Enhanced Middle Convolution

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1 Introduction

Let X be a complex manifold. The Riemann–Hilbert correspondence of M. Kashiwara ([Kas84]) establishes an equivalence

$$D^b_{\mathrm{rh}}(\mathscr{D}_X) \xrightarrow{\simeq} D^b_{\mathbb{C}-c}(\mathbb{C}_X), \quad \mathscr{M} \longmapsto DR_X(\mathscr{M}) = \Omega_X \overset{L}{\otimes}_{\mathscr{D}_X} \mathscr{M} \tag{1.1}$$

between the triangulated categories of regular holonomic $\mathscr{D}_X$-modules and $\mathbb{C}$-constructible sheaves on X. Recently, A. D'Agnolo and M. Kashiwara extended this result to the case of irregular holonomic $\mathscr{D}_X$-modules in [DK16b]. This enhanced Riemann–Hilbert correspondence provides a fully faithful embedding, in form of the enhanced de Rham functor DR^E, of $D^b_{\mathrm{hol}}(\mathscr{D}_X)$ into the category of so called *enhanced ind-sheaves* $E(X)$ (this category is denoted by $E^b(\mathrm{I}\mathbb{C}_X)$ in [DK16b]), together with a reconstruction functor that allows us to recover a holonomic $\mathscr{D}_X$-module from its associated enhanced ind-sheaf.

Let us denote by $\mathrm{Perv}(\mathbb{C}_X)$ the abelian category of perverse $\mathbb{C}_X$-sheaves. It is well known (e. g. [Bjö93, theorem 5.5.4]) that the Riemann–Hilbert-correspondence restricts to an equivalence $\mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X) \simeq \mathrm{Perv}(\mathbb{C}_X)$. In [DK16a], A. D'Agnolo and M. Kashiwara proved an analogue to this in the enhanced setting: The triangulated category $E_{\mathbb{R}-c}(X)$ of $\mathbb{R}$-constructible enhanced ind-sheaves admits a self-dual generalized t-structure

$$({}^{1/2}E^{\leqslant c}_{\mathbb{R}-c}(X), {}^{1/2}E^{\geqslant c}_{\mathbb{R}-c}(X))_{c\in\mathbb{R}},$$

and the enhanced de Rham functor DR^E is exact with respect to this (generalized) t-structure and the standard t-structure on $D^b_{\mathrm{hol}}(\mathscr{D}_X)$, i. e. $DR^E(\mathscr{M}) \in {}^{1/2}E^0_{\mathbb{R}-c}(X)$ for any $\mathscr{M} \in \mathrm{Hol}(\mathscr{D}_X)$. A noteworthy difference compared to the classical case – besides the fact that $DR^E\colon \mathrm{Hol}(\mathscr{D}_X) \to {}^{1/2}E^0_{\mathbb{R}-c}(X)$ still is not essentially surjective – is that ${}^{1/2}E^0_{\mathbb{R}-c}(X)$ is only a quasi-abelian category in general (cf. [DK16a; Sch98] and [Bri07, section 4]).

This thesis is motivated by the following line of thoughts: Recall that in [Kat95, section 5.2], N.M. Katz stated his main theorem on the structure of rigid local systems, which (roughly) says that one can reduce any cohomologically rigid l-adic sheaf $\mathscr{F}$ on $\mathbb{A}^1$ (over an algebraically closed field k with characteristic different from l) with generic rank at least 2 of a certain class[1] to a rigid sheaf of generic rank one by successively applying two invertible operations – one of these is the so called *(additive) middle convolution* of $\mathscr{F}$ with some Kummer-sheaf and the other is the *middle tensor product* of $\mathscr{F}$ with some appropriate (lisse, tamely ramified) rank one sheaf (this latter operation is essentially a tensor product, followed by a middle extension). Later, the techniques of [Kat95] have been applied in a range of different settings, including complex local systems (e. g.

[1] The $\overline{\mathbb{Q}}_l$-sheaf in question has to be a middle extension of a lisse and irreducible sheaf on a dense open subset $U \subset \mathbb{A}^1$, has to be tamely ramified at every point of $\mathbb{P}^1 \setminus U$ and it has to have at least two singularities in $\mathbb{A}^1$, cf. [Kat95, section 5.1].

[DR03]), see [Sim09] for an expository view on different versions of Katz' middle convolution algorithm. In particular, based on investigations of S. Bloch and H. Esnault ([BE04]) concerning the preservation of rigidity of (possibly irregular) meromorphic connections on $\mathbb{P}^1$ under Fourier transforms, D. Arinkin proved a Katz algorithm in the setting of rigid meromorphic connections on $\mathbb{P}^1$ with arbitrary singularities ([Ari10]). In this setting, and because the middle convolution of [Ari08; Ari10] may even transform regular meromorphic connections into irregular ones, this Katz-Arinkin algorithm is not accessible by the classical Riemann–Hilbert correspondence. With the emergence of the enhanced version though, it seems natural to ask if there is a counterpart to this in the setting of ($\mathbb{R}$-constructible) enhanced ind-sheaves. Certainly, one of the main ingredients of the Katz algorithm is the middle convolution operation, so in our thesis, we want to focus on establishing an enhanced version of this. Our approach is to stick to the motto stated in the introduction of [Sim09], that the geometric nature of the definition of Katz' middle convolution in [Kat95, section 2.6] allows for transferring it in basically any setting where one has a Grothendieck formalism and a category of perverse sheaves. Both of these prerequisites are satisfied for the case of $\mathbb{R}$-constructible enhanced ind-sheaves, so that we would like to define our enhanced middle convolution, in complete analogy to [Kat95], as

$$K \overset{E}{*}_{\mathrm{mid}} L := \mathrm{Im}\left(E\sigma_{!!}\left(K \overset{+}{\boxtimes} L\right) \to E\sigma_*\left(K \overset{+}{\boxtimes} L\right)\right)$$

for $K, L \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$, where $\mathbf{A}$ is the bordered space $(\mathcal{A}, \mathcal{P})$ with $\mathcal{A} := \mathbb{C} = (\mathbb{A}^1)^{\mathrm{an}}$ and $\mathcal{P} := (\mathbb{P}^1)^{\mathrm{an}} \simeq S^2$, and $\sigma\colon \mathbf{A} \to \mathbf{A}$ is the morphism of bordered spaces induced by

$$\sigma\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (a, b) \mapsto a + b.$$

We will call this construction the *enhanced middle convolution*. Furthermore we will introduce the shorthands $K \overset{E}{*}_! L := E\sigma_{!!}(K \overset{+}{\otimes} L)$ and $K \overset{E}{*}_* L := E\sigma_*(K \overset{+}{\otimes} L)$ for the above two convolution terms.

There are two main issues with this approach though. First, as we mentioned above, ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$ is only quasi-abelian, in particular image and coimage of $K \overset{E}{*}_! L \to K \overset{E}{*}_* L$ need not necessarily be isomorphic. So if we want to stay with our approach, there is no way around defining the dual version

$$K \overset{E}{*}_{\mathrm{co-mid}} L := \mathrm{Coim}\left(E\sigma_{!!}\left(K \overset{+}{\boxtimes} L\right) \to E\sigma_*\left(K \overset{+}{\boxtimes} L\right)\right)$$

as well, which we will refer to as the *(enhanced) co-middle convolution*. One of our main goals in this thesis will therefore be to find some criterion for when middle and co-middle convolution are actually isomorphic. Our criterion, theorem 3.14, will be obtained by transferring the ideas for the proofs of some classical results on the interplay between middle convolution and middle extensions [Kat95, section 2.8] and on the characterization of middle extension perverse sheaves [HTT08, section 8.2] to the enhanced setting. The second issue is that, in order to build middle or co-middle extension $K \overset{E}{*}_{\mathrm{mid}} L$ resp. $K \overset{E}{*}_{\mathrm{co-mid}} L$ for some pair (K, L) of objects in ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$ at all, we obviously first need to assure that the convolutions $K \overset{E}{*}_! L$ and $K \overset{E}{*}_* L$ are again enhanced perverse, i. e. objects of ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$. This part of the problem is already known from the classical setting in [Kat95], and in resemblance of the original notation we will say that a pair (K, L) as above has property $\mathfrak{P}$ if $K \overset{E}{*}_! L, K \overset{E}{*}_* L \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$. Our second task will then be to find some non-trivial (e. g. not coming from some pair of classical perverse sheaves which are known to have property $\mathfrak{P}$ in the original sense, cf. lemma 2.5) pair (K, L) with property $\mathfrak{P}$, which we will do in section 2.4.

The last section is then dedicated to investigating if our enhanced middle convolution (with the second argument fixed as a enhanced Kummer-sheaf L^E_λ, where $\lambda \in \mathbb{C} \setminus \mathbb{Z}$) is compatible with the Arinkin–Katz convolution (for the same λ) defined in [Ari10] via the enhanced Riemann–Hilbert correspondence. To be precise, when we denote the latter one by $\mathscr{M} *_{\mathrm{mid}} \mathcal{K}^\lambda$ in notation of [Ari10], then we would like to show that for some irreducible meromorphic connection $\mathscr{M}$ on $\mathbb{P}^1$ with singularities containing ∞ as in [Ari10], we have

$$Ej_{\mathbf{A}}^{-1} \mathcal{S}ol^E_{\mathcal{P}}(\mathscr{M} *_{\mathrm{mid}} \mathcal{K}^\lambda)[1] \simeq Ej_{\mathbf{A}}^{-1} \mathcal{S}ol^E_{\mathcal{P}}(\mathscr{M})[1] \overset{E}{*}_{\mathrm{mid}} L^E_\lambda[1],$$

and that middle and co-middle convolution agree in this case (conjecture 4.17). Here, $j_{\mathbf{A}} \colon \mathbf{A} \to \mathcal{P}$ is the bordered open embedding and $L^E_\lambda = \mathcal{S}ol^E_{\mathcal{P}}(\mathcal{K}^\lambda)$. In theorem 4.20, we will give a proof of this conjecture under the assumption that the Fourier transform transfers two specific canonical constructions into each other, cf. assumption 4.19.

In order to someday get a full version of a Katz algorithm for enhanced ind-sheaves, a lot more work would still have to be done. For example, finding a criterion to verify property $\mathfrak{P}$ for pairs (K, L) as in [Kat95, section 2.6] would be desirable, and, to get to a similar classification result as [Kat95] or [Ari10], an appropriate concept of rigidity for enhanced ind-sheaves would have to be found. However, both of these tasks seem to be out of the scope of these notes. With regard to this conclusion, we want to mention at least one more justification for our choice of the middle convolution operation as the starting point of our investigation, by pointing out that besides being in some way the centerpiece of Katz' algorithm, Katz' middle convolution has been used beyond that, in non-rigid cases as well, cf. e. g. [Sim09] for an overview of examples. For the rest of this first section, we will recall – mainly from [DK16b] and [DK16a] – some of the technical prerequisites we will use.

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1.1 Bordered spaces

Definition 1.1 (definition 3.2.1 of [DK16b]). A bordered space $\mathbf{X}$ is a pair $(X, \check{X})$ of good topological spaces, where $X \subset \check{X}$ is an open subset. A morphism of bordered spaces $(X, \check{X}) = \mathbf{X} \to \mathbf{Y} = (Y, \check{Y})$ is a continuous map $f\colon X \to Y$, such that, if we consider projections

$$\check{X} \xleftarrow{\mathrm{pr}_{\check{X}}} \check{X} \times \check{Y} \xrightarrow{\mathrm{pr}_{\check{X}}} \check{Y}$$

and label the closure of the graph Γ_f in $\check{X} \times \check{Y}$ with $\overline{\Gamma_f}$, the projection $\mathrm{pr}_{\check{X}}|_{\overline{\Gamma_f}}$ is proper.

Definition 1.2 (definition 2.3.5 of [DK16a]). A morphism $f\colon \mathbf{X} \to \mathbf{Y}$ of bordered spaces is called *semi-proper*, if $\mathrm{pr}_{\check{Y}}|_{\overline{\Gamma_f}}$ is proper. It is called proper if in addition the continuous map $f\colon X \to Y$ is proper.

Remark 1.3 (cf. section 3.2 of [DK16b]). The category of bordered spaces has a final object $(\{\mathrm{pt}\}, \{\mathrm{pt}\})$ and fiber products, which are, for $\mathbf{X} = (X, \check{X})$, $\mathbf{Y} = (Y, \check{Y})$ resp. $\mathbf{Z} = (Z, \check{Z})$ and morphisms $f\colon \mathbf{X} \to \mathbf{Z}$, $g\colon \mathbf{Y} \to \mathbf{Z}$, represented by

$$\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y} = (X \times_Z X, \overline{\Gamma_f} \times_{\check{Z}} \overline{\Gamma_g}).$$

Remark 1.4 (cf. remark 2.3.2 of [DK16a]). When we set $\mathring{\mathbf{X}} := X$ for a bordered space $\mathbf{X} = (X, \check{X})$, this defines a forgetful functor $(\mathring{\bullet})$ from bordered spaces to good topological spaces. It has a fully faithful left adjoint, given by $X \mapsto (X, X)$. For some morphism $f\colon \mathbf{X} \to \mathbf{Y}$ of bordered spaces we will, if the context is clear, often write $f\colon X \to Y$ when referring to $\mathring{f}\colon \mathring{\mathbf{X}} \to \mathring{\mathbf{Y}}$.

Remark 1.5 (cf. remark 3.2.4 of [DK16b]). The morphisms $\mathrm{Id}\colon X \to X$ and $j_X\colon X \to \check{X}$ induce morphisms of bordered spaces

$$(X, X) \longrightarrow \mathbf{X} \xrightarrow{j_{\mathbf{X}}} (\check{X}, \check{X}).$$

Definition 1.6 (cf. notation 2.3.3 of [DK16a]). For any locally closed $Z \subset X$, denote by Z_∞ the bordered space $(Z, \overline{Z})$, where $\overline{Z}$ is the closure of Z in $\check{X}$. The embedding $Z \subset X$ induces a morphism $i_{Z_\infty}\colon Z_\infty \to \mathbf{X}$ of bordered spaces.

Definition 1.7 (definition 2.3.6 of [DK16a]). An (open, closed, locally closed) subset of a bordered space $\mathbf{X} = (X, \check{X})$ is an (open, closed, locally closed) subset of X. Such a subset is called *relatively compact* if it is contained in a compact subset of $\check{X}$.

1.2 Ind-sheaves on bordered spaces

From now on let k be some field. For the theory of ind-sheaves cf. e. g. [KS01]. The bounded derived category of ind-sheaves (over the fixed base k) on a space X is denoted by $D(X)$ (short for $D^b(\mathrm{I}k_X)$, in notation of e. g. [KS01; DK16b] – the version we use here is that of [DK16a]). Let again $\mathbf{X}$ be a bordered space $(X, \check{X})$, and consider the continuous mappings

$$\check{X} \setminus X \xrightarrow{i} \check{X} \xleftarrow{j} X.$$

Then, i gives an embedding

$$Ri_* \simeq Ri_{!!} \colon D(\check{X} \setminus X) \subset D(\check{X}).$$

Definition 1.8 (cf. proposition 2.4.1 of [DK16a]). The derived category of ind-sheaves on the bordered space $\mathbf{X}$ may be defined as the quotient

$$D(\mathbf{X}) := D(\check{X})/D(\check{X} \setminus X).$$

Remark 1.9 (cf. section 2.4 of [DK16a]). In particular, there is the quotient functor

$$q_{\mathbf{X}} \colon D(\check{X}) \to D(\mathbf{X}).$$

It has left and right adjoints, $l_{\mathbf{X}}$ and $r_{\mathbf{X}}$, which satisfy

$$l_{\mathbf{X}} q_{\mathbf{X}} F \simeq k_X \otimes F, \quad r_{\mathbf{X}} q_{\mathbf{X}} F \simeq R\mathscr{I}hom(k_X, F).$$

Remark 1.10 (cf. remark 2.4.2 of [DK16a]). One has a canonical exact embedding

$$\iota_{\mathbf{X}} \colon D^b(k_X) \to D(\mathbf{X})$$

determined by the following commutative diagram:

$$\begin{tikzcd} D^b(k_X) \arrow[r, "\iota_{\mathbf{X}}"] \arrow[d, "\simeq"] & D(\mathbf{X}) \\ D^b(k_{\check{X}})/D^b(k_{\check{X}\setminus X}) \arrow[r, "\iota_{\check{X}}"] & D(\check{X})/D(\check{X} \setminus X) \arrow[u, "="'] \end{tikzcd}$$

Definition 1.11 (cf. section 3.4 of [DK16b]). The classical t-structure on $D(\mathbf{X})$ is denoted by $(D^{\leqslant 0}(\mathbf{X}), D^{\geqslant 0}(\mathbf{X}))$. We have

$$D^{\leqslant 0}(\mathbf{X}) = \{K \in D(\mathbf{X}) | Rj_{\mathbf{X},!!}K \in D^{\leqslant 0}(\check{X})\}$$
$$D^{\geqslant 0}(\mathbf{X}) = \{K \in D(\mathbf{X}) | Rj_{\mathbf{X},!!}K \in D^{\geqslant 0}(\check{X})\}$$

Definition 1.12 (cf. section 3.3 of [DK16b]). For a morphism $f \colon \mathbf{X} \to \mathbf{Y}$ of bordered spaces and $F, F' \in D(\check{X})$, $G \in D(\check{Y})$, one sets (with $\check{X} \overset{q_1}{\leftarrow} \check{X} \times \check{Y} \overset{q_2}{\rightarrow} \check{Y}$ the usual projections)

- $q_{\mathbf{X}}F \otimes q_{\mathbf{X}}F' := q_{\mathbf{X}}(F \otimes F')$,
- $R\mathscr{I}hom(q_{\mathbf{X}}F, q_{\mathbf{X}}F') := q_{\mathbf{X}}R\mathscr{I}hom(F, F')$,
- $Rf_{!!}q_{\mathbf{X}}F := q_{\mathbf{Y}}Rq_{2\,!!}(k_{\Gamma_f} \otimes q_1^{-1}F)$,
- $Rf_{*}q_{\mathbf{X}}F := q_{\mathbf{Y}}Rq_{2\,*}R\mathscr{I}hom(k_{\Gamma_f}, q_1^{!}F)$,
- $f^{-1}q_{\mathbf{Y}}G := q_{\mathbf{X}}Rq_{1\,!!}(k_{\Gamma_f} \otimes q_2^{-1}G)$,
- $f^{!}q_{\mathbf{Y}}G := q_{\mathbf{X}}Rq_{1\,*}R\mathscr{I}hom(k_{\Gamma_f}, q_2^{!}G)$.

Remark 1.13 (cf. proposition 3.4.4 of [DK16b]). For a morphism $f\colon \mathbf{X} \to \mathbf{Y}$ of bordered spaces, $Rf_{!!}$ and Rf_* are left exact and f^{-1} is exact. If $f^{-1}(y)$ has soft-dimension at most d, for every $y \in Y$, then in addition $Rf_{!!}[d]$ is right exact and $f^{!}[-d]$ is left exact.

Remark 1.14 (cf. remark 2.4.3 of [DK16a]). Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of bordered spaces. The natural embeddings $\iota_{\mathbf{X}}$ resp. $\iota_{\mathbf{Y}}$ commute with the operations $\otimes$, $R\mathscr{I}hom$ (actually, ι commutes $R\mathscr{I}hom$ with $R\mathscr{H}om$), Rf_*, f^{-1}, $f^{!}$. If f is semi-proper, ι commutes with $Rf_{!!}$ as well, i. e., denoting with $\mathring{f}\colon X \to Y$ the map underlying f, the diagram

$$\begin{tikzcd} D^b(k_X) \arrow[r, "\iota_{\mathbf{X}}"] \arrow[d, "R\mathring{f}_!"] & D(\mathbf{X}) \arrow[d, "Rf_{!!}"] \\ D^b(k_Y) \arrow[r, "\iota_{\mathbf{Y}}"] & D(\mathbf{Y}) \end{tikzcd}$$

(quasi-)commutes.

Remark 1.15 (cf. remark 2.4.4 of [DK16a]). One can express the quotient functor $q_{\mathbf{X}}$ and its adjoints, $l_{\mathbf{X}}$ and $r_{\mathbf{X}}$, in terms of the natural embedding $j_{\mathbf{X}}\colon \mathbf{X} \to \check{X}$, as

$$q_{\mathbf{X}} \simeq j_{\mathbf{X}}^{-1} \simeq j_{\mathbf{X}}^{!}, \quad l_{\mathbf{X}} \simeq j_{\mathbf{X}\,!!}, \quad r_{\mathbf{X}} \simeq j_{\mathbf{X}\,*}.$$

In particular, the quotient functor is exact.

1.3 Enhanced ind-sheaves

Let $\mathbf{X}$ be a bordered space and $\mathbb{R}_\infty$ the bordered space $(\mathbb{R}, \overline{\mathbb{R}})$, where

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$$

is the two-point-compactification of $\mathbb{R}$. Consider the following natural morphisms:

$$\mathbf{X} \xleftarrow{\pi} \mathbf{X} \times \mathbb{R}_\infty \xrightarrow{\tilde{j}_{\mathbb{R}_\infty}} \mathbf{X} \times \overline{\mathbb{R}} \xrightarrow{\overline{\pi}} \mathbf{X}$$

Definition 1.16 (cf. section 2.6 of [DK16a]). Now set $\mathcal{N} := \pi^{-1}D(\mathbf{X})$, then

$$E(\mathbf{X}) := D(\mathbf{X} \times \mathbb{R}_\infty)/\mathcal{N}$$

is called the category of *enhanced ind-sheaves* on the bordered space $\mathbf{X}$.

In particular, we get a quotient functor $Q_\mathbf{X} \colon D(\mathbf{X} \times \mathbb{R}_\infty) \to E(\mathbf{X})$, which has left and right adjoints, L^E resp. R^E. With $p_1, p_2, \mu \colon \mathbb{R}^2 \to \mathbb{R}$ the first and second projection and the sum map $(t_1, t_2) \mapsto t_1 + t_2$, respectively, and using the same labels for the induced maps $p_1, p_2, \mu \colon \mathbf{X} \times \mathbb{R}^2_\infty \to \mathbf{X} \times \mathbb{R}_\infty$, one defines functors

$$\begin{aligned} \overset{+}{\otimes} \colon D(\mathbf{X} \times \mathbb{R}_\infty) \times D(\mathbf{X} \times \mathbb{R}_\infty) &\to D(\mathbf{X} \times \mathbb{R}_\infty) \\ (K_1, K_2) &\mapsto K_1 \overset{+}{\otimes} K_2 := R\mu_{!!}(p_1^{-1}K_1 \otimes p_2^{-1}K_2) \end{aligned}$$

and

$$\begin{aligned} \mathscr{I}hom^+ \colon D(\mathbf{X} \times \mathbb{R}_\infty)^{\mathrm{op}} \times D(\mathbf{X} \times \mathbb{R}_\infty) &\to D(\mathbf{X} \times \mathbb{R}_\infty) \\ (K_1, K_2) &\mapsto \mathscr{I}hom^+(K_1, K_2) := Rp_{1,*}R\mathscr{I}hom(p_2^{-1}K_1, \mu^! K_2). \end{aligned}$$

These induce functors

$$\overset{+}{\otimes} \colon E(\mathbf{X}) \times E(\mathbf{X}) \to E(\mathbf{X}), \quad \mathscr{I}hom^+ \colon E(\mathbf{X})^{\mathrm{op}} \times E(\mathbf{X}) \to E(\mathbf{X}),$$

cf. definition 1.20 below. One can show (cf. [DK16a, section 2.6]) that

$$\begin{aligned} L^E Q_\mathbf{X} F &\simeq (k_{\{t \geqslant 0\}} \oplus k_{\{t \leqslant 0\}}) \overset{+}{\otimes} F, \\ R^E Q_\mathbf{X} F &\simeq \mathscr{I}hom^+(k_{\{t \geqslant 0\}} \oplus k_{\{t \leqslant 0\}}, F) \end{aligned}$$

for any $F \in D(\mathbf{X} \times \mathbb{R}_\infty)$.

Remark 1.17. We will consider a sheaf $F \in D^b(k_{X \times \mathbb{R}})$ as the enhanced ind-sheaf

$$Q_\mathbf{X} \iota_{\mathbf{X} \times \mathbb{R}_\infty}(F) \in E(\mathbf{X})$$

which we will often denote by F again, as long as the context is clear.

One may as well consider some $F \in D^b(k_X)$ resp. $D(\mathbf{X})$ as an enhanced ind-sheaf, as in the following

Remark 1.18 (cf. section 2.6 of [DK16a]). The functor $\epsilon \colon D(\mathbf{X}) \to E(\mathbf{X})$, defined by

$$F \mapsto Q_\mathbf{X}(k_{\{t=0\}} \otimes \pi^{-1}F),$$

is fully faithful.

Definition/Proposition 1.19 (cf. def. 2.6.1 and prop. 2.6.2 of [DK16a]). We get a t-structure on $E(\mathbf{X})$ by setting, for $n \in \mathbb{Z}$:

$$\begin{aligned} E^{\leqslant n}(\mathbf{X}) &:= \{K \in E(\mathbf{X}) | L^E K \in D^{\leqslant n}(\mathbf{X} \times \mathbb{R}_\infty)\}, \\ E^{\geqslant n}(\mathbf{X}) &:= \{K \in E(\mathbf{X}) | L^E K \in D^{\geqslant n}(\mathbf{X} \times \mathbb{R}_\infty)\}. \end{aligned}$$

The Grothendieck-operations for enhanced ind-sheaves Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of bordered spaces. Set $f_{\mathbb{R}_\infty} := f \times \mathrm{Id}_{\mathbb{R}_\infty}$.

Definition 1.20 (cf. section 2.7 of [DK16a]). The functors

$$\begin{aligned}
\overset{+}{\otimes}\colon E(\mathbf{X}) \times E(\mathbf{X}) &\to E(\mathbf{X}),\\
\mathscr{I}hom^{+}\colon E(\mathbf{X})^{\mathrm{op}} \times E(\mathbf{X}) &\to E(\mathbf{X}),\\
Ef_{!!}, Ef_{*}\colon E(\mathbf{X}) &\to E(\mathbf{Y}),\\
Ef^{-1}, Ef^{!}\colon E(\mathbf{Y}) &\to E(\mathbf{X})
\end{aligned}$$

are defined, for $K, K' \in D(\mathbf{X} \times \mathbb{R}_\infty)$ and $L \in D(\mathbf{Y} \times \mathbb{R}_\infty)$, as

$$\begin{aligned}
Q_{\mathbf{X}}K \overset{+}{\otimes} Q_{\mathbf{X}}K' &:= Q_{\mathbf{X}}(K \overset{+}{\otimes} K'),\\
\mathscr{I}hom^{+}(Q_{\mathbf{X}}K, Q_{\mathbf{X}}K') &:= Q_{\mathbf{X}}\mathscr{I}hom^{+}(K, K'),\\
Ef_{!!}Q_{\mathbf{X}}K &:= Q_{\mathbf{Y}}Rf_{\mathbb{R}_\infty\,!!}K,\\
Ef_{*}Q_{\mathbf{X}}K &:= Q_{\mathbf{Y}}Rf_{\mathbb{R}_\infty\,*}K,\\
Ef^{-1}Q_{\mathbf{Y}}L &:= Q_{\mathbf{X}}f^{-1}_{\mathbb{R}_\infty}L,\\
Ef^{!}Q_{\mathbf{Y}}L &:= Q_{\mathbf{X}}f^{!}_{\mathbb{R}_\infty}L.
\end{aligned}$$

The duality functor $\mathrm{D}^{Q}_{\mathbf{X}}$ is defined by

$$\mathrm{D}^{Q}_{\mathbf{X}}\colon E(\mathbf{X}) \to E(\mathbf{X})^{\mathrm{op}}, \quad K \mapsto \mathscr{I}hom^{+}(K, \omega^{Q}_{\mathbf{X}}),$$

for $\omega^{Q}_{\mathbf{X}} := \epsilon(\omega_{\mathbf{X}}) = Q_{\mathbf{X}}(k_{\{t=0\}} \otimes \pi^{-1}\omega_{\mathbf{X}}) \in E(\mathbf{X})$ with $\omega_{\mathbf{X}} := j^{!}_{\mathbf{X}}\omega_{\check{X}} \simeq j^{-1}_{\mathbf{X}}\omega_{\check{X}}$.

Note that the functors

$$\begin{aligned}
\pi^{-1}(\bullet) \otimes (\bullet)\colon D(\mathbf{X}) \times E(\mathbf{X}) &\to E(\mathbf{X}),\\
R\mathscr{I}hom(\pi^{-1}(\bullet), \bullet)\colon D(\mathbf{X})^{\mathrm{op}} \times E(\mathbf{X}) &\to E(\mathbf{X}),
\end{aligned}$$

are defined, for $L \in D(\mathbf{X})$ and $K \in D(\mathbf{X} \times \mathbb{R}_\infty)$, as

$$\begin{aligned}
\pi^{-1}L \otimes Q_{\mathbf{X}}K &:= Q_{\mathbf{X}}(\pi^{-1}L \otimes K),\\
R\mathscr{I}hom(\pi^{-1}L, Q_{\mathbf{X}}K) &:= Q_{\mathbf{X}}R\mathscr{I}hom(\pi^{-1}L, K),
\end{aligned}$$

cf.[DK16a, section 2.7].

1.3.1 Idempotent and stable objects

The category $D(\mathbf{X} \times \mathbb{R}_\infty)$ is a commutative tensor category with tensor product $\overset{+}{\otimes}$ and unit element $k_{\{t=0\}}$, cf. [DK16b, corollary 4.2.2].

Idempotent objects Consider the sheaves k_I on $X \times \mathbb{R}$ with, for some fixed $a \in \mathbb{R}$, $I = \{t \geqslant a\}$,$\{t \leqslant a\}$,$\{t > a\}$,$\{t < a\}$,$\{t = a\}$ or $I = X \times \mathbb{R}$. We will refer to these as objects in $D(\mathbf{X} \times \mathbb{R}_\infty)$, meaning the objects $\iota_{\mathbf{X}\times\mathbb{R}_\infty}(k_I)$.

Lemma 1.21 (cf. lemma 4.2.3 of [DK16b]). *The objects $k_{\{t\geqslant 0\}}$ (resp. $k_{\{t\leqslant 0\}}$), $k_{\{t>0\}}[1]$ (resp. $k_{\{t<0\}}[1]$), $k_{\{t\geqslant 0\}} \oplus k_{\{t\leqslant 0\}}$ and $k_{X\times\mathbb{R}}[1]$ are idempotents (with respect to $\overset{+}{\otimes}$) in $D(\mathbf{X} \times \mathbb{R}_\infty)$. Furthermore there are the following relations:*

$$\begin{aligned}
k_{\{t\geqslant 0\}} \overset{+}{\otimes} k_{\{t\leqslant 0\}} &\simeq 0,\\
k_{\{t>0\}}[1] \overset{+}{\otimes} k_{\{t<0\}}[1] &\simeq k_{X\times\mathbb{R}}[1],\\
k_{\{t\geqslant 0\}} \overset{+}{\otimes} k_{\{t>0\}}[1] &\simeq 0,\\
k_{\{t\geqslant 0\}} \overset{+}{\otimes} k_{X\times\mathbb{R}}[1] &\simeq 0,\\
k_{\{t>0\}}[1] \overset{+}{\otimes} k_{M\times\mathbb{R}}[1] &\simeq k_{X\times\mathbb{R}}[1],\\
k_{\{t\geqslant 0\}} \overset{+}{\otimes} k_{\{t<0\}}[1] &\simeq k_{\{t\geqslant 0\}}.
\end{aligned}$$

Applying $Q_{\mathbf{X}}$, we will interpret these k_I as objects of $E(\mathbf{X})$ as well. We will suppress $Q_{\mathbf{X}}$ as well as $\iota_{\mathbf{X}\times\mathbb{R}_\infty}$ in our notation if the context is clear. Then $k_{\{t\geqslant 0\}} \in E(\mathbf{X})$ has the following noteworthy property:

Lemma 1.22. *Let $K = Q_{\mathbf{X}}(K') \in E(\mathbf{X})$ with $K' \in D(\mathbf{X} \times \mathbb{R}_\infty)$. Then*

$$k_{\{t\geqslant 0\}} \overset{+}{\otimes} K \simeq \mathscr{I}hom^+(k_{\{t\geqslant 0\}}, K).$$

Proof. This is clear from the existence of the distinguished triangle

$$\pi^{-1}L \longrightarrow k_{\{t\geqslant 0\}} \overset{+}{\otimes} K' \longrightarrow \mathscr{I}hom^+(k_{\{t\geqslant 0\}}, K') \xrightarrow{+1}$$

in $D(\mathbf{X} \times \mathbb{R}_\infty)$, where $L \simeq R\pi_*(k_{\{t\geqslant 0\}} \overset{+}{\otimes} K')$, cf. [DK16b, proposition 4.3.10]. □

Stable objects Consider the following object in $D(\check{X} \times \overline{\mathbb{R}})$:

$$k_{\{t\gg 0\}} := \text{“}\varinjlim_{a\to+\infty}\text{”}\, k_{\{t\geqslant a\}}.$$

The corresponding object in $E(\mathbf{X})$ is denoted by

$$k_{\mathbf{X}}^E := Q_{\mathbf{X}}(k_{\{t\gg 0\}})$$

and is another idempotent object, i.e. $k_{\mathbf{X}}^E \overset{+}{\otimes} k_{\mathbf{X}}^E \simeq k_{\mathbf{X}}^E$, cf. [DK16a, section 2.8]. One defines the full subcategory $E_{\mathrm{st}}(\mathbf{X})$ of stable objects in $E(\mathbf{X})$ as

$$E_{\mathrm{st}}(\mathbf{X}) := \{K \in E(\mathbf{X}) | K \xrightarrow{\simeq} k_{\mathbf{X}}^E \overset{+}{\otimes} K\}.$$

1.3.2 The functor $e\colon D(\mathbf{X}) \to E_{\mathrm{st}}(\mathbf{X})$

The inclusion $E_{\mathrm{st}}(\mathbf{X}) \to E(\mathbf{X})$ has left adjoint $k^E_{\mathbf{X}} \overset{+}{\otimes} (\bullet)$ and right adjoint $\mathscr{I}hom^+(k^E_{\mathbf{X}}, \bullet)$, cf. [DK16a, section 2.8].

Lemma 1.23 (cf. lemma 2.8.2 of [DK16a])**.** *The endofunctor $k^E_{\mathbf{X}} \overset{+}{\otimes} (\bullet)$ on $E(\mathbf{X})$ is exact.*

Definition/Proposition 1.24 (cf. section 2.8 of [DK16a])*.* The embedding

$$e\colon D(\mathbf{X}) \to E_{\mathrm{st}}(\mathbf{X})$$

is defined as

$$e(F) := k^E_{\mathbf{X}} \otimes \pi^{-1}F = Q_{\mathbf{X}}(k_{\{t \gg 0\}} \otimes \pi^{-1}F)$$

for any $F \in D(\mathbf{X})$ and it is fully faithful and exact.

Remark 1.25. We have $e(\bullet) \simeq k^E_{\mathbf{X}} \overset{+}{\otimes} \epsilon(\bullet)$.

Definition 1.26. The duality for stable enhanced ind-sheaves is defined as

$$\mathrm{D}^E_{\mathbf{X}}\colon E_{\mathrm{st}}(\mathbf{X}) \to E_{\mathrm{st}}(\mathbf{X})^{\mathrm{op}}, \quad K \mapsto \mathscr{I}hom^+(K, \omega^E_{\mathbf{X}})$$

with dualizing object $\omega^E_{\mathbf{X}} := e(\omega_{\mathbf{X}})$.

1.3.3 $\mathbb{R}$-constructible enhanced ind-sheaves

From now on, let $\mathbf{X} = (X, \check{X})$ be a subanalytic[2] bordered space, cf. [DK16a, definition 3.1.1] (i. e. $\check{X}$ is a subanalytic space and X is an open subanalytic subset of $\check{X}$). Furthermore all morphisms $f\colon \mathbf{X} \to \mathbf{Y}$ of bordered spaces considered shall be subanalytic, meaning that their graph $\Gamma_f \subset \check{X} \times \check{Y}$ is a subanalytic subset.

Definition 1.27 (cf. definition 3.1.2 of [DK16a]). The category $D^b_{\mathbb{R}-c}(k_{\mathbf{X}})$ is the full subcategory of $D^b(k_X)$ consisting of the objects

$$D^b_{\mathbb{R}-c}(k_{\mathbf{X}}) := \{F \in D^b(k_X) | Ri_{X,!}F \in D^b_{\mathbb{R}-c}(k_{\check{X}})\},$$

where $i_X\colon X \to \check{X}$ is the open embedding. In particular, $D^b_{\mathbb{R}-c}(k_{\mathbf{X}})$ is a full subcategory of $D^b_{\mathbb{R}-c}(k_X)$, as i_X^{-1} preserves $\mathbb{R}$-constructibility.

The following result conveys the compatibility of this notion of $\mathbb{R}$-constructibility with external operations, which essentially is as one might expect from the case of $\mathbb{R}$-constructibility on non-bordered spaces.

Proposition 1.28 (cf. proposition 3.1.3 of [DK16a])**.** *Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of subanalytic bordered spaces, then*

[2] All bordered spaces that will appear in this thesis will actually be analytic, resp. complex bordered spaces in the sense of [KS16, definition 4.11]. For a definition of subanalytic sets, cf. e. g. [BM88].

i) f^{-1} *and* $f^!$ *induce functors* $D^b_{\mathbb{R}-c}(k_{\mathbf{Y}}) \to D^b_{\mathbb{R}-c}(k_{\mathbf{X}})$.

ii) If f *is semi-proper, then* $Rf_!$ *and* Rf_* *induce functors* $D^b_{\mathbb{R}-c}(k_{\mathbf{X}}) \to D^b_{\mathbb{R}-c}(k_{\mathbf{Y}})$.

Note that ii) is put slightly differently here compared to [DK16a, proposition 3.1.3], where $D^b_{\mathbb{R}-c}(k_{\mathbf{X}})$ is considered as a full subcategory of $D(\mathbf{X})$ via $\iota_{\mathbf{X}}$, but as we know that $\iota_{\mathbf{Y}} \circ Rf_! \simeq Rf_{!!} \circ \iota_{\mathbf{X}}$ in the given case that f is semi-proper ($\iota_{\mathbf{Y}} \circ Rf_* \simeq Rf_* \circ \iota_{\mathbf{X}}$ holds anyway), both formulations are clearly equivalent.

Lemma 1.29. D_X *induces an equivalence* $D^b_{\mathbb{R}-c}(k_{\mathbf{X}}) \xrightarrow{\simeq} D^b_{\mathbb{R}-c}(k_{\mathbf{X}})^{\mathrm{op}}$.

Proof. Let $F \in D^b_{\mathbb{R}-c}(k_{\mathbf{X}})$, then $j_{X,!}F \in D^b_{\mathbb{R}-c}(k_{\check{X}})$ by definition and so

$$G := \mathrm{D}_{\check{X}}(j_{X,!}F) \in D^b_{\mathbb{R}-c}(k_{\check{X}})$$

as well. Assume G is cohomologically constructible with respect to some locally finite covering $\check{X} = \bigcup_{i\in I} X_i$ of $\check{X}$ by subanalytic subsets. Then

$$j_{X,!}\,\mathrm{D}_X\, F \simeq j_{X,!}\,\mathrm{D}_X\, j_X^{-1}\,\mathrm{D}_{\check{X}}\, G \simeq G_X$$

is cohomologically constructible with respect to the (locally finite) subanalytic covering

$$\bigcup_{i\in I}((X_i \cap X) \cup (X_i \cap (\check{X} \setminus X)),$$

thus $\mathbb{R}$-constructible, as $(Rj_{X,!}\,\mathrm{D}_X\, F)_x \simeq 0$ for $x \in \check{X} \setminus X$ and $(Rj_{X,!}\,\mathrm{D}_X\, F)_x \simeq (\mathrm{D}_X\, F)_x$ if $x \in X$, which is a perfect complex because $\mathrm{D}_X\, F$ is known to be $\mathbb{R}$-constructible, as F was $\mathbb{R}$-constructible by hypothesis (cf. [KS90, definition 8.4.3]). □

Definition 1.30 (cf. definition 3.3.1 of [DK16a]). An object $K \in E(\mathbf{X})$ is called $\mathbb{R}$-constructible, if for any relatively compact subanalytic open subset U of $\mathbf{X}$, one has

$$Ei_{U_\infty}^{-1}K \simeq k^E_{U_\infty} \overset{+}{\otimes} Q_{U_\infty}\iota_{U_\infty\times\mathbb{R}_\infty}F \in E(U_\infty) \text{ for some } F \in D^b_{\mathbb{R}-c}(k_{U_\infty\times\mathbb{R}_\infty}).$$

The strictly full triangulated subcategory of $E(\mathbf{X})$ consisting of the $\mathbb{R}$-constructible objects is denoted by $E_{\mathbb{R}-c}(\mathbf{X})$.

In particular, $\mathbb{R}$-constructible enhanced ind-sheaves are stable objects in $E(\mathbf{X})$. Furthermore, with $j_{\mathbf{X}}\colon \mathbf{X} \to \check{X}$ as usual, for some $K \in E(\mathbf{X})$, one finds that $K \in E_{\mathbb{R}-c}(\mathbf{X})$ if and only if $Ej_{\mathbf{X},!!}K \in E_{\mathbb{R}-c}(\check{X})$, in analogy to the situation in definition 1.27, cf. [DK16a, lemma 3.3.2]. The following result encloses many other features of $E_{\mathbb{R}-c}(\mathbf{X})$ one might expect with regard to the case of usual sheaves.

