precise, some further notation is needed. In stochastic geometry, a k-cylinder in $\mathbb{R}^d$ is defined as Minkowski sum $B \oplus \mathbb{L}$ of a *direction space* $\mathbb{L} \in \mathcal{G}(d,k)$ (= the Grassmannian of k-dimensional subspaces of $\mathbb{R}^d$) and a compact *base* B in the orthogonal complement $\mathbb{L}^\perp$, see e.g. [19] or [20]. In the following we go along the line suggested in [8], [10] (which slightly differs from that in [14] and [20]) and identify $\mathbb{L}$ with a unique element $O_\mathbb{L}$ of the equivalence class $\mathbf{O}_\mathbb{L}$ of orthogonal matrices $O \in \mathbb{SO}_d$ (i.e. $O \in \mathbb{R}^{d\times d}$, $O^T = O^{-1}$ and $\det(O) = 1$) satisfying $O\,\mathbb{E}_k = \mathbb{L}$ (and $O\,\mathbb{E}_k^\perp = \mathbb{L}^\perp$), where $\mathbb{E}_k = \operatorname{span}\{e_{d-k+1},\ldots,e_d\}$, $\mathbb{E}_k^\perp = \operatorname{span}\{e_1,\ldots,e_{d-k}\}$ for $k = 1,\ldots,d-1$ with the usual orthonormal basis $\{e_1,\ldots,e_d\}$ of $\mathbb{R}^d$. In other words, two matrices O_1, O_2 belong to the compact set $\mathbf{O}_\mathbb{L} \subset \mathbb{SO}_d$ iff $O_1^T\,O_2$ belongs to the set of orthogonal block matrices $\mathbb{S}(\mathbb{O}_{d-k}\times\mathbb{O}_k)$ defined by

$$\left\{\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix} : A \in \mathbb{R}^{(d-k)\times(d-k)}, B \in \mathbb{R}^{k\times k}, A^T = A^{-1}, B^T = B^{-1}, \det(A) = \det(B)\right\}.$$

The element $O_\mathbb{L}$ can be chosen in a canonical way, e.g. as lexicographically smallest element of the set of matrices $\mathbf{O}_\mathbb{L}$. In this way we get a one-to-one correspondence between $\mathbb{SO}_{d,k} = \{O_\mathbb{L} := \operatorname{lex\,min}\mathbf{O}_\mathbb{L} : \mathbb{L} \in \mathcal{G}(d,k)\}$ and $\mathcal{G}(d,k)$ up to orientation of the subspaces. Note that for $k = 1$ (and analogously for $k = d-1$) the orthogonal matrix $O_\mathbb{L}$ can be chosen such that $\det(O_\mathbb{L}) = 1$ and $O_\mathbb{L} e_d = u$, where $u \in S^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ is expressed in terms of spherical coordinates and u and $-u$ are identified. For example, in the special cases $\mathbb{L} = \operatorname{span}\{(\cos\vartheta, \sin\vartheta)^T\} \in \mathcal{G}(2,1)$ and $\mathbb{L} = \operatorname{span}\{(\cos\vartheta_1\sin\vartheta_2, \sin\vartheta_1\sin\vartheta_2, \cos\vartheta_2)^T\} \in \mathcal{G}(3,1)$ we take the matrices

$$O_\mathbb{L}(\vartheta) = \begin{pmatrix} \sin\vartheta & \cos\vartheta \\ -\cos\vartheta & \sin\vartheta \end{pmatrix} \text{ and } O_\mathbb{L}(\vartheta_1,\vartheta_2) = \begin{pmatrix} \sin\vartheta_1 & \cos\vartheta_1\cos\vartheta_2 & \cos\vartheta_1\sin\vartheta_2 \\ -\cos\vartheta_1 & \sin\vartheta_1\cos\vartheta_2 & \sin\vartheta_1\sin\vartheta_2 \\ 0 & -\sin\vartheta_2 & \cos\vartheta_2 \end{pmatrix},$$

respectively, for $0 \le \vartheta < \pi$ and $(\vartheta_1,\vartheta_2) \in [0,2\pi)\times[0,\pi/2]$. In the particular case $\mathbb{L} = \operatorname{span}\{(\cos\vartheta_1\cos\vartheta_2, \sin\vartheta_1\cos\vartheta_2, -\sin\vartheta_2)^T, (-\sin\vartheta_1, \cos\vartheta_1, 0)^T\} \in \mathcal{G}(3,2)$ it is easily checked that $O^*_\mathbb{L}(\vartheta_1,\vartheta_2)\,\mathbb{E}_2 = \mathbb{L}$, where $O^*_\mathbb{L}(\vartheta_1,\vartheta_2)$ is obtained from $O_\mathbb{L}(\vartheta_1,\vartheta_2)$ by multiplying the first column of $O_\mathbb{L}(\vartheta_1,\vartheta_2)$ by -1 and exchanging it with the third column.

In this way, to each random subspace $\mathbb{L} \in \mathcal{G}(d,k)$ corresponds a (unique) random matrix $\Theta = \Theta(\mathbb{L}) \in \mathbb{SO}_{d,k}$ and vice versa. Throughout this paper, all random elements are defined on a common probability space $[\Omega, \sigma(\Omega), \mathbf{P}]$ and $\mathbf{E}$ denotes the expectation with respect to $\mathbf{P}$. Let $Q_{d,k}$ be a distribution on the Borel-σ-algebra of the *mark space* $\mathbb{M}_{d,k} := \mathbb{SO}_{d,k} \times \mathcal{K}'_{d-k}$, where $\mathcal{K}'_{d-k}$ is the space of all non-empty compact sets in $\mathbb{R}^{d-k}$ equipped with the Hausdorff metric, see e.g. [14]. For later use, we put $\mathcal{K}_d := \mathcal{K}'_d \cup \{\emptyset\}$ and denote by $\mathcal{C}_d$ the subfamily of convex sets in $\mathcal{K}_d$, whereas $\mathcal{B}_d$ signifies the Borel-σ-algebra generated by the family $\mathcal{F}_d$ of all closed in $\mathbb{R}^d$. Further, let $\mathbf{o}_\ell$ flag the origin (null vector) in $\mathbb{R}^\ell$ for $\ell \geq 1$.

Now, we are ready to introduce a stationary independently marked Poisson point process (see e.g. [3],[9], [19]) on $\mathbb{R}^{d-k}$ with mark space $\mathbb{M}_{d,k}$, intensity $\lambda > 0$ and mark distribution $Q_{d,k}$ as locally bounded counting measure $\Pi_{\lambda,Q_{d,k}} = \sum_{i\geq 1} \delta_{[X_i,(\Theta_i,\Xi_i)]}$ on the product space $\mathbb{R}^{d-k} \times \mathbb{M}_{d,k}$, i.e., for some random element (Θ_0, Ξ_0) in $\mathbb{M}_{d,k}$ (called *typical mark*) with distribution $Q_{d,k}$ the sequence $((\Theta_i, \Xi_i))_{i\geq 1}$ of independent copies of (Θ_0, Ξ_0) is independent of the unmarked stationary Poisson point process $\Pi_\lambda = \sum_{i\geq 1} \delta_{X_i}$ on $\mathbb{R}^{d-k}$ with intensity $\lambda = \mathbf{E}\,\#\{i \geq 1 : X_i \in [0,1]^{d-k}\}$.