Proposition 1.31 (proposition 3.3.3 of [DK16a])**.** *Let* $f\colon \mathbf{X} \to \mathbf{Y}$ *be a morphism of subanalytic bordered spaces.*

i) $E_{\mathbb{R}-c}(\mathbf{X})$ *is a triangulated subcategory of* $E(\mathbf{X})$.

ii) *The duality functor* $\mathrm{D}^E_{\mathbf{X}}$ *induces an equivalence* $E_{\mathbb{R}-c}(\mathbf{X})^{\mathrm{op}} \xrightarrow{\simeq} E_{\mathbb{R}-c}(\mathbf{X})$, *and there is a canonical isomorphism of functors*

$$\mathrm{Id}_{E_{\mathbb{R}-c}(\mathbf{X})} \xrightarrow{\simeq} \mathrm{D}^E_{\mathbf{X}} \circ \mathrm{D}^E_{\mathbf{X}}.$$

iii) *The functors* Ef^{-1} *and* $Ef^!$ *induce functors* $E_{\mathbb{R}-c}(\mathbf{Y}) \to E_{\mathbb{R}-c}(\mathbf{X})$ *and*

$$\begin{aligned} \mathrm{D}^E_{\mathbf{X}} \circ Ef^{-1} &\simeq Ef^! \circ \mathrm{D}^E_{\mathbf{Y}} \\ \mathrm{D}^E_{\mathbf{X}} \circ Ef^! &\simeq Ef^{-1} \circ \mathrm{D}^E_{\mathbf{Y}}. \end{aligned}$$

iv) *If* f *is semi-proper,* Ef_* *and* $Ef_{!!}$ *induce functors* $E_{\mathbb{R}-c}(\mathbf{X}) \to E_{\mathbb{R}-c}(\mathbf{Y})$ *and*

$$\begin{aligned} \mathrm{D}^E_{\mathbf{Y}} \circ Ef_* &\simeq Ef_{!!} \circ \mathrm{D}^E_{\mathbf{X}} \\ \mathrm{D}^E_{\mathbf{Y}} \circ Ef_{!!} &\simeq Ef_* \circ \mathrm{D}^E_{\mathbf{X}}. \end{aligned}$$

Remark 1.32. Note that $e(F) = k^E \otimes \pi^{-1}(F) = k^E \overset{+}{\otimes} (k_{\{t=0\}} \otimes F)$, showing that the embedding $e\colon D(\mathbf{X}) \to E_{\mathrm{st}}(\mathbf{X})$ from section 1.3.2 induces a functor

$$e_X := e \circ \iota_{\mathbf{X}} \colon D^b_{\mathbb{R}-c}(k_{\mathbf{X}}) \to E_{\mathbb{R}-c}(\mathbf{X}).$$

If the context is clear, we will write e again instead of e_X.

1.4 Some properties of enhanced ind-sheaves on bordered spaces

If not otherwise stated, we will assume all bordered spaces (and corresponding morphisms) to be subanalytic. Let $\mathbf{X}, \mathbf{Y}$ be two such bordered spaces.

Lemma 1.33 (cf. lemma 4.3.1 of [DK16b])**.** *For* $K_1, K_2 \in D(\mathbf{X} \times \mathbb{R}_\infty)$ *and* $L \in D(\mathbf{X})$ *one has*

i) $\pi^{-1}L \otimes (K_1 \overset{+}{\otimes} K_2) \simeq (\pi^{-1}L \otimes K_1) \overset{+}{\otimes} K_2$,

ii) $$\begin{aligned} R\mathscr{H}om(\pi^{-1}L, \mathscr{H}om^+(K_1, K_2)) &\simeq \mathscr{H}om^+(\pi^{-1}L \otimes K_1, K_2) \\ &\simeq \mathscr{H}om^+(K_1, R\mathscr{H}om(\pi^{-1}L, K_2)). \end{aligned}$$

Proof. It is enough to apply $j^{-1}_{\mathbf{X},\mathbb{R}_\infty}$ to [DK16b, lemma 4.3.1], where

$$j_{\mathbf{X},\mathbb{R}_\infty} \colon \mathbf{X} \times \mathbb{R}_\infty \to \check{X} \times \mathbb{R}_\infty$$

is the bordered open embedding induced by $j_{\mathbf{X}} \colon \mathbf{X} \to \check{X}$. $\square$

Lemma 1.34 (cf. lemma 4.3.2 of [DK16b])**.** *Let $a\colon \mathbf{X} \times \mathbb{R}_\infty \to \mathbf{X} \times \mathbb{R}_\infty$ be the map induced by $\mathbb{R} \to \mathbb{R}$, $t \mapsto -t$. For $K \in D(\mathbf{X} \times \mathbb{R}_\infty)$ and $L \in D(\mathbf{X})$ one has:*

i) $\pi^{-1}L \otimes K \simeq (\pi^{-1}L \otimes k_{\{t=0\}}) \overset{+}{\otimes} K$,

ii) $R\mathscr{H}om(\pi^{-1}L, K) \simeq \mathscr{H}om^+(\pi^{-1}L \otimes k_{\{t=0\}}, K)$,

iii) $a^{-1}R\mathscr{H}om(K, \pi^!L) \simeq \mathscr{H}om^+(K, \pi^{-1}L \otimes k_{\{t=0\}})$.

Proof. Apply $j^{-1}_{\mathbf{X},\mathbb{R}_\infty}$ to [DK16b, lemma 4.3.2]. □

Remark 1.35. As $\mathrm{D}^E_{\mathbf{X}}(k^E_{\mathbf{X}} \overset{+}{\otimes} Q_{\mathbf{X}}\iota(F)) \simeq k^E_{\mathbf{X}} \overset{+}{\otimes} Q_{\mathbf{X}}\iota(a^{-1}\,\mathrm{D}_{X\times\mathbb{R}}\,F)$ for some $F \in D^b(k_{X\times\mathbb{R}})$ is known from [DK16a, lemma 2.8.3], we can get the bordered space analogue to [DK16b, corollary 4.8.4] by repeating step by step the proof given there: Let $F \in D^b_{\mathbb{R}-c}(k_{\mathbf{X}})$, then

$$\begin{aligned}\mathrm{D}^E_{\mathbf{X}}(k^E_{\mathbf{X}} \otimes Q_{\mathbf{X}}\iota(\pi^{-1}F)) &\simeq \mathrm{D}^E_{\mathbf{X}}(k^E_{\mathbf{X}} \overset{+}{\otimes} Q_{\mathbf{X}}\iota(k_{\{t=0\}} \otimes \pi^{-1}F)) \\ &\simeq k^E_{\mathbf{X}} \overset{+}{\otimes} Q_{\mathbf{X}}\iota(a^{-1}\,\mathrm{D}_{X\times\mathbb{R}}(k_{\{t=0\}} \otimes \pi^{-1}F)) \\ &\simeq k^E_{\mathbf{X}} \overset{+}{\otimes} Q_{\mathbf{X}}\iota(k_{\{t=0\}} \otimes \pi^{-1}\,\mathrm{D}_X\,F) \\ &\simeq k^E_{\mathbf{X}} \overset{+}{\otimes} Q_{\mathbf{X}}\iota(\pi^{-1}\,\mathrm{D}_X\,F).\end{aligned}$$

For the third isomorphism, $\pi \circ a = \pi$ was used, as well as the facts that π is a topological submersion relative dimension 1, i. e. $\pi^!\,\mathrm{D}_X\,F \simeq \pi^{-1}(\mathrm{D}_X\,F)[1]$, and furthermore $i^!\pi^{-1}\,\mathrm{D}_X\,F \simeq i^{-1}\pi^{-1}\,\mathrm{D}_X\,F[-1]$ for the closed embedding $i\colon \{t = 0\} \to X \times \mathbb{R}$, cf. corollary 2.20. What was shown, in other words, is

$$e_X \circ \mathrm{D}_X \simeq \mathrm{D}^E_{\mathbf{X}} \circ e_X.$$

Lemma 1.36 (cf. [DK16b, Proposition 4.1.5])**.** *For $K_1, K_2, K_3 \in D(\mathbf{X} \times \mathbb{R}_\infty)$ one has*

i) $(K_1 \overset{+}{\otimes} K_2) \overset{+}{\otimes} K_3 \simeq K_1 \overset{+}{\otimes} (K_2 \overset{+}{\otimes} K_3)$,

ii) $\mathrm{Hom}_{D(\mathbf{X}\times\mathbb{R}_\infty)}(K_1 \overset{+}{\otimes} K_2, K_3) \simeq \mathrm{Hom}_{D(\mathbf{X}\times\mathbb{R}_\infty)}(K_1, \mathscr{H}om^+(K_2, K_3))$,

iii) $\mathscr{H}om^+(K_1 \overset{+}{\otimes} K_2, K_3) \simeq \mathscr{H}om^+(K_1, \mathscr{H}om^+(K_2, K_3))$.

Proof. The proofs of all three statements given in [DK16b] work out the very same way in the bordered setting. □

As [DK16a, lemma 3.3.2] states, $F \in E_{\mathbb{R}-c}(\mathbf{X})$ if and only if $Ej_{\mathbf{X}\,!!}F \in E_{\mathbb{R}-c}(\check{X})$, the latter referring to the usual enhanced sheaves in the sense of [DK16b]. We thus have

Lemma 1.37. *For $F \in E_{\mathbb{R}-c}(\mathbf{X})$, $G \in E_{\mathbb{R}-c}(\mathbf{Y})$, one has*

$$\mathrm{D}^E_{\mathbf{X}\times\mathbf{Y}}(F \overset{+}{\boxtimes} G) \simeq D^E_{\mathbf{X}} F \overset{+}{\boxtimes} \mathrm{D}^E_{\mathbf{Y}}\, G.$$

Proof. We observe that $Ej_{\mathbf{X}}^{-1} Ej_{\mathbf{X}\,!!} \simeq \mathrm{Id}_{E_{\mathbb{R}-c}(\mathbf{X})} \simeq Ej^!_{\mathbf{X}} Ej_{\mathbf{X}\,*}$ and thus, dropping the indexes of the duality functors as a shorthand,

$$\begin{aligned}
\mathrm{D}^E(F \overset{+}{\boxtimes} G) &\simeq \mathrm{D}^E(Ej_{\mathbf{X}}^{-1} Ej_{\mathbf{X}\,!!} F \overset{+}{\boxtimes} Ej_{\mathbf{Y}}^{-1} Ej_{\mathbf{Y}\,!!} G) \\
&\simeq \mathrm{D}^E\, Ej_{\mathbf{X}\times\mathbf{Y}}^{-1}(Ej_{\mathbf{X}\,!!} F \overset{+}{\boxtimes} Ej_{\mathbf{Y}\,!!} G) \\
&\simeq Ej^!_{\mathbf{X}\times\mathbf{Y}}\, \mathrm{D}^E(Ej_{\mathbf{X}\,!!} F \overset{+}{\boxtimes} Ej_{\mathbf{X}\,!!} G) \\
&\overset{(*)}{\simeq} Ej^!_{\mathbf{X}\times\mathbf{Y}}(\mathrm{D}^E\, Ej_{\mathbf{X}\,!!} F \overset{+}{\boxtimes} \mathrm{D}^E\, Ej_{\mathbf{X}\,!!} G) \\
&\simeq Ej^!_{\mathbf{X}\times\mathbf{Y}}(Ej_{\mathbf{X}\,*}\, \mathrm{D}^E\, F \overset{+}{\boxtimes} Ej_{\mathbf{Y}\,*}\, \mathrm{D}^E\, G) \\
&\simeq (Ej^!_{\mathbf{X}} Ej_{\mathbf{X}\,*}\, \mathrm{D}^E\, F) \overset{+}{\boxtimes} (Ej^!_{\mathbf{Y}} Ej_{\mathbf{Y}\,*}\, \mathrm{D}^E\, G) \\
&\simeq \mathrm{D}^E\, F \overset{+}{\boxtimes} \mathrm{D}^E\, G,
\end{aligned}$$

where we used the fact that, for a bordered open embedding j, one has $Ej^{-1} \simeq Ej^!$ and [DK16b, proposition 4.5.10], or [DK16b, proposition 4.9.22], respectively, and [DK16b, proposition 4.9.21] for step $(*)$. □

1.5 Quasi-abelian categories

While on the one hand, the concept of a quasi-abelian category is essential for all of the following, we will on the other hand not need any deeper insights into the corresponding theory for this thesis. Thus, let us only very quickly recall the basic definitions from [Sch98]. For the following, let $\mathscr{C}$ be an additive category with kernels and cokernels.

Definition 1.38 (definition 1.1.1 of [Sch98]). For some morphism $f\colon A \to B$ in $\mathscr{C}$, one defines

$$\begin{aligned}
\mathrm{Im}(f) &:= \ker(B \to \mathrm{Coker}(f)), \\
\mathrm{Coim}(f) &:= \mathrm{Coker}(\ker(f) \to A).
\end{aligned}$$

By the universal properties of kernel and cokernel, f induces a canonical morphism

$$\mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f).$$

Definition 1.39 (section 1.1.1 of [Sch98]). A morphism $f\colon A \to B$ in $\mathscr{C}$ is called *strict* if the canonical morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

Definition 1.40 (definition 1.1.3 of [Sch98]). The category $\mathscr{C}$ is called *quasi-abelian* if:

i) In any cartesian square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow & & \uparrow \\ A' & \xrightarrow{f'} & B', \end{array}$$

where f is a strict epimorphism, f' is also a strict epimorphism.

ii) In any cartesian square

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ \uparrow & & \uparrow \\ A & \xrightarrow{f} & B, \end{array}$$

where f is a strict monomorphism, f' is also a strict monomorphism.

One of the drawbacks of the existence of non-strict morphisms is that one now has to split the definition of exact sequences known from abelian categories to distinguish (strictly) exact and coexact sequences.

Definition 1.41 (definition 1.1.9 of [Sch98]). A sequence of the form

$$A \xrightarrow{f} B \xrightarrow{g} C$$

with $g \circ f = 0$ is called *strictly exact* (at B) if f is strict and the canonical morphism

$$\operatorname{Im}(f) \to \ker(g)$$

induced by the universal properties is an isomorphism. It is called *strictly coexact* if instead g is strict. A sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} A_n$$

is called strictly exact (resp. coexact) if it is so at every point A_j, $2 \leqslant j \leqslant n-1$.

However, it turns out ([Sch98, remark 1.1.10]) that a sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is strictly exact if and only if it is strictly coexact, if and only if f is a kernel of g and g is a cokernel of f. In particular, one may introduce the (non-split) notion of a *strict short exact sequence* in a quasi-abelian category. One of the main results in view of the appearance of quasi-abelian categories in the context of generalized t-structures is the following

Proposition 1.42 (lemma 4.2 of [Bri07]). *An additive category $\mathscr{C}$ is quasi-abelian if and only if there are abelian categories $\mathscr{C}^\sharp$ and $\mathscr{C}^\flat$ and fully faithful embeddings $\mathscr{C} \subset \mathscr{C}^\sharp$ and $\mathscr{C} \subset \mathscr{C}^\flat$ such that*

i) if $A \to C$ is a monomorphism in $\mathscr{C}^\sharp$ with $C \in \mathscr{C}$, then also $A \in \mathscr{C}$,

ii) if $C \to B$ is an epimorphism in $\mathscr{C}^\flat$ with $C \in \mathscr{C}$, then also $B \in \mathscr{C}$.

If i) and ii) hold, the strict short exact sequences in $\mathscr{C}$ are those sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

that are exact both in $\mathscr{C}^\sharp$ and in $\mathscr{C}^\flat$.

1.6 Generalized t-structures

Let T be a triangulated category.

Definition 1.43 (definition 1.2 of [Kas15]). For families $(T^{\leqslant c})_{c\in\mathbb{R}}$ and $(T^{\geqslant c})_{c\in\mathbb{R}}$ of strictly full subcategories of T, set $T^{<c} := \bigcup_{b<c} T^{\leqslant b}$ and $T^{>c} := \bigcup_{b>c} T^{\geqslant b}$. Then $(T^{\leqslant c}, T^{\geqslant c})_{c\in\mathbb{R}}$ is called a generalized t-structure (on T) if it satisfies the conditions

i) $T^{\leqslant c} = \bigcap_{b>c} T^{\leqslant b}$ and $T^{\geqslant c} = \bigcap_{b<c} T^{\geqslant b}$ for any $c \in \mathbb{R}$,

ii) $T^{\leqslant c+1} = T^{\leqslant c}[-1]$ and $T^{\geqslant c+1} = T^{\geqslant c}[-1]$ for any $c \in \mathbb{R}$,

iii) $\mathrm{Hom}_T(A, B) = 0$ for any $c \in \mathbb{R}$, $A \in T^{<c}$ and $B \in T^{>c}$,

iv) for any $A \in T$ and $c \in \mathbb{R}$, there exist distinguished triangles

$$A_{\leqslant c} \longrightarrow A \longrightarrow A_{>c} \xrightarrow{+1}$$
$$A_{<c} \longrightarrow A \longrightarrow A_{\geqslant c} \xrightarrow{+1},$$

where $A_{\leqslant c} \in T^{\leqslant c}$, $A_{<c} \in T^{<c}$, $A_{\geqslant c} \in T^{\geqslant c}$ and $A_{>c} \in T^{>c}$.

By i) – iii), the objects $A_{\leqslant c}$, $A_{<c}$ resp. $A_{\geqslant c}$, $A_{>c}$ in iv) are unique up to unique isomorphism and thus define truncation functors $\tau^{\leqslant c}$, $\tau^{<c}$, resp. $\tau^{\geqslant c}$, $\tau^{>c}$ that are right resp. left adjoint to the inclusion functors $T^{\leqslant c} \to T$, $T^{<c} \to T$, resp. $T^{\geqslant c} \to T$, $T^{>c} \to T$.

Definition 1.44 (cf. section 1.3 of [DK16a]). For some interval $I = [a, b] \subset \mathbb{R}$ (resp. $(a, b]$, $[a, b)$, (a, b)), one sets

$$T^I := T^{\leqslant b} \cap T^{\geqslant a} \quad (\text{resp. } T^{\leqslant b} \cap T^{>a},\ T^{<b} \cap T^{\geqslant a},\ T^{<b} \cap T^{>a}).$$

The functor

$$\tau^{\leqslant b} \circ \tau^{\geqslant a} \colon T \to T^I, \quad (\text{resp. } \tau^{\leqslant b} \circ \tau^{>a},\ \tau^{<b} \circ \tau^{\geqslant a},\ \tau^{<b} \circ \tau^{>a})$$

is denoted by H^I. For $I = \{c\}$ for some $c \in \mathbb{R}$, one writes $T^c := T^{\{c\}}$ and $H^c := H^{\{c\}}$.

Remark 1.45 (cf. section 1.2 of [DK16a]). One may show that condition iii) of definition 1.43 is equivalent to either of

iii)' $\mathrm{Hom}_T(T^{\leqslant c}, T^{>c}) = 0$ for any $c \in \mathbb{R}$,

iii)" $\mathrm{Hom}_T(T^{<c}, T^{\geqslant c}) = 0$ for any $c \in \mathbb{R}$.

Remark 1.46 (cf. section 1.2 of [DK16a]). For a triangulated category T as above, if $(T^{\leqslant 0}, T^{\geqslant 0})$ is a t-structure in the classical sense, then $(T^{\leqslant c}, T^{\geqslant c})_{c\in\mathbb{R}}$ with

$$\begin{aligned} T^{\leqslant c} :=& T^{\leqslant 0}[-\lfloor c \rfloor], \\ T^{\geqslant c} :=& T^{\geqslant 0}[-\lceil c \rceil] \end{aligned}$$

is a generalized t-structure on T. On the other hand, if $(T^{\leqslant c}, T^{\geqslant c})_{c\in\mathbb{R}}$ is a generalized t-structure on T, then $(T^{\leqslant c+1}, T^{>c})$ and $(T^{<c+1}, T^{\geqslant c})$ are classical t-structures for any $c \in \mathbb{R}$.

Definition 1.47 (cf. definition 1.2.4 of [DK16a]). Let $\Sigma \subset \mathbb{R}$ be discrete and such that $\Sigma + \mathbb{Z} = \Sigma$. Then, a generalized t-structure $(T^{\leqslant c}, T^{\geqslant c})_{c\in\mathbb{R}}$ is called *indexed by* Σ if $T^c = 0$ for any $c \in \mathbb{R} \setminus \Sigma$.

For example, the self-dual generalized t-structure on $D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ for some real manifold X from [Kas15] as well as the generalized self-dual t-structure on $E_{\mathbb{R}-c}(\mathbf{X})$ for some subanalytic bordered space $\mathbf{X}$ are 1/2-indexed.

Definition 1.48 (cf. definition 1.4.1 of [DK16a]). Let T, S be triangulated categories and $F : T \to S$ a triangulated functor. Then F is called

i) *left t-exact* if $F(T^{\geqslant c}) \subset S^{\geqslant c}$ for any $c \in \mathbb{R}$,

ii) *right t-exact* if $F(T^{\leqslant c}) \subset S^{\leqslant c}$ for any $c \in \mathbb{R}$, and

iii) *t-exact* if it is both left and right t-exact.

Some properties of generalized t-structures Let T be a triangulated category equipped with some generalized t-structure $(T^{\leqslant c}, T^{\geqslant c})_{c\in\mathbb{R}}$.

Proposition 1.49 (proposition 1.3.1 of [DK16a] resp. lemma 4.3 of [Bri07]). *Let $I \subset \mathbb{R}$ be some interval.*

i) If $I \to \mathbb{R}/\mathbb{Z}$ is injective, T^I is a quasi-abelian category and strict short exact sequences in T^I correspond (one-to-one) to distinguished triangles in T with all vertices in T^I.

ii) If $I \to \mathbb{R}/\mathbb{Z}$ is bijective, then T^I is an abelian category and the functor $H^I : T \to T^I$ is cohomological.

Let S be another triangulated category with a generalized t-structure $(S^{\leqslant c}, S^{\geqslant c})_{c\in\mathbb{R}}$. Consider the following result analogous to that in the case of a classical t-structure, cf. [HTT08, proposition 8.1.15].

Lemma 1.50. *Let $F\colon T \to S$ be a left t-exact functor and $I \subset \mathbb{R}$ an interval such that $I \to \mathbb{R}/\mathbb{Z}$ is injective. Then one has*

$$\tau^{\leqslant c} F(\tau^{\leqslant c}(A)) \simeq \tau^{\leqslant c} F(A)$$

for any $A \in T$. In particular, if $I = (a, b)$ or $I = (a, b]$ (resp. $I = [a, b)$ or $I = [a, b]$), then

$$H^I_S F(H^I_T A) \simeq H^I_S F(A)$$

for any object A in $T^{>a}$ (resp. $T^{\geqslant a}$).

If $I \to \mathbb{R}/\mathbb{Z}$ is bijective, so that T^I, S^I are abelian, $H^I F$ is a left exact functor $T^I \to S^I$.

Proof. The proof works the very same way as the one in [HTT08] for the case of classical t-structures. Recall that $\tau^{\leqslant c}\colon T \to T^{\leqslant c}$ is right adjoint to the inclusion functor $T^{\leqslant c} \to T$ (cf. [Kas15, section 1]). This allows us to show

$$\tau^{\leqslant c} F(\tau^{\leqslant c}(A)) \simeq \tau^{\leqslant c} F(A)$$

for any $A \in T$. Completely analogous to the reasoning in [HTT08], note that is is enough to prove

$$\mathrm{Hom}_{S^{\leqslant c}}\left(B, \tau^{\leqslant c} F(\tau^{\leqslant c}(A))\right) \simeq \mathrm{Hom}_{S^{\leqslant c}}\left(B, \tau^{\leqslant c} F(A)\right)$$

for any $B \in S^{\leqslant c}$.

By adjunction of $\tau^{\leqslant c}$ and the inclusion functor, one has the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{S^{\leqslant c}}\left(B, \tau^{\leqslant c} F(\tau^{\leqslant c}(A))\right) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{S^{\leqslant c}}\left(B, \tau^{\leqslant c} F(A)\right) \arrow[d, "\simeq"] \\
\mathrm{Hom}_S(B, F(\tau^{\leqslant c}(A)) \arrow[r, "b"] & \mathrm{Hom}_S(B, F(A))
\end{tikzcd} \tag{1.2}$$

(where the horizontal arrows are induced by the canonical morphism $\tau^{\leqslant c} A \to A$) and it thus suffices to show the lower map b is an isomorphism for any $B \in S^{\leqslant c}$. Now, by definition of a generalized t-structure,

$$\tau^{\leqslant c} A \longrightarrow A \longrightarrow \tau^{>c} A \xrightarrow{+1}$$

is a distinguished triangle in T, yielding a distinguished triangle

$$F(\tau^{\leqslant c} A) \longrightarrow F(A) \longrightarrow F(\tau^{>c} A) \xrightarrow{+1}$$

in S. As $\mathrm{Hom}(B, \bullet)$ is a cohomological functor by the definition of a triangulated category, one gets an exact sequence of abelian groups

$$\begin{aligned} \ldots \longrightarrow \mathrm{Hom}_S(B, F(\tau^{>c}A)[-1]) \longrightarrow \\ \longrightarrow \mathrm{Hom}_S(B, F(\tau^{\leqslant c}A)) \xrightarrow{b} \mathrm{Hom}_S(B, F(A)) \longrightarrow \\ \longrightarrow \mathrm{Hom}_S(B, F(\tau^{>c}A)) \longrightarrow \ldots \end{aligned}$$

and, as F is left exact by hypothesis, the first and the last of the shown terms in the above sequence vanish (recall that $B \in S^{\leqslant c}$). So b in (1.2) indeed is an isomorphism. The same works for $\tau^{<c}$ as well, as it is again right adjoint to the inclusion $T^{<c} \to T$. Now let $I = [a, b) \subset \mathbb{R}$, where one could as well have chosen an interval of the form $(a, b]$, (a, b) or $[a, b]$, including the case $[a, a]$, in which we denote the functor $H^{[a,a]}$ by H^a as usual. Then, as F is left t-exact, we have $F(\tau^{\geqslant a}A) \simeq \tau^{\geqslant a}F(\tau^{\geqslant a}A)$ and thus, if $A \in T^{\geqslant a}$, i. e. $\tau^{\geqslant a}A \simeq A$, we get

$$H^{[a,b)}F(A) \simeq \tau^{\geqslant a}\tau^{<b}F(A) \simeq \tau^{\geqslant a}\tau^{<b}F(\tau^{<b}\tau^{\geqslant a}A) \simeq H^{[a,b)}F(H^{[a,b)}A).$$

Now, suppose that $I := [a, a+1) \to \mathbb{R}/\mathbb{Z}$ is bijective (again everything works completely analogous for $I = (a, a+1]$), so that S^I is abelian and H^I is cohomological. Let us consider a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in T^I. This corresponds to a distinguished triangle

$$A \longrightarrow B \longrightarrow C \xrightarrow{+1}$$

in T (note that T^I is nothing but the heart of the classical t-structure $(T^{<a+1}, T^{\geqslant a})$ on T which is associated to the generalized t-structure $(T^{\leqslant c}, T^{\geqslant c})$, cf. remark 1.46 resp. [Bri07, section 3] and [DK16a, section 1.2]).

So, $F(A) \to F(B) \to F(C) \xrightarrow{+1}$ is a distinguished triangle in S and applying the cohomological functor $H := H^I_S$ gives us an exact sequence

$$\ldots \longrightarrow H^{-1}(F(C)) \longrightarrow H^0(F(A)) \longrightarrow H^0(F(B)) \longrightarrow H^0(F(C)) \longrightarrow \ldots$$

with $H^i = H_S^{[a+i,a+1+i)}[i] = \tau_S^{\geqslant a+i} \circ \tau_S^{<a+1+i}[i]$. But now $C \in T^I$, so, as F is left t-exact, $F(C) \in S^{\geqslant a}$, so $\tau^{<a}F(C) \simeq 0$, i. e.

$$H^{-1}(F(C)) = \tau_S^{\geqslant a-1} \circ \tau_S^{<a}F(C)[-1] \simeq 0,$$

showing $H^I_S F \colon T^I \to S^I$ is indeed left exact. $\square$

Remark 1.51. The very same proof works if we replace the left t-exact functor F with a right t-exact functor G and every appearance of "$\leqslant$" (resp. "$<$") with "$\geqslant$" (resp. "$>$") and vice versa. Later, we will have to use $\mathrm{Hom}_{S^{\geqslant c}}(\bullet, B)$ (instead of $\mathrm{Hom}_{S^{\leqslant}}(B, \bullet)$, using the covariant instead of the contravariant Yoneda-embedding), and replace the distinguished triangle above with

$$\tau^{<c}A \longrightarrow A \longrightarrow \tau^{\geqslant c}A \xrightarrow{+1} .$$

The very same reasoning then yields that for such right t-exact G, we have

$$\tau^{\geqslant c}(G(\tau^{\geqslant c}A)) \simeq \tau^{\geqslant c}G(A)$$

for any A and G induces a right exact functor

$$G^I : T^I \to S^I$$

if $I \to \mathbb{R}/\mathbb{Z}$ is bijective.

1.7 Enhanced perverse sheaves

Recall that for some (real analytic) manifold X and some perversity function $p\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}$, one defines the (classical) t-structure on $D^b_{\mathbb{R}-c}(k_X)$ corresponding to p by defining the properties

$$\begin{aligned}
&(p^{\leqslant 0})(F)\colon \quad \dim(\mathrm{Supp}(H^j F)) < m \text{ for any } j, k \text{ with } j > p(k)\\
&(p^{\geqslant 0})(F)\colon \quad H^j(i_Z^! F) = 0 \text{ for any } Z \in \mathrm{LCS}(X) \text{ with } j < p(\dim(Z))
\end{aligned}$$

of a $F \in D^b_{\mathbb{R}-c}(X)$ (notation inspired by the one in [DK16a]), where $\mathrm{LCS}(X)$ shall denote the locally closed subanalytic subsets of X, and then setting

$$\begin{aligned}
{}^p D^{\leqslant 0}_{\mathbb{R}-c}(k_X) :=& \{F \in D^b_{\mathbb{R}-c}(k_X) | (p^{\leqslant 0})(F) \text{ holds}\},\\
{}^p D^{\geqslant 0}_{\mathbb{R}-c}(k_X) :=& \{F \in D^b_{\mathbb{R}-c}(k_X) | (p^{\geqslant 0})(F) \text{ holds}\},
\end{aligned} \tag{1.3}$$

cf. [KS90, definition 10.2.1]. Note that the property $(p^{\leqslant 0})(F)$ may be reformulated in a way formally more similar to $(p^{\geqslant 0})(F)$: Let $X = \coprod_{a \in \mathfrak{A}} X_a$ be a subanalytic stratification of X, consisting of equidimensional strata, such that F has locally constant cohomologies with respect to $(X_a)_{a \in \mathfrak{A}}$. For every $a \in \mathfrak{A}$, let $i_a\colon X_a \to X$ denote the corresponding locally closed embedding. Then, $(p^{\leqslant 0})(F)$ is equivalent to

$$(p^{\leqslant 0})_{\mathfrak{A}}(F)\colon \quad H^j(i_a^{-1} F) = 0 \text{ for any } a, j \text{ with } j > p(\dim(X_a)),$$

cf. [KS90, proposition 10.2.4]. Under the very same assumptions, $(p^{\geqslant 0})(F)$ is equivalent to

$$(p^{\geqslant 0})_{\mathfrak{A}}(F)\colon \quad H^j(i_a^! F) = 0 \text{ for any } a, j \text{ with } j < p(\dim(X_a)),$$

again by [KS90, proposition 10.2.4]. For some perversity function p, one defines p^* by $p^*(n) = -p(n) - n$. Then, it is a well known fact that

$$\mathrm{D}_X({}^p D^{\leqslant 0}_{\mathbb{R}-c}(k_X)) \subset {}^{p^*} D^{\geqslant 0}_{\mathbb{R}-c}(k_X), \quad \mathrm{D}_X({}^p D^{\geqslant 0}_{\mathbb{R}-c}(k_X)) \subset {}^{p^*} D^{\leqslant 0}_{\mathbb{R}-c}(k_X), \tag{1.4}$$

with D_X denoting the Poincaré–Verdier dual on X as usual, cf. [KS90, proposition 10.2.13]. Further recall that, for a complex manifold X, one sets

$$\begin{aligned} {}^p D^{\leqslant 0}_{\mathbb{C}-c}(k_X) &:= {}^p D^{\leqslant 0}_{\mathbb{R}-c}(k_X) \cap D^b_{\mathbb{C}-c}(k_X), \\ {}^p D^{\geqslant 0}_{\mathbb{C}-c}(k_X) &:= {}^p D^{\geqslant 0}_{\mathbb{R}-c}(k_X) \cap D^b_{\mathbb{C}-c}(k_X), \end{aligned}$$

cf. [KS90, section 10.3]. For some $F \in D^b_{\mathbb{C}-c}(k_X)$, checking the criteria $(p^{\leqslant 0})_{\mathfrak{A}}(F)$, $(p^{\geqslant 0})_{\mathfrak{A}}(F)$ from above, we may choose every stratum X_a to be complex analytic and thus of even real dimension. This allows us to apply the above definition 1.3 to the so called *middle perversity function* $p_{1/2}\colon 2\mathbb{Z}_{\geqslant 0} \to \mathbb{Z}$, $n \mapsto -n/2$, yielding

$$\begin{aligned} {}^{p_{1/2}} D^{\leqslant 0}_{\mathbb{C}-c}(k_X) &= \{F \in D^b_{\mathbb{C}-c}(k_X) | \forall j\colon \dim(\operatorname{Supp} H^j F) \leqslant -j\} \\ {}^{p_{1/2}} D^{\geqslant 0}_{\mathbb{C}-c}(k_X) &= \{F \in D^b_{\mathbb{C}-c}(k_X) | \forall j\colon \dim(\operatorname{Cosupp}^j(F)) \leqslant j\}, \end{aligned}$$

where $j \in \mathbb{Z}$ and $\operatorname{Cosupp}^j(F) = \operatorname{Supp}(H^{-j}(\mathrm{D}_X F))$, cf. [Dim04, section 5.1], [KS90, section 10.3]. We will write $({}^{1/2}D^{\leqslant 0}_{\mathbb{C}-c}(k_X), {}^{1/2}D^{\geqslant 0}_{\mathbb{C}-c}(k_X))$ for the $p_{1/2}$-t-structure on $D^b_{\mathbb{C}-c}(k_X)$. As $p_{1/2}$ is characterized, amongst all perversity functions, by the property that $p^*_{1/2} = p_{1/2}$, the middle perversity t-structure, by (1.4), has the desirable property of being self-dual, i. e.

$$\mathrm{D}_X({}^{1/2}D^{\leqslant 0}_{\mathbb{C}-c}(k_X)) \subset {}^{1/2}D^{\geqslant 0}_{\mathbb{C}-c}(k_X), \quad \mathrm{D}_X({}^{1/2}D^{\geqslant 0}_{\mathbb{C}-c}(k_X)) \subset {}^{1/2}D^{\leqslant 0}_{\mathbb{C}-c}(k_X).$$

Furthermore, moving to the case $k = \mathbb{C}$, the de Rham functor

$$DR_X\colon D^b_{\mathrm{rh}}(\mathscr{D}_X) \to D^b_{\mathbb{C}-c}(\mathbb{C}_X)$$

is known to be exact with respect to the standard t-structure on $D^b_{\mathrm{rh}}(\mathscr{D}_X)$ and the middle perversity t-structure on $D^b_{\mathbb{C}-c}(\mathbb{C}_X)$. The heart

$${}^{1/2}D^{0}_{\mathbb{C}-c}(\mathbb{C}_X) = {}^{1/2}D^{\leqslant 0}_{\mathbb{C}-c}(\mathbb{C}_X) \cap {}^{1/2}D^{\geqslant 0}_{\mathbb{C}-c}(\mathbb{C}_X)$$

is an abelian category (as is the heart of any classical t-structure), whose objects are called *perverse sheaves.*

Clearly, this construction of a middle perversity t-structure does not work for case of $D^b_{\mathbb{R}-c}(k_X)$, due to the mere fact that $p_{1/2}$ takes non-integer values on odd numbers, conflicting with the definition of a classical perversity function. Actually, in view of (1.4), there is in general no peversity function p that would yield a self-dual (classical) t-structure on $D^b_{\mathbb{R}-c}(k_X)$. However there is a natural self-dual *generalized* t-structure on $D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ ([Kas15]), using the generalized perversity function

$$p_{1/2} \colon \mathbb{Z}_{\geqslant 0} \to \mathbb{R}, \quad n \mapsto -n/2.$$

To be precise, one may define (cf. [DK16a, definition 1.7.3, lemma 1.7.4 and proposition 1.7.5] – with notation as above), for any $c \in \mathbb{R}$,

$$\begin{aligned} {}^{1/2}D^{\leqslant c}_{\mathbb{R}-c}(k_X) &:= \{K \in D^b_{\mathbb{R}-c}(k_X) \,|\, \dim(\operatorname{Supp}(H^j K)) < k, \forall\, k \in \mathbb{Z}_{\geqslant 0}, \forall j \colon j > c + p_{1/2}(k)\} \\ {}^{1/2}D^{\geqslant c}_{\mathbb{R}-c}(k_X) &:= \{K \in D^b_{\mathbb{R}-c}(k_X) \,|\, \mathrm{D}_X\, K \in {}^{1/2}D^{\leqslant -c}_{\mathbb{R}-c}(k_X)\}. \end{aligned}$$

Building upon these ideas, in [DK16a], M. Kashiwara and A. D'Agnolo prove the existence of a self-dual generalized t-structure on $E_{\mathbb{R}-c}(\mathbf{X})$ for a subanalytic bordered space $\mathbf{X}$, again using the (generalized) perversity function $p_{1/2}$.

1.7.1 Enhanced ind-sheaf t-structure

From now on, if the context is clear, we would like to refer to a generalized t-structure simply as a t-structure and, analogously, to a generalized perversity function, i. e. a map $p \colon \mathbb{Z}_{\geqslant 0} \to \mathbb{R}$ such that p and p^* (which is again defined by $p^*(n) = -p(n) - n$) are both decreasing, as a perversity function. Let $\mathbf{X} = (X, \check{X})$ be a subanalytic bordered space, and let j_X denote the corresponding open embedding $X \to \check{X}$.

t-structure on $D^b_{\mathbb{R}-c}(k_{\mathbf{X}})$ Recall from [DK16a] that one may define a self-dual generalized t-structure on $D^b_{\mathbb{R}-c}(k_{\mathbf{X}})$, similar to the case of $D^b_{\mathbb{R}-c}(k_X)$ (cf. [Kas15]), the following way: Let $\mathrm{CS}_{\mathbf{X}}$ denote the closed subanalytic subsets of the bordered space $\mathbf{X}$. Furthermore write d_Z for the dimension of a $Z \in \mathrm{CS}_{\mathbf{X}}$ and set

$$\begin{aligned} \mathrm{CS}^{<k}_{\mathbf{X}} &:= \{Z \in \mathrm{CS}_{\mathbf{X}} \,|\, d_Z < k\} \\ \mathrm{CS}^{\leqslant k}_{\mathbf{X}} &:= \{Z \in \mathrm{CS}_{\mathbf{X}} \,|\, d_Z \leqslant k\}. \end{aligned}$$

Definition/Proposition 1.52 (cf. definition 3.1.5 of [DK16a]). Consider $D^b_{\mathbb{R}-c}(k_{\mathbf{X}})$ as a full subcategory of $D(\mathbf{X})$ via $\iota_{\mathbf{X}}$. For any perversity function p there are given the following two conditions in [DK16a]:

$$\begin{aligned} (Ip^{\leqslant c}_k) \colon \quad & i^{-1}_{(X \setminus Z)_\infty} F \in D^{\leqslant c + p(k)}((X \setminus Z)_\infty) \text{ for some } Z \in \mathrm{CS}^{<k}_{\mathbf{X}} \\ (Ip^{\geqslant c}_k) \colon \quad & i^!_{Z_\infty} F \in D^{\geqslant c + p(k)}(Z_\infty) \text{ for any } Z \in \mathrm{CS}^{\leqslant k}_{\mathbf{X}} \end{aligned} \tag{1.5}$$

These conditions yield the following full subcategories of $D(\mathbf{X})$:

$$\begin{aligned} {}^pD^{\leqslant c}(\mathbf{X}) &:= \{F \in D(\mathbf{X}) | (Ip^{\leqslant c}_k) \text{ holds for any } k \in \mathbb{Z}_{\geqslant 0}\} \\ {}^pD^{\geqslant c}(\mathbf{X}) &:= \{F \in D(\mathbf{X}) | (Ip^{\geqslant c}_k) \text{ holds for any } k \in \mathbb{Z}_{\geqslant 0}\} \end{aligned}$$

One defines

$$
\begin{aligned}
{}^{p}D^{\leqslant c}_{\mathbb{R}-c}(k_{\mathbf{X}}) &:= {}^{p}D^{\leqslant c}(\mathbf{X}) \cap D^{b}_{\mathbb{R}-c}(k_{\mathbf{X}}) \\
{}^{p}D^{\geqslant c}_{\mathbb{R}-c}(k_{\mathbf{X}}) &:= {}^{p}D^{\geqslant c}(\mathbf{X}) \cap D^{b}_{\mathbb{R}-c}(k_{\mathbf{X}}),
\end{aligned}
$$

and this finally gives a (generalized) t-structure on $D^{b}_{\mathbb{R}-c}(k_{\mathbf{X}})$ (note that, for $p = p_{1/2}$ and $\mathbf{X} = X$, this is just the self-dual t-structure on $D^{b}_{\mathbb{R}-c}(k_X)$ mentioned above, cf. [DK16a, lemma 1.7.4]).