Note that (Θ_0, Ξ_0) specifies direction and base of the *typical k-cylinder* $\Theta_0(\{(\xi, \mathbf{o}_k)^T : \xi \in \Xi_0\} \oplus \mathbb{E}_k)$ (expressed in short form by $\Theta_0(\,\Xi_0 \times \mathbb{R}^k\,)$) of the corresponding stationary *Poisson k-cylinder process* in $\mathbb{R}^d$ driven by $\Pi_{\lambda,Q_{d,k}}$ and defined by the countable family of random k-cylinders

$$\{\,\Theta_i(\,(\Xi_i + X_i) \times \mathbb{R}^k\,) = \Theta_i(\,\{(\xi + X_i, \mathbf{o}_k)^T : \xi \in \Xi_i\} \oplus \mathbb{E}_k\,)\,,\, i \geq 1\,\}\,. \tag{1.1}$$

In addition we assume that

$$\mathbf{E}\,\nu_{d-k}(\Xi_0 \oplus B_\varepsilon^{d-k}) < \infty \tag{1.2}$$

for some $\varepsilon > 0$, where $B_\varepsilon^{d-k} := \{x \in \mathbb{R}^{d-k} : \|x\| \leq \varepsilon\}$ and ν_{d-k} denotes the Lebesgue measure on $\mathbb{R}^{d-k}$ for $k = 0, 1, \ldots, d$.

Finally, we are in a position to present the following

Definition 1.1: A stationary P-k-CM $\Xi_{\lambda,Q_{d,k}}$ in $\mathbb{R}^d$ is defined to be the countable union over the Poisson-k-cylinder process (1.1),

$$\Xi_{\lambda,Q_{d,k}} := \bigcup_{i\geq 1} \Theta_i((\Xi_i + X_i) \times \mathbb{R}^k) \tag{1.3}$$

provided that (1.2) is satisfied which ensures the $\mathbf{P}$-a.s. closedness of $\Xi_{\lambda,Q_{d,k}}$.

Remark 1.1: In other words, $\Xi_{\lambda,Q_{d,k}}$ can be considered as random variable taking values in the measurable space $[\mathcal{F}_d, \sigma_f]$, where σ_f is the smallest σ-algebra containing all sets $\mathcal{F}_C := \{F \in \mathcal{F}_d :\; F \cap C \neq \emptyset\}$ for $C \in \mathcal{K}_d$, see [14] for details. The *capacity or hitting functional* of $\Xi_{\lambda,Q_{d,k}}$ is then given by

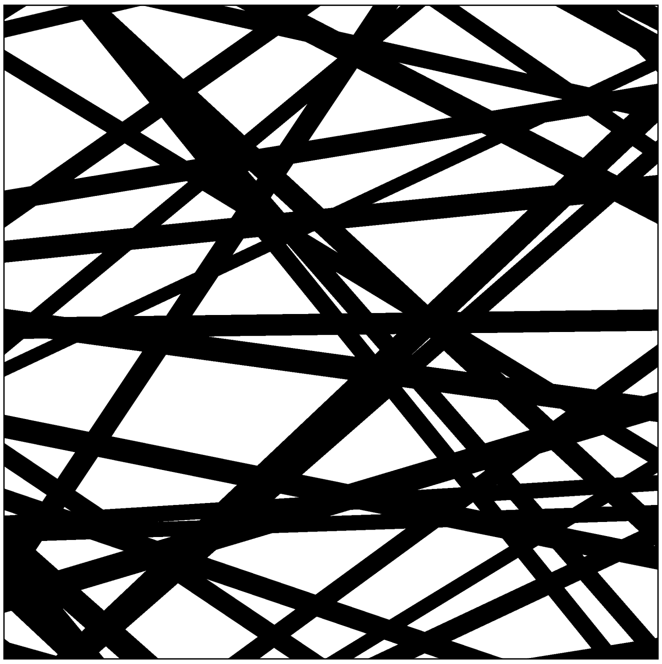

Figure 1: Realization of a planar stationary and isotropic Poisson 1-cylinder model

$$T_{\lambda,Q_{d,k}}(C) := \mathbf{P}(\Xi_{\lambda,Q_{d,k}} \in \mathcal{F}_C) = 1 - \exp\{-\lambda\,\mathbf{E}\,\nu_{d-k}(\Xi_0 \oplus \pi_{d-k}(-\Theta_0^T C))\} \tag{1.4}$$

for $C \in \mathcal{K}_d$, see [8], [10]. Here, $\pi_{d-k}(B) := \{\pi_{d-k}(x) : x \in B\}$ for any $B \subset \mathbb{R}^d$ and $\pi_{d-k}(x)$ denotes the projection on the first $d-k$ components of $x \in \mathbb{R}^d$. Notice that the probability space $[\Omega, \sigma(\Omega), \mathbf{P}]$ can be chosen in such way that the indicator function $\mathbb{R}^d \times \Omega \ni (x,\omega) \mapsto 1(x \in \Xi_{\lambda,Q_{d,k}}(\omega))$ is $\mathcal{B}_d \otimes \sigma(\Omega)$-measurable, see Appendix in [7] and [8].

Remark 1.2:

- The degenerate case $k = 0$ ($\mathbb{E}_0 = \{\mathbf{o}_d\}$ and $\Theta_0 = \text{id}$) yields the well-studied Boolean model, see e.g. [14], [3].
- In the special case $\Xi_0 = \{\mathbf{o}_{d-k}\}$ the RACS $\Xi_{\lambda,Q_{d,k}}$ coincides with (the union of) a stationary Poisson k-flat process, see [14], [16], [19].

Next, we recall the notion of ergodicity and various mixing properties of RACSs, see [6], [10] and [19] for details. For this we need a family of shift operators $\{S_x : x \in \mathbb{R}^d\}$ defined by $S_x F := \{y + x : y \in F\}$ for $F \in \mathcal{F}_d$, $S_x\mathcal{A} := \{S_x F : F \in \mathcal{A}\}$ for $\mathcal{A} \in \sigma_f$ and a suitable family of sets growing unboundedly in all directions.

Definition 1.2: (see [4], p. 196) A sequence of sets $(W_n)_{n\in\mathbb{N}}$ is called *convex averaging sequence* (short: CAS) in $\mathbb{R}^d$ if

1. $W_n \in \mathcal{C}_d$ and $W_n \subset W_{n+1}$ for each $n \in \mathbb{N}$,

2. $\varrho_n := \sup\{r > 0 : \, B_r^d + x \subseteq W_n \text{ for a } x \in W_n\} \xrightarrow[n\to\infty]{} \infty$.

It can be shown that (2) is equivalent to $\nu_{d-1}(\partial W_n)/\nu_d(W_n) \xrightarrow[n\to\infty]{} 0$, where $\nu_{d-1}(\partial W_n)$ denotes the surface content of W_n, see [9], p. 133.

Definition 1.2: A stationary RACS Ξ in $\mathbb{R}^d$ with distribution P_Ξ is said to be *ergodic*, *weakly mixing* resp. *mixing* if, for a CAS $(W_n)_{n\in\mathbb{N}}$ and all $\mathcal{A}_0, \mathcal{A}_1 \in \sigma_f$,

$$\frac{1}{\nu_d(W_n)} \int_{W_n} P_\Xi(\mathcal{A}_0 \cap S_x\mathcal{A}_1)\,\mathrm{d}x \xrightarrow[n\to\infty]{} P_\Xi(\mathcal{A}_0)\, P_\Xi(\mathcal{A}_1)\,, \tag{1.5}$$

$$\frac{1}{\nu_d(W_n)} \int_{W_n} \big|P_\Xi(\mathcal{A}_0 \cap S_x\mathcal{A}_1) - P_\Xi(\mathcal{A}_0)\, P_\Xi(\mathcal{A}_1)\big|\,\mathrm{d}x \xrightarrow[n\to\infty]{} 0 \tag{1.6}$$

$$\text{resp.} \qquad P_\Xi(\mathcal{A}_0 \cap S_x\mathcal{A}_1) \xrightarrow[\|x\|\to\infty]{} P_\Xi(\mathcal{A}_0)\, P_\Xi(\mathcal{A}_1)\,. \tag{1.7}$$

Furthermore, Ξ is said to be *mixing of order* $\ell\,(\geq 2)$ if for all $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_\ell \in \sigma_f$,

$$P_\Xi(\mathcal{A}_0 \cap S_{x_{n,1}}\mathcal{A}_1 \cap \cdots \cap S_{x_{n,\ell}}\mathcal{A}_\ell) \xrightarrow[n\to\infty]{} P_\Xi(\mathcal{A}_0)\, P_\Xi(\mathcal{A}_1) \,\cdots\, P_\Xi(\mathcal{A}_k) \tag{1.8}$$

as $\|x_{n,i}\| \xrightarrow[n\to\infty]{} \infty$ for $i = 1, \ldots, \ell$ in such a way that $\|x_{n,i} - x_{n,j}\| \xrightarrow[n\to\infty]{} \infty$ also for all $i \neq j$, see [4] (p. 215) for ℓth order mixing of random (counting) measures.

Obviously, mixing of order $\ell\,(\geq 2) \Longrightarrow$ mixing $\Longrightarrow$ weak mixing $\Longrightarrow$ ergodic but the reverse implications do not hold in general.

Remark 1.3: In view of Lemma 4 in [6] the sets $\mathcal{A}_0, \mathcal{A}_1$ in the limits (1.5) - (1.7) can be replaced by $\mathcal{F}^{C_i} := \{F \in \mathcal{F}_d : F \cap C_i = \emptyset\}$ for $i = 0, 1$ and all $C_0, C_1 \in \mathcal{K}_d$. In the same way the condition (1.8) can be reformulated with $\mathcal{F}^{C_i}$ for $C_i \in \mathcal{K}_d$ instead of $\mathcal{A}_i$ for $i = 0, 1, \ldots, \ell$.

2 Main Results on Mixing of Poisson k-Cylinder Models

Since a stationary P-0-CM can be identified with a stationary Boolean model which is always mixing (of any order), see [19] or [6], we only need to consider P-k-CMs for $k = 1, \ldots, d-1$.

Theorem 2.1. *For each $k = 1, \ldots, d-1$, the stationary P-k-CM* (1.3) *satisfying* (1.2) *is weakly mixing (and thus also ergodic).*

Proof. Let $P_{\lambda,Q_{d,k}}$ denote the distribution of the RACS $\Xi_{\lambda,Q_{d,k}}$. According to Remark 1.3 we need to prove (1.6) only for $\mathcal{A}_i = \mathcal{F}^{C_i}, i = 0,1$. Since $\mathcal{F}^{C_0} \cap S_x\mathcal{F}^{C_1} = \mathcal{F}^{C_0 \cup S_x C_1}$ and the relation $P_{\lambda,Q_{d,k}}(\mathcal{F}^C) = 1 - T_{\lambda,Q_{d,k}}(C)$ for any $C \in \mathcal{K}_d$, which follows from (1.4), we shall show the limit

$$\lim_{n\to\infty} \frac{1}{\nu_d(W_n)} \int\limits_{W_n} \big|1 - T_{\lambda,Q_{d,k}}(C_0 \cup S_x C_1) - (1 - T_{\lambda,Q_{d,k}}(C_0))(1 - T_{\lambda,Q_{d,k}}(C_1))\big| \, \mathrm{d}x = 0$$

for all $C_0, C_1 \in \mathcal{K}_d$, where $(W_n)_{n\in\mathbb{N}}$ is an arbitrary CAS in $\mathbb{R}^d$. For notational ease we use here and throughout Section 2 the abbreviations

$$\widetilde{K_i} := K \oplus \pi_{d-k}(-\theta^T C_i) \quad \text{for all} \quad (\theta, K) \in \mathbb{M}_{d,k} \quad \text{or} \quad \widetilde{\Xi}_i := \Xi_0 \oplus \pi_{d-k}(-\Theta_0^T C_i)$$

for all $i = 0,1$. An application of formula (1.4) expressing the capacity functional of $\Xi_{\lambda,Q_{d,k}}$ in combination with the identity

$$\begin{aligned} &\nu_{d-k}(K \oplus \pi_{d-k}(-\theta^T(C_0 \cup S_x C_1))) = \nu_{d-k}(\widetilde{K_0} \cup (\widetilde{K_1} - \pi_{d-k}(\theta^T x))) \\ &= \nu_{d-k}(\widetilde{K_0}) + \nu_{d-k}(\widetilde{K_1}) - \nu_{d-k}(\widetilde{K_0} \cap (\widetilde{K_1} - \pi_{d-k}(\theta^T x))) \end{aligned} \tag{2.1}$$

reveals that the previous limiting relation is equivalent to

$$R_n := \frac{1}{\nu_d(W_n)} \int\limits_{W_n} \Big(\exp\{\, \lambda\, \mathbf{E}\, \nu_{d-k}(\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_1 - \pi_{d-k}(\Theta_0^T x)))\,\} - 1\Big) \, \mathrm{d}x \xrightarrow[n\to\infty]{} 0\,. \tag{2.2}$$

The elementary inequality $e^y - 1 \le y\, e^y$ for $y \ge 0$ and

$$\mathbf{E}\, \nu_{d-k}(\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_1 - \pi_{d-k}(\Theta_0^T x))) \le \gamma := \min\Big\{\mathbf{E}\, \nu_{d-k}(\widetilde{\Xi}_0), \mathbf{E}\, \nu_{d-k}(\widetilde{\Xi}_1)\Big\} < \infty \tag{2.3}$$

yield the estimate

$$R_n \le \frac{\lambda\, e^{\lambda\gamma}}{\nu_d(W_n)} \int\limits_{W_n} \mathbf{E}\, \nu_{d-k}(\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_1 - \pi_{d-k}(\Theta_0^T x))) \, \mathrm{d}x = \lambda\, e^{\lambda\gamma}\, \mathbf{E}\, \widetilde{R}_n(\Theta_0, \Xi_0)\,,$$

where

$$\widetilde{R}_n(\theta, K) = \frac{1}{\nu_d(W_n)} \int\limits_{\theta^T W_n} \nu_{d-k}(\widetilde{K_0} \cap (\widetilde{K_1} - \pi_{d-k}(x))) \, \mathrm{d}x \quad \text{for} \quad (\theta, K) \in \mathbb{M}_{d,k}.$$

It is easily seen that $\widetilde{R}_n(\theta, K)$ is bounded by $\min\{\nu_{d-k}(\widetilde{K_0}), \nu_{d-k}(\widetilde{K_1})\}$ for all $(\theta, K) \in \mathbb{M}_{d,k}$ and this bound is integrable with respect to $Q_{d,k}$. Thus, in order to obtain (2.2) via Lebesgue's dominated convergence theorem it remains to show $\widetilde{R}_n(\theta, K) \xrightarrow[n\to\infty]{} 0$ for any fixed $(\theta, K) \in \mathbb{M}_{d,k}$.