Intermediate (not self-dual) t-structure on $E_{\mathbb{R}-c}(\mathbf{X})$ Let K be an object in $E(\mathbf{X})$, p some (generalized) perversity function, $c \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geqslant 0}$, as above.

Definition 1.53 (definition 3.2.1 of [DK16a]). Consider the following conditions (in analogy to (1.5)):

$$
\begin{aligned}
(Ep^{\leqslant c}_{k}):\quad & Ei^{-1}_{(X\setminus Z)_\infty} K \in E^{\leqslant c+p(k)}((X \setminus Z)_\infty) \text{ for some } Z \in \mathrm{CS}^{<k}_{\mathbf{X}} \\
(Ep^{\geqslant c}_{k}):\quad & Ei^{!}_{Z_\infty} K \in E^{\geqslant c+p(k)}(Z_\infty) \text{ for any } Z \in \mathrm{CS}^{\leqslant k}_{\mathbf{X}}
\end{aligned}
\tag{1.6}
$$

The corresponding strictly full subcategories of $E(\mathbf{X})$ are denoted by

$$
\begin{aligned}
{}_{p}E^{\leqslant c}(\mathbf{X}) &:= \{K \in E(\mathbf{X}) | (Ep^{\leqslant c}_{k}) \text{ holds for any } k \in \mathbb{Z}_{\geqslant 0}\}, \\
{}_{p}E^{\geqslant c}(\mathbf{X}) &:= \{K \in E(\mathbf{X}) | (Ep^{\geqslant c}_{k}) \text{ holds for any } k \in \mathbb{Z}_{\geqslant 0}\}.
\end{aligned}
$$

Remark 1.54. $({}_{p}E^{\leqslant c}(\mathbf{X}), {}_{p}E^{\geqslant c}(\mathbf{X}))$ is not a (generalized) t-structure, cf. [DK16a, section 3.2]. A useful note supplementing the conditions (1.6) (cf. [DK16a, Remark 3.2.2 (i)]) is that one has:

$$
\begin{aligned}
Ei^{-1}_{(X\setminus Z)_\infty} K \in E^{\leqslant c}((X \setminus Z)_\infty) &\Longleftrightarrow \pi^{-1} k_{X\setminus Z} \otimes K \in E^{\leqslant c}(\mathbf{X}), \\
Ei^{!}_{Z_\infty} K \in E^{\geqslant c}(Z_\infty) &\Longleftrightarrow R\mathscr{I}hom(\pi^{-1}k_Z, K) \in E^{\geqslant c}(\mathbf{X})
\end{aligned}
$$

Definition/Proposition 1.55 (cf. def. 3.3.11 and prop. 3.3.12 of [DK16a]). For a perversity function p and $c \in \mathbb{R}$ as above,

$$
\begin{aligned}
{}_{p}E^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X}) &:= {}_{p}E^{\leqslant c}(\mathbf{X}) \cap E_{\mathbb{R}-c}(\mathbf{X}), \\
{}_{p}E^{\geqslant c}_{\mathbb{R}-c}(\mathbf{X}) &:= {}_{p}E^{\geqslant c}(\mathbf{X}) \cap E_{\mathbb{R}-c}(\mathbf{X})
\end{aligned}
$$

defines a generalized t-structure on $E_{\mathbb{R}-c}(\mathbf{X})$.

However this t-structure still misses the property that D^{E} interchanges ${}_{p}E^{\leqslant c}_{\mathbb{R}-c}$ with ${}_{p^*}E^{\geqslant -c}_{\mathbb{R}-c}(\mathbf{X})$, which is the basis for obtaining a self-dual t-strucure by setting $p = p_{1/2}$. This issue is solved in [DK16a] by intersecting $({}_{p}E^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X}), {}_{p}E^{\geqslant c}_{\mathbb{R}-c}(\mathbf{X}))_{c\in\mathbb{R}}$ with its dual t-structure, to finally obtain a self-dual t-structure for $p = p_{1/2}$.

Dual intermediate enhanced t-structure Let p be a perversity function and $c \in \mathbb{R}$, as above.

Definition 1.56 (notation 3.4.1 in [DK16a]). One sets

$$
\begin{aligned}
{}_{p}'E^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X}) &:= \{K \in E_{\mathbb{R}-c}(\mathbf{X}) |\, \mathrm{D}^E_{\mathbf{X}}\, K \in {}_{p^*}E^{\geqslant -c}_{\mathbb{R}-c}(\mathbf{X})\\
{}_{p}'E^{\geqslant c}_{\mathbb{R}-c}(\mathbf{X}) &:= \{K \in E_{\mathbb{R}-c}(\mathbf{X}) |\, \mathrm{D}^E_{\mathbf{X}}\, K \in {}_{p^*}E^{\leqslant -c}_{\mathbb{R}-c}(\mathbf{X}).
\end{aligned}
$$

This again defines a t-structure on $E_{\mathbb{R}-c}(\mathbf{X})$, cf. [DK16a, proposition 3.4.2], and $\mathrm{D}^E_{\mathbf{X}}$ interchanges ${}_{p}E^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X})$ and ${}_{p^*}'E^{\geqslant -c}_{\mathbb{R}-c}(\mathbf{X})$ resp. ${}_{p}E^{\geqslant c}_{\mathbb{R}-c}(\mathbf{X})$ and ${}_{p^*}'E^{\leqslant -c}_{\mathbb{R}-c}(\mathbf{X})$, by definition.

Enhanced t-structure

Definition 1.57 (definition 3.5.1 of [DK16a]). For a perversity function p and $c \in \mathbb{R}$ as above, set

$$
\begin{aligned}
{}^{p}E^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X}) &:= {}_{p}E^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X}) \cap {}_{p}'E^{\leqslant c+1/2}_{\mathbb{R}-c}(\mathbf{X})\\
&= \{K \in E_{\mathbb{R}-c}(\mathbf{X}) | K \in {}_{p}E^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X}) \text{ and } \mathrm{D}^E_{\mathbf{X}}\, K \in {}_{p^*}E^{\geqslant -c-1/2}_{\mathbb{R}-c}(\mathbf{X})\},\\
{}^{p}E^{\geqslant c}_{\mathbb{R}-c}(\mathbf{X}) &:= {}_{p}E^{\geqslant c-1/2}_{\mathbb{R}-c}(\mathbf{X}) \cap {}_{p}'E^{\geqslant c}_{\mathbb{R}-c}(\mathbf{X})\\
&= \{K \in E_{\mathbb{R}-c}(\mathbf{X}) | K \in {}_{p}E^{\geqslant c-1/2}_{\mathbb{R}-c}(\mathbf{X}) \text{ and } \mathrm{D}^E_{\mathbf{X}}\, K \in {}_{p^*}E^{\leqslant -c}_{\mathbb{R}-c}(\mathbf{X})\}.
\end{aligned}
$$

In [DK16a] it is shown that one has

Theorem 1.58 (cf. theorem 3.5.2 of [DK16a])**.** *For a bordered space* $\mathbf{X}$ *as above,* $({}^{p}E^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X}), {}^{p}E^{\geqslant c}_{\mathbb{R}-c}(\mathbf{X}))_{c\in\mathbb{R}}$ *is a t-structure on* $E_{\mathbb{R}-c}(\mathbf{X})$*, and* $\mathrm{D}^E_{\mathbf{X}}$ *interchanges* ${}^{p}E^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X})$ *and* ${}^{p^*}E^{\geqslant -c}_{\mathbb{R}-c}(\mathbf{X})$*.*

In particular, setting $p = p_{1/2}$, the resulting generalized t-structure is self-dual.

Definition 1.59 (definition 3.5.8 of [DK16a]). The self-dual generalized t-structure on $E_{\mathbb{R}-c}(\mathbf{X})$ for the perversity $p_{1/2}$ is denoted by

$$
\left({}^{1/2}E^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X}), {}^{1/2}E^{\geqslant c}_{\mathbb{R}-c}(\mathbf{X})\right)_{c\in\mathbb{R}}
$$

and called the *enhanced middle perversity t-structure.*

Defining $p[d]$ by $p[d](n) = p(d+n)$, one can show

Proposition 1.60 (proposition 3.5.6 of [DK16a])**.** *Let* $f\colon \mathbf{X} \to \mathbf{Y}$ *be a morphism of bordered spaces, and* $d \in \mathbb{Z}_{\geqslant 0}$ *such that* $\dim(\mathring{f}^{-1}(y)) \leqslant d$ *for any* $y \in \mathring{\mathbf{Y}}$*. Then, for any* $c \in \mathbb{R}$*, one has*

i) $Ef^{-1}({}^{p[d]}E^{\leqslant c}_{\mathbb{R}-c}(\mathbf{Y})) \subset {}^{p}E^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X})$, *ii)* $Ef^!({}^{p[d]}E^{\geqslant c}_{\mathbb{R}-c}(\mathbf{Y})) \subset {}^pE^{\geqslant c-d}_{\mathbb{R}-c}(\mathbf{X})$,

iii) $E_{\mathbb{R}-c}(\mathbf{Y}) \cap Ef_*({}^pE^{\geqslant c}_{\mathbb{R}-c}(\mathbf{X})) \subset {}^{p[d]}E^{\geqslant c}_{\mathbb{R}-c}(\mathbf{Y})$,

iv) $E_{\mathbb{R}-c}(\mathbf{Y}) \cap Ef_{!!}({}^pE^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X})) \subset {}^{p[d]}E^{\leqslant c+d}_{\mathbb{R}-c}(\mathbf{Y})$.

Proposition 1.61 (proposition 3.5.7 of [DK16a]). *The embedding*

$$e\colon D^b_{\mathbb{R}-c}(k_{\mathbf{X}}) \to E_{\mathbb{R}-c}(\mathbf{X})$$

is exact with respect to the generalized t-structures $({}^pD^{\leqslant c}_{\mathbb{R}-c}(k_{\mathbf{X}}), {}^pD^{\geqslant c}_{\mathbb{R}-c}(k_{\mathbf{X}}))_{c\in\mathbb{R}}$ *resp.* $({}^pE^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X}), {}^pE^{\geqslant c}_{\mathbb{R}-c}(\mathbf{X}))_{c\in\mathbb{R}}$ *described above.*

1.7.2 Riemann–Hilbert correspondence

Let X be a complex manifold and $k = \mathbb{C}$.

Theorem 1.62 (cf. theorem 4.5.1 of [DK16a]). *The enhanced de Rham and (shifted) solution functors* $\mathcal{DR}^E_X$ *resp.* $\mathcal{S}ol^E_X[d^{\mathbb{C}}_X]$ *are exact with respect to the standard t-structure on* $D^b_{\mathrm{h}}(\mathscr{D}_X)$ *and the enhanced middle perversity t-structure on* $E_{\mathbb{R}-c}(\mathbf{X})$*. In particular, one has the following (quasi-)commutative diagrams*

$$\begin{array}{ccc} \mathrm{Mod}_{\mathrm{hol}}(\mathscr{D}_X) & \xrightarrow{\mathcal{DR}^E_X} & {}^{1/2}E^0_{\mathbb{R}-c}(X) \\ \uparrow & & \uparrow \\ \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X) & \xrightarrow{\mathcal{DR}_X} & {}^{1/2}D^0_{\mathbb{R}-c}(\mathbb{C}_X) \end{array} \qquad \begin{array}{ccc} \mathrm{Mod}_{\mathrm{hol}}(\mathscr{D}_X)^{\mathrm{op}} & \xrightarrow{\mathcal{S}ol^E_X} & {}^{1/2}E^{d^{\mathbb{C}}_X}_{\mathbb{R}-c}(X) \\ \uparrow & & \uparrow \\ \mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)^{\mathrm{op}} & \xrightarrow{\mathcal{S}ol_X} & {}^{1/2}D^{d^{\mathbb{C}}_X}_{\mathbb{R}-c}(\mathbb{C}_X), \end{array}$$

where $d^{\mathbb{C}}_X$ *denotes the complex dimension of* X.

1.8 Meromorphic connections

As stated in the introduction of [Sim09] and built on in [Ari10], the setting of irregular meromorphic connections on an algebraic variety provides a natural framework for applying the concept of Katz's middle convolution operation. As announced at the very beginning of this section, we would like to find an enhanced counterpart to this in this thesis. However, to make use of the enhanced Riemann–Hilbert correspondence later on, we would have to pass over to the analytic setting via the analytification functor described in [Ser56]. With that said, let us finally recall some basic facts about meromorphic connections, in the algebraic as well as in the analytic setting.

Algebraic case Let $D \subset X$ be a divisor on a smooth variety X and denote by $U := X \setminus D$ its complement, as well as by $j : U \to X$ the corresponding open embedding. Let us denote by $\mathscr{O}_X(*D) := j_*\mathscr{O}_U$ the sheaf of meromorphic functions on X with poles on D. Recall from [HTT08, section 5.3] that an *(algebraic) meromorphic connection* on X along D is a $\mathscr{D}_X$-module $\mathscr{M}$ that is isomorphic to some coherent $\mathscr{O}_X(*D)$-module as an $\mathscr{O}_X$-module. In particular, for such $\mathscr{M}$, one has that $j^\bigstar\mathscr{M} \simeq j^\dagger\mathscr{M} \simeq j^{-1}\mathscr{M}$ is an integrable connection on U. As in [HTT08], we denote the category of (algebraic) meromorphic connections on X with poles along D by $\mathrm{Conn}(X, D)$ and that of integrable connections on X by $\mathrm{Conn}(X)$. A result that distinguishes the algebraic case from the analytic case in a fundamental way is the following

Lemma 1.63 (lemma 5.3.1 of [HTT08])**.** *The functor j^{-1} establishes an equivalence of categories*

$$\mathrm{Conn}(X, D) \xrightarrow{\sim} \mathrm{Conn}(U)$$

with quasi-inverse j_*.

In particular, any algebraic meromorphic connection is holonomic (this of course is true in the analytic case as well).

Analytic case For the analytic case, we are referring to [HTT08, section 5.2] and [Bjö93, section III.6]. If X is a complex manifold and $D \subset X$ a divisor, we denote again by $\mathscr{O}_X(*D)$ the sheaf of (analytic) meromorphic functions on X with poles on D. An (analytic) meromorphic connection on X along D is a $\mathscr{D}_X$-module $\mathscr{M}$ such that $\mathscr{M}$ is isomorphic as an $\mathscr{O}_X$-module to some coherent $\mathscr{O}_X(*D)$-module, or equivalently, a holonomic $\mathscr{D}_X$-module $\mathscr{M}$ such that $\mathscr{M}|_{X\setminus D}$ is a integrable connection and $\mathscr{M} \simeq \mathscr{M}(*D)$, cf. [Bjö93, section 3.6.6]. We write $\mathrm{Conn}(X, D)$ again for the category of meromorphic connections on X along D.

As mentioned in [HTT08, section 5.3], on a projective smooth variety, the analytifications of algebraic meromorphic connections are analytic meromorphic connections,

Lemma 1.64. *Let X be a projective smooth variety, $D \subset X$ some divisor, as above, and $\mathscr{M} \in \mathrm{Conn}(X, D)$. Then $\mathscr{M}^{\mathrm{an}}$ is an analytic meromorphic connection on X^{an} along D^{an}, i. e. $\mathscr{M}^{\mathrm{an}} \in \mathrm{Conn}(X^{\mathrm{an}}, D^{\mathrm{an}})$.*

As is well known, no analog of lemma 1.63 exists in the analytic case.

Meromorphic connections and enhanced Riemann–Hilbert correspondence Let X be a complex manifold, $D \subset X$ some divisor and $U := X \setminus D$ as above, where we again denote by $j : U \to X$ the corresponding open embedding. The following is an observation used in [DHMS17, section 2].

Lemma 1.65. *In the above setting, let* $\mathscr{M}$ *be some* $\mathscr{D}_X$*-module satisfying* $\mathscr{M} \simeq \mathscr{M}(*D)$ *(e. g.* $\mathscr{M} \in \mathrm{Conn}(X, D)$*). Then*

$$DR_X^E(\mathscr{M}) \simeq R\mathscr{I}hom(\pi^{-1}\mathbb{C}_U, DR_X^E(\mathscr{M})) \quad \textit{resp.} \quad \mathcal{S}ol_X^E(\mathscr{M}) \simeq \pi^{-1}\mathbb{C}_U \otimes \mathcal{S}ol_X^E(\mathscr{M}).$$

Proof. Because the proof is omitted in [DHMS17], let us give a short sketch here for convenience. By [DK16a, lemma 2.4.5], both versions correspond to each other via duality, so it is certainly enough to prove the first one. Now, $\mathscr{O}_X(*D)$ is regular holonomic and $\mathcal{S}ol_X(\mathscr{O}_X(*D)) \simeq \mathbb{C}_U$, so[3] we get

$$\begin{aligned} DR_X^E(\mathscr{M}) \simeq DR_X^E(\mathscr{O}(*D) \overset{D}{\otimes} \mathscr{M}) &\simeq R\mathscr{I}hom(\pi^{-1}\mathcal{S}ol_X(\mathscr{O}(*D)), DR_X^E(\mathscr{M})) \\ &\simeq R\mathscr{I}hom(\pi^{-1}\mathbb{C}_U, DR_X^E(\mathscr{M})) \end{aligned}$$

by [DK16b, theorem 9.1.2 (iv)]. □

[3] To see this aforementioned equation, consider the distinguished triangle

$$R\Gamma_{[D]}(\mathscr{O}_X) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X(*D) \xrightarrow{+1}$$

from [Bjö93, section 2.5] and apply $\mathcal{S}ol_X(\bullet)$, together with

$$R\Gamma_{[D]}(\mathscr{O}_X) \simeq \mathscr{T}hom(\mathbb{C}_D, \mathscr{O}_X)$$

from [KS96, theorem 5.12], which means that

$$\mathcal{S}ol_X(R\Gamma_{[D]}(\mathscr{O}_X) \simeq \mathbb{C}_D.$$

2 Convolution operations

Recall that a group object G in the category of smooth algebraic varieties or complex manifolds is a variety or manifold equipped with the corresponding structure morphisms, i. e. a group operation $\sigma\colon G \times G \to G$, an identity $\{*\} \to G$ (where $\{*\}$ denotes the corresponding one-point terminal object), and inverses given by $\iota\colon G \to G$, subject to the ordinary group axiom diagrams.

The concept of the additive convolutions in [Kat95] is based on the additive group structure of $\mathbb{A}^1$. When proceeding to the enhanced setting, probably the first basic issue coming up is that, though $\mathcal{A} := (\mathbb{A}^1)^{\mathrm{an}}$ of course still is a group object in the category of manifolds, for some meromorphic connection $\mathscr{M}$ on $\mathbb{P}^1$ with a pole at ∞, obviously $\mathcal{S}ol^E_{\mathcal{P}}(\mathscr{M})|_{\mathcal{A}}$ (with $\mathcal{P} := (\mathbb{P}^1)^{\mathrm{an}}$) does not capture enough information to recover $\mathscr{M}$ – on the other hand, $\mathcal{P}$ is not a group object with respect to the appropriate additive structure. Our suggestion here is based on the following observation that emerges quite naturally: The bordered space $\mathbf{A} := (\mathcal{A}, \mathcal{P})$ is a group object in the category of (subanalytic, actually complex in the sense of [KS16, section 4.3]) bordered spaces, with respect to the sum map

$$\sigma\colon \mathbf{A} \times \mathbf{A} \to \mathbf{A}$$

that is induced by the group operation $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(a, b) \mapsto a + b$ on $\mathcal{A}$, with unit morphism $(\{\mathrm{pt}\}, \{\mathrm{pt}\}) \to \mathbf{A}$ induced by $\{\mathrm{pt}\} \to \mathcal{A}$, $\mathrm{pt} \mapsto 0$ and inverses morphism $\iota\colon \mathbf{A} \to \mathbf{A}$ determined by $\mathcal{A} \to \mathcal{A}$, $a \mapsto -a$. In addition, $\mathcal{S}ol^E_X(\mathscr{M})|_{\mathbf{A}}$ does keep the necessary information on $\mathscr{M}$, see lemma 1.65. In the course of this section, we will use this observation, together with the concept of enhanced perverse sheaves established in [DK16a], to define additive !- and $*$-convolutions on $E_{\mathbb{R}-c}(\mathbf{A})$ and, building on these, an enhanced middle convolution operation on ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$ (definition 2.8), in nearly complete analogy to the concepts of [Kat95, section 2.6]. In particular, our enhanced middle convolution operation will rely on a pair (K, L) of objects in ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$ satisfying some property $\mathfrak{P}$, similar to the one of [Kat95, section 2.6] for the classical case. The major part of this section is then devoted to finding some non-trivial pair (i. e. not both objects originating from classical perverse sheaves, cf. proposition 1.61) of objects in ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$ meeting this requirement.

Remark 2.1. Some observations concerning $\mathbf{A}$.

- The group operation $+\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ does indeed induce a morphism

  $$\sigma\colon \mathbf{A} \times \mathbf{A} \to \mathbf{A},$$

  as $\mathcal{P}$ and thus $\mathcal{P} \times \mathcal{P}$ is compact, so in particular, the restriction of the first projection $\mathrm{pr}_1|_{\overline{\Gamma_+}}\colon \overline{\Gamma_+} \to \mathcal{P}$ (cf. definition 1.1) is proper. The same argument of course works for identity and inverses morphism as well.

- For the same reason as above, $\mathrm{pr}_2|_{\overline{\Gamma_+}}\colon \overline{\Gamma_+} \to \mathcal{P}$ (for pr_2 the second projection $\mathcal{P}\times\mathcal{P}\to\mathcal{P}$) is proper as well, which makes σ a semi-proper morphism of bordered spaces (cf. definition 1.2).
- As $\mathbb{P}^1$ is compact and of dimension one, the analytification functor $(\bullet)^{\mathrm{an}}$ from [Ser56] gives an equivalence $D^b_{\mathrm{hol}}(\mathscr{D}_{\mathbb{P}^1}) \simeq D^b_{\mathrm{hol}}(\mathscr{D}_{\mathcal{P}})$ between algebraic and analytic holonomic $\mathscr{D}$-modules, cf. [Mal91, section I.4].
- The homeomorphism

  $$\alpha\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}\times\mathcal{A},\quad (a,b)\mapsto(a,a+b)$$

  induces an isomorphism of bordered spaces $\alpha\colon \mathbf{A}\times\mathbf{A}\xrightarrow{\simeq}\mathbf{A}\times\mathbf{A}$, such that

  $$p_2\circ\alpha=\sigma$$

  for the projection $p_2\colon \mathbf{A}\times\mathbf{A}\to\mathbf{A}$ on the second factor.

Definition 2.2 (Enhanced convolution). Let us now define two kinds of convolution operations $\overset{E}{*}_!, \overset{E}{*}_*\colon E_{\mathbb{R}-c}(\mathbf{A})\times E_{\mathbb{R}-c}(\mathbf{A})\to E_{\mathbb{R}-c}(\mathbf{A})$ (analogous to the definitions in [Kat95]) – let K, L be objects in $E_{\mathbb{R}-c}(\mathbf{A})$, denote by p_1, p_2 the projections

$$\mathbf{A}\xleftarrow{p_1}\mathbf{A}\times\mathbf{A}\xrightarrow{p_2}\mathbf{A}$$

and set[4]

$$\begin{aligned} K\overset{E}{*}_* L &:= E\sigma_*(K\overset{+}{\boxtimes}L) = E\sigma_*(Ep_1^{-1}K\overset{+}{\otimes}Ep_2^{-1}L),\\ K\overset{E}{*}_! L &:= E\sigma_{!!}(K\overset{+}{\boxtimes}L) = E\sigma_{!!}(Ep_1^{-1}K\overset{+}{\otimes}Ep_2^{-1}L).\end{aligned}$$

2.1 Compatibility with "classical" convolution

Recall the embedding $e_{\mathcal{A}}\colon D^b_{\mathbb{R}-c}(k_{\mathbf{A}})\to E_{\mathbb{R}-c}(\mathbf{A})$ (section 1.3.2). In this section, we want to assure that the above convolutions correspond to the "classical" ones, as defined e. g. in [Kat95], via this embedding e.

Lemma 2.3. *$E\sigma_{!!}$ and $E\sigma_*$ commute with e, to be precise:*

$$E\sigma_*\circ e_{\mathcal{A}\times\mathcal{A}}\simeq e_{\mathcal{A}}\circ R\sigma_*,\quad E\sigma_{!!}\circ e_{\mathcal{A}\times\mathcal{A}}\simeq e_{\mathcal{A}}\circ R\sigma_!.$$

Proof. As known from [DK16a, remark 2.4.3], we have $\iota_{\mathbf{A}}\circ R\sigma_*\simeq R\sigma_*\circ\iota_{\mathbf{A}\times\mathbf{A}}$, and $\iota_{\mathbf{A}}\circ R\sigma_!\simeq R\sigma_{!!}\circ\iota_{\mathbf{A}\times\mathbf{A}}$ as σ is semi-proper. Let us show the commutativity

$$E\sigma_{!!}\circ e\simeq e\circ R\sigma_{!!}.$$

[4] Recall that $E\sigma_{!!}$ and $E\sigma_*$ preserve $\mathbb{R}$-constructibility as σ is semi-proper, cf. [DK16a, proposition 3.3.3]

We set $\mathbf{X} := \mathbf{A} \times \mathbf{A}$ and $\sigma_{\mathbb{R}_\infty} := \sigma \times \mathrm{Id}_{\mathbb{R}_\infty}$ as shorthands and compute

$$\begin{aligned}E\sigma_{!!}(e(F)) =& E\sigma_{!!}(Q_{\mathbf{X}}(k^{\mathbf{X}}_{\{t\gg 0\}} \otimes \pi^{-1}F)) \simeq Q_{\mathbf{A}}(R\sigma_{\mathbb{R}_\infty\,!!}(\sigma^{-1}_{\mathbb{R}_\infty}(k^{\mathbf{A}}_{\{t\gg 0\}}) \otimes \pi^{-1}F))\\ \simeq& Q_{\mathbf{A}}(k^{\mathbf{A}}_{\{t\gg 0\}} \otimes R\sigma_{\mathbb{R}_\infty\,!!}\pi^{-1}F) \simeq Q_{\mathbf{A}}(k^{\mathbf{A}}_{\{t\gg 0\}} \otimes \pi^{-1}R\sigma_{!!}F) = e(R\sigma_{!!}(F))\end{aligned}$$

for some $F \in D(\mathbf{X})$, using the obvious fact that $\sigma^{-1}_{\mathbb{R}_\infty} k^{\mathbf{A}}_{\{t\gg 0\}} \simeq k^{\mathbf{X}}_{\{t\gg 0\}}$ (second step), [DK16b, proposition 3.3.13] (third step) and [DK16b, lemma 3.3.14] (fourth step). Finally, one has $\mathrm{D}^E_{\mathbf{A}} \circ E\sigma_{!!} \simeq E\sigma_* \circ \mathrm{D}^E_{\mathbf{X}}$ and $\mathrm{D}^E_{\mathbf{X}} \circ e \simeq e \circ \mathrm{D}_X$, as well as $\mathrm{D}_{\mathcal{A}} \circ R\sigma_! \simeq R\sigma_* \circ \mathrm{D}_X$ and $\mathrm{D}^E_{\mathbf{X}} \circ \mathrm{D}^E_{\mathbf{X}} \simeq \mathrm{Id}$, $\mathrm{D}_X \circ \mathrm{D}_X \simeq \mathrm{Id}$ (and of course all the same on $\mathbf{A}$), cf. [DK16a, Proposition 3.3.3], so we get

$$\begin{aligned}E\sigma_*(e(F)) \simeq& E\sigma_*(e(\mathrm{D}_X \circ \mathrm{D}_X(F))) \simeq E\sigma_*(\mathrm{D}^E_{\mathbf{X}}(e(\mathrm{D}_X F)))\\ \simeq& \mathrm{D}^E_{\mathbf{A}} \circ E\sigma_{!!}(e(\mathrm{D}_X F)) \simeq \mathrm{D}^E_{\mathbf{A}} \circ e(R\sigma_!(\mathrm{D}_X F)) \simeq e(\mathrm{D}_{\mathcal{A}} \circ R\sigma_!(\mathrm{D}_X F))\\ \simeq& e(R\sigma_*(\mathrm{D}_X \circ \mathrm{D}_X(F))) \simeq e(R\sigma_* F).\end{aligned}$$

□

Lemma 2.4. *e_X (with notation as above, i. e. $X = \mathcal{A} \times \mathcal{A}$) interchanges $\boxtimes$ with $\overset{+}{\boxtimes}$.*

Proof. It is enough to show this for e (instead of e_X) and for $\otimes$ and $\overset{+}{\otimes}$ (instead of $\boxtimes$ and $\overset{+}{\boxtimes}$), as e clearly commutes with inverse images. For any $K_1, K_2 \in D(\mathbf{X} \times \mathbb{R}_\infty)$, and $L \in D(\mathbf{X})$, one has (cf. [DK16b, lemma 4.3.1] resp. lemma 1.33)

$$\pi^{-1}L \otimes (K_1 \overset{+}{\otimes} K_2) \simeq (\pi^{-1}L \otimes K_1) \overset{+}{\otimes} K_2.$$

So, in particular we have

$$\begin{aligned}e(F) \overset{+}{\otimes} e(G) =& (k^E_{\mathbf{X}} \otimes \pi^{-1}F) \overset{+}{\otimes} (k^E_{\mathbf{X}} \otimes \pi^{-1}G) \simeq \pi^{-1}F \otimes (k^E_{\mathbf{X}} \overset{+}{\otimes} (k^E_{\mathbf{X}} \otimes \pi^{-1}G))\\ \simeq& \pi^{-1}F \otimes \pi^{-1}G \otimes (k^E_{\mathbf{X}} \overset{+}{\otimes} k^E_{\mathbf{X}}) \simeq \pi^{-1}(F \otimes G) \otimes k^E_{\mathbf{X}} = e(F \otimes G).\end{aligned}$$

□

Let us denote by $K *_! L$ and $K *_* L$ the classical additve convolutions from [Kat95]. With the above observations, we get

Lemma 2.5. *Let $F, G \in D^b_{\mathbb{R}-c}(k_{\mathbf{A}})$, then*

$$e(F *_\square G) \simeq e(F) \overset{E}{*}_\square e(G)$$

for $\square =!, *$.

2.2 Duality interchanges the two types of convolutions

Let $\mathbf{X} := \mathbf{A} \times \mathbf{A}$ be as above. We already used that $D_{\mathcal{A}} \circ R\sigma_! \simeq R\sigma_* \circ D_X$ and $D^E_{\mathbf{A}} \circ E\sigma_{!!} \simeq E\sigma_* \circ D^E_{\mathbf{X}}$. It is a well known fact that, for any reasonably good (cf. [KS90] for a precise description) topological spaces X and Y, and $F \in D^b_{\mathbb{R}-c}(k_X)$, $G \in D^b_{\mathbb{R}-c}(k_Y)$, we have

$$D_{X\times Y}(F \boxtimes G) \simeq D_X(F) \boxtimes D_Y(G).$$

The same was proven for the enhanced setting in [DK16b, Proposition 4.9.21], and the result immediately carries over to the enhanced setting (cf. lemma 1.37). We thus have

Lemma 2.6. *Let $K, L \in E_{\mathbb{R}-c}(\mathbf{A})$, then*

$$\mathrm{D}^E_{\mathbf{A}}(K \overset{E}{*}_! L) \simeq \mathrm{D}^E_{\mathbf{A}} K \overset{E}{*}_* \mathrm{D}^E_{\mathbf{A}} L,$$
$$\mathrm{D}^E_{\mathbf{A}}(K \overset{E}{*}_* L) \simeq D^E_{\mathbf{A}} K \overset{E}{*}_! \mathrm{D}^E_{\mathbf{A}} L.$$

2.3 Enhanced middle convolution

Before we state our definition, we would like to recall the following fact.

Lemma 2.7. *Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of bordered spaces, $K \in D(\mathbf{X})$. Then we have a canonical morphism*

$$Rf_{!!}K \to Rf_*K.$$

In particular, for $K \in E(\mathbf{X})$, this induces a canonical morphism

$$Ef_{!!}K \to Ef_*K.$$

Proof. Let us first note that in case f is an open (bordered) embedding j, we have $j^! \simeq j^{-1}$ and thus we get the morphism in question as

$$Rj_{!!}K \to Rj_*j^{-1}Rj_{!!}K \simeq Rj_*K, \tag{2.1}$$

using the unit of the $j^{-1} \dashv Rj_*$ adjunction and base change[5]. For the general case, by [DK16b, lemma 3.2.5], we may factor f as

$$(X, \check{X}) \underset{\simeq}{\xleftarrow{p_1}} (\Gamma_f, \overline{\Gamma_f}) \xrightarrow{p_2} (Y, \check{Y}),$$

where $\overline{\Gamma_f}$ is the closure of Γ_f in $\check{X} \times \check{Y}$ and the p_i are induced by the projections

$$\check{X} \xleftarrow{q_1} \overline{\Gamma_f} \xrightarrow{q_2} \check{Y},$$

[5] Note that this construction coincides with the canonical morphism obtained by

$$Rj_{!!}K \simeq Rj_{!!}j^{-1}Rj_*K \to Rj_*K.$$

in particular, q_1 is proper by definition of a morphism of bordered spaces, and q_2, likewise by definition, is proper if and only if f is semi-proper. As p_1 is an isomorphism of bordered spaces, we have, by [DK16b, corollary 3.3.11], that

$$Rf_{!!}K \simeq Rp_{2,!!}p_1^{-1}K, \quad Rf_*K \simeq Rp_{2,*}p_1^{-1}K. \tag{2.2}$$

As p_2 is induced by q_2, writing $j_{\mathbf{X}} \colon \mathbf{X} \to \check{X}$ and $j_{\mathbf{Y}} \colon \mathbf{Y} \to \check{Y}$ for the open embeddings as usual, we get

$$Rp_{2,!!}L \simeq j_{\mathbf{Y}}^{-1}Rq_{2,!!}Rj_{\mathbf{X},!!}L, \quad Rp_{2,*}L \simeq j_{\mathbf{Y}}^{-1}Rq_{2,*}Rj_{\mathbf{X},*}L$$

for any $L \in D((\Gamma_f, \overline{\Gamma_f}))$ by [DK16b, lemma 3.3.12]. We have a canonical morphism $Rq_{2,!!} \to Rq_{2,*}$ (cf. [KS01, proposition 5.2.6]) that actually is an isomorphism if q_2 is proper (i.e. if f is semi-proper). Together with (2.1) and (2.2) this gives the desired canonical morphism $Rf_{!!} \to Rf_*$. □

In particular, lemma 2.7 proves that we have a canonical morphism $K \overset{E}{*}_! L \to K \overset{E}{*}_* L$ for some pair (K, L) as above. Furthermore recall that ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$ is a quasi-abelian category.

Definition 2.8 (Enhanced middle convolution). For $K, L \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$, we want to say, in a slight alteration of the definition in [Kat95], that the pair (K, L) has property $\mathfrak{P}_!$ (resp. $\mathfrak{P}_*$, $\mathfrak{P}$), if $K \overset{E}{*}_! L \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$ (resp. $K \overset{E}{*}_* L \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$, resp. both). If (K, L) has property $\mathfrak{P}$, we set

$$K \overset{E}{*}_{\mathrm{mid}} L := \operatorname{Im}\left(K \overset{E}{*}_! L \to K \overset{E}{*}_* L\right) \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$$

and call this the *enhanced middle convolution* of K and L. Clearly there is no reason to prefer the image over the coimage here, so we introduce

$$K \overset{E}{*}_{\mathrm{co-mid}} L := \operatorname{Coim}\left(K \overset{E}{*}_! L \to K \overset{E}{*}_* L\right) \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A}),$$

which we will refer to as the *enhanced co-middle convolution* of a pair (K, L) with property $\mathfrak{P}$.

Immediately by the definitions, duality interchanges middle and co-middle convolution (cf. lemma 2.6), and the two versions coincide if and only if the canonical morphism $K \overset{E}{*}_! L \to K \overset{E}{*}_* L$ is strict.

2.4 A non-trivial pair (K, L) with property $\mathfrak{P}$

As announced in the course of the introduction, our definition of the enhanced middle convolutions raises (at least) two major issues. One point is if there are actually any non-trivial pairs (K, L) – where by this we mean, in the context of lemma 2.5, that the objects of such a pair should not both be in the essential image of e, i. e. not be coming from some ordinary perverse sheaves – that have property $\mathfrak{P}$. A positive answer to this seems to be indispensable in order to justify definition 2.8. The second question emerging naturally is about the existence of some criterion asserting $K \overset{E}{*}_{\mathrm{mid}} L \simeq K \overset{E}{*}_{\mathrm{co-mid}} L$ for a given pair (K, L). For the rest of this section, we would like to address the first of these two matters. Our main result here will be theorem 2.30, stating that the pair $(E^w[1], L_\lambda^E[1])$ has property $\mathfrak{P}$, where $E^w = \mathcal{S}ol_{\mathcal{P}}^E(\mathscr{E}^w)$ is the image under the enhanced solutions functor of the irregular exponential meromorphic connection $\mathscr{E}^w \in \mathrm{Conn}(\mathcal{P}, \{\infty\})$, cf. [DK16b, definition 6.1.1], where w is a local coordinate of the chart $\mathcal{A} \simeq \mathcal{P} \setminus \{\infty\}$, and L_λ^E is the enhanced ind-sheaf associated to a classical Kummer-sheaf for some $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ via the embedding e (cf. section 1.3.2). The rest of this section is devoted to the proof of this theorem.

2.4.1 Kummer-sheaves

Definition 2.9. Let z be the affine coordinate of $\mathbb{A}^1$. For $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ we define a rank one connection

$$\widetilde{\mathscr{K}^\lambda} := \left(\mathscr{O}_{\mathbb{A}^1 \setminus \{0\}}, \mathrm{d} + \lambda \frac{\mathrm{d}\, z}{z}\right)$$

on $U := \mathbb{A}^1 \setminus \{0\} \subset \mathbb{P}^1$, cf. e. g. [Ari10, section 2.3]. For $j \colon U \to \mathbb{P}^1$, let us denote by $\mathcal{K}_a^\lambda := j_* \widetilde{\mathscr{K}^\lambda}$ and $\mathscr{K}^\lambda := (\mathcal{K}_a^\lambda)^{\mathrm{an}} \in \mathrm{Hol}(\mathscr{D}_{\mathcal{P}})$ the extension of $\widetilde{\mathscr{K}^\lambda}$ to a regular (analytic) meromorphic connection on $\mathcal{P}$.

Let us denote by $\mathbb{A}^1 \simeq U := \mathbb{P}^1 \setminus \{\infty\}$ and $\mathbb{A}^1 \simeq V := \mathbb{P}^1 \setminus \{0\}$ the two standard charts of $\mathbb{P}^1$. Let us write w resp. z for the corresponding local coordinates and $j_U \colon U \to \mathbb{P}^1$ resp. $j_V \colon V \to \mathbb{P}^1$ for the associated open embeddings. Then, $\mathcal{K}^\lambda|_U \simeq \mathscr{D}_U / \mathscr{D}_U P_U$ and $\mathcal{K}^\lambda|_V \simeq \mathscr{D}_V / \mathscr{D}_V P_V$, for $P_U = w\partial_w + \lambda$ and $P_V = z\partial_z - \lambda$. In particular, we have

$$(j_U^{\mathrm{an}})^{-1} \mathbb{D}_{\mathcal{P}} \mathcal{K}^\lambda \simeq (\mathbb{D}_U \mathcal{K}_a^\lambda|_U)^{\mathrm{an}} \simeq (\mathscr{D}_U / \mathscr{D}_U P_U^*)^{\mathrm{an}}$$
$$(j_V^{\mathrm{an}})^{-1} \mathbb{D}_{\mathcal{P}} \mathcal{K}^\lambda \simeq (\mathbb{D}_V \mathcal{K}_a^\lambda|_V)^{\mathrm{an}} \simeq (\mathscr{D}_V / \mathscr{D}_V P_V^*)^{\mathrm{an}}$$

for P_U^* resp. P_V^* the transpose operators, cf. [HTT08, pages 70,71], i. e.

$$P_U^* = -\,\partial_w w + \lambda = -w\partial_w - (1 - \lambda),$$
$$P_V^* = -\,\partial_z z - \lambda = -z\partial_z + (1 - \lambda).$$

We naturally get $\mathbb{D}_{\mathbb{P}^1}\mathcal{K}_a^\lambda \simeq \widetilde{\mathcal{K}_a^{1-\lambda}}$ and thus $\mathbb{D}_{\mathcal{P}}\mathcal{K}^\lambda \simeq \mathcal{K}^{1-\lambda}$. Furthermore, there is a canonical isomorphism $\widetilde{\mathcal{K}^{1-\lambda}} \simeq \widetilde{\mathcal{K}^{-\lambda}}$, cf. lemma 4.22, so we have

$$\mathbb{D}_{\mathcal{P}}\mathcal{K}^\lambda \simeq \mathcal{K}^{-\lambda}.$$

Let us denote by $\widetilde{L_\lambda} := \mathcal{S}ol(\widetilde{\mathscr{K}^\lambda}) \in D^b(\mathbb{C}_{\mathcal{A}\setminus\{0\}})$ the local system corresponding to $\widetilde{\mathscr{K}^\lambda}$. Due to the algebraic origin of $\mathscr{K}^\lambda$, we have (cf. [HTT08, theorem 7.1.1])

$$DR(\mathscr{K}^\lambda) \simeq Rj_*DR(\widetilde{\mathscr{K}^\lambda}), \quad L_\lambda := \mathcal{S}ol(\mathscr{K}^\lambda) \simeq j_!\mathcal{S}ol(\widetilde{\mathscr{K}^\lambda}) = j_!\widetilde{L_\lambda}.$$

We will call L_λ the Kummer-sheaf corresponding to $\lambda \in \mathbb{C}\setminus\mathbb{Z}$. Clearly, $\widetilde{L_\lambda}$ is a rank one local system with monodromies $e^{-2\pi i\lambda} \neq 1$ (resp. $e^{2\pi i\lambda}$) around 0 (resp. ∞) – and write

$$L_\lambda^E := \mathcal{S}ol^E(\mathscr{K}^\lambda) = e(L_\lambda).$$

If the context is clear, we will consider L_λ as an object $L^\lambda|_{\mathbf{A}} \in D^b_{\mathbb{R}-c}(\mathbb{C}_{\mathbf{A}}) \subset D^b_{\mathbb{R}-c}(\mathbb{C}_{\mathcal{A}})$. Note that $L_\lambda^E[1] = e(L_\lambda[1]) \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathcal{P})$ by [DK16a, proposition 3.5.7 and theorem 4.5.1.].

2.4.2 The pair (K, L)

On the other hand, consider the irregular meromorphic connection

$$\mathscr{E}^w := \mathscr{E}^w_{\mathcal{P}\setminus\{\infty\}|\mathcal{P}} \in \mathrm{Hol}(\mathscr{D}_{\mathcal{P}})$$

(notation as in [DK16b]), where w is a local coordinate on $\mathcal{A} = \mathcal{P}\setminus\{\infty\}$. We know that

$$\mathcal{S}ol^E_{\mathcal{P}}(\mathscr{E}^w) = \mathbb{C}^E \overset{+}{\otimes} \mathbb{C}_{\{t=-\operatorname{Re}(w)\}} =: E^w$$

by [DK16b, corollary 9.4.12]. By theorem 1.62, $E^w[1] \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathcal{P})$. We will consider $L^E_\lambda[1]$ and $E^w[1]$ as objects of ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$ via $Ej^{-1}_{\mathbf{A}}$. Note that, for the following calculations, we would like to use some general meromorphic function $\varphi(w)$ instead of w as long as this does not complicate things too much, where we will assume, without loss of generality, that φ has one of its poles at $\infty \in \mathcal{P}$. Analogous to our notation above, we will write

$$E^\varphi := \mathcal{S}ol^E_{\mathcal{P}}(\mathscr{E}^\varphi) = \mathbb{C}^E \overset{+}{\otimes} \mathbb{C}_{\{t=-\operatorname{Re}(\varphi)\}}.$$

Let us also point out that choosing one part of a pair (K, L) as above to be a Kummer-sheaf seems natural with regard to the fact that the main use of the classical middle convolution construction as e. g. in [Kat95] and [Ari10] lies in their application within the framework of the corresponding middle convolution algorithms, for which convolutions with Kummer-sheaves resp. their $\mathscr{D}$-module counterparts $\mathcal{K}^\lambda$ are distinctive.

While $L^E_\lambda[1]$ is in the essential image of e by construction, $E^w[1]$ certainly is not, so the pair $(K, L) := (E^w[1], L^E_\lambda[1])$ is non-trivial in the sense we mentioned above. Let us show that $(E^w[1], L^E_\lambda[1])$ has property $\mathfrak{P}$.

2.4.3 Reduction to the case of usual sheaves, part I

Recall the choice for the bordered space $\mathbf{A} = (\mathcal{A}, \mathcal{P})$ from above. Let us denote

$$
\begin{aligned}
K_! &:= E^{\varphi} \overset{E}{*}_! L^E_\lambda = E\sigma_{!!}(E^{\varphi} \overset{+}{\boxtimes} L^E_\lambda), \\
K_* &:= E^{\varphi} \overset{E}{*}_! L^E_\lambda = E\sigma_{*}(E^{\varphi} \overset{+}{\boxtimes} L^E_\lambda)
\end{aligned}
$$

as shorthands. For the following calculations, for a bordered space $\mathbf{X} = (X, \check{X})$, let as before $\iota_{\mathbf{X}} \colon D^b_{\mathbb{R}-c}(\mathbb{C}_{\mathbf{X}}) \to D(\mathbf{X})$ denote the embedding of sheaves to ind-sheaves, $Q_{\mathbf{X}} \colon D(\mathbf{X} \times \mathbb{R}_\infty) \to E(\mathbf{X})$ the quotient functor and $\pi \colon D(\mathbf{X} \times \mathbb{R}_\infty) \to D(\mathbf{X})$ the projection (the restriction of π to the full subcategory of usual sheaves will be denoted by π again). If the situation is clear from the context we will often drop the indices. Finally let us recall the notation L^E, R^E for the left resp. right adjoints to the quotient functor Q. We will write

$$
\mathbf{A} \xleftarrow{p_1} \mathbf{A} \times \mathbf{A} \xrightarrow{p_2} \mathbf{A}
$$

for the projections, and, if there is no risk of confusion, we will use the same labels for the corresponding projections $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ resp. $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$. We will denote the coordinates on $\mathcal{A} \times \mathcal{A}$ by (z_1, z_2). So we may write (with $\varphi(z_1) := \varphi \circ p_1$)

$$
\begin{aligned}
K_! &= E\sigma_{!!}(E^{\varphi} \overset{+}{\boxtimes} L^E_\lambda) \\
&= E\sigma_{!!}\left(Ep_1^{-1}(\mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} \mathbb{C}_{\{t=-\operatorname{Re}(\varphi)\}}) \overset{+}{\otimes} Ep_2^{-1}(\mathbb{C}^E_{\mathbf{A}} \otimes \pi^{-1} L_\lambda) \right) \\
&\simeq E\sigma_{!!}\left(\mathbb{C}^E_{\mathbf{A}\times\mathbf{A}} \overset{+}{\otimes} \big(Ep_1^{-1}\mathbb{C}_{\{t=-\operatorname{Re}(\varphi)\}} \overset{+}{\otimes} Ep_2^{-1}(\mathbb{C}_{\{t=0\}} \otimes \pi^{-1} L_\lambda)\big) \right) \\
&\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} E\sigma_{!!}\big((\mathbb{C}_{\{t=-\operatorname{Re}(\varphi(z_1))\}} \overset{+}{\otimes} \mathbb{C}_{\{t=0\}}) \otimes p_2^{-1}\pi^{-1} L_\lambda \big) \\
&\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} \big(R\sigma_{\mathbb{R},!}(\mathbb{C}_{\{t=-\operatorname{Re}(\varphi(z_1))\}} \otimes p_2^{-1}\pi^{-1} L_\lambda)\big) .
\end{aligned}
$$

where we write $\sigma_{\mathbb{R}}$ to denote the morphisms $\sigma \times \mathrm{Id}_{\mathbb{R}}$ and omit the functors Q and ι, as we will often do if the context is clear. Furthermore, as σ is semi-proper, we get

$$
\begin{aligned}
K_* &= E\sigma_*(E^{\varphi} \overset{+}{\boxtimes} L^E_\lambda) \\
&\simeq D^E_{\mathbf{A}} E\sigma_{!!} \mathrm{D}^E_{\mathbf{A}\times\mathbf{A}} \left(\mathbb{C}^E_{\mathbf{A}\times\mathbf{A}} \overset{+}{\otimes} \big(Ep_1^{-1}\mathbb{C}_{\{t=-\operatorname{Re}(\varphi)\}} \overset{+}{\otimes} Ep_2^{-1}(\mathbb{C}_{\{t=0\}} \otimes \pi^{-1} L_\lambda)\big) \right) \\
&\simeq \mathrm{D}^E_{\mathbf{A}}\, E\sigma_{!!} \left(\mathbb{C}^E_{\mathbf{A}\times\mathbf{A}} \overset{+}{\otimes} \mathrm{D}_{\mathcal{A}\times\mathcal{A}\times\mathbb{R}}\, a^{-1} \big(p_1^{-1}\mathbb{C}_{\{t=-\operatorname{Re}(\varphi)\}} \overset{+}{\otimes} p_2^{-1}(\mathbb{C}_{\{t=0\}} \otimes \pi^{-1} L_\lambda)\big) \right) \\
&\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} \left(\mathrm{D}_{\mathcal{A}\times\mathbb{R}}\, a^{-1} R\sigma_{\mathbb{R},!}\, \mathrm{D}_{\mathcal{A}\times\mathcal{A}\times\mathbb{R}}\, a^{-1} \big(p_1^{-1}\mathbb{C}_{\{t=-\operatorname{Re}(\varphi)\}} \overset{+}{\otimes} p_2^{-1}(\mathbb{C}_{\{t=0\}} \otimes \pi^{-1} L_\lambda)\big) \right) \\
&\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} \big(R\sigma_{\mathbb{R},*}(\mathbb{C}_{\{t=-\operatorname{Re}(\varphi(z_1))\}} \otimes p_2^{-1}\pi^{-1} L_\lambda)\big) .
\end{aligned}
$$

Remark 2.10. We could without any further changes have replaced any appearance of $\mathbb{C}_{\{t=0\}}$ with $\mathbb{C}_{\{t\geqslant 0\}}$ in the above lines, as for any $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_{\mathbf{X}})$ we have

$$\mathbb{C}^E \overset{+}{\otimes} (\mathbb{C}_{\{t=0\}} \otimes \pi^{-1}F) \simeq (\mathbb{C}^E \overset{+}{\otimes} \mathbb{C}_{\{t=0\}}) \otimes \pi^{-1}F \simeq \mathbb{C}^E \otimes \pi^{-1}F = \\ = (\mathbb{C}^E \overset{+}{\otimes} \mathbb{C}_{\{t\geqslant 0\}}) \otimes \pi^{-1}F \simeq \mathbb{C}^E \overset{+}{\otimes} (\mathbb{C}_{\{t\geqslant 0\}} \otimes \pi^{-1}F).$$

Remark 2.11. In this context, let us also verify, for the sake of completeness, that

$$\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} \mathbb{C}_{\{t=-\operatorname{Re}(\varphi)\}} \simeq \mathbb{C}_{\{t\geqslant -\operatorname{Re}(\varphi)\}},$$

as one would expect. Consider the obvious morphism $\mathbb{C}_{\{t\geqslant -\operatorname{Re}(\varphi)\}} \to \mathbb{C}_{\{t=-\operatorname{Re}(\varphi)\}}$. We want to convince ourselves that this induces, applying $\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} (\bullet)$, a canonical morphism

$$\mathbb{C}_{\{t\geqslant -\operatorname{Re}(\varphi)\}} \to \mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} \mathbb{C}_{\{t=-\operatorname{Re}(\varphi)\}},$$

which we may then easily prove to be an isomorphism by checking stalks (and using lemma 1.21). To do so we would like to verify that the canonical morphism

$$\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} \mathbb{C}_{\{t\geqslant -\operatorname{Re}(\varphi)\}} \to \mathbb{C}_{\{t\geqslant -\operatorname{Re}(\varphi)\}}$$

induced by $\mathbb{C}_{\{t\geqslant 0\}} \to \mathbb{C}_{\{t=0\}}$ (together with the fact that $\mathbb{C}_{\{t=0\}} \overset{+}{\otimes} K \simeq K$ for any K) is an isomorphism. Note that the latter morphism fits into the standard distinguished triangle

$$\mathbb{C}_{\{t>0\}} \longrightarrow \mathbb{C}_{\{t\geqslant 0\}} \longrightarrow \mathbb{C}_{\{t=0\}} \xrightarrow{+1}$$

so we may equivalently prove that $\mathbb{C}_{\{t>0\}} \overset{+}{\otimes} \mathbb{C}_{\{t\geqslant -\operatorname{Re}(\varphi)\}} \simeq 0$. We can check this on stalks again: For some $y \in \mathcal{A}$ at which φ is defined, let $i_y \colon \{y\} \to \mathcal{A}$ and $i_{y,\mathbb{R}} \colon \{y\} \times \mathbb{R} \to \mathcal{A} \times \mathbb{R}$ denote the canonical closed embeddings. Then, obviously

$$i_{y,\mathbb{R}}^{-1}(\mathbb{C}_{\{t>0\}} \overset{+}{\otimes} \mathbb{C}_{\{t\geqslant -\operatorname{Re}(\varphi)\}}) \simeq \mathbb{C}_{\{t>0\}} \overset{+}{\otimes} \mathbb{C}_{\{t\geqslant -\operatorname{Re}(\varphi(y))\}}.$$

Now, setting $a := -\operatorname{Re}(\varphi(y)) \in \mathbb{R}$ and $\mu_a \colon \mathbb{R} \to \mathbb{R},\ t \mapsto t+a$ the translation map (where, as usual, we label any map $X \times \mathbb{R} \to X \times \mathbb{R}$ that is induced by μ_a with μ_a again), we get that

$$\mathbb{C}_{\{t\geqslant a\}} \simeq R\mu_{a,*}\mathbb{C}_{\{t\geqslant 0\}} \simeq \mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} \mathbb{C}_{\{t=a\}},$$

cf. [DK16b, section 4.6], and so

$$\mathbb{C}_{\{t>0\}} \overset{+}{\otimes} \mathbb{C}_{\{t\geqslant a\}} \simeq \mathbb{C}_{\{t>0\}} \overset{+}{\otimes} (\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} \mathbb{C}_{\{t=a\}}) \simeq (\mathbb{C}_{\{t>0\}} \overset{+}{\otimes} \mathbb{C}_{\{t\geqslant 0\}}) \overset{+}{\otimes} \mathbb{C}_{\{t=a\}} \simeq 0,$$

cf. lemma 1.21 resp. [DK16b, section 4.6].

Finally, as for any subanalytic space $\mathbf{X}$ and any $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_{\mathbf{X}\times\mathbb{R}_\infty})$, we have

$$\mathrm{D}^E_{\mathbf{X}}(\mathbb{C}^E_{\mathbf{X}} \otimes Q\iota(F)) \simeq \mathbb{C}^E_{\mathbf{X}} \otimes Q\iota(\mathrm{D}_{X\times\mathbb{R}}\, a^{-1}F)$$

(cf. [DK16a]), as we already used above, the dual cases $\mathrm{D}^E_{\mathbf{A}}\, K_!$ and $\mathrm{D}^E_{\mathbf{A}}\, K_*$ can be handled the same way.

2.4.4 Enhanced perversity conditions

By definition, we have $E^\varphi[1] \overset{E}{*_!} L^E_\lambda[1] = K_![2]$ and $E^\varphi[1] \overset{E}{*_*} L^E_\lambda[1] = K_*[2]$. For proving $K_![2], K_*[2] \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$, in view of [DK16a, definition 3.5.1] (cf. section 1.7.1 for a short summary), what we have to show is

$$K_![2], K_*[2], \mathrm{D}^E(K_![2]), D^E(K_*[2]) \in {}_{1/2}E^{[-1/2,0]}_{\mathbb{R}-c}(\mathbf{A}).$$

Filling in the definitions, this amounts to showing that, for

$$G \in \{K_![2], K_*[2], \mathrm{D}^E(K_![2]), \mathrm{D}^E(K_*[2])\},$$

we have (notation as in [DK16a, section 3] resp. section 1.7.1)

i) for any $k \in \mathbb{Z}_{\geqslant 0}$, there exists a $Z \in \mathrm{CS}^{<k}_{\mathbf{A}}$ such that

$$Ei^{-1}_{(\mathcal{A}\setminus Z)_\infty} G \in E^{\leqslant -k/2}((\mathcal{A}\setminus Z)_\infty),$$

ii) for any $k \in \mathbb{Z}_{\geqslant 0}$, for any $Z \in \mathrm{CS}^{\leqslant k}_{\mathbf{A}}$ one has

$$Ei^!_{Z_\infty} G \in E^{\geqslant -(k+1)/2}(Z_\infty).$$

Remark 2.12 (Summary of what to show). Item i) is trivially true for $k \geqslant 3$, as we may chose $Z = \mathcal{A}$ in this case. Considering $k = 0$, i) implies that we must have

$$G \in E^{\leqslant 0}_{\mathbb{R}-c}(\mathbf{A}).$$

Analogously, ii), applied for $k \geqslant 2$, implies that we must have

$$G \in E^{\geqslant -1}_{\mathbb{R}-c}(\mathbf{A}).$$

If we manage to show these two, as the standard t-structure on $E_{\mathbb{R}-c}(\mathbf{A})$ is 1-indexed, what remains to check is that there is a $Z \in \mathrm{CS}^{<1}(\mathbf{A})$ such that

$$Ei^{-1}_{(\mathcal{A}\setminus Z)_\infty} G \in E^{\leqslant -1}((\mathcal{A}\setminus Z)_\infty)$$

and that, for all $Z \in \mathrm{CS}^{\leqslant 0}(\mathbf{A})$, we have

$$Ei^!_{Z_\infty} G \in E^{\geqslant 0}(Z_\infty),$$

cf. [DK16a, rem. 3.2.2 and prop. 2.7.3 (iv,v)].

2.4.5 Reduction to the case of usual sheaves, part II

Now, let us consider how cohomology of enhanced ind-sheaves of the type $\mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} F$ for some $F \in D^b_{\mathbb{R}-c}(C_{\mathbf{A}\times\mathbb{R}_\infty})$ can be computed from the cohomology of F (or rather $\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} F$). To do so, recall the definition

$$E^{\leqslant n}(\mathbf{A}) := \{K \in E(\mathbf{A}) | L^E K \in D^{\leqslant n}(\mathbf{A} \times \mathbb{R}_\infty)\}$$
$$E^{\geqslant n}(\mathbf{A}) := \{K \in E(\mathbf{A}) | L^E K \in D^{\geqslant n}(\mathbf{A} \times \mathbb{R}_\infty)\},$$

cf. [DK16a, definition 2.6.1]. Furthermore,

$$D^{\leqslant n}(\mathbf{A} \times \mathbb{R}_\infty) = \{K \in D(\mathbf{A} \times \mathbb{R}_\infty) | Rj_{\mathbf{A}\times\mathbb{R}_\infty,!!} K \in D^{\leqslant n}(\mathcal{P} \times \overline{\mathbb{R}})\},$$
$$D^{\geqslant n}(\mathbf{A} \times \mathbb{R}_\infty) = \{K \in D(\mathbf{A} \times \mathbb{R}_\infty) | Rj_{\mathbf{A}\times\mathbb{R}_\infty,!!} K \in D^{\geqslant n}(\mathcal{P} \times \overline{\mathbb{R}})\},$$

for $j_{\mathbf{A}\times\mathbb{R}_\infty} \colon \mathbf{A} \times \mathbb{R}_\infty \to \mathcal{P} \times \overline{\mathbb{R}}$ the canonical morphism of bordered spaces, as usual, cf. [DK16b, section 3.4]. In particular, as $L^E \circ Ej_{\mathbf{A},!!} = Rj_{\mathbf{A}\times\mathbb{R}_\infty,!!} \circ L^E$, this implies

$$E^{\leqslant n}(\mathbf{A}) = \{K \in E(\mathbf{A}) | Ej_{\mathbf{A},!!} K \in E^{\leqslant n}(\mathcal{P})\}$$
$$E^{\geqslant n}(\mathbf{A}) = \{K \in E(\mathbf{A}) | Ej_{\mathbf{A},!!} K \in E^{\geqslant n}(\mathcal{P})\},$$

so that we may apply the results of [DK16b, section 4.6]. In particular, we are going to use that, for some $K \in E(\mathcal{P})$ (resp. $K \in E(\mathbf{A})$, as $Ej_{\mathbf{A}}^{-1}$ is exact, compare the above characterization), one has

$$u(K) = Q(u(L^E K)) \quad \text{for } u = H^n, \tau^{\leqslant n}, \tau^{\geqslant n},$$

cf. [DK16b, section 4.6]. Recalling [DK16b, notation 4.4.5] resp. [DK16a, section 2.6], if, for some bordered subanalytic space $\mathbf{X}$, we write $K \in E(\mathbf{X})$ as $K = Q(G)$ for some $G \in \mathrm{D}(\mathbf{X} \times \mathbb{R}_\infty)$, then

$$L^E K = L^E Q(G) = (\mathbb{C}_{\{t\geqslant 0\}} \oplus \mathbb{C}_{\{t\leqslant 0\}}) \overset{+}{\otimes} G.$$

We know that $\mathbb{C}_{\{t\gg 0\}} \overset{+}{\otimes} \mathbb{C}_{\{t\geqslant 0\}} \simeq \mathbb{C}_{\{t\gg 0\}}$ and $\mathbb{C}_{\{t\gg 0\}} \overset{+}{\otimes} \mathbb{C}_{\{t\leqslant 0\}} \simeq 0$, so

$$\begin{aligned} H^n(\mathbb{C}^E_{\mathbf{M}} \overset{+}{\otimes} Q(F)) \simeq & Q(H^n(L^E(\mathbb{C}^E \overset{+}{\otimes} Q(F)))) \\ \simeq & Q(H^n(\mathbb{C}_{\{t\gg 0\}} \overset{+}{\otimes} F)) \simeq \mathbb{C}^E \overset{+}{\otimes} Q(H^n(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} F)), \end{aligned}$$

cf. [DK16b, proposition 4.7.2]. As $D^b_{\mathbb{R}-c}(\mathbb{C}_{\mathbf{X}\times\mathbb{R}_\infty})$ is closed under $\overset{+}{\otimes}$, this reduces computation of cohomologies to the case of usual sheaves for the case that $F \in D^b_{\mathbb{R}-c}(\mathbb{C}_{\mathbf{X}\times\mathbb{R}_\infty})$ as it occurs in our example.

2.4.6 Characteristic Varieties and μ-stratifications

With regard to the subsequent calculations, let us recall some more tools and definitions from [KS90] and [Dim04]. Let X be a n-dimensional (real) manifold, $M \subset X$ a closed submanifold. Then, the normal resp. conormal bundle $T_M X$ resp. $T_M^* X$ to M in X are defined by the short exact sequences of vector bundles on M (cf. [KS90, section A.2])

$$\begin{aligned} 0 \longrightarrow TM \longrightarrow M \times_X TX \longrightarrow T_M X \longrightarrow 0, \\ 0 \longrightarrow T_M^* X \longrightarrow M \times_X T^* X \longrightarrow T^* M \longrightarrow 0. \end{aligned}$$

Keeping the notation of [KS90], one defines the *normal deformation* of M in X (cf. [KS90, section 4.1]), to be a manifold $\widetilde{X_M}$ with maps $p\colon \widetilde{X_M} \to X$ and $t\colon \widetilde{X_M} \to \mathbb{R}$, characterized by

$$\begin{aligned} p^{-1}(X \setminus M) \simeq& (X \setminus M) \times (\mathbb{R} \setminus \{0\}), \\ t^{-1}(\mathbb{R} \setminus \{0\}) \simeq& X \times (\mathbb{R} \setminus \{0\}), \\ t^{-1}(\{0\}) \simeq& T_M X. \end{aligned}$$

Definition 2.13 (normal cones, definition 4.1.1 of [KS90]). Consider the open subset $\Omega := t^{-1}(\mathbb{R}_{>0}) \subset \widetilde{X_M}$. Let $\tilde{p}\colon \Omega \to X$ denote the restriction of p to Ω and let s be the isomorphism $T_M X \overset{\simeq}{\to} t^{-1}(\{0\})$.

i) For a subset $S \subset X$, define the *normal cone to S along M* by

$$C_M(S) := s^{-1}\left(t^{-1}(\{0\}) \cap \overline{\tilde{p}^{-1}(S)}\right).$$

ii) For two subsets $S_1, S_2 \subset X$, define the *normal cone* $C(S_1, S_2)$ as

$$C(S_1, S_2) := C_{\Delta_X}(S_1 \times S_2),$$

for $\Delta_X \subset X \times X$ the diagonal subset.

Now, for a given manifold X, consider its cotangent bundle T^*X and two subsets $A, B \subset T^*X$. If the context is clear, we will identify X with its image Δ_X under the diagonal embedding, again following the notation of [KS90, section 6.2]. Note that the cone $C(A, B^a)$ (where B^a is the image of B under the antipodal map for bundles) is a subset of $T_{T_X^*(X\times X)}T^*(X \times X) \simeq T^*T_X^*(X \times X)$ (cf. [KS90, p. 259] for the isomorphism). Denoting with q the induced projection $q\colon T_X^*(X \times X) \to X$, we can consider T^*X as a subset of $T_{T_X^*(X\times X)}T^*(X \times X)$ via

$$T^*X \simeq X \times_X T^*X \overset{\iota \times \mathrm{Id}_{T^*X}}{\longrightarrow} T_X^*(X \times X) \times_X T_X^* \overset{{}^t q'}{\longrightarrow} T^*T_X^*(X \times X) \simeq T_{T_X^*(X\times X)}T^*(X \times X),$$

where $\iota\colon X \to T_X^*(X \times X)$ is the zero section.

Definition 2.14 (cf. definition 6.2.3 resp. proposition 6.2.4 of [KS90]). For conic subsets $A, B \subset T^*X$ as above, one defines

$$A \widehat{+} B := q_\pi {}^t q'^{-1}(C(A, B^a)) = T^*X \cap C(A, B^a).$$

Let us state some properties of $\widehat{+}$ that are obvious from the definitions but helpful for the forthcoming calculations.

Lemma 2.15. *Let $A, B \subset T^*X$ be two conic subsets.*

i) If $A' \subset T^*X$ is a conic subset such that $A \subset A'$, then $A \widehat{+} B \subset A' \widehat{+} B$ (and of course the same for $B \subset B'$).

ii) If $A = \coprod_{i=1}^n A_i$ is a disjoint union of conic subsets $A_i \subset T^*X$, then

$$A \widehat{+} B = \bigcup_{i=1}^n (A_i \widehat{+} B)$$

(and of course the same for $B = \coprod_{i=1}^n B_i$).

Definition 2.16 (μ-stratification, cf. definition 8.3.19 of [KS90]). Let $X = \coprod_{a \in \mathfrak{A}} X_a$ be a subanalytic stratification of X.

i) Consider two submanifolds M, N of X. The μ-condition for the pair (M, N) is

$$(T^*_M X \widehat{+} T^*_N X) \cap \pi_X^{-1}(N) \subset T^*_N X, \tag{μ}$$

where $\pi_X \colon T^*X \to X$ is the projection.

ii) The stratification $(X_a)_{a \in \mathfrak{A}}$ is called a *μ-stratification* if for all pairs $(a, b) \in \mathfrak{A} \times \mathfrak{A}$ such that $X_b \subset \overline{X_a} \setminus X_a$, the pair (X_a, X_b) satisfies the condition (μ).

[KS90, lemma 6.2.1 resp. proposition 6.2.4] give a useful criterion for verifying the μ-condition for a given stratification.

Proposition 2.17 (cf. Proposition 6.2.4 (iii)(a) of [KS90]). *Let (x, ξ) be a system of local coordinates on T^*X (where (x) is a set of local coordinates on X) and consider two conic subsets $A, B \subset T^*X$. Then (x_0, ξ_0) is in $A \widehat{+} B$ if and only if there is a sequence $\{(x_n, \xi_n), (y_n, \eta_n)\}$ in $A \times B^a$ such that*

$$\left.\begin{aligned} & x_n \underset{n\to\infty}{\longrightarrow} x_0, \\ & y_n \underset{n\to\infty}{\longrightarrow} x_0, \\ & \eta_n - \xi_n \underset{n\to\infty}{\longrightarrow} \xi_0, \\ & |x_n - y_n| \cdot |\xi_n| \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}\right\} \tag{2.3}$$

Note that, for the third line of (2.3), we formally identify $T^*_{x_n}X$ with $T^*_{y_n}X$ resp. $T^*_{x_0}X$. To be more precise, we consider $(x_n, y_n, \xi_n, \eta_n)$ as a sequence in $T^*(X \times X)$ with local coordinates (x, y, ξ, η) induced by the given local coordinates (x, ξ) resp. (y, η) on both factors T^*X of $T^*(X \times X) \simeq T^*X \times T^*X$. Then, we apply the change of coordinates $(x, y) \mapsto (x-y, y)$ on $X \times X$ and consider the corresponding new coordinates (x', y', ξ', η') on $T^*(X \times X)$. Let $(x'_n, y'_n, \xi'_n, \eta'_n)$ be the above sequence in these new coordinates (i. e. $x'_n = x_n - y_n$ etc.) – then, criterion (2.3) may be formulated as

$$\begin{aligned} (x'_n, y'_n, \xi'_n, \eta'_n) &\underset{n\to\infty}{\longrightarrow} (0, x_0, \tilde{\xi}', \xi_0) \text{ for some } \tilde{\xi}', \\ |x'_n| \cdot |\xi'_n| &\underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

cf. [KS90, proof of proposition 6.2.4].

Example 2.18. Let $M, N \subset X$ be submanifolds such that $M \subset X$ is open with $\overline{M} = X$ and $N \subset X \setminus M$. We would like to show that (M, N) satisfies the condition (μ). So let us use proposition 2.17 on the conic subsets $A = T^*_M X$ and $B = T^*_N X$ of T^*X. By assumption, $T^*_M X = M \times_X T^*_X X$, so $\xi_n = 0$ for all n in any sequence $\{(x_n, \xi_n, y_n, \eta_n)\}$ as in proposition 2.17. In particular, the fourth property of (2.3) is trivially satisfied. The first two lines of (2.3) are realizable if and only if $x_0 \in \overline{N}$. The third line then finally states that such (x_0, ξ_0) with $x_0 \in \overline{N}$ is in $A \widehat{+} B$ if and only if for some appropriate sequence $y_n \in N$ with $y_n \underset{n\to\infty}{\longrightarrow} x_0$, there is a sequence $\eta_n \in ((T^*_N X)^a)_{y_n}$ such that $\eta_n \underset{n\to\infty}{\longrightarrow} \xi_0$. Let (y_n, η_n) be the subset of these sequences. To verify condition (μ), we may assume $x_0 \in N$ and have to show that, for such x_0, for any (y_n, η_n) as above with $y_n \underset{n\to\infty}{\longrightarrow} x_0 \in N$, we have $\xi_0 = \lim_{n\to\infty} \eta_n$ is in $T^*_N X$. By choosing a chart $U \ni x_0$ around x_0 of X such that N is closed in U and choosing compatible trivializations of T^*X and $T^*_N X$ within U, this is clear.

Now consider a complex $F \in D^b(\mathbb{C}_X)$ and recall the definition of the *characteristic variety* $\mathrm{CV}(F)$ from [Dim04, definition 4.3.1] resp. [KS90, section 5.1] (where it is also called *singular support of* F and denoted $\mathrm{SS}(F)$), as well as the notation of [KS90, section A.2], associating to a morphism $f\colon Y \to X$ of manifolds the maps

$$T^*Y \xleftarrow{{}^t f'} Y \times_X T^*X \xrightarrow{f_\pi} T^*X.$$

A morphism $f\colon Y \to X$ is called *non-characteristic for* F if

$$f_\pi^{-1}(\mathrm{CV}(F)) \cap T^*_Y X \subset Y \times_X T^*_X X,$$

(where $T^*_Y X$ is the kernel of ${}^t f'$ and f_π is the canonical projection from above), cf. [Dim04, definition 4.3.4] resp. [KS90, definition 5.4.12]. What we are going to use extensively in the next section is the following consequence of a morphism being non-characteristic:

Proposition 2.19 (cf. proposition 4.3.6 of [Dim04] resp. proposition 5.4.13 of [KS90])**.** *Let F be an object of $D^b(\mathbb{C}_X)$ and $f\colon Y \to X$ be non-characteristic for F. Then, the natural morphism $f^{-1}(F) \otimes \omega_{Y/X} \to f^!(F)$ is an isomorphism.*

More precisely, we want to use the following

Corollary 2.20 (cf. corollary 4.3.7 of [Dim04])**.** *Let X be a (real, orientable) manifold and $Y \subset X$ a locally closed (real, orientable) submanifold of X of codimension r. Denote by $i\colon Y \to X$ the embedding. Let $F \in D^b(\mathbb{C}_X)$ be cohomologically constructible with respect to a μ-stratification $X = \coprod_{a\in\mathfrak{A}} X_a$ – which we will denote by $\mathcal{S}$ – and assume Y is transversal to this stratification. Then*

$$i^!(F) \simeq i^{-1}(F)[-r].$$

Proof. As the formulation of the corollary differs – though very slightly – from the one in [Dim04, corollary 4.3.7], let us give a sketch of a proof. With exactly the same reasoning as in [Dim04, proposition 3.2.11], we have $\omega_{Y/X} = \mathbb{C}_X[-r]$, so the only thing that remains to show is that i is non-characteristic for F. By [KS90, proposition 8.4.1] we have

$$\mathrm{CV}(F) \subset \coprod_{a\in\mathfrak{A}} T^*_{X_a}X.$$

Now, Y being transversal to $\mathcal{S}$ by definition means nothing but $T_xY + T_xX_a = T_xX$ for all a and all $x \in Y \cap X_a$. We may of course equivalently consider dual spaces, so we find, denoting with $i\colon Y \to X$ and $i_a\colon X_a \to X$ the embeddings and with $i'\colon TY \to TX$ resp. ${}^ti'\colon T^*X \to T^*Y$ the corresponding tangent space resp. cotangent space map (and the same for i_a), that we have $\ker({}^ti'_x) \cap \ker({}^ti'_{a,x}) = \{(x,0)\}$ for any $x \in X$, i. e.

$$(Y \cap X_a) \times_X (T^*_Y X \cap T^*_{X_a}X) = (Y \cap X_a) \times_X T^*_X X$$

and so (where again we consider $T^*_{X_a}X$ and $T^*_Y X$ as subsets of T^*X)

$$\begin{aligned} i_\pi^{-1}(\mathrm{CV}(F)) \cap T^*_Y X \subset \coprod_{a\in\mathfrak{A}} i_\pi^{-1}(T^*_{X_a}X) \cap T^*_Y X = \\ = \coprod_{a\in\mathfrak{A}} (Y \cap X_a) \times_X (T^*_{X_a}X \cap T^*_Y X) = Y \times_X T^*_X X, \end{aligned}$$

in other words, i is non-characteristic for F. □

2.4.7 Duality

Recall the exponential meromorphic connection $\mathscr{E}^\varphi$ and our notation

$$E^\varphi := \mathcal{S}ol^E(\mathscr{E}^\varphi) = \mathbb{C}^E \overset{+}{\otimes} \mathbb{C}_{\{t=-\operatorname{Re}(\varphi)\}}$$

from above. Let $U \subset \mathcal{A} \subset \mathcal{P}$ denote the open subset on which φ is defined. Again we will denote by w resp. (z_1, z_2) the coordinates on $\mathcal{A}$ resp. $\mathcal{A} \times \mathcal{A}$ and write $\varphi(z_1)$ for the meromorphic function $\varphi \circ p_1 \colon (z_1, z_2) \mapsto \varphi(z_1)$ on $\mathcal{A} \times \mathcal{A}$. Consider the enhanced ind-sheaf (notation from section 2.4.3)

$$\left.\begin{aligned} K_! = E^{\varphi} \overset{E}{*_!} L^E_\lambda &\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota\big(R\sigma_{\mathbb{R},!}(\mathbb{C}_{\{t=-\operatorname{Re}(\varphi(z_1))\}} \otimes p_2^{-1}\pi^{-1}L_\lambda)\big) \\ &\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} \Big(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} Q\iota\big(R\sigma_{\mathbb{R},!}(\mathbb{C}_{\{t=-\operatorname{Re}(\varphi(z_1))\}} \otimes p_2^{-1}\pi^{-1}L_\lambda)\big)\Big) \\ &\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota\big(R\sigma_{\mathbb{R},!}(\mathbb{C}_{\{t\geqslant -\operatorname{Re}(\varphi(z_1))\}} \otimes p_2^{-1}\pi^{-1}L_\lambda)\big) \end{aligned}\right\} \tag{2.4}$$

(see calculations in section 2.4.3 and 2.4.5). Now let us compute the dual $\mathrm{D}^E_{\mathbf{A}}(K_![2])$. As remarked in section 2.4.5, this can be done on the level usual sheaves, by – using the first line of the equivalent formulations (2.4) –

$$\begin{aligned} \mathrm{D}^E_{\mathbf{A}}(K_![2]) &\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota\big(\mathrm{D}_{\mathcal{A}\times\mathbb{R}}\, R\sigma_{\mathbb{R},!}(\mathbb{C}_{\{t=\operatorname{Re}(\varphi(z_1))\}} \otimes p_2^{-1}\pi^{-1}L_\lambda)[2]\big) \\ &\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota\big(R\sigma_{\mathbb{R},*}R\mathscr{H}om(\mathbb{C}_{\{t=\operatorname{Re}(\varphi(z_1))\}}, \mathrm{D}_{\mathcal{A}\times\mathcal{A}\times\mathbb{R}}(p_2^{-1}\pi^{-1}L_\lambda[2]))\big) \end{aligned}$$

Here, note that we may – in a slight misuse of notation – write $p_2^{-1}\pi^{-1}L_\lambda \simeq \pi^{-1}p_2^{-1}L_\lambda$, referring to $\pi\colon \mathcal{A}\times\mathbb{R} \to \mathcal{A}$ and $p_2\colon \mathcal{A}\times\mathcal{A}\times\mathbb{R} \to \mathcal{A}\times\mathbb{R}$ on the left hand side and to $\pi\colon \mathcal{A}\times\mathcal{A}\times\mathbb{R} \to \mathcal{A}\times\mathcal{A}$ and $p_2\colon \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ on the right hand side, respectively. We can then continue the calculation with

$$\begin{aligned} &\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota\big(R\sigma_{\mathbb{R},*}R\mathscr{H}om(\mathbb{C}_{\{t=\operatorname{Re}(\varphi(z_1))\}}, \pi^!\,\mathrm{D}_{\mathcal{A}\times\mathcal{A}}(p_2^{-1}L_\lambda[2]))\big) \\ &\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota\big(R\sigma_{\mathbb{R},*}R\mathscr{H}om(\mathbb{C}_{\{t=\operatorname{Re}(\varphi(z_1))\}}, \pi^{-1}p_2^{-1}L_{-\lambda})[3]\big), \end{aligned}$$

where for the last step, we used the following calculations: By definition, $L_\lambda = \mathcal{S}ol_{\mathcal{P}}(\mathcal{K}^\lambda)$ (cf. section 2.4.1). As K^λ is regular singular, we can apply the results from [HTT08, section 7.1] to the effect that

$$\begin{aligned} \mathrm{D}_{\mathcal{P}\times\mathcal{P}}(p_2^{-1}\mathcal{S}ol_{\mathcal{P}}(\mathcal{K}^\lambda)[2]) \simeq \mathrm{D}_{\mathcal{P}\times\mathcal{P}}(p_2^!\mathcal{S}ol_{\mathcal{P}}(\mathcal{K}^\lambda)) &\simeq \mathrm{D}_{\mathcal{P}\times\mathcal{P}}(p_2^!\mathcal{DR}_{\mathcal{P}}(\mathbb{D}_{\mathcal{P}}\mathcal{K}^\lambda)[-1]) \\ &\simeq \Big(\mathrm{D}_{\mathcal{P}\times\mathcal{P}}(\mathcal{DR}_{\mathcal{P}\times\mathcal{P}}(p_2^\dagger\mathcal{K}^{-\lambda}))\Big)[1] \\ &\simeq \Big(\mathcal{DR}_{\mathcal{P}\times\mathcal{P}}(\mathbb{D}_{\mathcal{P}\times\mathcal{P}}(p_2^\dagger\mathcal{K}^{-\lambda})\Big)[1] \\ &\simeq \Big(\mathcal{DR}_{\mathcal{P}\times\mathcal{P}}(p_2^\bigstar\mathbb{D}_{\mathcal{P}}(\mathcal{K}^{-\lambda}))\Big)[1] \\ &\simeq \Big(p_2^{-1}\mathcal{DR}_{\mathcal{P}}(\mathcal{K}^\lambda)\Big)[1] \simeq p_2^{-1}\mathcal{S}ol_{\mathcal{P}}(\mathcal{K}^{-\lambda})[2], \end{aligned}$$

where we are additionally using [KS90, proposition 3.3.2], together with the fact that p_2 is a topological submersion of fiber dimension 2.

Let us call $W := \{t = \operatorname{Re}(\varphi(z_1))\} \subset \mathcal{A} \times \mathcal{A} \times \mathbb{R} \subset \mathcal{P} \times \mathcal{P} \times \overline{\mathbb{R}}$. By definition, W is a closed subset of $U_\mathbb{R} := U \times \mathcal{A} \times \mathbb{R}$. As a graph of the smooth function $\operatorname{Re}(\varphi(z_1))|_{U_\mathbb{R}}$, it is actually a closed submanifold of $U_\mathbb{R}$, i. e. a locally closed submanifold of $\mathcal{A} \times \mathcal{A} \times \mathbb{R}$, of (real) codimension 1. Denote by j_W (resp. j_W^U, j_U) the locally closed (resp. closed, open) embeddings $W \to \mathcal{A} \times \mathcal{A} \times \mathbb{R}$ (resp. $W \to U_\mathbb{R}$, $U_\mathbb{R} \to \mathcal{A} \times \mathcal{A} \times \mathbb{R}$). By definition

$$p_2^{-1}\pi^{-1}L_{-\lambda} = p_2^{-1}\pi^{-1}j_! \widetilde{L_{-\lambda}} = \tilde{j}_!\pi^{-1}p_2^{-1}\widetilde{L_{-\lambda}},$$

for $j\colon \mathcal{A} \setminus \{0\} \to \mathcal{A}$ resp. $\tilde{j}\colon \mathcal{A} \times (\mathcal{A} \setminus \{0\}) \times \mathbb{R} \to \mathcal{A} \times \mathcal{A} \times \mathbb{R}$ the open embeddings. In particular, writing $V := \mathcal{A} \setminus \{0\}$, we know that $p_2^{-1}\pi^{-1}L_{-\lambda}|_{U_\mathbb{R}}$ is locally constant on the stratification $\mathcal{S}_V|_{U_\mathbb{R}} := \big((\mathcal{A} \times V \times \mathbb{R}) \cap U_\mathbb{R}, (\mathcal{A} \times \{0\} \times \mathbb{R}) \cap U_\mathbb{R}\big)$ on $U_\mathbb{R}$ (which is induced by the stratification $\mathcal{S}_V := (\mathcal{A} \times V \times \mathbb{R}, \mathcal{A} \times \{0\} \times \mathbb{R})$ on $\mathcal{A} \times \mathcal{A} \times \mathbb{R}$).