Since the support of the function $\mathbb{R}^d \ni x \mapsto \nu_d\big(\widetilde{K}_0 \cap (\widetilde{K}_1 - \pi_{d-k}(x))\big)$ is unbounded, we split $\mathbb{R}^d$ into the orthogonal subspaces $\mathbb{E}_k^\perp$ and $\mathbb{E}_k$. For this purpose, let $\nu_{\mathbb{L}}$ denote the Lebesgue measure on an affine subspace $\mathbb{L}$ of $\mathbb{R}^d$ which can be identified with ν_p if $p = \dim \mathbb{L}$. By applying Fubini's theorem we obtain that

$$\begin{aligned}\widetilde{R}_n(\theta, K) &= \int\limits_{\mathbb{E}_k^\perp}\int\limits_{\mathbb{E}_k} \frac{1_{\theta^T W_n}(y+z)}{\nu_d(W_n)} \nu_{d-k}\big(\widetilde{K}_0 \cap (\widetilde{K}_1 - \pi_{d-k}(y+z))\big)\, \nu_{\mathbb{E}_k}(\mathrm{d}z)\, \nu_{\mathbb{E}_k^\perp}(\mathrm{d}y) \\ &= \int\limits_{\mathbb{E}_k^\perp} \frac{\nu_{\mathbb{E}_k}\big((\theta^T W_n - y) \cap \mathbb{E}_k\big)}{\nu_d(W_n)} \nu_{d-k}\big(\widetilde{K}_0 \cap (\widetilde{K}_1 - y)\big)\, \nu_{\mathbb{E}_k^\perp}(\mathrm{d}y).\end{aligned}$$

Although it seems to be intuitively clear that $\nu_{\mathbb{E}_k}((\theta^T W_n - y) \cap \mathbb{E}_k)/\nu_d(\theta^T W_n) \longrightarrow 0$ as $n \to \infty$, we give a rigouros reasoning for this by employing the following result proved in [15]: For any $C \in \mathcal{C}_d$ and affine subspaces $\mathbb{L}_1, \dots, \mathbb{L}_m$ of $\mathbb{R}^d$ with $\dim \mathbb{L}_j = d_j \geq 1$ such that $d_1 + \cdots + d_m = d$, the inequality

$$\nu_d(C) \geq \frac{d_1! \cdots d_m!}{d!} \nu_{d_1}(C \cap \mathbb{L}_1) \cdots \nu_{d_m}(C \cap \mathbb{L}_m)$$

holds so that

$$\frac{\nu_{E_k}\big((\theta^T W_n - y) \cap \mathbb{E}_k\big)}{\nu_d(\theta^T W_n)} \leq \frac{\binom{d}{k}}{\nu_{\mathbb{E}_k^\perp}\big((\theta^T W_n - y) \cap \mathbb{E}_k^\perp\big)} = \frac{\binom{d}{k}}{\nu_{d-k}(\theta^T W_n \cap \mathbb{E}_k^\perp)}$$

for all $y \in \mathbb{E}_k^\perp$ and $\theta \in \mathbb{SO}_{d,k}$. Since $\theta^T W_n$ is a CAS in $\mathbb{R}^d$ it follows from Definition 1.2 that

$$\nu_{d-k}(\theta^T W_n \cap \mathbb{E}_k^\perp) \geq \nu_{d-k}(B_{\varrho_n}^{d-k}) \longrightarrow \infty \quad \text{as} \quad n \to \infty.$$

Finally, together with

$$\int_{\mathbb{E}_k^\perp} \nu_{d-k}\big(\widetilde{K}_0 \cap (\widetilde{K}_1 - y)\big)\, \nu_{E_k^\perp}(\mathrm{d}y) = \nu_{d-k}(\widetilde{K}_0) \nu_{d-k}(\widetilde{K}_1)$$

we arrive at

$$\widetilde{R}_n(\theta, K) \leq \binom{d}{k} \frac{\nu_{d-k}(\widetilde{K}_1)\, \nu_{d-k}(\widetilde{K}_2)}{\nu_{d-k}(B_{\varrho_n}^{d-k})} \xrightarrow[n\to\infty]{} 0\,.$$

This completes the proof of Theorem 2.1. □

It is well-know, see Theorem 10.5.3 in [19], that a stationary Poisson hyperplane process (= P-$(d-1)$-CM with $\Xi_0 = \{\mathbf{o}_{d-1}\}$) is mixing if its spherical directional distribution (defined on S^{d-1}) vanishes on every great subsphere $S^{d-1} \cap \mathbb{L}$ for $\mathbb{L} \in \mathcal{G}(d, d-1)$. A corresponding generalization of this result for any stationary P-k-CMs is given in the following

Theorem 2.2. *For each $k = 1, \ldots, d-1$, the stationary P-k-CM* (1.3) *satisfying* (1.2) *is mixing if and only if the directional distribution $Q^{(0)}_{d,k}(\,\cdot\,) := Q_{d,k}(\,\cdot\, \times \mathcal{K}'_{d-k})$ fulfills the condition*

$$Q^{(0)}_{d,k}(\{\theta \in \mathbb{SO}_{d,k} : \, u \in \theta\,\mathbb{E}_k\}) = 0 \quad \textit{for all} \quad u \in S^{d-1}. \tag{2.4}$$

Proof. We use the notation introduced in the proof of Theorem 2.1. Taking into account Remark 1.3, the shape of the capacity functional (1.4), the decomposition (2.1), and (2.2) we recognize that $\Xi_{\lambda,Q_{d,k}}$ is mixing if and only if

$$\exp\{\,\lambda\,\mathbf{E}\,\nu_{d-k}(\widetilde{\Xi}_0 \cap (\widetilde{\Xi}_1 - \pi_{d-k}(\Theta_0^T x_n)))\,\} - 1 \xrightarrow[n\to\infty]{} 0 \quad \text{for all} \quad C_0, C_1 \in \mathcal{K}_d$$

and any sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ satisfying $\|x_n\| \xrightarrow[n\to\infty]{} \infty$, or equivalently

$$\lim_{n\to\infty} \int_{\mathbb{M}_{d,k}} \nu_{d-k}(\widetilde{K_0} \cap (\widetilde{K_1} - \pi_{d-k}(\theta^T x_n)))\, Q_{d,k}(\mathrm{d}(\theta, K)) = 0. \tag{2.5}$$

Let us first show that (2.4) implies (2.5). By (2.3) and Lebesgue's dominated convergence theorem it suffices to show that, for any fixed $K \in \mathcal{K}'_{d-k}$ and $C_0, C_1 \in \mathcal{K}_d$

$$\lim_{n\to\infty} \nu_{d-k}(\widetilde{K_0} \cap (\widetilde{K_1} - \pi_{d-k}(\theta^T x_n))) = 0 \quad \text{for} \quad Q^{(0)}_{d,k} - \text{almost all} \quad \theta \in \mathbb{SO}_{d,k}. \tag{2.6}$$

Obviously, due to $\widetilde{K_0}, \widetilde{K_1} \in \mathcal{K}'_{d-k}$, (2.6) holds true if $\|\pi_{d-k}(\theta^T x_n)\| \to \infty$ as $n \to \infty$ for $Q^{(0)}_{d,k}$-almost all $\theta \in \mathbb{SO}_{d,k}$. Suppose there is some Borel set $B \subset \mathbb{SO}_{d,k}$ such that $Q^{(0)}_{d,k}(B) > 0$ and $\liminf_{n\to\infty} \|\pi_{d-k}(\theta^T x_n)\| < \infty$ for $\theta \in B$. Thus, putting $u_n := x_n/\|x_n\| \in S^{d-1}$ it follows together with $\pi_{d-k}(\theta^T u_n) = \pi_{d-k}(\theta^T x_n)/\|x_n\|$ that $\liminf_{n\to\infty} \|\pi_{d-k}(\theta^T u_n)\| = 0$ for $\theta \in B$. Since $S^{d-1} \in \mathcal{K}'_d$ there exists a subsequence $(u_{n_m})_{m\in\mathbb{N}}$ having the limit $u \in S^{d-1}$ as $m \to \infty$ satisfying $\pi_{d-k}(\theta^T u) = \mathbf{o}_{d-k}$ (i.e. $u \in \theta\,\mathbb{E}_k$) for $\theta \in B$. But this is a contradiction to condition (2.4). Hence, (2.4) implies the mixing property of the RACS $\Xi_{\lambda,Q_{d,k}}$.

To prove the reverse direction we assume the contrary of (2.4), i.e. there exists an $u_0 \in S^{d-1}$ such that $Q^{(0)}_{d,k}(\{\theta \in \mathbb{SO}_{d,k} : \; \pi_{d-k}(\theta^T u_0) = \mathbf{o}_{d-k}\}) = \varepsilon > 0$. Choosing $C_0 = C_1 = B_1^d$ and $x_n = n\,u_0$ for all $n \in \mathbb{N}$ we conclude that

$$\begin{aligned}
&\int_{\mathbb{M}_{d,k}} \nu_{d-k}(\widetilde{K_0} \cap (\widetilde{K_1} - n\pi_{d-k}(\theta^T u_0)))\, Q_{d,k}(\mathrm{d}(\theta, K)) \\
\geq &\int_{\{\theta:\pi_{d-k}(\theta^T u_0)=\mathbf{o}_{d-k}\}\times\mathcal{K}'_{d-k}} \nu_{d-k}(\widetilde{K_0} \cap (\widetilde{K_1} - n\pi_{d-k}(\theta^T u_0)))\, Q_{d,k}(\mathrm{d}(\theta, K)) \\
= &\int_{\{\theta:\pi_{d-k}(\theta^T u_0)=\mathbf{o}_{d-k}\}\times\mathcal{K}'_{d-k}} \nu_{d-k}(K \oplus B_1^{d-k})\, Q_{d,k}(\mathrm{d}(\theta, K)) \\
\geq &\varepsilon\, \nu_{d-k}(B_1^{d-k}) > 0 \quad \text{for all} \quad n \in \mathbb{N}\,.
\end{aligned}$$

But this means that (2.5) does not hold and thus the P-k-CM $\Xi_{\lambda,Q_{d,k}}$ is not mixing. In other words, (2.4) is necessary to ensure the mixing property (1.7) for $\Xi_{\lambda,Q_{d,k}}$. This completes the proof of Theorem 2.2. □

Theorem 2.3. *For each* $1 \le k \le d-1$*, the stationary P-*k*-CM* (1.3) *satisfying* (1.2) *and the condition* (2.4) *is mixing of any order* $\ell \ge 2$*.*

Proof. First, we rewrite (1.8) according to Remark 1.3 in terms of the hitting functional $T_{\lambda,Q_{d,k}}(C) = 1 - \exp\{-\mu(C)\}$ with $\mu(C) := \lambda\,\mathbf{E}\,\nu_{d-k}(\Xi_0 \oplus \pi_{d-k}(-\Theta_0^T C))$. This means we need to prove that, for any $C_0, C_1, \ldots C_\ell \in \mathcal{K}_d$ and sequences $x_{n,0} \equiv \mathbf{o}_d$, $x_{n,1}, \ldots, x_{n,\ell}$ satisfying $\|x_{n,i} - x_{n,j}\| \underset{n\to\infty}{\longrightarrow} \infty$ for $0 \le i < j \le \ell$,

$$\Delta_n(C_0, \ldots, C_\ell) := 1 - T_{\lambda,Q_{d,k}}\big(\bigcup_{i=0}^{\ell} S_{x_{n,i}} C_i\big) - \prod_{i=0}^{\ell}(1 - T_{\lambda,Q_{d,k}}(C_i)) \underset{n\to\infty}{\longrightarrow} 0\,.$$

It is easily seen that $\Delta_n(C_0, \ldots, C_\ell) \ge 0$ and

$$\begin{aligned}\Delta_n(C_0, \ldots, C_\ell) &= \exp\Big\{-\mu\big(\bigcup_{i=0}^{\ell} S_{x_{n,i}} C_i\big)\Big\} - \exp\Big\{-\sum_{i=0}^{\ell}\mu(C_i)\Big\}\\ &\le \exp\Big\{\sum_{i=0}^{\ell}\mu(C_i) - \mu\big(\bigcup_{i=0}^{\ell} S_{x_{n,i}} C_i\big)\Big\} - 1\\ &\le \exp\Big\{\sum_{0\le i<j\le\ell} \lambda\,\mathbf{E}\,\nu_{d-k}\Big(\widetilde{\Xi}_i \cap \big(\widetilde{\Xi}_j - \pi_{d-k}(\Theta_0(x_{n,j} - x_{n,i}))\big)\Big)\Big\} - 1\,,\end{aligned}$$

where $\widetilde{\Xi}_j := \Xi_0 \oplus \pi_{d-k}(-\Theta_0^T C_j)$ for $j = 0, 1, \ldots, \ell$. The last bound results from the additivity of the Lebesgue measure ν_{d-k} combined with its translation-invariance yielding, among others, $\mu(S_{x_{n,i}}\, C_i) = \mu(C_i)$. Finally, repeating the proof of (2.5) leads to the limits

$$\mathbf{E}\,\nu_{d-k}(\widetilde{\Xi}_i \cap (\widetilde{\Xi}_j - \pi_{d-k}(\Theta_0^T(x_{n,j} - x_{n,i})))) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{if} \quad \|x_{n,i} - x_{n,j}\| \underset{n\to\infty}{\longrightarrow} \infty$$

for $0 \le i < j \le \ell$. Thus, $\Delta_n(C_0, \ldots, C_\ell) \underset{n\to\infty}{\longrightarrow} 0$ for any $\ell \ge 2$ which provides the assertion of Theorem 2.3. □

Remark 2.1: The shape of the hitting functional (1.4) with $\mu(C) \in [0, \infty)$ (being a completely alternating semicontinuous capacity on $\mathcal{K}_d$ such that $\mu(\emptyset) = 0$) reveals that every P-k-CM $\Xi_{\lambda,Q_{d,k}}$ (satisfying (1.2)) is an union-infinite divisible stationary RACS in $\mathbb{R}^d$ without fixed points, see Theorem 2.3.3 in [19] and Chapt. 4.1 in [18].