By example 2.18, $\mathcal{S}_V$ resp. $\mathcal{S}_V|_{U_\mathbb{R}}$ are μ-stratifications. We want to convince ourselves that j_W^U is transversal to $\mathcal{S}_V|_{U_\mathbb{R}}$ (or, equivalently, j_W is transversal to $\mathcal{S}_V$). It is obviously enough to show that W intersects transversally with $U \times \{0\} \times \mathbb{R}$, which is clear as for any $p := (x, 0, t) \in U \times \{0\} \times \mathbb{R}$ with $\operatorname{Re}(\varphi(x)) = t$, we have $T_p(U \times \{0\} \times \mathbb{R}) \simeq \mathbb{R}^2 \times \{0\} \times \mathbb{R}$ and $\{0\} \times \mathbb{R}^2 \times \{0\} \subset T_pW$. So we can proceed with the calculation of $\mathrm{D}^E_\mathbf{A}(K_![2])$, where we would like to set $L := \pi^{-1}p_2^{-1}L_\lambda \in \mathcal{A} \times \mathcal{A} \times \mathbb{R}$ as a further shorthand, and then write

$$W^a := \{t = -\operatorname{Re}(\varphi(z_1))\}, \quad L^a := \pi^{-1}p_2^{-1}L_{-\lambda}.$$

With this we may write

$$\begin{aligned}
\mathrm{D}^E_\mathbf{A}(K_![2]) &\simeq \mathbb{C}^E_\mathbf{A} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},*}R\mathscr{H}om(\mathbb{C}_W, L^a[3])) \\
&\simeq \mathbb{C}^E_\mathbf{A} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},*}Rj_{U,*}j^U_{W,*}j^!_W L^a)[3] \\
&\simeq \mathbb{C}^E_\mathbf{A} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},*}Rj_{U,*}j^U_{W,!}j^{-1}_W L^a)[2] \\
&\simeq \mathbb{C}^E_\mathbf{A} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},*}Rj_{U,*}(j^{-1}_U L^a)_W)[2].
\end{aligned}$$

By the very same calculations we further get

$$\begin{aligned}
K_*[2] :=& E^\varphi[1] \overset{E}{*}_* L^E_\lambda[1] \simeq \mathbb{C}^E_\mathbf{A} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},*}(L)_{W^a})[2], \\
\mathrm{D}^E_\mathbf{A}(K_*[2]) \simeq& \mathbb{C}^E_\mathbf{A} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},!}Rj_{U,*}(j^{-1}_U L^a)_W)[2].
\end{aligned}$$

In the case of $\varphi(z_1) = z_1$ – or slightly more generally the case that φ has only one pole (which we may assume is at ∞) on $\mathcal{P}$ – we have $U_\mathbb{R} = \mathcal{A} \times \mathcal{A} \times \mathbb{R}$, so the above may be simplified further, to

$$\left.\begin{aligned}
K_![2] &\simeq \mathbb{C}^E_\mathbf{A} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},!}L_{W^a})[2], \\
\mathrm{D}^E_\mathbf{A}(K_![2]) &\simeq \mathbb{C}^E_\mathbf{A} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},*}L^a_W)[2], \\
K_*[2] &\simeq \mathbb{C}^E_\mathbf{A} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},*}L_{W^a})[2], \\
\mathrm{D}^E_\mathbf{A}(K_*[2]) &\simeq \mathbb{C}^E_\mathbf{A} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},!}L^a_W)[2].
\end{aligned}\right\} \tag{2.5}$$

For further reference, let us quickly consider the following reformulation of 2.5: We may write $\sigma = q_2 \circ u \circ \alpha$, for $u\colon \mathcal{A} \times \mathcal{A} \to \mathcal{P} \times \mathcal{A}$ the open embedding,

$$\alpha\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A} \times \mathcal{A}, \quad (a,b) \mapsto (a, a+b)$$

and $q_2\colon \mathcal{P} \times \mathcal{A} \to \mathcal{A}$ the (now proper) projection on the second factor, where we will use the same letters for the induced morphisms $f \times \mathrm{Id}_R$ (instead of writing $f_{\mathbb{R}}$ as usual) for $f = u, \alpha, q_2$. Then we have (with $\alpha(W) = W$ resp. $\alpha(W^a) = W^a$)

$$\left.\begin{aligned} K_![2] &\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota(Rq_{2,!}Ru_!(\alpha_* L)_{W^a})[2], \\ \mathrm{D}^E_{\mathbf{A}}(K_![2]) &\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota(Rq_{2,!}Ru_*(\alpha_* L^a)_{W})[2], \\ K_*[2] &\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota(Rq_{2,!}Ru_*(\alpha_* L)_{W^a})[2], \\ \mathrm{D}^E_{\mathbf{A}}(K_*[2]) &\simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota(Rq_{2,!}Ru_!(\alpha_* L^a)_{W})[2]. \end{aligned}\right\} \tag{2.6}$$

2.4.8 Cohomology computations

We will keep the notation of the last section, only we will use $\varphi(w) = w$ from here on. Recall from section 2.4.5 that for some $F \in D^b(\mathbb{C}_{\mathbf{X}\times\mathbb{R}_\infty})$ we have

$$H^n(\mathbb{C}^E_{\mathbf{X}} \overset{+}{\otimes} Q\iota(F)) \simeq \mathbb{C}^E_{\mathbf{X}} \overset{+}{\otimes} Q\iota H^n(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} F)$$

and note that, for F as above and some morphism $u\colon \mathbf{X} \to \mathbf{Y}$ of bordered spaces

$$Ru_!(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} F) \simeq \mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} Ru_! F,$$

cf. [DK16b, porposition 3.3.13]. Furthermore we are going to use the fact that

$$\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} L_W = \mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} (\mathbb{C}_W \otimes L) \simeq (\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} \mathbb{C}_W) \otimes L$$

(and the same of course for any replacement of L resp. W with L^a resp. W^a). With regard to our previous notation, we will write

$$W_{\geqslant} := \{t \geqslant \mathrm{Re}(z_1)\}, \quad W^a_{\geqslant} := \{t \geqslant -\mathrm{Re}(z_1)\},$$

and we then know that

$$\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} \mathbb{C}_W \simeq \mathbb{C}_{W_\geqslant} \quad \text{resp.} \quad \mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} \mathbb{C}_{W^a} \simeq \mathbb{C}_{W^a_\geqslant},$$

cf. remark 2.11. To determine the vanishing of the cohomologies of (2.5) resp. (2.6), it is thus enough to consider the cohomologies

$$\begin{gathered} H^n(Rp_{2,!}(\alpha_* L)_{W^a_\geqslant}), \\ H^n(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} Rp_{2,*}(\alpha_* L^a)_W) \simeq H^n(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} Rq_{2,!}Ru_*(\alpha_* L^a)_W), \\ H^n(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} Rp_{2,*}(\alpha_* L)_{W^a}) \simeq H^n(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} Rq_{2,!}Ru_*(\alpha_* L)_{W^a}), \\ H^n(Rp_{2,!}(\alpha_* L^a)_{W_\geqslant}). \end{gathered}$$

Without loss of generality (interchanging λ with $-\lambda$ resp. w with $-w$), it is enough to handle the first two of these cases. With regard to considering stalks at $(y,s) \in \mathcal{A} \times \mathbb{R}$, let us denote by $i_{(y,s)} \colon \{y\} \times \{s\} \to \mathcal{A} \times \mathbb{R}$ the corresponding embedding. The following diagram will recall resp. introduce the associated labeling we will use for the rest of this section (all squares are obviously cartesian).

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{A} \times \mathbb{R} \arrow[r, "\alpha"] & \mathcal{A} \times \mathcal{A} \times \mathbb{R} \arrow[r, "u"] \arrow[rr, bend left, dotted, "p_2"] & \mathcal{P} \times \mathcal{A} \times \mathbb{R} \arrow[r, "q_2"] & \mathcal{A} \times \mathbb{R} \\
& \mathcal{A} \times \{y\} \times \mathbb{R} \arrow[u, "\tilde{i}^{\mathcal{A}}_{\mathbb{R}}"] \arrow[r, "\tilde{u}_y"] \arrow[rr, bend left, dotted, "\tilde{p}_2"] & \mathcal{P} \times \{y\} \times \mathbb{R} \arrow[u, "\tilde{i}^{\mathcal{P}}_{\mathbb{R}}"] \arrow[r, "\tilde{q}_2"] & \{y\} \times \mathbb{R} \arrow[u, "i_{\mathbb{R}}"] \\
& \mathcal{A} \times \{y\} \times \{s\} \arrow[u, "\tilde{i}^{\mathcal{A}}"] \arrow[r, "\tilde{u}_{(y,s)}"] \arrow[uu, bend left=60, dotted, "i_{\mathcal{A}}"] & \mathcal{P} \times \{y\} \times \{s\} \arrow[u, "\tilde{i}^{\mathcal{P}}"] \arrow[r, "\mathrm{pt}"] & \{y\} \times \{s\} \arrow[u, "i"] \arrow[uu, bend right=60, dotted, "i_{(y,s)}"']
\end{tikzcd} \tag{2.7}$$

Now let us consider the first case, i. e. we want to determine the vanishing of the cohomologies

$$H^n(Rp_{2,!}(\alpha_* L)_{W^a_{\gtrless}}). \tag{2.8}$$

From now on, if the point $(y,s) \in \mathcal{A} \times \mathbb{R}$ is specified, we will often write $\mathcal{A}$ instead of $\mathcal{A} \times \{y\} \times \{s\}$ resp. $\mathcal{A} \times \mathbb{R}$ instead of $\mathcal{A} \times \{y\} \times \mathbb{R}$ (and the same of course for $\mathcal{A}$ replaced with $\mathcal{P}$) for the sake of notational brevity. Considering stalks at $(y,s) \in \mathcal{A} \times \mathbb{R}$, we get

$$(H^n(Rp_{2,!}(\alpha_* L)_{W^a_{\gtrless}}))_{(y,s)} \simeq H^n_c(\mathcal{A}, i_{\mathcal{A}}^{-1}(\alpha_* L)_{W^a_{\gtrless}}),$$

so let us have a look on $i_{\mathcal{A}}^{-1}(\alpha_* L)_{W^a_{\gtrless}}$. Recall that we defined $L := \pi^{-1} p_2^{-1} L_\lambda$, where we have $L_\lambda = j_{0,!}\widetilde{L_\lambda}$ for the open embedding $j_0 \colon \mathcal{A} \setminus \{0\} \to \mathcal{A}$ (cf. section 2.4.1) and $\widetilde{L_\lambda}$ is a local system on $\mathcal{A} \setminus \{0\}$, with monodromy $e^{-2\pi i \lambda} \in \mathbb{C} \setminus \{1\}$ around 0. So L is a local system on $\mathcal{A} \times (\mathcal{A} \setminus \{0\}) \times \mathbb{R}$ with monodromy $e^{-2\pi i\lambda}$ around $D := \mathcal{A} \times \{0\} \times \mathbb{R}$ and $\alpha_* L$ is a local system on

$$V := (\mathcal{A} \times \mathcal{A} \times \mathbb{R}) \setminus \alpha(D) = (\mathcal{A} \times \mathcal{A} \times \mathbb{R}) \setminus (\Delta_{\mathcal{A}} \times \mathbb{R}),$$

where $\Delta_{\mathcal{A}} := \{(w,w) | w \in \mathcal{A}\} \subset \mathcal{A} \times \mathcal{A}$ is the diagonal, with monodromy $e^{-2\pi i\lambda}$ around $D' := \Delta_{\mathcal{A}} \times \mathbb{R}$. Note that $D' \cap \mathcal{A} \times \{y\} \times \{s\} = \{y\} \times \{y\} \times \{S\}$, i. e.

$$V \cap (\mathcal{A} \times \{y\} \times \{s\}) = (\mathcal{A} \setminus \{y\}) \times \{y\} \times \{s\} \simeq \mathcal{A} \setminus \{y\},$$

and with the labeling as depicted in the square

$$\begin{tikzcd}
V \arrow[r, "i_V"] & \mathcal{A} \times \mathcal{A} \times \mathbb{R} \\
\mathcal{A} \setminus \{y\} \arrow[u, "\tilde{i}_{\mathcal{A}}"] \arrow[r, "\tilde{i}_V"] & \mathcal{A} \times \{y\} \times \{s\}, \arrow[u, "i_{\mathcal{A}}"]
\end{tikzcd}$$

we may write
$$\tilde{L} := i_{\mathcal{A}}^{-1}(\alpha_* L) = \tilde{i}_{V,!}\tilde{i}_{\mathcal{A}}^{-1} i_V^{-1}(\alpha_* L),$$
where $\mathcal{L} := \tilde{i}_{\mathcal{A}}^{-1} i_V^{-1}(\alpha_* L) \in D^0(\mathbb{C}_{\mathcal{A}\setminus\{y\}})$ is a local system, with monodromy $e^{-2\pi i\lambda}$ around y. Analogously we define $\tilde{L}^a := i_{\mathcal{A}}^{-1}(\alpha_* L^a)$ and $\mathcal{L}^a := \tilde{i}_{\mathcal{A}}^{-1} i_V^{-1}(\alpha_* L^a)$. Finally, writing
$$\tilde{W}^a_{\geqslant} := i_{\mathcal{A}}^{-1} W^a_{\geqslant} = \{z \in \mathcal{A} | s \geqslant -\operatorname{Re}(z)\} \subset \mathcal{A},$$
we get
$$i_{\mathcal{A}}^{-1}(\alpha_* L)_{W^a_{\geqslant}} = \tilde{L}_{\tilde{W}^a_{\geqslant}}$$
with $\tilde{L} = \tilde{i}_{V,!}\mathcal{L}$. Now we may state that
$$H_c^0(\mathcal{A}, i_{\mathcal{A}}^{-1}(\alpha_* L)_{W^a_{\geqslant}}) \simeq H_c^0(\tilde{W}^a_{\geqslant}, \tilde{L}|_{\tilde{W}^a_{\geqslant}}) \simeq \begin{cases} H_c^0(\tilde{W}^a_{\geqslant}, \mathcal{L}|_{\tilde{W}^a_{\geqslant}}) & \text{if } s < -\operatorname{Re}(y), \\ H_c^0(\tilde{W}^a_{\geqslant} \setminus \{y\}, \mathcal{L}|_{\tilde{W}^a_{\geqslant}\setminus\{y\}}) & \text{if } s \geqslant -\operatorname{Re}(y) \end{cases}$$
which obviously vanishes in both cases, as neither $\tilde{W}^a_{\geqslant}$ nor $\tilde{W}^a_{\geqslant} \setminus \{y\}$ is compact. On the other hand,
$$H_c^n(\mathcal{A}, \tilde{L}) \simeq H_c^n(\mathcal{A}, \tilde{i}_{V,!}\mathcal{L}) \simeq H_c^n(\mathcal{A} \setminus \{y\}, \mathcal{L}) \simeq H^{2-n}(\mathcal{A} \setminus \{y\}, \mathcal{L}^\vee)^\vee$$
by Poincaré–Verdier duality, where $(\bullet)^\vee$ as usual denotes the dual of vector spaces resp. local systems. In particular, as $\mathcal{L}^\vee$ has monodromy $e^{2\pi i\lambda} \in \mathbb{C} \setminus \{1\}$ around y, we have $H_c^n(\mathcal{A}, \tilde{L}) \simeq 0$ for all n. We denote the complement of $\tilde{W}^a_{\geqslant}$ by
$$\tilde{W}^a_{<} := \mathcal{A} \setminus \tilde{W}^a_{\geqslant}.$$
The standard distinguished triangle
$$\tilde{L}_{\tilde{W}^a_{<}} \longrightarrow \tilde{L} \longrightarrow \tilde{L}_{\tilde{W}^a_{\geqslant}} \xrightarrow{+1}$$
induces the long exact sequence on cohomologies
$$\longrightarrow H_c^1(\mathcal{A}, \tilde{L}) \longrightarrow H_c^1(\mathcal{A}, \tilde{L}_{\tilde{W}^a_{\geqslant}}) \longrightarrow H_c^2(\mathcal{A}, \tilde{L}_{\tilde{W}^a_{<}}) \longrightarrow H_c^2(\mathcal{A}, \tilde{L}) \longrightarrow H_c^2(\mathcal{A}, \tilde{L}_{\tilde{W}^a_{\geqslant}}) \longrightarrow 0,$$
which in turn implies $H_c^2(\mathcal{A}, \tilde{L}_{\tilde{W}^a_{\geqslant}}) \simeq 0$ because of $H_c^2(\mathcal{A}, \tilde{L}) \simeq 0$. So we have shown as a first intermediate step the following

Lemma 2.21. *We have*
$$K_![2] \simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},!} L_{W^a})[2] \in E^{-1}_{\mathbb{R}-c}(\mathbf{A}).$$

When addressing the second case, concerning the cohomologies

$$H^n(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} Rp_{2,*}(\alpha_* L^a)_W) \simeq H^n(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} Rq_{2,!}Ru_*(\alpha_* L^a)_W),$$

some additional work has to be done. We want to start by reducing dimensions so as to simplify the calculations. Recall the notation of diagram (2.7). Let us choose a point $y \in \mathcal{A}$ and consider

$$i_{\mathbb{R}}^{-1} H^n(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} Rq_{2,!}Ru_*(\alpha_* L^a)_W) \simeq H^n(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} R\tilde{q}_{2,!}(\tilde{i}_{\mathbb{R}}^{\mathcal{P}})^{-1} Ru_*(\alpha_* L^a)_W).$$

Lemma 2.22. *$Ru_*((\alpha_* L^a)_W)|_{\{\infty\}\times\mathcal{A}\times\mathbb{R}}$ is a locally constant sheaf.*

Let us denote by i_∞ the closed embedding $\{\infty\} \times \mathcal{A} \times \mathbb{R} \to \mathcal{P} \times \mathcal{A} \times \mathbb{R}$. Before we start proving the lemma, let us quickly note some elementary facts about sheaves that we will use later on.

Remark 2.23. Let X, Y be topological spaces, $f\colon Y \to X$ a continuous map.

i) For any presheaf $F \in \mathrm{PSh}(X)$ let $f^{-1,p}(F)\colon U \mapsto \varinjlim_{V\supset f(U)} F(V)$ denote the presheaf inverse image (note that $f^{-1,p} \dashv f_*$ as functors on the presheaf categories, by definition). With this notation we have

$$f^{-1}F^s = (f^{-1,p}F)^s,$$

where $(\bullet)^s$ denotes the sheafification functor. For a proof consider the canonical morphism $f^{-1,p}F \to f^{-1,p}F^s \to f^{-1}F^s$ of presheaves. For some sheaf G on Y, we have

$$\mathrm{Hom}_{\mathrm{PSh}}(f^{-1,p}F, G) \simeq \mathrm{Hom}_{\mathrm{PSh}}(F, f_*G) \simeq \mathrm{Hom}_{\mathrm{Sh}}(F^s, f_*G) \simeq \mathrm{Hom}_{\mathrm{Sh}}(f^{-1}F^s, G),$$

so $f^{-1}F^s$ satisfies the universal property of $(f^{-1,p}F)^s$ and $(f^{-1,p}F)^s \simeq f^{-1}F^s$.

ii) For $F, G \in \mathrm{PSh}(X)$ we may in particular consider F, G as presheaves on a base $\mathcal{B}$ of X. If we have a morphism $F \to G$ of presheaves on the base $\mathcal{B}$ that is an isomorphism on stalks, then $F^s \simeq G^s$ (by definition of the sheafification functor, the morphism of the presheaves on a base induces a morphism of the sheafifications, which, being an isomorphism on stalks, is then indeed an isomorphism).

iii) If F is a local system on X and $R \underset{\iota}{\overset{\tau}{\leftrightarrows}} X$ is a deformation retract of X, then

$$F \simeq \tau^{-1}(\iota^{-1}F),$$

cf. [Dim04, remark 2.5.12] resp. [MeNM90, proposition I.3.4]. In particular, if X is contractible, $G \in \mathrm{Sh}(X)$ is a local system if and only if $G \simeq \mathrm{pt}^{-1}G_{x_0}$ for any $x_0 \in X$, with $\mathrm{pt}\colon X \to \{x_0\}$.

Proof of lemma 2.22. We know that $R^i u_*(\alpha_* L^a)_W$ is the sheafification of the presheaf

$$V \mapsto H^i(u^{-1}(V), (\alpha_* L^a)_W|_{u^{-1}(V)}) = H^i(V \cap W, (\alpha_* L^a)|_{V\cap W}).$$

Let us denote this presheaf by F^i and choose a base $\mathcal{B}$ of the standard topology on $\{\infty\} \times \mathcal{A} \times \mathbb{R}$ as

$$\mathcal{B} := \{\{\infty\} \times B_r(x) \times (a,b) | x \in \mathcal{A},\ r \in \mathbb{R}_{>0},\ (a,b) \subset \mathbb{R}\}$$

and denote $V_{x,r,a,b} := \{\infty\} \times B_r(x) \times (a,b)$. Analogously we consider a basis of open neighborhoods of $\{\infty\} \in \mathcal{P}$ given by $\{B_q(\infty) | q \in \mathbb{R}_{>0}\}$, where

$$B_q(\infty) := \{z \in \mathcal{P}|\ |\infty - z| < q\} = \mathcal{P} \setminus \overline{B_{1/q}(0)},$$

means a standard open disc around ∞.

Furthermore write $\mathcal{K} := \alpha_* L^a|_V$ for the local system that $\alpha_* L^a$ is away from $\Delta_{\mathcal{A}} \times \mathbb{R}$ (recall $V := (\mathcal{A} \times \mathcal{A} \times \mathbb{R}) \setminus (\Delta_{\mathcal{A}} \times \mathbb{R})$) and set $P := (\infty, 0, 0) \in \{\infty\} \times \mathcal{A} \times \mathbb{R}$ and $\mathrm{pt}_P\colon \{\infty\} \times \mathcal{A} \times \mathbb{R} \to \{P\}$. For the presheaf F^i we then may observe – with denoting $V_{q,r,s} := B_q(\infty) \times B_r(0) \times (-s,s)$ – that

$$\begin{aligned}(R^i u_*(\alpha_* L^a)_W)_P \simeq F^i_P &= \varinjlim_{V \ni P} H^i(V \cap W, (\alpha_* L^a)|_{V\cap W}) \\ &= \varinjlim_{r,q,s \to 0} H^i(V_{q,r,s} \cap W, (\alpha_* L^a)|_{V_{q,r,s}\cap W})\end{aligned}$$

As soon as $q < 1/2$, $s < 1$, we in particular have $2q < 1/s$ and $V_{q,r,s} \cap W$ consists of two disjoint parts, which we will refer to as $W^+_{q,r,s}, W^-_{q,r,s}$. If $r < 2$, we additionally have $r < 1/q$, which implies that $((B_q(\infty) \setminus \{\infty\}) \times B_r(0)) \cap \Delta_{\mathcal{A}} = \emptyset$ such that

$$\alpha_* L^a|_{(V_{q,r,s}\cap W)} = \mathcal{K}|_{V_{q,r,s}\cap W}$$

and we may continue the above lines with

$$\begin{aligned}&= \varinjlim_{\substack{q,r,s \to 0 \\ q,r,s < 1/2}} H^i(V_{q,r,s} \cap W, (\alpha_* L^a)|_{V_{q,r,s}\cap W}) \\ &\simeq \varinjlim_{\substack{q,r,s \to 0 \\ q,r,s < 1/2}} (H^i(W^+_{q,r,s}, \mathcal{K}|_{W^+_{q,r,s}}) \oplus H^i(W^-_{q,r,s}, \mathcal{K}|_{W^-_{q,r,s}})).\end{aligned}$$

Note that in particular we get $(R^i u_*(\alpha_* L^a)_W)_P \simeq 0$ if $i \neq 0$, actually $R^i u_*(\alpha_* L^a)_W \simeq 0$ for $i \neq 0$, as the very same argument clearly works as well for any choice $(x,y) \in \mathcal{A} \times \mathbb{R}$ (instead of $(0,0)$) for P. On the other hand, using the same reasoning and notation – but introducing furthermore

$$V_{q,x,r,a,b} := B_q(\infty) \times B_r(x) \times (a,b),$$

and $\tilde{F}^i := F^i|_{\{\infty\}\times\mathcal{A}\times\mathbb{R}}$ – we may, for some $V_{x,r,a,b} \in \mathcal{B}$, consider

$$\begin{aligned}
\tilde{F}^i(V_{x,r,a,b}) &= \varinjlim_{q\to 0} H^i(V_{q,x,r,a,b} \cap W, (\alpha_* L^a)|_{V_{q,x,r,a,b}\cap W}) \\
&= \varinjlim_{\substack{q\to 0 \\ q<\min\left(\frac{1}{|x|+r},\frac{1}{\max(|a|,|b|)}\right)}} H^i(V_{q,x,r,a,b} \cap W, (\alpha_* L^a)|_{V_{q,x,r,a,b}\cap W}) \\
&\simeq \varinjlim_{\substack{q\to 0 \\ q<\min\left(\frac{1}{|x|+r},\frac{1}{\max(|a|,|b|)}\right)}} H^i(V_{q,x,r,a,b} \cap W, \mathcal{K}|_{V_{q,x,r,a,b}\cap W}) \\
&\simeq \varinjlim_{\substack{q\to 0 \\ q<\min\left(\frac{1}{|x|+r},\frac{1}{\max(|a|,|b|)}\right)}} H^i(V_{q,|x|+r,\max(|a|,|b|)} \cap W, \mathcal{K}|_{V_{q,|x|+r,\max(|a|,|b|)}\cap W}) \\
&\simeq \varinjlim_{\substack{q\to 0 \\ q<\min\left(\frac{1}{|x|+r},\frac{1}{\max(|a|,|b|)}\right)}} (H^i(W^+_{q,r,\max(|a|,|b|)}, \mathcal{K}|_{W^+_{q,r,\max(|a|,|b|)}}) \oplus \\
&\qquad \oplus H^i(W^-_{q,r,\max(|a|,|b|)}, \mathcal{K}|_{W^-_{q,r,\max(|a|,|b|)}})) \\
&\longrightarrow \tilde{F}^i_P = F^i_P \simeq (R^i u_*(\alpha_* L^a)_W)_P,
\end{aligned}$$

which is clearly compatible with restrictions and thus gives us a morphism of presheaves on the base $\mathcal{B}$ from the presheaf $\tilde{F}^i$ to the constant presheaf $\tilde{F}^i_P$. Also this is clearly an isomorphism on stalks, so that we indeed get

$$R^i u_*(\alpha_* L^a)_W|_{\{\infty\}\times\mathcal{A}\times\mathbb{R}} \simeq (\tilde{F}^i)^s \simeq \tilde{(F^i_P)}^s,$$

that is, $R^i u_*(\alpha_* L^a)_W|_{\{\infty\}\times\mathcal{A}\times\mathbb{R}}$ is locally constant for all i (recall we already showed that $R^i u_*(\alpha_* L^a)_W \simeq 0$ for $i \neq 0$). □

Remark 2.24. Note that the fact that W splits into two disjoint parts when restricted to some small enough $V_{q,r,s}$ doesn't matter for the above proof, but only that from a certain index in the directed colimit on, $(\alpha_* L^a)|_{V_{q,r,s}}$ is the restriction of some local system and all subsequent $W \cap V_{q,r,s}$ are homotopy equivalent to each other. In particular, by lemma 1.22, we might have done the proof for $W_\geqslant$ instead of W which would have been even easier, as $V_{q,r,s} \cap W_\geqslant$ is just homotopy-equivalent to a point (as soon as the indices are small enough, in the sense of the above proof). This may seem a little weird at first glance, as replacing W with $W_\geqslant$ changes the stalks – as just sketched – from $\mathbb{C}^2$ to $\mathbb{C}$, but becomes more reasonable in view of the fact that lemma 1.22 only holds in the enhanced setting, together with the observation stated in the proof of lemma 3.15 (that, for some local system on $X \times \mathbb{R}$, for some space X, one obviously has $Q\iota(L) \simeq 0$ by the very definition of the category of enhanced ind-sheaves).

Now let us use lemma 2.22 to observe that $Ru_*(\alpha_* L^a)_W$ is cohomologically constructible with respect to the following stratification $\mathcal{S}$ of $\mathcal{P} \times \mathcal{A} \times \mathbb{R} = \coprod_{i\in\mathfrak{A}} S_i$, where $\mathfrak{A} = \{1, \ldots, 6\}$, given by:

$$\mathcal{S} := \begin{cases} \mathcal{S}_0 := \{(a,b,t)|a \neq b \in \mathcal{A},\ t > \mathrm{Re}(a)\} \\ \mathcal{S}_1 := \{(a,b,t)|a \neq b \in \mathcal{A},\ t = \mathrm{Re}(a)\} \\ \mathcal{S}_2 := \{(a,b,t)|a \neq b \in \mathcal{A},\ t < \mathrm{Re}(a)\} \\ \mathcal{S}_3 := \{(a,b,t)|a = b \in \mathcal{A},\ t > \mathrm{Re}(a)\} \\ \mathcal{S}_4 := \{(a,b,t)|a = b \in \mathcal{A},\ t = \mathrm{Re}(a)\} \\ \mathcal{S}_5 := \{(a,b,t)|a = b \in \mathcal{A},\ t < \mathrm{Re}(a)\} \\ \mathcal{S}_6 := \{(a,b,t)|a = \infty\} \end{cases} \tag{2.9}$$

In order to verify this is indeed a stratification, it is enough to remark that $\mathcal{S}_6 \subset \overline{\mathcal{S}_i}$ for $i = 0, 1, 2$ – to be a little more precise: $\overline{\mathcal{S}_0} \setminus \mathcal{S}_0 = \mathcal{S}_1 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_6$. The situation is the very same for $\overline{\mathcal{S}_2} \setminus \mathcal{S}_2$. Then $\overline{\mathcal{S}_1} \setminus \mathcal{S}_1 = \mathcal{S}_4 \cup \mathcal{S}_6$ and $\overline{\mathcal{S}_3} \setminus \mathcal{S}_3 = \mathcal{S}_4$ (and the same for $\mathcal{S}_5$).

In view of corollary 2.20, we would like to show that $\mathcal{S}$ is a μ-stratification. As it will turn out in the course of our proof below, the steps concerning the stratum $\mathcal{S}_6$ are a little cumbersome, so we will weaken our objective to verifying that $\tilde{i}^{\mathcal{P}}_{\mathbb{R}}$ is non-characteristic for the sheaf $Ru_*(\alpha L^a)_W$ (and $\tilde{i}^{\mathcal{A}}_{\mathbb{R}}$ is non-characteristic for $(\alpha_* L^a)_W$). This is enough to apply [KS90, proposition 5.4.13], telling us that $(\tilde{i}^{\mathcal{P}}_{\mathbb{R}})^! Ru_*(\alpha L^a)_{W\geqslant} \simeq (\tilde{i}^{\mathcal{P}}_{\mathbb{R}})^{-1} Ru_*(\alpha L^a)_W[-2]$ (and the same for $\tilde{i}^{\mathcal{A}}_{\mathbb{R}}$ and $(\alpha_* L^a)_W$).

Due to symmetry, and taking into account example 2.18, we only need to check condition (μ) for the pairs $(\mathcal{S}_1, \mathcal{S}_4)$, $(\mathcal{S}_1, \mathcal{S}_6)$ and $(\mathcal{S}_3, \mathcal{S}_4)$. As in example 2.18, we want to use proposition 2.17. Let us write $a = x + iy$ and $b = z + iw$ for a, b in (2.9) and let (x, y, z, w, t) be the corresponding local coordinates of the real manifold $X := \mathcal{P} \times \mathcal{A} \times \mathbb{R}$ (the actual coordinates for the chart $\mathcal{A} \times \mathcal{A} \times \mathbb{R}$ excluding $\{\infty\} \times \mathcal{A} \times \mathbb{R}$). The induced coordinates on T^*X are denoted by $(x, y, z, w, t, \xi_x, \xi_y, \xi_z, \xi_w, \xi_t)$.

- To the pair $(\mathcal{S}_1, \mathcal{S}_4)$. In the above coordinates,

$$T^*_{\mathcal{S}_1} X = \{(x, y, z, w, x, \xi_x, 0, 0, 0, -\xi_x)|(x, y) \neq (z, w)\}$$

  and

$$T^*_{\mathcal{S}_4} X = \{(x, y, x, y, x, \xi_x, \xi_y, \xi_z, -\xi_y, -\xi_x - \xi_z)\}.$$

  Using proposition 2.17, we know that some $(x_0, y_0, z_0, w_0, t_0, \xi_{x,0}, \xi_{y,0}, \xi_{z,0}, \xi_{w,0}, \xi_{t,0})$ is in $T^*_{\mathcal{S}_1} X \widehat{+} T^*_{\mathcal{S}_4} X^a$ if and only if there are sequences

$$(a_{1,n}, a_{2,n}, a_{3,n}, a_{4,n}, a_{1,n}, \xi_{1,n}, 0, 0, 0, -\xi_{1,n})$$

  in $T^*_{\mathcal{S}_1} X$ and

$$(b_{1,n}, b_{2,n}, b_{1,n}, b_{2,n}, b_{1,n}, -\eta_{1,n}, -\eta_{2,n}, -\eta_{3,n}, \eta_{2,n}, \eta_{1,n} + \eta_{3,n})$$

in $T^*_{\mathcal{S}_4}X^a$ such that

$$(a_{1,n}, a_{2,n}, a_{3,n}, a_{4,n}, a_{1,n}) \underset{n\to\infty}{\longrightarrow} (x_0, y_0, z_0, w_0, t_0) \underset{n\to\infty}{\longleftarrow} (b_{1,n}, b_{2,n}, b_{1,n}, b_{2,n}, b_{1,n}),$$

in particular $(x_0, y_0, z_0, w_0, t_0) = (x, y, x, y, x) \in \mathcal{S}_4$ for some $x, y \in \mathbb{R}$, as $\mathcal{S}_4$ is closed, furthermore

$$|(a_{1,n}, a_{2,n}, a_{3,n}, a_{4,n}, a_{1,n}) - (b_{1,n}, b_{2,n}, b_{1,n}, b_{2,n}, b_{1,n})| \cdot |(\xi_{1,n}, 0, 0, 0, -\xi_{1,n})| \underset{n\to\infty}{\longrightarrow} 0,$$

as well as

$$(-\eta_{1,n}, -\eta_{2,n}, -\eta_{3,n}, \eta_{2,n}, \eta_{1,n} + \eta_{3,n}) - (\xi_{1,n}, 0, 0, 0, -\xi_{1,n}) \\ \underset{n\to\infty}{\longrightarrow} (\xi_{x,0}, \xi_{y,0}, \xi_{z,0}, \xi_{w,0}, \xi_{t,0}).$$

In this last condition, we observe that

$$(\xi_{x,n}, \xi_{y,n}, \xi_{z,n}, \xi_{w,n}, \xi_{t,n}) := (-\eta_{1,n} - \xi_{1,n}, -\eta_{2,n}, -\eta_{3,n}, \eta_{2,n}, \eta_{1,n} + \eta_{3,n} + \xi_{1,n})$$

has the property

$$\xi_{y,n} + \xi_{w,n} = 0 = (-\eta_{1,n} - \xi_{1,n}) + (-\eta_{3,n}) + (\eta_{1,n} + \eta_{3,n} + \xi_{1,n}) = \xi_{x,n} + \xi_{z,n} + \xi_{t,n}$$

for all n, so we get

$$(x_0, y_0, z_0, w_0, t_0, \xi_{x,0}, \xi_{y,0}, \xi_{z,0}, \xi_{w,0}, \xi_{t,0}) = \\ = \lim_{n\to\infty} (b_{1,n}, b_{2,n}, b_{1,n}, b_{2,n}, b_{1,n}, \xi_{x,n}, \xi_{y,n}, \xi_{z,n}, \xi_{w,n}, \xi_{t,n}) \in T^*_{\mathcal{S}_4}X,$$

proving that $(\mathcal{S}_1, \mathcal{S}_4)$ satisfies condition (μ).

- To the pair $(\mathcal{S}_3, \mathcal{S}_4)$. In the same local coordinates as above,

  $$T^*_{\mathcal{S}_3}X = \{(x, y, x, y, t, \xi_x, \xi_y, -\xi_x, -\xi_y, 0) | t > x\}.$$

  By the very same argument as before, we find

  $$(T^*_{\mathcal{S}_3}X \widehat{+} T^*_{\mathcal{S}_4}X) \cap \pi_X^{-1}(\mathcal{S}_4) \subset T^*_{\mathcal{S}_4}X,$$

  so $(\mathcal{S}_3, \mathcal{S}_4)$ satisfies (μ) as well (where π_X denotes the projection $T^*X \to X$).

So far, we know that $\mathcal{S}$, restricted to $\mathcal{A} \times \mathcal{A} \times \mathbb{R}$ is a μ-stratification. The remaining part to check is:

- To the pair $(\mathcal{S}_1, \mathcal{S}_6)$. Here we have to switch to a chart around $\infty \in \mathcal{P}$. We chose the standard chart $\mathcal{A} \simeq \mathcal{P}\backslash\{0\}$ to get new local coordinates (x', y', z, w, t) of $\mathcal{P} \times \mathcal{A} \times \mathbb{R}$, where the relation $t = x = \mathrm{Re}(x + iy)$ becomes $t = \mathrm{Re}\left(\frac{1}{x'+iy'}\right) = \frac{x'}{x'^2+y'^2}$. With respect to these coordinates,

$$\mathcal{S}_6 = \{(0,0,z,w,t)\}$$

and

$$\mathcal{S}_1 = \left\{(x', y', z, w, t) \middle| (0,0) \neq (x', y'), \left(\frac{x'}{x'^2+y'^2}, \frac{y'}{x'^2+y'^2}\right) \neq (z, w),\ t = \frac{x'}{x'^2+y'^2}\right\}.$$

We will again denote $X := \mathcal{P} \times \mathcal{A} \times \mathbb{R}$ and write $(x', y', z, w, t, \xi_{x'}, \xi_{y'}, \xi_z, \xi_w, \xi_t)$ for the induced local coordinate on T^*X. Then,

$$T^*_{\mathcal{S}_6}X = \{(0,0,z,w,t,\xi_{x'},\xi_{y'},0,0,0)\} \subset T^*X$$

and

$$T^*_{\mathcal{S}_1}X = \left\{ \left(x', y', z, w, \frac{x'}{x'^2+y'^2}, \frac{x'^2-y'^2}{(x'^2+y'^2)^2}\xi_t, \frac{2x'y'}{(x'^2+y'^2)^2}\xi_t, 0, 0, \xi_t\right) \middle| (0,0) \neq (x', y'), \left(\frac{x'}{x'^2+y'^2}, \frac{y'}{x'^2+y'^2}\right) \neq (z, w)\right\}.$$

In particular, we need to be careful when applying proposition 2.17. Let

$$(X_n, \Xi_n) := \left(x'_n, y'_n, \tilde{z}_n, \tilde{w}_n, \frac{x'_n}{x'^2_n+y'^2_n}, \frac{x'^2_n-y'^2_n}{(x'^2_n+y'^2_n)^2}\xi_{t,n}, \frac{2x'_ny'_n}{(x'^2_n+y'^2_n)^2}\xi_{t,n}, 0, 0, \xi_{t,n}\right)$$

and

$$(Y_n, \Theta_n) := (0,0,z_n,w_n,t_n,\xi_{x',n},\xi_{y',n},0,0,0),$$

$n \in \mathbb{N}$, be sequences of the appropriate form. Because of $\lim_{n\to\infty} X_n \in \mathcal{S}_6$ we must have that $x'_n, y'_n \overset{n\to\infty}{\longrightarrow} 0$ in such a way that $|\lim_{n\to\infty} x'_n/(x'^2_n + y'^2_n)| < \infty$. Here, proposition 2.17 still clearly yields

$$T^*_{\mathcal{S}_1}X \widehat{+} T^*_{\mathcal{S}_6}X \subset \{0,0,z,w,t,\xi_{x'},\xi_{y'},0,0,\xi_t\} \subset T^*X.$$

This will be sufficient for our purposes.

Remark 2.25. Let us conclude the above observations for later reference: The given stratification $\mathcal{S}$, restricted to $\mathcal{A}\times\mathcal{A}\times\mathbb{R}$ (using the chart $\mathcal{A} \simeq \mathcal{P}\backslash\{\infty\}$), is a μ-stratification, and $(\mathcal{P} \times \mathcal{A} \times \mathbb{R}) \setminus (\mathcal{A} \times \mathcal{A} \times \mathbb{R}) = \mathcal{S}_6$ satisfies

$$(T^*_{\mathcal{S}_i}X \widehat{+} T^*_{\mathcal{S}_6}X) \cap \pi^{-1}(\mathcal{S}_6) \subset T^*_{\mathcal{S}_6}X$$

for $i \in \{0, 2\}$ and

$$T^*_{\mathcal{S}_1} X \widehat{+} T^*_{\mathcal{S}_6} X \subset \underbrace{\{0, 0, z, w, t, \xi_{x'}, \xi_{y'}, 0, 0, \xi_t\}}_{= \mathcal{S}_6 \times_X T^*_{\{\infty\} \times \mathcal{A} \times \{0\}} X} \subset T^* X$$

with respect to the local coordinates on the chart $\mathcal{A} \simeq \mathcal{P} \setminus \{0\}$ around $\infty \in \mathcal{P}$ given above.

In particular, we have already shown

$$(\tilde{i}^{\mathcal{A}}_{\mathbb{R}})^! (\alpha_* L^a)_W \simeq (\tilde{i}^{\mathcal{A}}_{\mathbb{R}})^{-1} (\alpha_* L^a)_W [-2],$$

by corollary 2.20, as $\mathcal{A} \times \{y\} \times \mathbb{R}$ is clearly transversal to $\mathcal{S}|_{\mathcal{A} \times \mathcal{A} \times \mathbb{R}}$ – recall the morphism labeling from diagram (2.7). In view of [KS90, proposition 5.4.13] resp. proposition 2.19 we now want to show that $\tilde{i}^{\mathcal{P}}_{\mathbb{R}} \colon \mathcal{P} \times \{y\} \times \mathbb{R} \to \mathcal{P} \times \mathcal{A} \times \mathbb{R}$ is non-characteristic for $Ru_*(\alpha_* \hat{L}^a)_{W_{\geqslant}}$. As a shorthand we will write $X := \mathcal{P} \times \mathcal{A} \times \mathbb{R}$ and $Y := \mathcal{A} \times \mathcal{A} \times \mathbb{R}$, as well as $\tilde{i} := \tilde{i}^{\mathcal{P}}_{\mathbb{R}}$. Recall the morphisms

$$T^* Y \underset{=}{\overset{{}^t u'}{\longleftarrow}} Y \times_X T^* X \xrightarrow{u_\pi} T^* X$$

induced by $u \colon Y \to X$. As u is an open embedding, we know that

$$\begin{aligned} \mathrm{CV}((\alpha_* L^a)_W) &= \mathrm{CV}(u^{-1} Ru_* (\alpha_* L^a)_W) \\ &= {}^t u' (u_\pi^{-1} \mathrm{CV}(Ru_*(\alpha_* L^a)_W)) = \pi_X^{-1}(Y) \cap \mathrm{CV}(Ru_*(\alpha_* L^a)_W), \end{aligned}$$

where $\pi_X \colon T^* X \to X$ is the usual projection, cf. [KS90, proposition 5.4.5]. On the other hand, we can tell from [KS90, proposition 6.3.2] that

$$\mathrm{CV}(Ru_*(\alpha_* L^a)_W) \cap \pi_X^{-1}(X \setminus Y) \subset \mathrm{CV}((\alpha_* L^a)_W) \widehat{+} T^*_{X \setminus Y} X.$$

Note that $X \setminus Y = \mathcal{S}_6$ and, as $(\alpha_* L^a)_W$ is cohomologically constructible with respect to the μ-stratification $\mathcal{S} \cap Y$ of Y, we have

$$\mathrm{CV}((\alpha_* L^a)_W) \subset \coprod_{i=0}^{5} T^*_{\mathcal{S}_i} Y$$

by [KS90, proposition 8.4.1], with $T^*_{\mathcal{S}_i} X = T^*_{\mathcal{S}_i} Y$ for $i = 0, \dots, 5$, as $\mathcal{S}_i \subset Y$. This shows

$$\begin{aligned} \mathrm{CV}(Ru_*(\alpha_* L^a)_W) \cap \pi_X^{-1}(\mathcal{S}_6) &\subset \mathrm{CV}((\alpha_* L^a)_W) \widehat{+} T^*_{\mathcal{S}_6} X \\ &\subset \left(\coprod_{i=0}^{5} T^*_{\mathcal{S}_i} Y \right) \widehat{+} T^*_{\mathcal{S}_6} X \end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{i=0}^{5} (T^*_{\mathcal{S}_i} X \widehat{+} T^*_{\mathcal{S}_6} X) \\
&\subset \mathcal{S}_6 \times_X T^*_{\{\infty\}\times\mathcal{A}\times\{0\}} X
\end{aligned}
$$

by the observations summarized in remark 2.25. But $\mathcal{P} \times \{y\} \times \mathbb{R}$ clearly is transversal to $\{\infty\} \times \mathcal{A} \times \{0\}$, in particular

$$\tilde{i}_\pi^{-1}(\mathrm{CV}(Ru_*(\alpha_* L^a)_W)) \cap T^*_{\mathcal{P}\times\{y\}\times\mathbb{R}} X \subset (\mathcal{P} \times \{y\} \times \mathbb{R}) \times_X T^*_X X$$

by the very same argument as in the proof of corollary 2.20. So $\tilde{i}$ is non-characteristic for $Ru_*(\alpha_* L^a)_W$, as desired, and we have shown

$$
\begin{aligned}
(\tilde{i}^{\mathcal{P}}_{\mathbb{R}})^{-1}(Ru_*(\alpha_* L^a)_W) &\simeq (\tilde{i}^{\mathcal{P}}_{\mathbb{R}})^!(Ru_*(\alpha_* L^a)_W)[2] \simeq \\
&\simeq R\tilde{u}_{y,*}(\tilde{i}^{\mathcal{A}}_{\mathbb{R}})^!(\alpha_* L^a)_W)[2] \simeq R\tilde{u}_{y,*}(\tilde{i}^{\mathcal{A}}_{\mathbb{R}})^{-1}(\alpha_* L^a)_W
\end{aligned}
\tag{2.10}
$$

Let us again introduce some shorthand notation. Recall that we write $\mathcal{A}\times\mathbb{R}$ resp. $\mathcal{P}\times\mathbb{R}$ instead of $\mathcal{A}\times\{y\}\times\mathbb{R}$ resp. $\mathcal{P}\times\{y\}\times\mathbb{R}$ if the context is clear. Let us furthermore set

$$\tilde{V} := (\mathcal{P}\times\mathbb{R}) \setminus (\{y\}\times\mathbb{R}) = (\tilde{i}^{\mathcal{P}}_{\mathbb{R}})^{-1}((\mathcal{P}\times\mathcal{A}\times\mathbb{R}) \setminus (\Delta_{\mathcal{A}}\times\mathbb{R})),$$

$V := \tilde{V} \cap (\mathcal{A}\times\mathbb{R})$ and $D := \{y\}\times\mathbb{R} = (\tilde{i}^{\mathcal{P}}_{\mathbb{R}})^{-1}(\Delta_{\mathcal{A}}\times\mathbb{R})$. Let $i_V \colon V \to \mathcal{A}\times\mathbb{R}$ be the open embedding and $\hat{L}^a := (\tilde{i}^{\mathcal{A}}_{\mathbb{R}})^{-1}(\alpha_* L^a)_W$ (and the same of course for $\hat{L}$ with L^a replaced by L) as well as $\hat{\mathcal{K}}^a := i_V^{-1}\hat{L}^a$ (and the same for $\hat{\mathcal{K}} := i_V^{-1}\hat{L}$), then

$$\hat{L}^a = i_{V,!}\hat{\mathcal{K}}^a,$$

where $\hat{\mathcal{K}}^a$ is a local system on V with monodromy $e^{2\pi i\lambda}$ around D. Finally, let us set $\hat{W} := (\tilde{i}^{\mathcal{A}}_{\mathbb{R}})^{-1}W = \{(z,t) \in \mathcal{A}\times\mathbb{R} | t = \mathrm{Re}(z)\}$, and $\tilde{u} := \tilde{u}_y \colon \mathcal{A}\times\mathbb{R} \to \mathcal{P}\times\mathbb{R}$. We now want to determine, for $\tilde{q}_2 \colon \mathcal{P}\times\mathbb{R} \to \mathbb{R}$ the second projection as in diagram (2.7), the cohomologies

$$H^n(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} R\tilde{q}_{2,!}R\tilde{u}_*\hat{L}^a_{\hat{W}}) \simeq H^n(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}}).$$

With embeddings labeled as in the following diagram

$$
\begin{array}{ccccc}
 & & \hat{W} & \xrightarrow{i_{\hat{W}}} & \mathcal{A}\times\mathbb{R} \\
 & & {\scriptstyle \tilde{i}_V}\uparrow & & {\scriptstyle i_V}\uparrow \\
\hat{W} \setminus \{(y, \mathrm{Re}(y))\} & = & \hat{W} \cap V & \xrightarrow{\tilde{i}_{\hat{W}}} & V,
\end{array}
$$

we have that

$$\hat{L}^a|_{\hat{W}} \simeq \tilde{i}_{V,!}\hat{\mathcal{K}}^a|_{\hat{W}\cap V}. \tag{2.11}$$

As $\hat{\mathcal{K}}^a$ has monodromy $e^{2\pi i\lambda} \in \mathbb{C}\setminus\{1\}$ around $(y, \mathrm{Re}(y)) \in \hat{W}$, we have the following

Lemma 2.26. *The restrictions* $\mathcal{K}_< := R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}}|_{(-\infty,\mathrm{Re}(y))}$ *and* $\mathcal{K}_> := R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}}|_{(\mathrm{Re}(y),\infty)}$ *are local systems and*

$$R\tilde{q}_{2,!}R\tilde{u}_*\hat{L}^a_{\hat{W}} \simeq R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}} \simeq L_> \oplus L_<,$$

where $L_> := i_{(\mathrm{Re}(y),\infty),!}\mathcal{K}_>$ *and* $L_< := i_{(-\infty,\mathrm{Re}(y)),!}\mathcal{K}_<$, *for* $i_{(a,b)}\colon (a,b) \to \mathbb{R}$ *the obvious embeddings.*

Proof. First, recall that $R^i\tilde{p}_{2,*}\hat{L}^a_{\hat{W}}$ is the sheaf associated to the presheaf

$$\begin{aligned}\mathbb{R} \supset U \mapsto & H^i(\tilde{p}_2^{-1}(U), \hat{L}^a_{\hat{W}}|_{\tilde{p}_2^{-1}(U)}) \\ =& H^i(\mathcal{A} \times U, \hat{L}^a_{\hat{W}}|_{\mathcal{A}\times U}) \\ \simeq& H^i((\mathcal{A} \times U) \cap \hat{W}, \hat{L}^a|_{(\mathcal{A}\times U)\cap\hat{W}}) \simeq \begin{cases}\mathbb{C} & \text{if } i = 0 \text{ and } \mathrm{Re}(y) \notin U \\ 0 & \text{else,}\end{cases}\end{aligned}$$

cf. (2.11) and [Dim04, theorem 3.4.4]. This in particular shows that $(R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}})_{\mathrm{Re}(y)} \simeq 0$, which means $R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}} \simeq L_> \oplus L_<$ as was claimed above, where we still have to show that $\mathcal{K}_<, \mathcal{K}_>$ are local systems. To do so, we might argue exactly as in the proof of lemma 2.22, or slightly shorten the proof by the following observation: Let $U := (-\infty, \mathrm{Re}(y))$ (the case for $U = (\mathrm{Re}(y), \infty)$ works completely analogous) and $V := \{z |\, \mathrm{Re}(z) \in U\} \subset \mathcal{A}$. Let us furthermore write $Y := V \times U$ and denote the projections by $\mathrm{pr}_U\colon Y \to U$, $\mathrm{pr}_V\colon Y \to V$. Finally, let $\mathrm{pt}_V\colon V \to \{*\}$ and $\mathrm{pt}_U\colon U \to \{*\}$ be the canonical morphisms to a point, as depicted in the following (cartesian) square:

$$\begin{tikzcd} Y \arrow[r, "\mathrm{pr}_U"] \arrow[d, "\mathrm{pr}_V"'] & U \arrow[d, "\mathrm{pt}_U"] \\ V \arrow[r, "\mathrm{pt}_V"'] & \{*\} \end{tikzcd}$$

Clearly, $\hat{L}^a$ is a local system on Y of the form $\hat{L}^a = \mathrm{pr}_V^{-1}F$ for some local system F on V. From the cartesian square above, we may thus read off

$$R\mathrm{pr}_{U,*}\hat{L}^a|_Y \simeq R\mathrm{pr}_{U,*}\mathrm{pr}_V^{-1}F \simeq R\mathrm{pr}_{U,*}\mathrm{pr}_V^![-1] \simeq \mathrm{pt}_U^![-1]R\mathrm{pt}_{V,*}F \simeq \mathrm{pt}_U^{-1}(\mathrm{pt}_{V,*}F)$$

is a local system (note that pr_V resp. pt_U are compositions of open embeddings and topological submersions of fiber dimension 1, so we used $j^{-1} \simeq j^!$ for any open embedding and [KS90, proposition 3.3.2]).

The proof is then finished by observing that, for $Z := (\mathcal{A} \times U) \cap \hat{W} = Y \cap \hat{W}$, denoting by $i_Z\colon Z \to Y$ the closed embedding and $\tilde{\mathrm{pr}}_U\colon Z \to U$ the projection induced by pr_U, we have

$$\mathrm{pr}_{U,*}\hat{L}^a_Z \simeq \tilde{\mathrm{pr}}_{U,*}\hat{L}^a|_Z,$$

and for any $U' \subset U$, the closed embedding $i_Z|_{U'}\colon \tilde{\mathrm{pr}}_U^{-1}(U') \to \mathrm{pr}_U^{-1}(U')$ clearly is a homotopy equivalence, so in the above calculation,

$$H^i(\tilde{\mathrm{pr}}_U^{-1}(U'), \hat{L}^a|_{\tilde{\mathrm{pr}}_U^{-1}(U')}) \simeq H^i(\mathrm{pr}_U^{-1}(U'), \hat{L}^a|_{\mathrm{pr}_U^{-1}(U')})$$

for all i (compatible with restrictions). Thus

$$\mathcal{K}_< \simeq R\mathrm{pr}_{U,*}\hat{L}^a_Z \simeq R\tilde{\mathrm{pr}}_{U,*}\hat{L}^a|_Z \simeq R\mathrm{pr}_{U,*}\hat{L}^a|_Y$$

is a local system. $\square$

By the above lemma, $R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}} \simeq L_> \oplus L_<$ is concentrated in degree 0. In particular, $\iota_{\mathbf{A}\times\mathbb{R}_\infty}(R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}}) \in D^0(\mathbf{A}\times\mathbb{R}_\infty)$. Recall however that $\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} (\bullet)$ is not an exact functor, but we know

$$\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} \iota_{\mathbf{A}\times\mathbb{R}_\infty}(R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}}) \simeq \iota_{\mathbf{A}\times\mathbb{R}_\infty}(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}}) \in D^{[0,1]}(\mathbf{A}\times\mathbb{R}_\infty).$$

Lemma 2.27. *We have*

$$\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}} \in D^1_{\mathbb{R}-c}(\mathbb{C}_{\mathbf{A}\times\mathbb{R}_\infty})$$

(resp. $\iota_{\mathbf{A}\times\mathbb{R}_\infty}(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}}) \in D^1(\mathbf{A}\times\mathbb{R}_\infty)$*).*

Proof. Recall from lemma 2.26 that $R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}} \simeq L_< \oplus L_>$. We apply $\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} (\bullet)$ to the distinguished triangle associated to the split short exact sequence

$$0 \longrightarrow L_< \longrightarrow L_< \oplus L_> \longrightarrow L_> \longrightarrow 0$$

and consider the associated cohomology long exact sequence

$$\begin{aligned} \ldots \longrightarrow H^{i-1}(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} L_>) \longrightarrow H^i(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} L_<) \longrightarrow H^i(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} (L_< \oplus L_>)) \longrightarrow \\ \longrightarrow H^i(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} L_>) \longrightarrow H^{i+1}(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} L_<) \longrightarrow \ldots \end{aligned}$$

We find that $H^i(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} R\tilde{p}_{2,*}\hat{L}^a_{\hat{W}}) \simeq H^i(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} (L_< \oplus L_>))$ is caught between the terms $H^i(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} L_\square)$ for $\square \in \{<,>\}$. It thus obviously suffices to prove $H^0(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} L_\square) \simeq 0$ for $\square \in \{<,>\}$. To do so, we might either explicitly consider stalks of

$$\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} L_\square = R\mu_!(p_1^{-1}\mathbb{C}_{\{t\geqslant 0\}} \otimes p_2^{-1}L_\square),$$

or, a little faster, observe the following: First consider $L_<$. Set $U_< := (-\infty, \mathrm{Re}(y))$ and denote by $i_{U_<}: U_< \to \mathbb{R}$ the open embedding. Then, by definition, $L_< = i_{U_<,!}\mathcal{K}_<$ (recall the notation from lemma 2.26). For the morphisms

$$\begin{tikzcd} U_< \arrow[r, "i_{U_<}"] \arrow[dr, "\mathrm{pt}_{U_<}"'] & \mathbb{R} \arrow[d, "\pi"] \\ & \{*\} \end{tikzcd}$$

where $\{*\}$ denotes the one point set, let us identify $\{*\}$ with some point $s \in U_<$ and write $K := (\mathcal{K}_<)_s$. Then, as a local system on $U_<$, $\mathcal{K}_< \simeq \mathrm{pt}_{U_<}^{-1} K$ (cf. remark 2.23), in particular

$$\mathcal{K}_< = i_{U_<}^{-1}\tilde{\mathcal{K}}_< \quad \text{with} \quad \tilde{\mathcal{K}}_< := \pi^{-1}K.$$

So we get

$$L_< = i_{U_<,!}\mathcal{K}_< \simeq i_{U_<,!}i_{U_<}^{-1}\tilde{\mathcal{K}}_< \simeq \mathbb{C}_{U_<} \otimes \pi^{-1}K$$

and finally, with $\mathbb{C}_{U_<} = \mathbb{C}_{\{t<\mathrm{Re}(y)\}} \simeq \mathbb{C}_{\{t=\mathrm{Re}(y)\}} \overset{+}{\otimes} \mathbb{C}_{\{t<0\}}$,

$$\begin{aligned} \mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} L_< &\simeq \mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} (\mathbb{C}_{U_<} \otimes \pi^{-1}K) \simeq \\ &\simeq (\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} \mathbb{C}_{U_<}) \otimes \pi^{-1}K \simeq (\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} (\mathbb{C}_{\{t=\mathrm{Re}(y)\}} \overset{+}{\otimes} \mathbb{C}_{\{t<0\}})) \otimes \pi^{-1}K \simeq \\ &\simeq ((\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} \mathbb{C}_{\{t<0\}}) \overset{+}{\otimes} \mathbb{C}_{\{t=\mathrm{Re}(y)\}}) \otimes \pi^{-1}K \simeq \\ &\simeq (\mathbb{C}_{\{t\geqslant \mathrm{Re}(y)\}} \otimes \pi^{-1}K)[-1] \in D^1_{\mathbb{R}-c}(\mathbb{C}_{\mathbf{A}\times\mathbb{R}_\infty}), \end{aligned}$$

where we used lemma 1.21 and the fact that

$$\mathbb{C}_{\{t=\mathrm{Re}(y)\}} \overset{+}{\otimes} (\bullet) \simeq R\mu_{\mathrm{Re}(y),*}(\bullet)$$

is exact, where μ_a for some $a \in \mathbb{R}$ denotes the translation $\mu_a\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto t + a$, cf. [DK16b, lemma 4.2.1]. Completely analogous we get, for $U_> := (\mathrm{Re}(y), \infty)$, $i_{U_>}$ the associated embedding and $K' := (\mathcal{K}_>)_s$ for some $s \in U_>$, that

$$L_> \simeq i_{U_>}^{-1}\tilde{\mathcal{K}}_> \quad \text{with} \quad \tilde{\mathcal{K}}_> := \pi^{-1}K'$$

and thus

$$\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} L_> \simeq ((\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} \mathbb{C}_{\{t>0\}}) \overset{+}{\otimes} \mathbb{C}_{\{t=\mathrm{Re}(y)\}}) \otimes \pi^{-1}K' \simeq 0,$$

cf. lemma 1.21. □

To conclude, we have proven the following

Lemma 2.28. *We have*

$$\mathrm{D}^E_{\mathbf{A}}(K_![2]) \simeq \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},*}L^a_W)[2] \in E^{-1}_{\mathbb{R}-c}(\mathbf{A}).$$

Note that, as we already stated, the two remaining cases, $K_*[2], \mathrm{D}^E_{\mathbf{A}}(K_*[2]) \in E^{-1}_{\mathbb{R}-c}(\mathbf{A})$ work completely analogous, so in view of remark 2.12 in section 2.4.4, all that remains to show is

Lemma 2.29. *For any $Z \in \mathrm{CS}^{\leqslant 0}(\mathbf{A})$, we have*

$$Ei^!_{Z_\infty} G \in E^{\geqslant 0}_{\mathbb{R}-c}(Z_\infty)$$

for

i) $G = K_![2] = \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},!}L_{W^a})[2]$,

ii) $G = \mathrm{D}^E_{\mathbf{A}}(K_![2]) = \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},*}L^a_W)[2]$,

iii) $G = K_*[2] = \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota(R\sigma_{\mathbb{R},*}L_{W^a})[2]$,

iv) $G = \mathrm{D}^E_{\mathbf{A}}(K_*[2]) = \mathbb{C}^E_{\mathbf{A}} \overset{+}{\otimes} Q\iota(R\sigma_{R,*}L^a_W)[2]$.

Proof. Again it clearly suffices to consider the cases i) and ii). Without loss of generality we may furthermore assume that $Z = \{y\}$ is a single point. Recall the labeling from (2.7). In particular, we again will denote the closed embedding $Z = \{y\} \to \mathcal{A}$ by i and write $i_{\mathbb{R}} \colon \{y\} \times \mathbb{R} \to \mathcal{A} \times \mathbb{R}$ for the induced embedding, as in (2.7). From what we have shown above, we already know that $\tilde{i}^{\mathcal{P}}_{\mathbb{R}}$ (resp. $\tilde{i}^{\mathcal{A}}_{\mathbb{R}}$) is non-characteristic for $Ru_*\alpha_*L^a_W$ and $Ru_!\alpha_*L_{W^a_\geqslant}$ (resp. $\alpha_*L^a_W$), in particular

$$(\tilde{i}^{\mathcal{P}}_{\mathbb{R}})^! Ru_*(\alpha_*L^a_W) \simeq (\tilde{i}^{\mathcal{P}}_{\mathbb{R}})^{-1} Ru_*L^a_W)[-2],$$

so

$$\begin{aligned}
i^!_{\mathbb{R}} \left(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} R\sigma_{\mathbb{R},*}(\alpha_*L^a_W) \right) &\simeq \mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} i^!_{\mathbb{R}} Rp_{2,*}\alpha_*L^a_) \simeq \\
&\simeq \mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} R\tilde{p}_{2,*}((\tilde{i}^{\mathcal{A}}_{\mathbb{R}})^! \alpha_*L^a_W) \simeq \mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} R\tilde{p}_{2,*}((\tilde{i}^{\mathcal{A}}_{\mathbb{R}})^{-1}\alpha_*L^a_W[-2]) \simeq \\
&\simeq \left(\mathbb{C}_{\{t\geqslant 0\}} \overset{+}{\otimes} R\tilde{p}_{2,*}(\tilde{i}^{\mathcal{A}}_{\mathbb{R}})^{-1}\alpha_*L^a_W \right)[-2] \in D^{\geqslant 2}_{\mathbb{R}-c}(\mathbb{C}_{\mathbf{A}\times\mathbb{R}_\infty}),
\end{aligned}$$

which proves ii). Similarly,

$$i^!_{\mathbb{R}}(Rq_{2,!}Ru_!\alpha_*L_{W^a_\geqslant}) \simeq R\tilde{q}_{2,!}(\tilde{i}^{\mathcal{P}}_{\mathbb{R}})^{-1}Ru_!\alpha_*L_{W^a_\geqslant}[-2] \in D^{\geqslant 2}_{\mathbb{R}-c}(\mathbb{C}_{\mathbf{A}\times\mathbb{R}_\infty})$$

shows i). $\square$

So, we have finally proven all parts (cf. remark 2.12) of

Theorem 2.30. *The pair* $(E^w[1], L_\lambda^E[1])$ *has property* $\mathfrak{P}$.

Proof. We combine lemmata 2.21, 2.28 and 2.29. $\square$

3 Enhanced middle extensions

In sections 2.7 and 2.8 of [Kat95], the construction of a so called "middle direct image" is introduced and the middle convolution is expressed in terms of it. In the context of (enhanced) sheaves we will restrict ourselves to the case of (bordered) open embeddings and refer to the corresponding type of construction as a *middle extension*, which is kind of a mixture between the terminology cited above and the one of [HTT08], where the term "minimal extension" is used (for the case of locally closed embeddings).

To be precise, we would like to consider the following situation: For a bordered space $\mathbf{X} = (X, \check{X})$, let $j\colon U_\infty \to \mathbf{X}$ denote the bordered open embedding associated to some open $U \subset X$. For $K \in {}^{1/2}E^0_{\mathbb{R}-c}(U_\infty)$ we may define

$$\begin{aligned} Ej_{!*}K &:= \mathrm{Im}({}^{1/2}H^0 Ej_{!!}K \to {}^{1/2}H^0 Ej_*K), \\ Ej^{\mathrm{co}}_{!*}K &:= \mathrm{Coim}({}^{1/2}H^0 Ej_{!!}K \to {}^{1/2}H^0 Ej_*K), \end{aligned}$$

where the splitting in middle and co-middle extension again is due to the fact that image and coimage do not necessarily need to coincide in the quasi-abelian category ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{X})$. This is the very same phenomenon that we already encountered in the definition of the enhanced middle convolution in the previous section. Recall that the remaining of the two main issues about this definition of our enhanced middle convolution, as stated in the introduction or at the beginning of section 2.4, was if there can be given some criterion for when enhanced middle and co-middle convolution agree for some given pair (K, L) (with property $\mathfrak{P}$). Adapting the techniques of [Kat95, sections 2.7, 2.8] to the enhanced setting, we will be able to reduce this problem to examining if a certain enhanced middle extension coincides with its co-middle version – to be precise, theorem 3.14 will prove that for the (bordered) open embedding

$$u\colon \mathbf{A} \times \mathbf{A} \to \mathcal{P} \times \mathbf{A}$$

and a pair (K, L) with property $\mathfrak{P}$ such that $Eu_{!!}(K \overset{+}{\boxtimes} L), Eu_*(K \overset{+}{\boxtimes} L) \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathcal{P} \times \mathbf{A})$ and $Eu_{!*}(K \overset{+}{\boxtimes} L) \simeq Eu^{\mathrm{co}}_{!*}(K \overset{+}{\boxtimes} L)$, we have

$$K \overset{E}{*}_{\mathrm{mid}} L \simeq K \overset{E}{*}_{\mathrm{co-mid}} L.$$

Despite, at first glance, deciding if splitting occurs in the middle extension situation seems to be a similar problem, compared to the original question about splitting in the case of enhanced middle convolutions, it turns out that theorem 3.14 indeed simplifies things substantially, as, with proposition 3.7, we will be able to transfer some well known characterization result concerning classical middle extensions of perverse sheaves to the enhanced setting, giving us a criterion for when precisely some enhanced perverse sheaf is the enhanced middle resp. co-middle extension of its restriction to some bordered open subspace, which in particular naturally contains a description of the special case of coincidence of middle and co-middle version (corollary 3.8). Finally, we will apply these results to our example pair $(E^w[1], L^E_\lambda[1])$ of section 2.4.

As there is an immediate interplay between the middle extension in the setting of classical perverse sheaves on the one, and the minimal extension of regular integral connections on the other hand, provided by the classical Riemann–Hilbert correspondence, one might at first sight hope for some similar relation in the irregular resp. enhanced setting. This can not happen though, for reasons that will become obvious along the definitions of the middle extensions given below (cf. remark 3.2), so it seems that the further benefits of this enhanced (co-)middle extension on its own are questionable.

3.1 Definition

Consider the classical construction of the middle extension of a perverse sheaf: For the open embedding $j\colon U \to X$ of a Zariski-open (dense) subset of some irreducible analytic space (or algebraic variety) and some perverse sheaf L on U such that $Rj_!L$ and Rj_*L are constructible again, the middle extension $j_{!*}L$ of L is defined as

$$j_{!*}L := \operatorname{Im}\left({}^{1/2}H^0 Rj_! L \to {}^{1/2}H^0 Rj_* L\right) \in {}^{1/2}D^0_{\mathbb{C}-c}(X).$$

For the purpose of better coping with the (co-)image splitting in the enhanced setting, let us recall the idea behind the proof that ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{X})$ (or, more generally the heart of any generalized t-structure) is quasi-abelian (cf. [DK16a, proposition 1.3.1] resp. [Bri07, lemma 4.3]). The truncation functors $\tau^{<c}$ for some c, associated to the generalized t-structure $({}^{1/2}E^{\leqslant c}_{\mathbb{R}-c}(\mathbf{X}), {}^{1/2}E^{\geqslant c}_{\mathbb{R}-c}(\mathbf{X}))$ are right adjoint to the embedding

$${}^{1/2}E^{<c}_{\mathbb{R}-c}(\mathbf{X}) \to E_{\mathbb{R}-c}(\mathbf{X})$$

(cf. [Kas15, section 1]). In particular, if $K \to L$ is a monomorphism in ${}^{1/2}E^{(-1,0]}_{\mathbb{R}-c}(\mathbf{X})$ with $L \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{X})$, then

$$\tau^{<0}K \to \tau^{<0}L \simeq 0$$

is a monomorphism in ${}^{1/2}E^{-1/2}_{\mathbb{R}-c}(\mathbf{X}) \subset {}^{1/2}E^{<0}_{\mathbb{R}-c}(\mathbf{X})$ and thus $\tau^{<0}K \simeq 0$ and K is in ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{X})$ as well.

Analogously, if $L \to K$ is an epimorphism in ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$ with $L \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{X})$, then K is in ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{X})$. As ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$ and $E^{(-1,0]}_{\mathbb{R}-c}(\mathbf{X})$ are abelian, this proves ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{X})$ is quasi-abelian (cf. [Bri07, lemma 4.2]) and images and cokernels (resp. coimages and kernels) can be computed in $E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$ (resp. ${}^{1/2}E^{(-1,0]}_{\mathbb{R}-c}(\mathbf{X})$), cf. [Sch98, lemma 1.2.34] and [DK16a]. In particular, t-exact endofunctors on $E_{\mathbb{R}-c}(\mathbf{X})$ preserve images (and coimages) in $E^0_{\mathbb{R}-c}(\mathbf{X})$, cf. lemma 1.50.

Let $\mathbf{X} = (X, \check{X})$ be a bordered space, $U \subset X$ open, and denote by $j\colon U_\infty \to \mathbf{X}$ the induced embedding of bordered spaces. As j is semi-proper, $Ej_{!!}$ and Ej_* preserve $\mathbb{R}$-constructibility (cf. [DK16a, proposition 3.3.3]).

Definition 3.1 (Enhanced middle extensions). In the above setting, for $L \in {}^{1/2}E^0_{\mathbb{R}-c}(U_\infty)$, we call

$$Ej_{!*}L := \mathrm{Im}({}^{1/2}H^0 Ej_{!!}L \to {}^{1/2}H^0 Ej_* L)$$

the *enhanced middle extension* of L to $\mathbf{X}$. Dually we would like to refer to

$$Ej^{\mathrm{co}}_{!*}L := \mathrm{Coim}({}^{1/2}H^0 Ej_{!!}L \to {}^{1/2}H^0 Ej_* L)$$

as the *enhanced co-middle extension.* By definition, both versions agree if and only if ${}^{1/2}H^0 Ej_{!!}L \to {}^{1/2}H^0 Ej_* L$ is a strict morphism in ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{X})$.

Remark 3.2. Recall the following well known compatibility of middle extensions and the Riemann–Hilbert correspondence in the classical setting, cf. [HTT08, remark 7.2.10]. Let X be a smooth complex variety and $U \subset X$ open (actually the same correspondence would work for U some open dense subset of the regular part of some locally closed subvariety $Z \subset X$, as described in [HTT08]). Let furthermore $\mathscr{L}$ be a regular integrable connection on U and let L denote the local system on U^{an} corresponding to $\mathscr{L}$ via the Riemann–Hilbert correspondence (i. e. $DR_{U^{\mathrm{an}}}(\mathscr{L}^{\mathrm{an}}) = L[d^{\mathbb{C}}_U] \in \mathrm{Perv}(\mathbb{C}_{U^{\mathrm{an}}})$). Writing $j\colon U \to X$ for the open embedding, we then have

$$DR_{X^{\mathrm{an}}}((Dj_{!*}\mathscr{L})^{\mathrm{an}}) \simeq j^{\mathrm{an}}_{!*}(L[d^{\mathbb{C}}_U]) = j^{\mathrm{an}}_{!*} DR_{U^{\mathrm{an}}}(\mathscr{L}^{\mathrm{an}}). \tag{3.1}$$

In the irregular setting, the situation is very different. For example, consider $X = \mathbb{A}^1$ and $U = \mathbb{A}^1 \setminus \{0\}$. Let us denote by x the affine coordinate on $\mathbb{A}^1$. Then,

$$\mathscr{L}_1 := \mathscr{D}_U / \mathscr{D}_U \partial_x \simeq \mathscr{O}_U$$

clearly is a regular integrable connection on U. On the other hand, let

$$\mathscr{L}_2 := \mathscr{D}_U / \mathscr{D}_U (x^2\partial_x - 1),$$

which is an irregular integrable connection on U (both examples are taken from [HTT08, example 5.1.24]). In particular, with $j\colon \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ denoting the open embedding,

$$(j_*\mathscr{L}_1)^{\mathrm{an}} \simeq (Dj_{!*}\mathscr{L}_1)^{\mathrm{an}} \not\simeq (Dj_{!*}\mathscr{L}_2)^{\mathrm{an}} \simeq (j_*\mathscr{L}_2)^{\mathrm{an}},$$

as the left hand side is regular while the right hand side is irregular. But we also have $\mathscr{L}_1^{\mathrm{an}} \simeq \mathscr{L}_2^{\mathrm{an}}$ (cf. [HTT08, example 5.1.24]), so in particular

$$DR^E_{U^{\mathrm{an}}}(\mathscr{L}_1^{\mathrm{an}}) \simeq DR^E_{U^{\mathrm{an}}}(\mathscr{L}_2^{\mathrm{an}}),$$

which shows that there can clearly be no enhanced analog of (3.1).

Example 3.3. Let $\mathbf{X} := \mathbb{R}$ and $U = \mathbb{R} \setminus \{0\}$. Consider the following example taken from [DK16a, example 3.5.10]: Set $S := \{x > 0, 0 \leqslant t < 1/x\} \cup \{x = 0, t \geqslant 0\} \subset X \times \mathbb{R}$ and

$$K := \mathbb{C}^E_S[1] = \mathbb{C}^E \overset{+}{\otimes} Q\iota(\mathbb{C}_S)[1].$$

Then, as calculated in [DK16a], we have $K \in {}^{1/2}E^0_{\mathbb{R}-c}(R)$, as well as

$$Ei^!_{\{0\}}K \simeq 0, \quad Ei^{-1}_{\{0\}}K \simeq \mathbb{C}^E_{\{0\}}[1] \in {}^{1/2}E^{-1}_{\mathbb{R}-c}(\{0\})$$

for $i_{\{0\}} \colon \{0\} \to R$ the corresponding closed embedding (cf. also [DK16a, example 3.5.9]). By corollary 3.8 below, K is the enhanced middle extension of its quotient, to be precise,

$$K \simeq Ei_{U_\infty,!*}Ei^{-1}_{U_\infty}K \simeq Ei^{\mathrm{co}}_{U_\infty,!*}Ei^{-1}_{U_\infty}K.$$

Now consider the following slight variation of this example: Set

$$S' := \{x < 0, 1/x \leqslant t < 0\} \cup \{x = 0, t < 0\}$$

and

$$L := \mathbb{C}^E_{S'}[1] = \mathbb{C}^E \overset{+}{\otimes} Q\iota(\mathbb{C}_{S'})[1] \in E^{[-1,0]}_{\mathbb{R}-c}(\mathbb{R}).$$

To compute $\mathrm{D}^E L$, we consider the distinguished triangle

$$a^{-1}\mathbb{C}_{S'} \longrightarrow \mathbb{C}_{\{x=0,t\geqslant 0\}\cup\{x<0,-1/x\geqslant t\geqslant 0\}} \longrightarrow \mathbb{C}_{\{x\leqslant 0,t=0\}} \xrightarrow{+1}$$

(where a, as usual, is the antipodal map induced by $\mathbb{R} \to \mathbb{R}$, $t \mapsto -t$) and observe that $\mathbb{C}_{\{x=0,t\geqslant 0\}\cup\{x<0,-1/x\geqslant t\geqslant 0\}} \simeq Rj_*\mathbb{C}_{\{x<0,-1/x>t>0\}}$ for $j \colon \{x < 0, 1/x > t > 0\} \to \mathbb{R}^2$, such that the dual triangle is

$$\mathrm{D}(\mathbb{C}_{\{x\leqslant 0,t=0\}}) \longrightarrow \mathbb{C}_{\{x<0,-1/x>t>0\}}[2] \longrightarrow a^{-1}\,\mathrm{D}\,\mathbb{C}_{S'} \xrightarrow{+1} . \tag{3.2}$$

It remains to determine $\mathrm{D}(\mathbb{C}_{\{x\leqslant 0,t=0\}})$. For that, take the distinguished triangle

$$\mathbb{C}_{\{x<0,t=0\}} \longrightarrow \mathbb{C}_{\{x\leqslant 0,t=0\}} \longrightarrow \mathbb{C}_{\{x=0,t=0\}} \xrightarrow{+1}$$

which, by applying duality, gives us a triangle

$$\underbrace{\mathrm{D}\,\mathbb{C}_{\{x=0,t=0\}}}_{\simeq\mathbb{C}_{\{x=0,t=0\}}} \longrightarrow \mathrm{D}\,\mathbb{C}_{\{x\leqslant 0,t=0\}} \longrightarrow \underbrace{\mathrm{D}\,\mathbb{C}_{\{x<0,t=0\}}}_{\simeq\mathbb{C}_{\{x\leqslant 0,t=0\}}[1]} \xrightarrow{+1},$$

with $i^!_{\mathbb{R}}\mathbb{C}_{\mathbb{R}^2} \simeq i^{-1}_{\mathbb{R}}\mathbb{C}_{\mathbb{R}^2}[-1]$ for the embedding $i_{\mathbb{R}} \colon \mathbb{R} \simeq \{x, t = 0\} \to \mathbb{R}^2$, and analogously for the embeddings of $\{x < 0, t = 0\}$ and the point $\{x = 0, t = 0\}$. Thus we get a long exact sequence

$$\ldots \longrightarrow 0 \longrightarrow H^{-1}(\mathrm{D}\,\mathbb{C}_{\{x\leqslant 0,t=0\}}) \longrightarrow \mathbb{C}_{\{x\leqslant 0,t=0\}} \longrightarrow \mathbb{C}_{\{x=0,t=0\}} \longrightarrow$$
$$\longrightarrow H^{0}(\mathrm{D}\,\mathbb{C}_{\{x\leqslant 0,t=0\}}) \longrightarrow 0 \longrightarrow \ldots$$

where we may observe that $H^0(\mathrm{D}\,\mathbb{C}_{\{x\leqslant 0,t=0\}}) = \big(\Gamma_{\{x\leqslant 0,t=0\}}(\mathbb{C}_{\mathbb{R}^2})\big)_{\{x\leqslant 0,t=0\}} \simeq 0$ and get $\mathrm{D}\,\mathbb{C}_{\{x\leqslant 0,t=0\}} \simeq \mathbb{C}_{\{x<0,t=0\}}[1]$. So the long exact cohomology sequence associated to (3.2) is

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{C}_{\{x<0,-1/x>t>0\}} \longrightarrow H^{-2}(a^{-1}\,\mathrm{D}\,\mathbb{C}_{S'}) \longrightarrow \mathbb{C}_{\{x<0,t=0\}} \longrightarrow 0 \longrightarrow \ldots,$$

showing that $a^{-1}\,\mathrm{D}\,\mathbb{C}_{S'} = \mathbb{C}_{\{x<0,-1/x>t\geqslant 0\}}[2]$ and thus

$$\mathrm{D}^E\,L \simeq \mathbb{C}^E_{\{x<0,-1/x>t\geqslant 0\}}[1] = \mathbb{C}^E \overset{+}{\otimes} Q\iota(\mathbb{C}_{\{x<0,-1/x>t\geqslant 0\}})[1]$$

(cf. [DK16b, proposition 4.8.3]). Now $Ei^{-1}_{\{x\}}\,\mathrm{D}^E\,L \simeq 0$ for all $x \in \mathbb{R}$ (cf. [DK16a, example 3.5.9]), i. e.

$$Ei^!_{\{x\}}L \simeq \mathrm{D}^E\,Ei^{-1}_{\{x\}}\,\mathrm{D}^E\,L \simeq 0$$

for all $x \in X$, and $L \in E^{-1}_{\mathbb{R}-c}(X)$. On the other hand, $\mathrm{D}^E\,L \in E^{-1}_{\mathbb{R}-c}(X)$ as well, and for all $x \in \mathbb{R}$, we have

$$Ei^!_{\{x\}}\,\mathrm{D}^E\,L \simeq \mathrm{D}^E\,Ei^{-1}_{\{x\}}L \simeq \begin{cases} 0, & \text{if } x \neq 0 \\ \mathrm{D}^E(\mathbb{C}^E_{\{0\}}) \simeq \mathbb{C}^E_{\{0\}} \in E^0_{\mathbb{R}-c}(X) & \text{if } x = 0, \end{cases}$$

(cf. lemma 1.21), so $L \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbb{R})$. But here, as we just stated,

$$Ei^{-1}_{\{0\}}L \simeq \mathbb{C}^E_{\{0\}} \in {}^{1/2}E^0_{\mathbb{R}-c}(\{0\})$$

by construction, so proposition 3.7 below tells us that L can not be the enhanced middle (or co-middle) extension of its quotient on $(X \setminus \{0\})_\infty$, that is,

$$Ei_{(\mathbb{R}\setminus\{0\})_\infty,!*}Ei^{-1}_{(\mathbb{R}\setminus\{0\})_\infty}L \not\simeq L \not\simeq Ei^{\mathrm{co}}_{(\mathbb{R}\setminus\{0\})_\infty,!*}Ei^{-1}_{(\mathbb{R}\setminus\{0\})_\infty}L.$$

3.2 Characterization of enhanced middle extensions

Let X and $U \subset X$, resp. $\mathbf{X}$ and $U_\infty \subset \mathbf{X}$, be as above. As a preparation for the rest of this section, we would like to state a simple analogon to the following classical result (cf. [HTT08, proposition 8.2.3]):

Proposition 3.4. *For any $L \in \mathrm{Perv}(\mathbb{C}_U)$, one has $\mathrm{D}_X(j_{!*}L) \simeq j_{!*}(\mathrm{D}_U\,L)$.*

Let us have a look at the proof given in [HTT08] and check to what extent it carries over to the enhanced setting. By definition, $Ej_{!*}L$ can be computed as the image of

${}^{1/2}H^0 Ej_{!!}L \to {}^{1/2}H^0 Ej_* L$ in ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$ and thus is determined by the image factorization

$$ {}^{1/2}H^0 Ej_{!!}L \twoheadrightarrow Ej_{!*}L \hookrightarrow {}^{1/2}H^0 Ej_* L $$

in ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$. By definition of the enhanced middle perversity t-structure (cf. [DK16a, definition 3.5.8]), $\mathrm{D}^E_{\mathbf{X}}$ is an exact functor from ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})^{\mathrm{op}}$ to ${}^{1/2}E^{(-1,0]}_{\mathbb{R}-c}(\mathbf{X})$. So

$$ \underbrace{\mathrm{D}^E_{\mathbf{X}}\, {}^{1/2}H^0 Ej_*(L)}_{\simeq {}^{1/2}H^0 Ej_{!!}(\mathrm{D}^E_{U_\infty} L)} \twoheadrightarrow \mathrm{D}^E_{\mathbf{X}}\, Ej_{!*}L \hookrightarrow \underbrace{\mathrm{D}^E_{\mathbf{X}}\, {}^{1/2}H^0 Ej_* L}_{\simeq {}^{1/2}H^0 Ej_*(\mathrm{D}^E_{U_\infty} L)} $$

is an image factorization in ${}^{1/2}E^{(-1,0]}_{\mathbb{R}-c}(\mathbf{X})$, immediately proving the following

Lemma 3.5. *For $U \subset X$ and $L \in {}^{1/2}E^0_{\mathbb{R}-c}(U_\infty)$ as above, we have:*

i) $\mathrm{D}^E_{\mathbf{X}}(Ej_{!*}L) \simeq Ej^{\mathrm{co}}_{!*}(\mathrm{D}^E_{U_\infty} L)$.

ii) The canonical morphism

$$ \mathrm{D}^E_{\mathbf{X}}(Ej_{!*}L) \to Ej_{!*}(\mathrm{D}^E_{U_\infty} L) $$

is an isomorphism if and only if the natural morphism ${}^{1/2}H^0 Ej_{!!}L \to {}^{1/2}H^0 Ej_* L$ is strict.