Corollary 2.4. *For each* $k = 1, \ldots, d-1$*, the P-k-CM* $\Xi_{\lambda, Q_{d,k}}$ *is not mixing if the directional distribution* $Q_{d,k}^{(0)}$ *has atoms.*

Proof. Let $Q_{d,k}^{(0)}(\{\vartheta_0\}) > 0$ for some $\vartheta_0 \in \mathbb{SO}_{d,k}$. Then $Q_{d,k}^{(0)}(\{\theta \in \mathbb{SO}_{d,k} : u \in \theta \mathbb{E}_k\}) \geq Q_{d,k}^{(0)}(\{\vartheta_0\}) > 0$ for all $u \in S^{d-1} \cap \vartheta_0 \, \mathbb{E}_k$. □

Now, let $\mu_{d,k}$ denote the restriction of the unique normalized rotation invariant (Haar) measure μ_d on $\mathbb{SO}_d$, see Chapt. 13.2 in [19], to $\mathbb{SO}_{d,k}$. Two linear subspaces $\mathbb{L}$ and $\mathbb{L}'$ of $\mathbb{R}^d$ are said to be in *special position* (in *general position* otherwise) if

$$\operatorname{span}(\mathbb{L} \cup \mathbb{L}') \neq \mathbb{R}^d \quad \text{and} \quad \dim(\mathbb{L} \cap \mathbb{L}') > 0\,.$$

Corollary 2.5. *For each* $k = 1, \ldots, d-1$*, the stationary P-k-CM* (1.3) *satisfying* (1.2) *is mixing iff*

$$Q_{d,k}^{(0)}\big(\{\theta \in \mathbb{SO}_{d,k} : \theta \, \mathbb{E}_k \textit{ and } \mathbb{L} \textit{ are in special position}\}\big) = 0 \quad \textit{for all} \quad \mathbb{L} \in \mathcal{G}(d,1)\,.$$

In particular $\Xi_{\lambda, Q_{d,k}}$ *is mixing if* $Q_{d,k}^{(0)}$ *is absolute continuous w.r.t.* $\mu_{d,k}$.

Proof. It is easily seen that, for all $u \in S^{d-1}$ and $\theta \in \mathbb{SO}_{d,k}$,

$$u \in \theta \, \mathbb{E}_k \quad \text{iff} \quad \operatorname{span}(u) \text{ and } \theta \, \mathbb{E}_k \quad \text{are in special position}\ .$$

On the other hand, from Lemma 13.1.2 in [19] we know that
$\mu_d\big(\{\theta \in \mathbb{SO}_d : \theta \, \mathbb{E}_k$ and $\mathbb{L}$ are in special position $\}\big) = 0$. □

In general, condition (2.4) turns out to be stronger than $Q_{d,k}^{(0)}(\{\theta\}) = 0$ for all $\theta \in \mathbb{SO}_{d,k}$. However, in the particular case $d = 2, k = 1$ both conditions are equivalent.

Example: Let $Q_0^{(d)}$ denote the image measure of $Q_{d,d-1}^{(0)}$ under the mapping
$\mathbb{SO}_d \ni \theta \mapsto \theta \, e_1 \in S^{d-1}$. Then $Q_0^{(d)}$ is a probability measure on the sphere S^{d-1} and condition (2.4) can be expressed as

$$Q_0^{(d)}(S^{d-1} \cap \mathbb{L}) = 0 \quad \text{for all} \quad \mathbb{L} \in \mathcal{G}(d, d-1) \tag{2.7}$$

confirming once more the above-mentioned result in [19], p. 517.

To study weak dependence properties of a stationary RACS Ξ in $\mathbb{R}^d$ which go beyond mixing, see e.g. [12], [13] in case of STIT tessellations, we consider the *tail-*σ*-algebra* $\sigma_f^\infty(\Xi) :=$

$\bigcap_{n\in\mathbb{N}} \sigma_f(\Xi \cap \{x \in \mathbb{R}^d : \|x\| \geq n\})$, where $\sigma_f(\Xi')$ is the smallest σ-algebra containing all events $\{\Xi' \in \mathcal{F}_C\} = \{\Xi' \cap C \neq \emptyset\}$ for $C \in \mathcal{K}_d$.

It is a well-known fact, see [4] for stationary point processes, that the triviality of the tail-σ-algebra $\sigma_f^\infty(\Xi)$, i.e. $\mathbf{P}(A) \in \{0,1\}$ for all tail events A, implies that Ξ is mixing (even of any order). On the other hand, the reverse implication is false in general. Following the terminology in [11], a stationary RACS Ξ in $\mathbb{R}^d$ having (non-)trivial tail-σ-algebra $\sigma_f^\infty(\Xi)$ is said to have *(long) short range correlations* or *(long) short range dependences*.

Remark 2.2: For each $k = 1,\ldots,d-1$, the stationary P-k-CM $\Xi_{\lambda,Q_{d,k}}$ has long range correlations. It is easily checked (and already mentioned in [10]) that the events $A_\varepsilon := \{\Xi_{\lambda,Q_{d,k}} \cap B_\varepsilon^d = \emptyset\}$ belong to $\sigma_f(\Xi_{\lambda,Q_{d,k}} \cap \{x \in \mathbb{R}^d : \|x\| \geq n\})$ for all $n \in \mathbb{N}$ and $\varepsilon > 0$, but $\mathbf{P}(A_\varepsilon) = 1 - T_{\lambda,Q_{d,k}}(B_\varepsilon^d) \in (0,1)$.

3 A Remarkable Property of Cells Generated by a P-(d-1)-CM

Throughout, in this section we consider exclusively $\mathbf{P}$-$(d-1)$-CMs satisfying $\mathbf{P}(\Theta_0\, e_1 \in \mathbb{L}) = Q_0^{(d)}(S^{d-1} \cap \mathbb{L}) < 1$ for all $\mathbb{L} \in \mathcal{G}(d,d-1)$ (in particular if (2.7) holds) with typical base $\Xi_0 \in \mathcal{C}_1$ satisfying (1.2) for $k = d-1$, i.e. Ξ_0 is a closed interval with finite mean length $\mathbf{E}\,\nu_1(\Xi_0)$ so that the $(d-1)$-cylinders can be regarded as randomly dilated hyperplanes in $\mathbb{R}^d$ and the complement of their union $\Xi^c_{\lambda,Q_{d,d-1}}$ consists of pairwise disjoint open bounded convex polytopes. By taking the closure of each of these open polytopes we we obtain a family $\{Z_i, i \geq 1\}$ of random compact convex polytopes satisfying $Z_i \cap Z_j = \emptyset$ or $\nu_d(Z_i \cap Z_j) = 0$ otherwise for all $i \neq j$. Let $\mathcal{P}'_d$ denote the subset of non-empty polytopes in $\mathcal{C}_d$.