Now, consider the following well known characterization result on classical middle extension perverse sheaves (cf. [HTT08, proposition 8.2.5]).

Proposition 3.6. *Let $U \subset X$ and $L \in \mathrm{Perv}(\mathbb{C}_U)$ be as above, $G = j_{!*}L$ be the middle extension of L to X and finally denote by $i\colon Z := X \setminus U \to X$ the closed embedding of the complement of U in X. Then*

i) $G|_U \simeq L$,

ii) $i^{-1}G \in {}^{1/2}D^{\leqslant -1}_{\mathbb{C}-c}(Z)$,

iii) $i^! G \in {}^{1/2}D^{\geqslant 1}_{\mathbb{C}-c}(Z)$.

Any other perverse sheaf on X satisfying i) – iii) is canonically isomorphic to G.

Let us find the analogon to this characterization in the enhanced setting (so we replace X by a bordered space $\mathbf{X} = (X, \check{X})$ and $U \subset X$ with the induced embedding $U_\infty \subset \mathbf{X}$) by adapting the proof of proposition 3.6 given in [HTT08]. Let $L \in {}^{1/2}E^0_{\mathbb{R}-c}(U_\infty)$ and recall the definition of the enhanced middle extensions,

$$ \begin{aligned} Ej_{!*}L &:= \mathrm{Im}({}^{1/2}H^0 Ej_{!!}L \to {}^{1/2}H^0 Ej_* L), \\ Ej^{\mathrm{co}}_{!*}L &:= \mathrm{Coim}({}^{1/2}H^0 Ej_{!!}L \to {}^{1/2}H^0 Ej_* L), \end{aligned} $$

from above. Then, as $Ej^{-1} \simeq Ej^!$ is t-exact ([DK16a, Proposition 3.5.6]), and furthermore $Ej^{-1}Ej_{!!} \simeq \mathrm{Id} \simeq Ej^!Ej_*$, one gets

$$\begin{aligned}(Ej_{!*}L)|_{U_\infty} = Ej^{-1}Ej_{!*}L =& Ej^{-1}\,\mathrm{Im}({}^{1/2}H^0Ej_{!!}L \to {}^{1/2}H^0Ej_*L)\\ \simeq& \mathrm{Im}({}^{1/2}H^0Ej^{-1}Ej_{!!}L \to {}^{1/2}H^0Ej^{-1}Ej_*L)\\ \simeq& \mathrm{Im}(L \xrightarrow{\sim} L) \simeq L\end{aligned}$$

and the same of course for $(Ej^{\mathrm{co}}_{!*}L)|_{U_\infty} \simeq L$. Now, set $G := Ej_{!*}L$ and consider the distinguished triangle

$$\mathbb{C}_U \longrightarrow \mathbb{C}_X \longrightarrow \mathbb{C}_Z \xrightarrow{+1} \tag{3.3}$$

in $D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, which induces a distinguished triangle

$$\pi^{-1}\mathbb{C}_U \otimes G \longrightarrow \pi^{-1}\mathbb{C}_X \otimes G \longrightarrow \pi^{-1}\mathbb{C}_Z \otimes G \xrightarrow{+1}$$

with $\pi^{-1}\mathbb{C}_U \otimes G \simeq Ej_{!!}Ej^{-1}G$ by [DK16a, lemma 2.4.5], as well as $\pi^{-1}\mathbb{C}_X \otimes G \simeq G$ and $\pi^{-1}\mathbb{C}_Z \otimes G \simeq Ei_{!!}Ei^{-1}G \simeq Ei_*Ei^{-1}G$, for i the closed embedding $i\colon Z_\infty \to \mathbf{X}$ and j the open embedding $U_\infty \to \mathbf{X}$. Recall that the functor

$${}^{1/2}H^{[0,1)}\colon E_{\mathbb{R}-c}(\mathbf{X}) \to {}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$$

is cohomological (cf. [DK16a, Proposition 1.3.1]). Denoting this functor by H, and writing $H^i := H(\bullet[i]) = {}^{1/2}H^{[i,i+1)}(\bullet)[i]$, we get a long exact sequence

$$\ldots \longrightarrow H^0Ej_{!!}Ej^{-1}G \longrightarrow H^0G \longrightarrow H^0Ei_*Ei^{-1}G \longrightarrow H^1Ej_{!!}Ej^{-1}G \longrightarrow \ldots$$

in ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$, where $H^0G \simeq G = Ej_{!*}L$ by definition of G, $H^0Ej_{!!}Ej^{-1}G \simeq H^0Ej_{!!}L$ by the above, and $H^1Ej_{!!}Ej^{-1}G = 0$, as Ej^{-1} is t-exact, and $Ej_{!!}$ is right t-exact by [DK16a, Proposition 3.5.6], so $Ej_{!!}Ej^{-1}G \in {}^{1/2}E^{\leqslant 0}_{\mathbb{R}-c}(\mathbf{X})$ and $H^1 = {}^{1/2}H^{[1,2)}(\bullet)[1]$. So the above exact sequence is of the form

$$\ldots \longrightarrow H^0Ej_{!!}L \longrightarrow Ej_{!*}L \longrightarrow H^0Ei_*Ei^{-1}G \longrightarrow 0 \tag{3.4}$$

and, as $Ej_{!!}$ is right t-exact, we have $H^0Ej_{!!}L \simeq {}^{1/2}H^0Ej_{!!}L$, so that the morphism $H^0Ej_{!!}L \to Ej_{!*}L$ from above is surjective by definition of $Ej_{!*}L$, so $H^0Ei_*Ei^{-1}G \simeq 0$. Because Ei^{-1} is right t-exact (again by [DK16a, proposition 3.5.6]) and $Ei_* \simeq Ei_{!!}$, this implies

$$Ei^{-1}G \in {}^{1/2}E^{<0}_{\mathbb{R}-c}(Z_\infty),$$

which can be easily seen by applying $H^0 \circ Ei^{-1}$ and using lemma 1.50.

Now let us review the above lines and replace G by $\tilde{G} := \mathrm{D}^E\,G = Ej^{\mathrm{co}}_{!*}\,\mathrm{D}^E\,L$ and ${}^{1/2}H^{[0,1)}$ by $\tilde{H} := {}^{1/2}H^{(-1,0]}$ where $\tilde{H}^i$ is defined analogously to H^i above. We will furthermore write $\tilde{L} := \mathrm{D}^E\,L$. Then we still have $\tilde{H}^1(Ej_{!!}Ej^{-1}\tilde{G}) \simeq 0$ due to right t-exactness of $Ej_{!!}$ (recall $\tilde{H}^1 = {}^{1/2}H^{(0,1]}(\bullet)[1]$) and arrive at the long exact sequence analogous to (3.4)

$$\ldots \longrightarrow \tilde{H}^0 Ej_{!!}\tilde{L} \longrightarrow Ej_{!*}^{\mathrm{co}}\tilde{L} \longrightarrow \tilde{H}^0 Ei_* Ei^{-1}G \longrightarrow 0.$$

Recall that by [DK16a, proposition 3.5.5], there is a distinguished triangle

$$\tau^{<0}K \longrightarrow K \longrightarrow \tau^{\geqslant 0}K \xrightarrow{+1}$$

in $E_{\mathbb{R}-c}(\mathbf{X})$ for any $K \in E_{\mathbb{R}-c}(\mathbf{X})$. Applying this to $K = Ej_{!!}\tilde{L}$ and taking the $\tilde{H}$-cohomologies gives us an exact sequence

$$\tilde{H}^{-1}\tau^{\geqslant 0}Ej_{!!}\tilde{L} \simeq 0 \longrightarrow \tilde{H}^0\tau^{<0}Ej_{!!}\tilde{L} \longrightarrow \tilde{H}^0 Ej_{!!}\tilde{L} \longrightarrow \underbrace{\tilde{H}^0\tau^{\geqslant 0}Ej_{!!}\tilde{L}}_{={}^{1/2}H^0 Ej_{!!}\tilde{L}} \longrightarrow 0 \simeq \tilde{H}^1\tau^{<0}Ej_{!!}\tilde{L}$$

in ${}^{1/2}E^{(-1,0]}_{\mathbb{R}-c}(\mathbf{X})$. The very same applies to $K = Ei_*Ei^{-1}\tilde{G}$ and, trivially, to $\tilde{G} = Ej^{\mathrm{co}}_{!*}\tilde{L}$, so we get a diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & {}^{1/2}H^0 Ej_{!!}\tilde{L} \arrow[r, "a"] \arrow[u] & Ej^{\mathrm{co}}_{!*}\tilde{L} \arrow[r, "b"] \arrow[u] & {}^{1/2}H^0 Ei_*Ei^{-1}\tilde{G} \arrow[u] & \\
\ldots \arrow[r] & \tilde{H}^0 Ej_{!!}\tilde{L} \arrow[r, "d"] \arrow[u, "c"] & Ej^{\mathrm{co}}_{!*}\tilde{L} \arrow[r, "e"] \arrow[u, "="] & \tilde{H}^0 Ei_*Ei^{-1}\tilde{G} \arrow[r] \arrow[u, "f"'] & 0 \\
 & \tilde{H}^0\tau_{<0}Ej_{!!}\tilde{L} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \tilde{H}^0\tau_{<0}Ei_*Ei^{-1}\tilde{G} \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

in ${}^{1/2}E^{(-1,0]}_{\mathbb{R}-c}(\mathbf{X})$, where all columns and the middle row are exact. Now a is an epimorphism by construction of the co-middle extension and c is an epimorphism by exactness of the first column, so d is an epimorphism as well. This implies $e = 0$ resp. $0 \simeq \tilde{H}^0 Ei_*Ei^{-1}\tilde{G}$ as e is the cokernel of d by exactness of the middle row. Again, right t-exactness of Ei^{-1}, $Ei_* \simeq Ei_{!!}$ and the application of $\tilde{H}^0 \circ Ei^{-1}$ together with lemma 1.50 thus give us $Ei^{-1}\tilde{G} \in {}^{1/2}E^{\leqslant -1}_{\mathbb{R}-c}(Z_\infty)$ or, equivalently,

$$Ei^! G \simeq \mathrm{D}^E\, Ei^{-1}\,\mathrm{D}^E\, G \simeq \mathrm{D}^E\, Ei^{-1}\tilde{G} \in {}^{1/2}E^{\geqslant 1}_{\mathbb{R}-c}(Z_\infty).$$

With these observations, we have already done half of the work on proving

Proposition 3.7. *Let $G \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{X})$, $L \in {}^{1/2}E^0_{\mathbb{R}-c}(U_\infty)$, and $j\colon U_\infty \to \mathbf{X}$ the bordered open embedding as above. Then $G \simeq Ej_{!*}L$ if and only if G satisfies*

i) $Ej^{-1}G \simeq L$,

ii) $Ei^{-1}G \in {}^{1/2}E^{<0}_{\mathbb{R}-c}(Z_\infty)$,

iii) $Ei^!G \in {}^{1/2}E^{\geqslant 1}_{\mathbb{R}-c}(Z_\infty)$.

On the other hand, we have the dual version: $G \simeq Ej^{\mathrm{co}}_{!*}L$ if and only if

i) $Ej^{-1}G \simeq L$,

ii)' $Ei^{-1}G \in {}^{1/2}E^{\leqslant -1}_{\mathbb{R}-c}(Z_\infty)$,

iii)' $Ei^!G \in {}^{1/2}E^{>0}_{\mathbb{R}-c}(Z_\infty)$.

In particular, as an obvious but useful consequence we get

Corollary 3.8. *Let $G \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{X})$ and $L \in {}^{1/2}E^0_{\mathbb{R}-c}(U_\infty)$ as in proposition 3.7. Suppose G satisfies*

i) $Ej^{-1}G \simeq L$,

ii) $Ei^{-1}G \in {}^{1/2}E^{\leqslant -1}_{\mathbb{R}-c}(Z_\infty)$,

iii) $Ei^!G \in {}^{1/2}E^{\geqslant 1}_{\mathbb{R}-c}(Z_\infty)$.

Then $G \simeq Ej_{!*}L \simeq Ej^{\mathrm{co}}_{!*}L$.

Proof of proposition 3.7. Cf. [HTT08, proof of proposition 8.2.5]. Without loss of generality, let us focus on the case of characterizing the middle extension, as the situation for the co-middle version corresponds to this via duality (cf. lemma 3.5). Note that above we have already shown that $Ej_{!*}L$ satisfies i)–iii). Now let us recall that the canonical morphism $Ej_{!!}L \to Ej_*L$ may be constructed (cf. lemma 2.7) using the adjunction unit $\mathrm{Id} \to Ej_*Ej^{-1}$ and $\mathrm{Id} \xrightarrow{\sim} Ej^{-1}Ej_{!!} \simeq Ej^!Ej_{!!}$, by

$$Ej_{!!}L \to Ej_*Ej^{-1}Ej_{!!}L \xrightarrow{\sim} Ej_*L.$$

In particular, for G satisfying i), the canonical morphism

$$Ej_{!!}L \simeq Ej_{!!}Ej^{-1}G \to G \to Ej_*Ej^{-1}G \simeq Ej_*L$$

constructed from the respective counit and unit is the canonical morphism from above, as the unit $\mathrm{Id} \to Ej_*Ej^{-1}$ induces a commutative square

$$\begin{tikzcd} Ej_{!!}Ej^{-1}G \arrow[r] \arrow[d] & G \arrow[d] \\ Ej_*Ej^{-1}Ej_{!!}Ej^{-1}G \arrow[r, "\simeq"] & Ej_*Ej^{-1}G. \end{tikzcd}$$

So, in order to prove the result, it remains to show that

$$ {}^{1/2}H^0 Ej_{!!}L \to G \to {}^{1/2}H^0 Ej_* L \tag{3.5} $$

induced by the above is an image factorization in ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$.

Let us assume G satisfies ii) and iii) as well. First, we would like to show that ${}^{1/2}H^0 Ej_{!!}L \to G$ is an epimorphism in ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$. Let $C \in {}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$ denote the cokernel, then, by hypothesis i), $Ej^{-1}C \simeq 0$, so, by considering the distinguished triangle

$$ Ej_{!!}Ej^{-1}C \longrightarrow C \longrightarrow Ei_* Ei^{-1}C \xrightarrow{+1} $$

in $E_{\mathbb{R}-c}(\mathbf{X})$, we find $C \simeq Ei_*K$ for some $K \in {}^{1/2}E^{\leqslant 0}_{\mathbb{R}-c}(Z_\infty)$. So the cokernel sequence is

$$ {}^{1/2}H^0 Ej_{!!}L \longrightarrow G \longrightarrow Ei_*K \longrightarrow 0. $$

Using the fact that Ei^{-1} is right t-exact and thus induces a right exact functor on ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$, applying ${}^{1/2}H^{[0,1)}Ei^{-1}$ to the above cokernel sequence gives an exact sequence

$$ 0 \underset{ii)}{\simeq} {}^{1/2}H^{[0,1)}Ei^{-1}G \longrightarrow {}^{1/2}H^{[0,1)}K \longrightarrow 0 $$

in ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(Z_\infty)$, which shows ${}^{1/2}H^{[0,1)}K \simeq 0$ and thus

$$ Ei_*K \simeq {}^{1/2}H^{[0,1)}Ei_*K \simeq {}^{1/2}H^{[0,1)}Ei_*\, {}^{1/2}H^{[0,1)}K \simeq 0 $$

(cf. lemma 1.50), which means ${}^{1/2}H^0 Ej_{!!}L \to G$ is an epimorphism in ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$. To show $G \to {}^{1/2}H^0 Ej_*L$ is a monomorphism in ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$, let us denote its kernel by $C \in {}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$. As before, by hypothesis i), we get $Ej^{-1}C \simeq 0$ and $C \simeq Ei_*K \simeq Ei_{!!}K$ for some $K \in {}^{1/2}E^{\leqslant 0}_{\mathbb{R}-c}(Z_\infty)$. So the kernel sequence is of the form

$$ 0 \longrightarrow Ei_*K \longrightarrow G \longrightarrow {}^{1/2}H^0 Ej_*G. $$

Recalling that $Ei^!$ is left t-exact and thus induces a left exact functor on ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(\mathbf{X})$, applying ${}^{1/2}H^{[0,1)}Ei^!$ to the kernel sequence yields an exact sequence

$$ 0 \longrightarrow {}^{1/2}H^{[0,1)}K \longrightarrow {}^{1/2}H^{[0,1)}Ei^!G \underset{iii)}{\simeq} 0 $$

in ${}^{1/2}E^{[0,1)}_{\mathbb{R}-c}(Z_\infty)$, so ${}^{1/2}H^{[0,1)}K \simeq 0$ and

$$ Ei_*K \simeq {}^{1/2}H^{[0,1)}Ei_*K \simeq {}^{1/2}H^{[0,1)}Ei_*\, {}^{1/2}H^{[0,1)}K \simeq 0 $$

(cf. lemma 1.50), which implies $G \to {}^{1/2}H^0 Ej_*L$ is indeed a monomorphism. $\square$

3.3 Minimal extensions of holonomic $\mathscr{D}_U$-modules

In this section, we want to quickly recall the definition of the minimal extension of an algebraic holonomic $\mathscr{D}$-module and observe that a characterization result completely analogous to that of corollary 3.8, resp. [HTT08, proposition 8.2.5], holds if we assume that the open subset in question has a smooth complement. Exactness of the enhanced de Rham functor then implies that minimal extensions of holonomic $\mathscr{D}$-modules correspond, via the enhanced Riemann–Hilbert correspondence, to enhanced ind-sheaves which are the enhanced middle extensions of their quotients – in particular, as we can apply corollary 3.8 and the standard t-structure on the category of $\mathscr{D}$-modules trivially is 1-indexed, middle and co-middle extension agree for this special class of enhanced perverse sheaves that are in the essential image of algebraic holonomic $\mathscr{D}$-module minimal extensions (cf. corollary 3.11).

Consider $j\colon U \to X$ as above (i. e. Zariski-open and dense) and let $\mathscr{M}$ be an algebraic holonomic $\mathscr{D}_U$-module. When speaking of the enhanced solutions resp. de Rham complex of an algebraic $\mathscr{M}$, we are of course referring to the functors $\mathcal{S}ol^E_{\mathrm{an}} := \mathcal{S}ol^E \circ (\bullet)^{\mathrm{an}}$ resp. $DR^E_{\mathrm{an}} := DR^E \circ (\bullet)^{\mathrm{an}}$. The well known minimal extension of $\mathscr{M}$ to a $\mathscr{D}_X$-module is defined as

$$Dj_{!*}\mathscr{M} := \mathrm{Im}\left(\int_{j!} \mathscr{M} \to \int_j \mathscr{M}\right) \in \mathrm{Hol}(\mathscr{D}_X)$$

(cf. [HTT08, definition 3.4.1]).

Lemma 3.9. *Let $j\colon U \to X$ and $i\colon Z := X \setminus U \to X$ be an affine open resp. a closed embedding as above, i. e. such that Z is smooth, and $\mathscr{M} \in \mathrm{Hol}(\mathscr{D}_U)$. Then $\mathscr{N} := Dj_{!*}\mathscr{M}$ has the properties*

i) $j^{\bigstar}\mathscr{N} \simeq \mathscr{M}$,

ii) $i^{\bigstar}\mathscr{N} \in D^{\leqslant -1}_{\mathrm{hol}}(Z)$

iii) $i^{\dagger}\mathscr{N} \in D^{\geqslant 1}_{\mathrm{hol}}(Z)$,

(where $i^{\bigstar} := \mathbb{D}_Z \circ i^{\dagger} \circ \mathbb{D}_X$ in notation of [HTT08]).

Proof. The proof works completely analogous to the one in section 3.2, resp. to that of [HTT08, proposition 8.2.5]. Nevertheless, we would like to give a sketch of proof for convenience. By the basechange theorem [HTT08, theorem 1.7.3], applied to $j^{\dagger}$ and $\int_j$ (which yields $j^{\dagger} \circ \int_j \simeq \mathrm{Id}$), and exactness of $j^{\dagger} \simeq j^{-1} \simeq j^{\bigstar}$ (cf. [HTT08, example 1.5.12]), one has

$$j^{\dagger}\mathscr{N} \simeq \mathrm{Im}\left(j^{\dagger}\mathbb{D}_X \int_j \mathbb{D}_U\mathscr{M} \to j^{\dagger}\int_j \mathscr{M}\right) \simeq \mathrm{Im}(\mathscr{M} \xrightarrow{\sim} \mathscr{M}) \simeq \mathscr{M},$$

where for the second step [HTT08, theorem 2.7.1] was applied (as open embeddings are smooth and thus non-characteristic), which shows i). Then, consider the distinguished triangle

$$\int_i i^\dagger \mathscr{N} \longrightarrow \mathscr{N} \longrightarrow \int_j j^\dagger \mathscr{N} \xrightarrow{+1}$$

from [HTT08, Proposition 1.7.1] (by hypothesis, Z is a smooth variety). Taking cohomologies, this yields a long exact sequence

$$\ldots \longrightarrow H^{-1}\left(\int_j j^\dagger \mathscr{N}\right) \longrightarrow H^0\left(\int_i i^\dagger \mathscr{N}\right) \longrightarrow \mathscr{N} \longrightarrow H^0\left(\int_j j^\dagger \mathscr{N}\right) \longrightarrow \ldots$$

As $\int_j \simeq Rj_* \simeq j_*$ (cf. [HTT08, example 1.5.22]) is left exact and $j^\dagger$ is exact, we have $H^{-1}\left(\int_j j^\dagger \mathscr{N}\right) \simeq 0$. Furthermore $j^\dagger \mathscr{N} \simeq \mathscr{M}$ by i) and

$$\mathscr{N} \longrightarrow H^0\left(\int_j j^\dagger \mathscr{N}\right) \simeq H^0\left(\int_j \mathscr{M}\right) \simeq j_*\mathscr{M}$$

is injective by definition of $\mathscr{N} = Dj_{!*}\mathscr{M}$. So we get

$$H^0\left(\int_i i^\dagger \mathscr{N}\right) = 0 \tag{3.6}$$

and thus, as $\int_i$ is exact (cf. e. g. [HTT08, proposition 1.5.24]) and $i^\dagger \simeq Ri^\natural$ is left exact (cf. [HTT08, propositions 1.5.24 and 1.5.16]), by applying $H^0 i^\dagger$ to (3.6) we get that $i^\dagger \mathscr{N} \in D^{\geqslant 1}_{\mathrm{hol}}(Z)$, which means iii) holds. Finally, ii) follows from using property iii) for

$$i^\dagger Dj_{!*}(\mathbb{D}_U \mathscr{M}) \simeq i^\dagger \mathbb{D}_X Dj_{!*}\mathscr{M} \simeq \mathbb{D}_Z i^\bigstar \mathscr{N}$$

and the fact that $\mathbb{D}_Z$ maps $D^{\geqslant 1}_{\mathrm{hol}}(Z)$ to $D^{\leqslant -1}_{\mathrm{hol}}(Z)$ – here we used $\mathbb{D}_X D_{j!*}\mathscr{M} \simeq Dj_{!*}(\mathbb{D}_U\mathscr{M})$, cf. [HTT08, proposition 3.4.3]. □

Thanks to Kashiwara's equivalence (cf. e. g. [HTT08, theorem 1.6.1]), the converse to lemma 3.9 holds as well, finishing the announced characterization.

Lemma 3.10. *For an affine open embedding $j\colon U \to X$ as above, such that its complement $Z = X \setminus U$ is a smooth variety and $\mathscr{M} \in \mathrm{Hol}(\mathscr{D}_U)$ as in lemma 3.9, any holonomic $\mathscr{D}_X$-module $\mathscr{N} \in \mathrm{Hol}(\mathscr{D}_X)$ satisfying conditions i) – iii) of lemma 3.9 is naturally isomorphic to $Dj_{!*}\mathscr{M}$.*

Proof. Again, the proof of [HTT08, proposition 8.2.5] may be transferred to the $\mathscr{D}$-module setting virtually without any changes. We would like to show that the factorization

$$\int_{j!} \mathscr{M} \longrightarrow \mathscr{N} \longrightarrow \int_j \mathscr{M}$$

of the canonical morphism $\int_{j!}\mathscr{M} \to \int_j \mathscr{M}$ obtained from the adjunctions $\int_{j!} \dashv j^\dagger$ and $j^\bigstar \dashv \int_j$ and the isomorphism $j^\dagger \simeq j^\bigstar$ is an image factorization. Let $\mathscr{A}$ be the kernel

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{N} \longrightarrow \int_j \mathscr{M}.$$

By assuming i), and with $j^\bigstar \int_{j!} \simeq \mathrm{Id}$, it is clear that $\mathscr{A}$ is supported on Z, hence of the form $\int_i \mathscr{A}'$ for some $\mathscr{A}' \in \mathrm{Hol}(\mathscr{D}_Z)$ by Kashiwara's equivalence (note that $\int_i$ is exact, cf. [HTT08, proposition 1.5.24]). Applying $H^0 \circ i^\dagger$ (note that $i^\dagger \simeq Ri^\natural$ is left exact with respect to the standard t-structure, cf. [HTT08, proposition 1.5.16]) to this sequence, we get an exact sequence

$$0 \longrightarrow \mathscr{A}' \longrightarrow H^0(i^\dagger \mathscr{N}) \underset{iii)}{\simeq} 0$$

and thus $\mathscr{A}' = 0$, meaning $\mathscr{A} = 0$, so $\mathscr{N} \to \int_j \mathscr{M}$ is injective. By the same reasoning, applying $i^\bigstar$ to the cokernel sequence

$$\int_{j!} \mathscr{M} \longrightarrow \mathscr{N} \longrightarrow \mathscr{B} \longrightarrow 0$$

one finds that $\int_{j!} \mathscr{M} \to \mathscr{N}$ is surjective. $\square$

Note that, in notation of [DK16b], $(f^{\mathrm{an}})^\dagger = D(f^{\mathrm{an}})^*[d_X - d_Y]$ for some $f\colon X \to Y$ and d_X, d_Y the dimensions of X and Y, respectively. In particular, [DK16b, theorem 9.1.2] and [DK16b, corollary 8.4.10], in the $\mathscr{D}$-module notation from [HTT08] that we are using here, are saying that, for any $\mathscr{N} \in D^b_{\mathrm{hol}}(Y)$, one has

$$\begin{aligned} DR^E_X((f^\dagger \mathscr{N})^{\mathrm{an}}) &\simeq Ef^! DR^E_Y \mathscr{N}, \\ DR^E_X((f^\bigstar \mathscr{N})^{\mathrm{an}}) &\simeq Ef^{-1} DR^E_Y \mathscr{N}. \end{aligned} \tag{3.7}$$

In the setting of $j\colon U \to X$ and $i\colon Z := X \setminus U \to X$ as above, let $\mathscr{M} \in \mathrm{Hol}(\mathscr{D}_U)$ and $K := \mathcal{DR}^E((Dj_{!*}\mathscr{M})^{\mathrm{an}}) \in {}^{1/2}E^0_{\mathbb{R}-c}(X^{\mathrm{an}})$ (recall that [DK16a, theorem 4.5.1] proves DR^E is t-exact). By t-exactness of DR^E and (3.7), we furthermore have

$$\begin{aligned} Ei^! K \simeq& \mathcal{DR}^E((i^\dagger Dj_{!*}\mathscr{M})^{\mathrm{an}}) \in {}^{1/2}E^{\geqslant 1}_{\mathbb{R}-c}(Z^{\mathrm{an}}), \\ Ei^{-1} K \simeq& \mathcal{DR}^E((i^\bigstar Dj_{!*}\mathscr{M})^{\mathrm{an}}) \in {}^{1/2}E^{\leqslant -1}_{\mathbb{R}-c}(Z^{\mathrm{an}}), \end{aligned}$$

where we used that for $\mathbf{X} = X^{\mathrm{an}} = (X^{\mathrm{an}}, X^{\mathrm{an}})$ and $Z^{\mathrm{an}} \subset X^{\mathrm{an}}$ closed, $Z^{\mathrm{an}}_\infty = Z^{\mathrm{an}}$ by definition. This concludes our example by showing the following immediate consequence of corollary 3.8, aiming towards a setting as in e. g. [BE04] or [Ari10]:

Corollary 3.11. *For some smooth complex variety, with notation as above, we again denote by j the bordered open embedding $U_\infty \simeq (U, X) \to X$. Let $\mathscr{M} \in \mathrm{Hol}(\mathscr{D}_X)$ be the* *middle extension of its restriction* $j^\dagger\mathscr{M} \in \mathrm{Hol}(\mathscr{D}_U)$*. Then, with* $G := DR^E_X(\mathscr{M}^{\mathrm{an}})$ *and* $L := Ej^{-1}G \in {}^{1/2}E^0_{\mathbb{R}-c}(U^{\mathrm{an}}_\infty)$*, we have that*

$$G \simeq Ej_{!*}L \simeq Ej^{\mathrm{co}}_{!*}L$$

is the enhanced middle extension of its quotient.

3.4 Enhanced middle convolution and middle extension

In this section, we finally use the observations made on the enhanced middle extension to give a first criterion for the coincidence of middle and co-middle convolution, which we then apply to our example of section 2.4.

Consider the following situation, which is precisely the one of [Kat95, proposition 2.7.2], adapted to the bordered enhanced setting: Let $\mathbf{X} = (X, \check{X})$ and $\mathbf{Y}$ be (subanalytic) bordered spaces, $U \subset X$ open and $j\colon U_\infty \to \mathbf{X}$ the induced bordered open embedding, set $Z := X \setminus U$ and let $\overline{f}\colon \mathbf{X} \to \mathbf{Y}$ be a proper morphism of bordered spaces with the additional property that $\overline{f}|_{Z_\infty}\colon Z_\infty \to \mathbf{Y}$ is a finite morphism, which shall mean here a proper morphism such that the underlying map $Z \to \mathring{\mathbf{Y}}$ has finite fibers (for example this situation occurs in the case of the bordered version of the analytification of the original situation in [Kat95, proposition 2.7.2], cf. [GR71, proposition 3.2]). Let $f := \overline{f} \circ j$. All these morphisms are depicted in the following diagram (cf. [Kat95, section 2.7]).

$$\begin{tikzcd} U_\infty \arrow[r, "j"] \arrow[dr, "f"'] & \mathbf{X} \arrow[d, "\overline{f}"] & Z_\infty \arrow[l, "i"'] \arrow[dl, "\overline{f}|_{Z_\infty}"] \\ & \mathbf{Y} & \end{tikzcd}$$

Proposition 3.12 (cf. proposition 2.7.2 of [Kat95])**.** *Assume* $K \in {}^{1/2}E^0_{\mathbb{R}-c}(U_\infty)$ *has the following properties:*

i) $Ef_{!!}K \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{Y})$ *and* $Ef_*K \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{Y})$*,*

ii) $Ej_{!!}K \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{X})$ *and* $Ej_*K \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{X})$*,*

iii) $Ej_{!*}K \simeq Ej^{\mathrm{co}}_{!*}K$*.*

Then

$$E\overline{f}_*Ej_{!*}K \simeq \mathrm{Im}(Ef_{!!}K \to Ef_*K) \simeq \mathrm{Coim}(Ef_{!!}K \to Ef_*K).$$

Proof. The proof is almost literally the same as the one of [Kat95, proposition 2.7.2], except for some changes in terminology that are due to the enhanced resp. quasi-abelian setting. Let us write down the details anyway. By hypothesis, we know that the morphism $Ej_{!!}K \to Ej_*K$ is strict. In particular, we get strictly exact kernel rep. cokernel sequences in ${}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{X})$,

$$0 \longrightarrow A \longrightarrow Ej_{!!}K \longrightarrow Ej_{!*}K \longrightarrow 0$$
$$0 \longrightarrow Ej_{!*}K \longrightarrow Ej_*K \longrightarrow B \longrightarrow 0,$$

which thus correspond to distinguished triangles

$$\begin{aligned} A \longrightarrow Ej_{!!}K \longrightarrow Ej_{!*}K \xrightarrow{+1} \\ Ej_{!*}K \longrightarrow Ej_*K \longrightarrow B \xrightarrow{+1} \end{aligned} \tag{3.8}$$

in $E_{\mathbb{R}-c}(\mathbf{X})$, where, by proposition 3.7, we have $Ej^{-1}A \simeq 0 \simeq Ej^{-1}B$. So

$$A \simeq Ei_{!!}Ei^{-1}A \simeq Ei_*Ei^!A,$$

(cf. [DK16a, lemmata 2.7.6 and 2.7.7]) and we get

$$Ei^!A \simeq Ei^{-1}Ei_{!!}Ei^!A \simeq Ei^{-1}Ei_*Ei^!A \simeq Ei^{-1}A \in {}^{1/2}E^0_{\mathbb{R}-c}(Z_\infty),$$

i. e. $A \simeq Ei_{!!}F \simeq Ei_*F$ for some $F \in {}^{1/2}E^0_{\mathbb{R}-c}(Z_\infty)$ and the very same of course for B. By the finiteness assumption made on $\overline{f}|_{Z_\infty}$, we furthermore have

$$E\overline{f}|_{Z_\infty,*}F \simeq E\overline{f}_*A \simeq E\overline{f}_{!!}A \simeq E\overline{f}_{!!}Ei_{!!}F \simeq E\overline{f}|_{Z_\infty,!!}F \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{Y})$$

and, by the same argument, $E\overline{f}_*B \simeq E\overline{f}_{!!}B \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{Y})$. Now, applying $E\overline{f}_{!!} \simeq E\overline{f}_*$ to the distinguished triangles (3.8), we obtain distinguished triangles

$$\begin{aligned} E\overline{f}_*A \longrightarrow Ef_{!!}K \longrightarrow E\overline{f}_*Ej_{!*}K \xrightarrow{+1}, \\ E\overline{f}_*Ej_{!*}K \longrightarrow Ef_*K \longrightarrow E\overline{f}_*B \xrightarrow{+1}, \end{aligned} \tag{3.9}$$

where $Ef_{!!}K \to E\overline{f}_*Ej_{!*}K \to Ef_*K$ is a factorization of the canonical morphism $Ef_{!!}K \to Ef_*K$, which we want to prove is indeed a strict image factorization. So far, we know that $E\overline{f}_*A, Ef_{!!}K, Ef_*K, E\overline{f}_*B \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{Y})$. Let us apply the cohomological functors $H := {}^{1/2}H^{[0,1)}$ resp. $\tilde{H} := {}^{1/2}H^{(-1,0]}$ to the triangles (3.9) and write $H^i(\bullet) := H(\bullet[i])$ resp. $\tilde{H}^i(\bullet) := \tilde{H}(\bullet[i])$ as usual.

- Applying H to the first triangle of (3.9) yields an exact sequence

$$\ldots \longrightarrow 0 \longrightarrow H^{-1}(E\overline{f}_*Ej_{!*}K) \longrightarrow E\overline{f}_*A \longrightarrow Ef_{!!}K \longrightarrow$$
$$\longrightarrow H^0(E\overline{f}_*Ej_{!*}K) \longrightarrow 0 \longrightarrow \ldots$$

which tells us that $E\overline{f}_*Ej_{!*}K \in {}^{1/2}E^{[-1,1)}_{\mathbb{R}-c}(\mathbf{Y})$. Similarly, applying $\tilde{H}$ to the same triangle gives an exact sequence

$$\dots Ef_{!!}K \longrightarrow \tilde{H}^0(E\overline{f}_*Ej_{!*}K) \longrightarrow 0 \longrightarrow \dots$$

and thus shows $\tilde{H}^1(E\overline{f}_*Ej_{!*}K) \simeq 0$, i.e. $E\overline{f}_*Ej_{!*}K \in {}^{1/2}E^{\leqslant 0}_{\mathbb{R}-c}(\mathbf{Y})$, so, overall, $E\overline{f}_*Ej_{!*}K \in {}^{1/2}E^{[-1,0]}_{\mathbb{R}-c}(\mathbf{Y})$.

- Analogously, applying $\tilde{H}$ to the second triangle of (3.9) gives us the exact sequence

$$\begin{aligned}\dots \longrightarrow 0 \longrightarrow \tilde{H}^0(E\overline{f}_*Ej_{!*}K) \longrightarrow Ef_*K \longrightarrow E\overline{f}_*B \longrightarrow \\ \longrightarrow \tilde{H}^1(E\overline{f}_*Ej_{!*}K) \longrightarrow 0 \longrightarrow \dots\end{aligned}$$

  showing $E\overline{f}_*Ej_{!*}K \in {}^{1/2}E^{(-1,1]}_{\mathbb{R}-c}(\mathbf{Y})$ and applying H results in the observation that $E\overline{f}_*Ej_{!*}K \in {}^{1/2}E^{\geqslant 0}_{\mathbb{R}-c}(\mathbf{Y})$, so $E\overline{f}_*Ej_{!*}K \in {}^{1/2}E^{[0,1]}_{\mathbb{R}-c}(\mathbf{Y})$.

Putting together both of these points, we have

$$E\overline{f}_*Ej_{!*}K \in {}^{1/2}E^{[-1,0]}_{\mathbb{R}-c}(\mathbf{Y}) \cap {}^{1/2}E^{[0,1]}_{\mathbb{R}-c}(\mathbf{Y}) = {}^{1/2}E^{0}_{\mathbb{R}-c}(\mathbf{Y}).$$

In particular, the two distinguished triangles (3.9) correspond to strict short exact sequences in ${}^{1/2}E^{0}_{\mathbb{R}-c}(\mathbf{Y})$, proving that $Ef_{!!}K \to Ef_*K$ is a strict morphism and

$$E\overline{f}_*Ej_{!*}K \simeq \operatorname{Im}(Ef_{!!}K \to Ef_*K) \simeq \operatorname{Coim}(Ef_{!!}K \to Ef_*K).$$

□

Remark 3.13. Note that condition iii) is indispensable here, in particular the above proof does not work for either middle or co-middle extension considered separately in the case of $Ej_{!*}K \not\simeq Ej^{\mathrm{co}}_{!*}K$.