To start with, we derive a formula for the *contact distribution function* $0 \leq r \mapsto H_S(r)$ of $\Xi := \Xi_{\lambda,Q_{d,d-1}}$, see e.g. [3],

$$H_S(r) := \mathbf{P}(\Xi \cap r\,S \neq \emptyset \,|\, \mathbf{o}_d \notin \Xi) = 1 - \frac{1 - \mathbf{P}(\mathbf{o}_d \in \Xi \oplus (-r\,S))}{1 - \mathbf{P}(\mathbf{o}_d \in \Xi)} \tag{3.1}$$

where the "structuring element" $S \in \mathcal{K}'_d$ is star-shaped w.r.t. $\mathbf{o}_d \in S$. Straightforward calculations carried out in [8] and [10], see also [20] for a different approach, yield that $p(r) := \mathbf{P}(\mathbf{o}_d \in \Xi \oplus (-r\,S)) = 1 - \exp\{-\lambda\, \mathbf{E}\,\nu_1(\Xi_0 \oplus r\,\pi_1(-\Theta_0^T\,S))\}$ and the expression $p(0) = \mathbf{E}\,\nu_d(\Xi \cap [0,1]^d) = 1 - \exp\{-\lambda\,\mathbf{E}\,\nu_1(\Xi_0)\}$ for the *volume fraction* of Ξ. Inserting these formulas in (3.1) and taking into account that $\pi_1(-\Theta_0^T\,S)$ is an intervall we arrive at $H_C(r) = 1 - \exp\{-r\,\lambda\,\mathbf{E}\,\nu_1(\pi_1(-\Theta_0^T\,S))\}$ for $r \geq 0$ which shows an exponential distribution function being always the same regardless of how $\nu_1(\Xi_0)$ is distributed. This interesting observation proves useful in the statistical analysis of $\Xi_{\lambda,Q_{d,d-1}}$ and is the consequence of an invariance property of the so-called *zero cell* $Z_{\mathbf{o}}$ which coincides with the unique polytope Z_i whose interior $\mathrm{int}(Z_i)$ contains the origin $\mathbf{o}_d$ conditional on $\mathbf{o}_d \notin \Xi_{\lambda,Q_{d,d-1}}$.

A simple statistical application is the following: Let Ξ be observed in a CAS W_n, see Definition 1.2. Then $\widehat{p}_n(r) := \nu_d(W_n \cap \Xi \oplus (-r\,S))/\nu_d(W_n)$ is unbiased and strongly consistent estimator for $p(r)$, where the consistency results from Theorem 2.1 and the spatial ergodic theorem, see Chapt. 12.2 in [4]. Hence, the empirical contact distribution function $\widehat{H}_{S,n}(r)$ turns out to be strongly consistent (even uniformly),

$$\widehat{H}_{S,n}(r) := 1 - \frac{1-\widehat{p}_n(r)}{1-\widehat{p}_n(0)} \xrightarrow[n\to\infty]{\mathbf{P}-\text{a.s.}} H_S(r) \quad \text{for} \quad r \geq 0$$

such that, for $S = B_1^d$ and $r > 0$, $\widehat{\lambda}_n := -\log(1-\widehat{H}_{S,n}(r))/2\,r \xrightarrow[n\to\infty]{\mathbf{P}-\text{a.s.}} \lambda$.

The above-mentioned invariance property was already mentioned in [16] and [17]. But neither there nor elsewhere – to the best of authors' knowledge – this rather surprising property of the stationary *particle process* $\{Z_i, i \geq 1\}$ has been precisely formulated and rigorously proved.

The family $\{Z_i, i \geq 1\}$ can be regarded as a stationary tessellation / mosaic, see Chapt. 10 in [19], with "thick boundaries". In Figure 1 the white polygons coincide with the interior of the closed cells Z_i and the black strips form their boundaries. In accordance with the above definition the zero cell $Z_{\mathbf{o}}$ is a random element in $\mathcal{P}'_d$ with (conditional) distribution

$$P_{\mathbf{o}}(\mathcal{A}) := \frac{P^*_{\mathbf{o}}(\mathcal{A} \cap \{F \in \mathcal{F}_d :\, \mathbf{o}_d \in F\})}{P^*_{\mathbf{o}}(\{F \in \mathcal{F}_d :\, \mathbf{o}_d \in F\})} \quad \text{for} \quad \mathcal{A} \in \sigma_f \cap \mathcal{P}'_d\,, \tag{3.2}$$

where $P^*_{\mathbf{o}}$ denotes the distribution of the random compact convex polytope

$$Z^*_{\mathbf{o}} := \begin{cases} \bigcup_{i\geq 1} 1(\mathbf{o}_d \in \mathrm{int}(Z_i))\, Z_i & \text{if} \quad \mathbf{o}_d \notin \Xi_{\lambda,Q_{d,d-1}} \\ \emptyset & \text{if} \quad \mathbf{o}_d \in \Xi_{\lambda,Q_{d,d-1}} \end{cases}.$$

On the other hand, the *typical cell* $\widehat{Z}_{\mathbf{o}}$ associated with the tessellation $\{Z_i, i \geq 1\}$ is defined via the *Palm mark distribution* $\widehat{P}_{\mathbf{o}}$ of the stationary marked point process $\Psi_\alpha := \sum_{i\geq 1} \delta_{[\alpha(Z_i), Z_i - \alpha(Z_i)]}$ on $\mathbb{R}^d$ with measurable mark space $[\mathcal{P}'_d, \sigma_f \cap \mathcal{P}'_d]$, where $\alpha \,|\, \mathcal{K}'_d \mapsto \mathbb{R}^d$ is some measurable mapping with $\alpha(K+x) = \alpha(K) + x$ for all $x \in \mathbb{R}^d$ and $K \in \mathcal{K}'_d$, for example $\alpha(K) = \mathrm{lex}\max(K)$ in what follows. From the theory of stationary marked point process, see Chapt. 3.2 in [19] or [4], we use the factorization of the intensity measure $\mathbf{E}\,\Psi_\alpha(\cdot)$ which implies the existence of a unique probability measure

$$\widehat{P}_{\mathbf{o}}(\mathcal{A}) = \frac{1}{\gamma_d}\, \mathbf{E}\,\Psi_\alpha([0,1)^d \times \mathcal{A}) \quad \text{for} \quad \mathcal{A} \in \sigma_f \cap \mathcal{P}'_{d,\mathbf{o}} \tag{3.3}$$

concentrated on $\mathcal{P}'_{d,\mathbf{o}} := \{C \in \mathcal{P}'_d : \operatorname{lex}\max(C) = \mathbf{o}_d\}$ with the intensity $\gamma_d := \mathbf{E}\,\#\{i \geq 1 : \operatorname{lex}\max(Z_i) \in [0,1)^d\}$. Now, we are ready to formulate the announced properties of $Z_\mathbf{o}$ and $\widehat{Z}_\mathbf{o}$:

Theorem 3.1. *Under the assumptions made at the beginning of Sect. 3, it holds:*

1. *The distribution $P_\mathbf{o}$ of the zero cell $Z_\mathbf{o}$ does not depend on the distribution of Ξ_0.*
2. *For any translation-invariant functional $h : \mathcal{P}'_d \mapsto [0,\infty)$ the expectation $\mathbf{E}\,h(\widehat{Z}_\mathbf{o}) = \int_{\mathcal{P}'_{d,\mathbf{o}}} h(C)\,\widehat{P}_\mathbf{o}(\mathrm{d}C)$ does not depend on the distribution of Ξ_0.*

Proof. For all $i \geq 1$, the sets Z_i and thus the zero cell are regular closed RACS, i.e. $Z_\mathbf{o} = \operatorname{cl}(\operatorname{int} Z_\mathbf{o})$ $\mathbf{P}$-a.s. As shown in [18], Chapt. 1.4.2, the distribution $P_\mathbf{o}$ is therefore determined if the *inclusion functional* $I(L) := P_\mathbf{o}(\{F \in \mathcal{F}_d : L \subseteq F\})$ is known for every finite set L. By the definition (3.2) and $P^*_\mathbf{o}(\{F \in \mathcal{F}_d : \mathbf{o}_d \in F\}) = \mathbf{P}(\mathbf{o}_d \notin \Xi_{\lambda,Q_{d,d-1}}) = 1 - p(0) = \exp\{-\lambda\,\mathbf{E}\,\nu_1(\Xi_0)\}$, it follows that

$$I(L) = P_\mathbf{o}(\{F \in \mathcal{F}_d : L \subseteq F\}) = P^*_\mathbf{o}(\{F \in \mathcal{F}_d : L \subseteq F\,,\, \mathbf{o}_d \in F\})/(1-p(0))$$
$$= \mathbf{P}(L \subseteq Z^*_\mathbf{o}\,,\, \mathbf{o}_d \notin \Xi_{\lambda,Q_{d,d-1}})/(1-p(0)). \qquad (3.4)$$

Since $Z^*_\mathbf{o} \in \mathcal{P}'_d$ iff $\mathbf{o}_d \notin \Xi_{\lambda,Q_{d,d-1}}$, it is obvious that $L \subseteq Z^*_\mathbf{o}$ for a finite set L implies that $Z^*_\mathbf{o}$ contains the convex hull $C_L := \operatorname{conv}(L \cup \{\mathbf{o}_d\})$ and vice versa. Hence, it suffices to show that $I(C_L)$ does not depend on the distribution of Ξ_0. It is immediately clear that $C_L \subseteq Z^*_\mathbf{o}$ iff the relative interior $\operatorname{relint}(C_L)$ of the polytope C_L is contained in the ($\mathbf{P}$-a.s.) open set $\Xi^c_{\lambda,Q_{d,d-1}}$. Further, due to the stationarity of the P-$(d-1)$-CM $\Xi_{\lambda,Q_{d,d-1}}$, the probability that at least one of the at most $\#L+1$ vertices of C_L lies in the boundary $\partial\Xi_{\lambda,Q_{d,d-1}}$ is zero so that the events $\{C_L = \operatorname{cl}(\operatorname{relint}(C_L)) \subset \Xi^c_{\lambda,Q_{d,d-1}}\}$ and $\{C_L \subseteq Z^*_\mathbf{o}\}$ have the same probability. Therefore, by applying (1.4) and noting that $\pi_1(-\Theta_0^T C_L)$ is an interval, we have

$$\mathbf{P}(C_L \subseteq Z^*_\mathbf{o}\,,\, \mathbf{o}_d \notin \Xi_{\lambda,Q_{d,d-1}}) = \mathbf{P}(C_L \cap \Xi_{\lambda,Q_{d,d-1}} = \emptyset) = 1 - T_{\lambda,Q_{d,d-1}}(C_L)$$
$$= (1-p(0))\,\exp\{-\lambda\;\mathbf{E}\,\nu_1(\pi_1(-\Theta_0^T C_L))\}\,.$$

This combined with (3.4) gives $I(L) = I(C_L) = \exp\{-\lambda\;\mathbf{E}\,\nu_1(\pi_1(-\Theta_0^T C_L))\}$ for any finite set $L \subset \mathbb{R}^d$. Thus, the first part of Theorem 3.1 is proved.

To prove the second part, we note that the intensity γ_d of Ψ_α with $\alpha(Z_i) = \operatorname{lex}\max(Z_i)$ can be expressed as product $\gamma_d = (1-p(0))\,\nu_d(Z(\lambda, Q_0^{(d)}))$, where

$$\nu_d(Z(\lambda, Q_0^{(d)})) = \frac{\lambda^d}{d!}\int_{(S^{d-1})^d} |\det(u_1,\ldots,u_d)|\,Q_0^{(d)}(\mathrm{d}u_1)\cdots Q_0^{(d)}(\mathrm{d}u_d) \qquad (3.5)$$

and $Z(\lambda, Q_0^{(d)})$ denotes the associated zonoid connected with a stationary Poisson hyperplane process with intensity λ and spherical directional distribution $Q_0^{(d)}$, see [19]. A detailed proof of the above shape of γ_d can be found among others in [2], see also [1]. Now, for any translation-invariant functional $g : \mathcal{P}'_d \mapsto [0,\infty)$ we integrate $g(\cdot)$ w.r.t. the probability measure (3.2). For doing this, we need to apply the Campbell theorem for stationary marked point processes, see Chapt. 3.5 in [19], which implies that

$$\begin{aligned}\mathbf{E}\, g(Z_{\mathbf{o}}) &= \frac{1}{\mathbf{P}(\mathbf{o}_d \notin \Xi_{\lambda,Q_{d,d-1}})}\; \mathbf{E}\Big[\sum_{i\geq 1} 1(\mathbf{o}_d \in \mathrm{int}(Z_i))\, g(Z_i)\Big] \\ &= \frac{\gamma_d}{1-p(0)} \int_{\mathcal{P}'_{d,\mathbf{o}}} \int_{\mathbb{R}^d} g(C)\; 1(\mathbf{o}_d \in x + C)\,\mathrm{d}x\, \widehat{P}_{\mathbf{o}}(\mathrm{d}C) = \gamma_d\, \frac{\mathbf{E}[g(\widehat{Z}_{\mathbf{o}})\, \nu_d(\widehat{Z}_{\mathbf{o}})]}{1-p(0)}\end{aligned}$$

Finally, replacing $g(\cdot)$ by $h(\cdot)/\nu_d(\cdot)$ for an arbitrary translation-invariant functional $h : \mathcal{P}'_d \mapsto [0,\infty)$ reveals that

$$\mathbf{E}\, h(\widehat{Z}_{\mathbf{o}}) = \frac{1-p(0)}{\gamma_d}\; \mathbf{E}\big[h(Z_{\mathbf{o}})/\nu_d(Z_{\mathbf{o}})\big] = \frac{1}{\nu_d(Z(\lambda, Q_0^{(d)}))}\, \mathbf{E}\big[h(Z_{\mathbf{o}})/\nu_d(Z_{\mathbf{o}})\big]. \tag{3.6}$$

The first part of Theorem 3.1 and (3.5) show that the right-hand side of (3.6), and thus also the expectation on the left-hand side, does not depend on the distribution of Ξ_0. Hence, the proof of Theorem 3.1 is complete. □

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