Now, let us recall the situation of our enhanced middle convolution from section 2: Let $K, L \in {}^{1/2}E^{0}_{\mathbb{R}-c}(\mathbf{A})$ be such that (K, L) has property $\mathfrak{P}$, that is we require that

$$E\sigma_{!!}(K \overset{+}{\boxtimes} L), E\sigma_*(K \overset{+}{\boxtimes} L) \in {}^{1/2}E^{0}_{\mathbb{R}-c}(\mathbf{A})$$

(notation as in section 2). Consider the morphisms (with $Z := \{\infty\} \times \mathcal{A}$)

$$\begin{tikzcd} \mathbf{A} \times \mathbf{A} \arrow[r, "u"] \arrow[dr, "p_2"'] & \mathcal{P} \times \mathbf{A} \arrow[d, dashed, "\overline{p_2}"] & Z_\infty \arrow[l, hook', "i"'] \arrow[dl, "\overline{p_2}|_{Z_\infty}", "\simeq"'] \\ & \mathbf{A}, & \end{tikzcd}$$

where $\overline{p_2}$ is proper and $\overline{p_2}|_{Z_\infty}$ is clearly finite, and recall from section 2 that, with

$$\alpha\colon \mathbf{A} \times \mathbf{A} \overset{\simeq}{\to} \mathbf{A} \times \mathbf{A}, \quad (a,b) \mapsto (a, a+b)$$

we have

$$E\sigma_{!!}(K \overset{+}{\boxtimes} L) \simeq Ep_{2,!!}E\alpha_*(K \overset{+}{\boxtimes} L) \simeq E(\overline{p_2})_* Eu_{!!}E\alpha_*(K \overset{+}{\boxtimes} L),$$
$$E\sigma_*(K \overset{+}{\boxtimes} L) \simeq Ep_{2,*}E\alpha_*(K \overset{+}{\boxtimes} L) \simeq E(\overline{p_2})_* Eu_*E\alpha_*(K \overset{+}{\boxtimes} L).$$

So, by proposition 3.12, we immediately get

Theorem 3.14. *If, in the above situation,* $Eu_{!!}E\alpha_*(K \overset{+}{\boxtimes} L) \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathcal{P} \times \mathbf{A})$ *and* $Eu_*E\alpha_*(K \overset{+}{\boxtimes} L) \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathcal{P} \times \mathbf{A})$ *and furthermore* $Eu_{!*}E\alpha_*(K \overset{+}{\boxtimes} L) \simeq Eu^{\mathrm{co}}_{!*}E\alpha_*(K \overset{+}{\boxtimes} L)$, *then we have*

$$K \overset{E}{*}_{\mathrm{mid}} L \simeq K \overset{E}{*}_{\mathrm{co-mid}} L.$$

Proof. By hypothesis and proposition 3.12, we have

$$K \overset{E}{*}_{\mathrm{mid}} L \simeq E(\overline{p_2})_* Eu_{!*}E\alpha_*(K \overset{+}{\boxtimes} L) \simeq K \overset{E}{*}_{\mathrm{co-mid}} L.$$

□

Consider our example from section 2.4, that is $K = E^w[1]$ (with w a local coordinate on $\mathcal{A} = \mathcal{P} \setminus \{\infty\}$) and $L = L^E_\lambda[1]$. We already showed that $K \overset{E}{*}_* L, K \overset{E}{*}_! L \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A})$ in section 2.4. Now let us write $G := \alpha_*(K \overset{+}{\boxtimes} L)$ as a shorthand and prove the following

Lemma 3.15. *In the situation of the example in section 2.4, we have (with notation as above):*

i) $Eu_{!!}G \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathcal{P} \times \mathbf{A})$ *and* $Eu_*G \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathcal{P} \times \mathbf{A})$,

ii) $Eu_{!*}G \simeq Eu^{\mathrm{co}}_{!*}G$.

Proof. We will actually show $Ei^{-1}Eu_*G \simeq 0 \simeq Ei^!Eu_{!!}G$, for $i\colon \{\infty\} \times \mathcal{A} \to \mathcal{P} \times \mathcal{A}$, that is

$$Eu_{!!}G \simeq Eu_*G \simeq Eu_{!*}G \simeq Eu^{\mathrm{co}}_{!*}G.$$

So, technically, it turns out we do not really need corollary 3.8 for this particularly simple case – still it does obviously apply in a trivial way. We will only show $Ei^{-1}Eu_*G \simeq 0$ (the other case corresponds to this one via duality). Recall from section 2.4 that (we will write α and u again instead of $\alpha_\mathbb{R} := \alpha \times \mathrm{Id}_\mathbb{R}$ or $u_\mathbb{R}$, as in section 2)

$$Eu_*G \simeq \mathbb{C}^E \overset{+}{\otimes} Q\iota(Ru_*(\alpha_*L)_{W^a}),$$

and that $F := Ru_*(\alpha_* L)_{W^a}|_{Z\times\mathbb{R}}$ is a local system on $Z \times \mathbb{R}$, for $Z := \{\infty\} \times \mathcal{A}$ (lemma 2.22). In particular, by remark 2.23, as Z clearly is a deformation retract

$$Z \underset{\rho}{\overset{\pi}{\leftrightarrows}} Z \times \mathbb{R}$$

of $Z \times \mathbb{R}$, we have $F = \pi^{-1}(\rho^{-1} F)$ and thus

$$Ei^{-1} Eu_* G \simeq Ei^{-1}(\mathbb{C}^E \overset{+}{\otimes} Q\iota(Ru_*(\alpha_* L)_{W^a})) \simeq \mathbb{C}^E \overset{+}{\otimes} Q\iota F \simeq 0.$$

For the dual case, we use (with notation as in section 2.4.7)

$$Ei^! Eu_{!!} G \simeq \mathrm{D}^E\, Ei^{-1} Eu_*\, \mathrm{D}^E\, G \simeq \mathrm{D}^E\, Ei^{-1}(\mathbb{C}^E \overset{+}{\otimes} Q\iota(Ru_*(\alpha_* L^a)_W)) \simeq 0.$$

□

Corollary 3.16. *We have*

$$E^w[1] \overset{E}{*}_{\mathrm{mid}} L^E_\lambda[1] \simeq E^w[1] \overset{E}{*}_{\mathrm{co-mid}} L^E_\lambda[1].$$

Proof. This is an immediate application of theorem 3.14 to the above observation. □

4 Arinkin–Katz convolution and enhanced middle convolution

As we already stated at the very beginning, the studies carried out in this thesis are largely emerging from the idea of finding a way towards an enhanced counterpart to Arinkin's version of the classical Katz' algorithm for irregular meromorphic connections on $\mathbb{P}^1$ ([Ari10]). In this final section of our notes, we would thus like to show that our enhanced middle convolution is compatible with the the middle convolution for irreducible holonomic modules on $\mathbb{P}^1$ as in [Ari08; Ari10] via the enhanced Riemann–Hilbert correspondence, cf. conjecture 4.17. After making some effort to connect the algebraic setting of minimal extension $\mathscr{D}$-modules to the one of enhanced ind-sheaves on complex bordered spaces, we will be able to give a prove of this conjecture, under assumption 4.19, cf. theorem 4.20.

4.1 Holonomic $\mathscr{D}$-modules on (projective) algebraic bordered spaces

Definition 4.1 (cf. definition 3.2.1 of [DK16b]). Let us call $(X, \check{X})$ an *algebraic bordered space* if $\check{X}$ is a smooth complex variety and $X \subset \check{X}$ is an open subvariety such that the embedding $j\colon X \to \check{X}$ is affine. We say that $(X, \check{X})$ is projective if $\check{X}$ is so. A *morphism of algebraic bordered spaces* $f\colon \mathbf{X} \to \mathbf{Y}$ is a morphism $f\colon X \to Y$ of smooth varieties such that the projection $\overline{\Gamma_f} \to \check{X}$ is a proper morphism of algebraic varieties (where $\overline{(\bullet)}$ of course refers to the scheme-theoretic closure).

Remark 4.2. For the purpose of feeling more confident about this scheme-theoretic version of the classical bordered space setup of [DK16b, section 3.2], let us recall, for our algebraic setting, the proof of [DK16b, lemma 3.2.3], stating that the composition of morphisms of bordered spaces is well defined. So, let $f\colon X \to Y$ and $g\colon Y \to Z$ induce morphisms $\mathbf{X} \to \mathbf{Y}$ resp. $\mathbf{Y} \to \mathbf{Z}$ of algebraic bordered spaces. For a morphism $f\colon X \to Y$ of varieties inducing a morphism $\mathbf{X} \to \mathbf{Y}$ of bordered spaces, let us introduce some notation concerning the graph of f, as depicted in the following diagram (by definition, Γ_f is the scheme-theoretic image of $X \overset{(\mathrm{Id},f)}{\longrightarrow} X \times Y$).

$$\begin{tikzcd}
 & & & X \\
X \arrow[r, "\gamma_f"] & \Gamma_f \arrow[r, "i_f"] \arrow[urr, bend left, "p_1^f"] \arrow[drr, bend right, "p_2^f"'] & X \times Y \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\
 & & & Y
\end{tikzcd}$$

Here, of course, $i_f \circ \gamma_f = (\mathrm{Id}, f)$. If the context is clear, we will use the same labels for the induced projections of the closure $\overline{\Gamma_f} \subset \check{X} \times \check{Y}$, i. e. $p_1^f\colon \overline{\Gamma_f} \to \check{X}$ and $p_2^f\colon \overline{\Gamma_f} \to \check{Y}$.

We would like to prove that $g \circ f\colon X \to Z$ gives a morphism $\mathbf{X} \to \mathbf{Z}$, i. e. the projection $p_1^{g\circ f}\colon \overline{\Gamma_{g\circ f}} \to \check{X}$ is proper. By hypothesis, $p_1^f\colon \overline{\Gamma_f} \to \check{X}$ and $p_1^g\colon \overline{\Gamma_g} \to \check{Y}$ are proper, so we know

$$\varphi\colon \overline{\Gamma_f} \times_{\check{Y}} \overline{\Gamma_g} \xrightarrow{(\mathrm{Id}, p_1^g)} \overline{\Gamma_f} \times_{\check{Y}} \check{Y} \simeq \overline{\Gamma_f} \xrightarrow{p_1^f} \check{X}$$

is proper ([Har77, corollary 4.8.(c)]). As we may trivially factor φ as

$$\overline{\Gamma_f} \times_{\check{Y}} \overline{\Gamma_g} \xrightarrow{(\varphi, p_2^g)} \check{X} \times \check{Z} \xrightarrow{\mathrm{pr}_{\check{X}}} \check{X}$$

(with $\mathrm{pr}_{\check{X}}$ the first projection), we get that $(\varphi, p_2^g)\colon \overline{\Gamma_f} \times_{\check{Y}} \overline{\Gamma_g} \to \check{X} \times \check{Z}$ is proper, by [Har77, corollary 4.8.(e)]. Let $A \subset \check{X} \times \check{Z}$ be the image of $\tilde{\varphi} := (\varphi, p_2^g)$ and denote the corresponding closed embedding by $i_A\colon A \to \check{X} \times \check{Z}$. Then, note that $\overline{\Gamma_{g\circ f}}$ is the image of the morphism

$$X \xrightarrow{(\mathrm{Id}, g\circ f)} X \times Z \longrightarrow \check{X} \times \check{Z}.$$

By definition, this morphism $X \to \check{X} \times \check{Z}$ factors as

$$X \xrightarrow{a\times b} \overline{\Gamma_f} \times_{\check{Y}} \overline{\Gamma_g} \xrightarrow{\tilde{\varphi}} \check{X} \times \check{Y},$$

where $a\colon X \xrightarrow{i_f} \Gamma_f \to \overline{\Gamma_f}$ and $b\colon X \xrightarrow{i_f} \Gamma_f \xrightarrow{p_2^f} Y \xrightarrow{i_g} \Gamma_g \to \overline{\Gamma_g}$. In particular, we have $\overline{\Gamma_{g\circ f}} \subset A$ by definition of the scheme-theoretic image. As φ is proper, we know that the induced morphism $A \xrightarrow{i_A} \check{X} \times \check{Z} \to \check{X}$ is proper, by [Stacks, tag 01W0, lemma 28.39.9]. So $p_1^{g\circ f}$ is a composition of a closed embedding $\overline{\Gamma_{g\circ f}} \subset A$ and a proper morphism $A \to \check{X}$, hence itself proper.

Remark 4.3. For an algebraic bordered space $(X, \check{X})$ as above, the analytification

$$(X, \check{X})^{\mathrm{an}} := (X^{\mathrm{an}}, \check{X}^{\mathrm{an}})$$

is a complex bordered space in the sense of [KS16, section 4.3] and the same holds true for morphisms (cf. [GR71, proposition 3.2]).

Let us denote by $j\colon X \to \check{X}$ the open embedding and by j^{an} its analytification. Due to the assumption that j is affine, we know that $\int_j = j_*$ is exact, as is $j^\dagger \simeq j^{-1}$ (because j is an open embedding), cf. [HTT08, examples 1.5.12 and 1.5.22]. As before we will write $\mathbf{X} := (X, \check{X})$ and $\mathbf{X}^{\mathrm{an}} := (X, \check{X})^{\mathrm{an}}$. In view of [Ari10], let us make the following

Definition 4.4. For an algebraic bordered space $\mathbf{X}$ as above, set

$$\mathrm{Hol}(\mathbf{X}) := \{\mathscr{M} \in \mathrm{Hol}(\mathscr{D}_{\check{X}}) | \mathscr{M} \simeq Dj_{!*}(j^{-1}\mathscr{M})\}.$$

Note that $\mathbb{D}_{\check{X}}$ induces a duality

$$\mathbb{D}_{\mathbf{X}}\colon \mathrm{Hol}(\mathbf{X})^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Hol}(\mathbf{X}).$$

Definition 4.5. For algebraic bordered spaces $\mathbf{X} = (X, \check{X})$ and $\mathbf{Y} = (Y, \check{Y})$, where we will denote the corresponding open embeddings by $j_X \colon X \to \check{X}$ and $j_Y \colon Y \to \check{Y}$, and a morphism $f \colon \mathbf{X} \to \mathbf{Y}$ of algebraic bordered spaces, we define operations

$$\underline{\int_f^0}, \ \underline{\int_{f!}^0} \colon \operatorname{Hol}(\mathbf{X}) \to \operatorname{Hol}(\mathbf{Y})$$
$$\underline{f^{\dagger,0}}, \ \underline{f^{\bigstar,0}} \colon \operatorname{Hol}(\mathbf{Y}) \to \operatorname{Hol}(\mathbf{X})$$

which are given, for $\mathscr{M} \in \operatorname{Hol}(\mathbf{X})$ and $\mathscr{N} \in \operatorname{Hol}(\mathbf{Y})$, by

$$\begin{aligned}
\underline{\int_f^0}(\mathscr{M}) &= Dj_{Y,!*}H^0 \int_f (j_X^{-1}\mathscr{M}) \\
\underline{\int_{f!}^0}(\mathscr{M}) &= Dj_{Y,!*}H^0 \int_{f!} (j_X^{-1}\mathscr{M}) \\
\underline{f^{\dagger,0}}(\mathscr{N}) &= Dj_{X,!*}H^0 f^\dagger j_Y^{-1}\mathscr{N} \\
\underline{f^{\bigstar,0}}(\mathscr{N}) &= Dj_{X,!*}H^0 f^\bigstar j_Y^{-1}\mathscr{N}.
\end{aligned}$$

Let us recall from [KS16, section 4.3] the definition of $\mathscr{D}$-modules on the complex bordered space $\mathbf{X}^{\mathrm{an}} = (X^{\mathrm{an}}, \check{X}^{\mathrm{an}})$,

$$D^b_{\mathrm{hol}}(\mathscr{D}_{\mathbf{X}^{\mathrm{an}}}) := D^b_{\mathrm{hol}}(\check{X}^{\mathrm{an}}) / \{\mathscr{M} \in D^b_{\mathrm{hol}}(\mathscr{D}_{\check{X}^{\mathrm{an}}}) | \operatorname{Supp}(\mathscr{M}) \subset \check{X}^{\mathrm{an}} \setminus X^{\mathrm{an}}\}.$$

Lemma 4.6. *By analytification,* $\operatorname{Hol}(\mathbf{X})$ *becomes a full subcategory of* $D^b_{\mathrm{hol}}(\mathscr{D}_{\mathbf{X}^{\mathrm{an}}})$.

Proof. Suppose that for $\mathscr{M}, \mathscr{N} \in \operatorname{Hol}(\mathbf{X})$, we have a morphism $f \colon \mathscr{M} \to \mathscr{N}$ fitting into a distinguished triangle

$$\mathscr{M}^{\mathrm{an}} \xrightarrow{f^{\mathrm{an}}} \mathscr{N}^{\mathrm{an}} \longrightarrow C \xrightarrow{+1} \tag{4.1}$$

in $D^b_{\mathrm{hol}}(\mathscr{D}_{\check{X}^{\mathrm{an}}})$, where $\operatorname{Supp}(C) \subset \check{X}^{\mathrm{an}} \setminus X^{\mathrm{an}}$. Then we know that $C \simeq \tilde{C}^{\mathrm{an}}$ for some $\tilde{C} \in D^b_{\mathrm{hol}}(\mathscr{D}_{\check{X}})$ with $\operatorname{Supp}(\tilde{C}) \subset \check{X} \setminus X$. From (4.1) we thus get a cohomology exact sequence

$$0 \longrightarrow H^{-1}\tilde{C} \longrightarrow \mathscr{M} \xrightarrow{f} \mathscr{N} \longrightarrow H^0\tilde{C} \longrightarrow 0$$

in $\operatorname{Hol}(\mathscr{D}_{\check{X}})$. By applying the exact functors j_*j^{-1} resp. $j_!j^{-1}$ and using the image factorizations

$$j_!j^{-1}\mathscr{M} \twoheadrightarrow \underbrace{Dj_{!*}j^{-1}\mathscr{M}}_{\simeq \mathscr{M}} \hookrightarrow j_*j^{-1}\mathscr{M}, \quad j_!j^{-1}\mathscr{N} \twoheadrightarrow \underbrace{Dj_{!*}j^{-1}\mathscr{N}}_{\simeq \mathscr{N}} \hookrightarrow j_*j^{-1}\mathscr{N}$$

we get a diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[r] & j_*j^{-1}\mathscr{M} \arrow[r, "\simeq"] & j_*j^{-1}\mathscr{N} \arrow[r] & 0 & \\
0 \arrow[r] & \tilde{H}^{-1}\tilde{C} \arrow[r] & \mathscr{M} \arrow[r, "f"] \arrow[u, hook] & \mathscr{N} \arrow[r] \arrow[u, hook] & H^0\tilde{C} \arrow[r] & 0 \\
 & 0 \arrow[r] & j_!j^{-1}\mathscr{M} \arrow[r, "\simeq"] \arrow[u, two heads] & j_!j^{-1}\mathscr{N} \arrow[r] \arrow[u, two heads] & 0, &
\end{tikzcd}$$

proving that f has indeed already been an isomorphism in $\mathrm{Hol}(\mathbf{X})$. □

Before we continue, we want to recall one more concept concerning $\mathscr{D}$-modules on bordered spaces from [KS16, section 4.3]. Let X be some smooth complex variety, $Z \subset X$ a closed subset and $U = X \setminus Z$ its open complement. Let I be the defining sheaf of ideals of Z. In this situation, the *algebraic cohomology functor* is, for some $\mathscr{O}_X$-module F, defined by

$$\Gamma_{[Z]}(F) := \varinjlim_{k} \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X/I^k, F),$$

cf. [KS96, section 5] (this functor is also referred to as the *temperate supports functor*, e. g. in [Bjö93, section II.5], where it is denoted by $\mathcal{H}^0_{[Z]}(F)$). For the open complement $U \subset X$, on the other hand, we may define[6]

$$\Gamma_{[U]}(F) := \varinjlim_{k} \mathscr{H}om_{\mathscr{O}_X}(I^k, F)$$

(which is denoted by $\mu(*Z)(F)$ in [Bjö93, remark 2.5.12]) and, for some locally closed $W = V \cap K \subset X$, where $V \subset X$ is open and $K \subset X$ is closed, set

$$\Gamma_{[W]}(F) = \Gamma_{[V]}(\Gamma_{[K]}(F)).$$

We will denote the right derived functors of these by $R\Gamma_{[Z]}$ and $R\Gamma_{[U]}$, resp. $R\Gamma_{[W]}$, cf. [KS96; Bjö93] (in [Bjö93], $R\Gamma_{[U]}(F)$ is denoted by $F(*Z)$). Concerning the definition of $R\Gamma_{[W]}$ from above, note that $\Gamma_{[K]}$ maps injective objects to stalkwise injective objects ([Bjö93, proposition 2.5.7]) and that, as I is $\mathscr{O}_X$-coherent ([Har77, proposition II.5.9]), we have

$$\mathscr{E}xt^i_{\mathscr{O}_X}(I^k, F)_x \simeq \mathrm{Ext}^i_{\mathscr{O}_{X,x}}(I^k_x, F_x)$$

for all i, F and $x \in X$ ([Har77, proposition III.6.8]), which means that stalkwise injective objects are $\Gamma_{[V]}$-acyclic. So, in conclusion, $R\Gamma_{[W]} = R(\Gamma_{[V]} \circ \Gamma_{[K]}) \simeq R\Gamma_{[V]} \circ R\Gamma_{[K]}$. This applies analogously to the setting of a complex manifold X, when $Z \subset X$ is a complex analytic subset and I its (coherent, cf. [GR84, section 4.2]) defining sheaf of ideals. If X

[6] Here, I'm much obliged to M. Kashiwara for clarifying this to me.

is the analytification of some projective smooth complex variety (as it will be the case in our situation), both versions obviously correspond to each other via analytification ([Ser56, section 3, proposition 10 and theorem 3]). If F is a (left) $\mathscr{D}$-module then so is $R\Gamma_{[W]}(F)$, cf. [Bjö93, section II.5].

Lemma 4.7. *Let X be a smooth complex variety (resp. a complex manifold). If $U \subset X$ is open, where $j\colon U \to X$ is the open embedding, and $W \subset X$ is a locally closed subset (resp. a locally closed complex analytic subset, in case X is a manifold), then*

$$j^{-1} \circ R\Gamma_{[W]} \simeq R\Gamma_{[j^{-1}(W)]} \circ j^{-1}.$$

Proof. The point here simply is that j^{-1} is compatible with quotients and directed colimits (as it has left and right adjoint) and with $\mathscr{H}om$, and it preserves injectives (and of course stalkwise injectivity), so it is enough to state that, for $W = K \cap V$ as above, I the defining ideal of $X \setminus V$ and J the defining ideal of K, and for any F, we have

$$\begin{aligned} j^{-1} \varinjlim_k \mathscr{H}om(I^k, \varinjlim_l \mathscr{H}om(\mathscr{O}_X/J^l, F)) \simeq \\ \simeq \varinjlim_k \mathscr{H}om((j^{-1}I)^k, \varinjlim_l \mathscr{H}om(\mathscr{O}_U/(j^{-1}J)^l, j^{-1}F)). \end{aligned}$$

□

Definition 4.8 (cf. section 4.3 of [KS16]). Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of complex (or algebraic) bordered spaces $\mathbf{X} = (X, \check{X})$ and $\mathbf{Y} = (Y, \check{Y})$. As usual let $\mathrm{pr}_{\check{X}}\colon \check{X} \times \check{Y} \to \check{X}$ and $\mathrm{pr}_{\check{Y}}\colon \check{X} \times \check{Y} \to \check{Y}$ be the projections. Then, for $\mathscr{M} \in D^b_{\mathrm{hol}}(\mathscr{D}_{\mathbf{X}})$ and $\mathscr{N} \in D^b_{\mathrm{hol}}(\mathscr{D}_{\mathbf{Y}})$, we choose representing objects $\tilde{\mathscr{M}} \in D^b_{\mathrm{hol}}(\mathscr{D}_{\check{X}})$ and $\tilde{\mathscr{N}} \in D^b_{\mathrm{hol}}(\mathscr{D}_{\check{Y}})$ and set

$$\begin{aligned} \underline{\int_f} \mathscr{M} &:= \int_{\mathrm{pr}_{\check{Y}}} \left(R\Gamma_{[\Gamma_f]}(\mathscr{O}_{\check{X}\times\check{Y}})[d_Y^{\mathbb{C}}] \overset{D}{\otimes} D\mathrm{pr}^*_{\check{X}} \tilde{\mathscr{M}} \right) \\ \underline{Df^*} \mathscr{N} &:= \int_{\mathrm{pr}_{\check{X}}} \left(R\Gamma_{[\Gamma_f]}(\mathscr{O}_{\check{X}\times\check{Y}})[d_Y^{\mathbb{C}}] \overset{D}{\otimes} D\mathrm{pr}^*_{\check{Y}} \tilde{\mathscr{N}} \right) \end{aligned}$$

If, in the above situation, $\mathbf{X}$ and $\mathbf{Y}$ are projective bordered spaces, algebraic and analytic construction correspond to each other via analytification, i. e. we have

$$\left(\underline{\int_f} (\bullet) \right)^{\mathrm{an}} \simeq \underline{\int_{f^{\mathrm{an}}}} (\bullet)^{\mathrm{an}}, \quad \left(\underline{Df^*}(\bullet) \right)^{\mathrm{an}} \simeq \underline{D(f^{\mathrm{an}})^*}(\bullet)^{\mathrm{an}}.$$

Remark 4.9. In [KS16, definition 4.14], bordered versions of the enhanced de Rham and solutions functors are established, denoted by $DR^E_{\mathbf{X}}$ resp. $\mathscr{S}ol^E_{\mathbf{X}}$, for some complex bordered space $\mathbf{X}$. By [KS16, proposition 4.15], these are compatible with the enhanced direct resp. inverse images in the usual way, in particular, for $\mathscr{M} \in D^b_{\mathrm{hol}}(\mathscr{D}_{\mathbf{X}})$ and $\tilde{\mathscr{M}} \in D^b_{\mathrm{hol}}(\mathscr{D}_{\check{X}})$ some representing object, one has

$$DR^E_{\mathbf{X}}(\mathscr{M}) \simeq Ej^{-1}_{\mathbf{X}} DR^E_{\check{X}}(\tilde{\mathscr{M}}),$$

where $j_{\mathbf{X}} \colon \mathbf{X} \to \check{X}$ is the bordered open embedding, as usual.

Lemma 4.10. *Let $f \colon X \to Y$ be a morphism of smooth complex varieties (resp. manifolds). We can consider this a morphism of algebraic (resp. complex) bordered spaces $\mathbf{X} = (X, X)$ and $\mathbf{Y} = (Y, Y)$. Then, the bordered versions of the external operations above agree with the usual ones, i. e.*

$$\int_{\underline{f}} \mathscr{M} \simeq \int_f \mathscr{M}, \quad \underline{Df^*}\mathscr{N} \simeq Df^*\mathscr{N}$$

for any $\mathscr{M} \in D^b_{\mathrm{hol}}(\mathscr{D}_X)$, $\mathscr{N} \in D^b_{\mathrm{hol}}(\mathscr{D}_Y)$.

Proof. Let us denote by i the closed embedding $\Gamma_f \to X \times Y$ and consider the projections

$$X \xleftarrow{\mathrm{pr}_X} X \times Y \xrightarrow{\mathrm{pr}_Y} Y.$$

Let us only write down the direct image case, the other one working completely analogous. Before we start, let us state the following observation: By hypothesis, we have that $\Gamma_f^{\mathrm{an}} \subset (X \times Y)^{\mathrm{an}}$ is a complex analytic (closed) subset. So, by [KS96, theorem 5.12], we know

$$R\Gamma_{[\Gamma_f^{\mathrm{an}}]}(\mathscr{O}_{(X\times Y)^{\mathrm{an}}}) \simeq \mathscr{T}\!hom(\mathbb{C}_{\Gamma_f^{\mathrm{an}}}, \mathscr{O}_{(X\times Y)^{\mathrm{an}}}).$$

On the other hand, $\mathscr{T}\!hom(\bullet, \mathscr{O}_{(X\times Y)^{\mathrm{an}}})$ is nothing but the quasi-inverse to the solution functor

$$\mathcal{S}ol_{(X\times Y)^{\mathrm{an}}} \colon D^b_{\mathrm{rh}}(\mathscr{D}_{(X\times Y)^{\mathrm{an}}}) \longrightarrow D^b_{\mathbb{C}-c}(\mathbb{C}_{(X\times Y)^{\mathrm{an}}})$$

of the classical Riemann–Hilbert-correspondence ([Kas84]), also cf. paragraph 1.4 of the introduction of [DK16b] for a very short summary. With

$$\mathcal{S}ol\left(\int_i (\bullet)\right) \simeq Ri^{\mathrm{an}}_! \mathcal{S}ol(\bullet)[d^{\mathbb{C}}_{\Gamma_f} - d^{\mathbb{C}}_{X\times Y}] = Ri^{\mathrm{an}}_! \mathcal{S}ol(\bullet)[-d^{\mathbb{C}}_Y],$$

which holds algebraically – for $\int_i$ – as well as analytically – i. e. for $\int_{i^{\mathrm{an}}}$ – as i is proper as a closed embedding, we thus find

$$R\Gamma_{[\Gamma_f^{\mathrm{an}}]}(\mathscr{O}_{(X\times Y)^{\mathrm{an}}}) \simeq \left(\int_i \mathscr{O}_{\Gamma_f}\right)^{\mathrm{an}} [-d^{\mathbb{C}}_Y],$$

resp.

$$R\Gamma_{[\Gamma_f]}(\mathscr{O}_{X\times Y}) \simeq \int_i \mathscr{O}_{\Gamma_f}[-d^{\mathbb{C}}_Y].$$

Now we can calculate

$$\begin{aligned}\underline{\int_f}\mathscr{M} &\simeq \int_{\mathrm{pr}_Y}\left(R\Gamma_{[\Gamma_f]}(\mathscr{O}_{X\times Y})[d_Y^{\mathbb{C}}]\overset{D}{\otimes} D\mathrm{pr}_X^*\mathscr{M}\right)\\
&\simeq \int_{\mathrm{pr}_Y}\left(\int_i \mathscr{O}_{\Gamma_f}\overset{D}{\otimes} D\mathrm{pr}_X^*\mathscr{M}\right)\\
&\simeq \int_{\mathrm{pr}_Y}\int_i\left(\mathscr{O}_{\Gamma_f}\overset{D}{\otimes} Di^*D\mathrm{pr}_X^*\mathscr{M}\right)\\
&\simeq \int_{\mathrm{pr}_Y\circ i} D(\mathrm{pr}_X\circ i)^*\mathscr{M}\\
&\simeq \int_{\mathrm{pr}_Y\circ i}\int_{(\mathrm{pr}_X\circ i)^{-1}}\mathscr{M}\simeq\int_f\mathscr{M},\end{aligned}$$

where we used the projection formula [HTT08, corollary 1.7.5]. □

Remark 4.11. In the notation of [KS16, section 3.1], $D^b_{\mathrm{g-hol}}(\mathscr{D}_X)$ is denoting the objects in $D^b(\mathscr{D}_X)$ with good holonomic cohomology, and $D^b_{\mathrm{g-hol}}(\mathscr{D}_{\mathbf{X}})$ is defined accordingly. We have that $\underline{Df^*}$ preserves goodness (as well as holonomicity) and $\underline{\int_f}$ preserves the property of being good and holonomic, that is $\underline{\int_f}$ induces a functor

$$\underline{\int_f}\colon D^b_{\mathrm{g-hol}}(\mathscr{D}_{\mathbf{X}})\to D^b_{\mathrm{g-hol}}(\mathscr{D}_{\mathbf{Y}}),$$

cf. [KS16, lemma 4.13]. From here on, we will assume that all appearing $\mathscr{D}$-modules have good cohomologies.

Remark 4.12. We used the unshifted $\mathscr{D}$-module inverse image Dp^* for $p=\mathrm{pr}_{\check{X}},\mathrm{pr}_{\check{Y}}$ above. As in [HTT08, section 1.5], we would like to write $\underline{f^\dagger}:=\underline{Df^*}[d^{\mathbb{C}}_{\mathbf{X}}-d^{\mathbb{C}}_{\mathbf{Y}}]$ in this setting, where $d^{\mathbb{C}}_{\mathbf{X}}:=d^{\mathbb{C}}_X$ for the bordered space $\mathbf{X}=(X,\check{X})$, and define $\underline{f^\bigstar}:=\mathbb{D}_{\mathbf{X}}\underline{f^\dagger}\mathbb{D}_{\mathbf{Y}}$ as well as $\underline{\int_{f!}}:=\mathbb{D}_{\mathbf{Y}}\underline{\int_f}\mathbb{D}_{\mathbf{X}}$, where $\mathbb{D}_{\mathbf{X}}$, for some bordered space $\mathbf{X}=(X,\check{X})$, shall refer to the duality functor on $D^b_{\mathrm{hol}}(\mathscr{D}_{\mathbf{X}})$ induced by $\mathbb{D}_{\check{X}}$.

Proposition 4.13. *Suppose that $f\colon\mathbf{X}\to\mathbf{Y}$ is a morphism of projective algebraic bordered spaces, f^{an} its analytification, and let $\mathscr{M}\in\mathrm{Hol}(\mathbf{X})$ and $\mathscr{N}\in\mathrm{Hol}(\mathbf{Y})$. Then*

$$\begin{aligned}j^{-1}_{Y^{\mathrm{an}}}\underline{\int_{f^{\mathrm{an}}}}\mathscr{M}^{\mathrm{an}}&\simeq\Big(\int_f j_X^{-1}\mathscr{M}\Big)^{\mathrm{an}},\\
j^{-1}_{X^{\mathrm{an}}}\underline{(f^{\mathrm{an}})^\dagger}\mathscr{N}^{\mathrm{an}}&\simeq(f^\dagger j_Y^{-1}\mathscr{N})^{\mathrm{an}}.\end{aligned}$$

Proof. By definition of the minimal extension, we have exact sequences

$$0\longrightarrow Dj_{X,!*}j_X^{-1}\mathscr{M}\longrightarrow j_{X,*}j_X^{-1}\mathscr{M}\longrightarrow A\longrightarrow 0,$$

$$0 \longrightarrow Dj_{Y,!*}j_Y^{-1}\mathscr{N} \longrightarrow j_{Y,*}j_Y^{-1}\mathscr{N} \longrightarrow B \longrightarrow 0,$$

where clearly $\operatorname{Supp}(A) \subset \check{X} \setminus X$ and $\operatorname{Supp}(B) \subset \check{Y} \setminus Y$. So we have

$$\begin{aligned}
\mathscr{M}^{\mathrm{an}} &\simeq (Dj_{X,!*}j_X^{-1}\mathscr{M})^{\mathrm{an}} \simeq (j_{X,*}j_X^{-1}\mathscr{M})^{\mathrm{an}} \in D^b_{\mathrm{hol}}(\mathscr{D}_{\mathbf{X}^{\mathrm{an}}}),\\
\mathscr{N}^{\mathrm{an}} &\simeq (Dj_{Y,*}j_Y^{-1}\mathscr{N})^{\mathrm{an}} \simeq (j_{Y,*}j_Y^{-1}\mathscr{N})^{\mathrm{an}} \in D^b_{\mathrm{hol}}(\mathscr{D}_{\mathbf{Y}^{\mathrm{an}}}).
\end{aligned}$$

In particular,

$$\begin{aligned}
j_{Y^{\mathrm{an}}}^{-1}\underline{\int_{f^{\mathrm{an}}}}\mathscr{M}^{\mathrm{an}} &\simeq j_{Y^{\mathrm{an}}}^{-1}\underline{\int_{f^{\mathrm{an}}}}(j_{X,*}j_X^{-1}\mathscr{M})^{\mathrm{an}},\\
j_{X^{\mathrm{an}}}^{-1}\underline{(f^{\mathrm{an}})^{\dagger}}\mathscr{N}^{\mathrm{an}} &\simeq j_{X^{\mathrm{an}}}^{-1}\underline{(f^{\mathrm{an}})^{\dagger}}(j_{Y,*}j_Y^{-1}\mathscr{N})^{\mathrm{an}}.
\end{aligned}$$

Note that, as $\mathbf{X}$ and $\mathbf{Y}$ are projective, we have

$$j_{Y^{\mathrm{an}}}^{-1}\underline{\int_{f^{\mathrm{an}}}}\mathscr{M}^{\mathrm{an}} \simeq \left(j_Y^{-1}\underline{\int_f}\mathscr{M}\right)^{\mathrm{an}}, \quad j_{X^{\mathrm{an}}}^{-1}\underline{(f^{\mathrm{an}})^{\dagger}}\mathscr{N}^{\mathrm{an}} \simeq \left(j_X^{-1}\underline{f^{\dagger}}\mathscr{N}\right)^{\mathrm{an}}.$$

Let us introduce (resp. recall) some notation, depicted in the following diagram, in which both sides and the top quadrangle are pullbacks:

$$\begin{tikzcd}
 & & X\times Y \arrow[dll, bend right, "i_X"'] \arrow[d, "j_{X\times Y}"] \arrow[drr, bend left, "i_Y"] & & \\
X\times\check{Y} \arrow[rr, "\tilde{j}_X"] \arrow[d, "\mathrm{pr}_X"] & & \check{X}\times\check{Y} \arrow[dl, "\mathrm{pr}_{\check{X}}"'] \arrow[dr, "\mathrm{pr}_{\check{Y}}"] & & \check{X}\times Y \arrow[ll, "\tilde{j}_Y"'] \arrow[d, "\mathrm{pr}_Y"] \\
X \arrow[r, "j_X"] & \check{X} & & \check{Y} & Y \arrow[l, "j_Y"']
\end{tikzcd}$$

By repeatedly using lemma 4.7, base change ([HTT08, theorem 1.7.3]) and the projection formula ([HTT08, corollary 1.7.5]), we get

$$\begin{aligned}
j_Y^{-1}\underline{\int_f}\mathscr{M} &\simeq j_Y^{-1}\int_{\mathrm{pr}_{\check{Y}}}\left(R\Gamma_{[\Gamma_f]}(\mathscr{O}_{\check{X}\times\check{Y}})[d_Y^{\mathbb{C}}] \overset{D}{\otimes} D\mathrm{pr}_{\check{X}}^{*}(j_{X,*}j_X^{-1}\mathscr{M})\right)\\
&\simeq \int_{\mathrm{pr}_Y}\left(\tilde{j}_Y^{-1}R\Gamma_{[\Gamma_f]}(\mathscr{O}_{\check{X}\times\check{Y}})[d_Y^{\mathbb{C}}] \overset{D}{\otimes} \tilde{j}_Y^{-1}\tilde{j}_{X,*}D\mathrm{pr}_X^{*}(j_X^{-1}\mathscr{M})\right)\\
&\simeq \int_{\mathrm{pr}_Y}\left(R\Gamma_{[\Gamma_f]}(\mathscr{O}_{\check{X}\times Y})[d_Y^{\mathbb{C}}] \overset{D}{\otimes} i_{Y,*}i_X^{-1}D\mathrm{pr}_X^{*}(j_X^{-1}\mathscr{M})\right)\\
&\simeq \int_{\mathrm{pr}_Y} i_{Y,*}\left(i_Y^{-1}R\Gamma_{[\Gamma_f]}(\mathscr{O}_{\check{X}\times Y})[d_Y^{\mathbb{C}}] \overset{D}{\otimes} i_X^{-1}D\mathrm{pr}_X^{*}(j_X^{-1}\mathscr{M})\right)\\
&\simeq \int_{p_2}\left(R\Gamma_{[\Gamma_f]}(\mathscr{O}_{X\times Y})[d_Y^{\mathbb{C}}] \overset{D}{\otimes} Dp_1^{*}(j_X^{-1}\mathscr{M})\right) \simeq \int_f (j_X^{-1}\mathscr{M}),
\end{aligned}$$

where, for the last line, we denoted by p_1, p_2 the canonical projections of $X \times Y$ so that, by definition, $p_1 = \mathrm{pr}_X \circ i_X$ and $p_2 = \mathrm{pr}_Y \circ i_Y$, and used lemma 4.10. Analogously, we get

$$\begin{aligned} j_X^{-1} \underline{f}^\dagger \mathscr{N} \simeq & j_X^{-1} \int_{\mathrm{pr}_{\check{X}}} \left(R\Gamma_{[\Gamma_f]}(\mathscr{O}_{\check{X}\times\check{Y}})[d_Y^{\mathbb{C}}] \overset{D}{\otimes} D\mathrm{pr}_{\check{Y}}^*(j_{Y,*}(j_Y^{-1}\mathscr{M}) \right) \\ \simeq & \int_{\mathrm{pr}_X} \left(R\Gamma_{[\Gamma_f]}(\mathscr{O}_{X\times\check{Y}})[d_Y^{\mathbb{C}}] \overset{D}{\otimes} \tilde{j}_X^{-1} \tilde{j}_{Y,*} D\mathrm{pr}_Y^*(j_Y^{-1}\mathscr{N}) \right) \\ \simeq & \int_{\mathrm{pr}_X} i_{X,*} \left(R\Gamma_{[\Gamma_f]}(\mathscr{O}_{X\times Y})[d_Y^{\mathbb{C}}] \overset{D}{\otimes} i_Y^{-1} D\mathrm{pr}_Y^*(j_Y^{-1}\mathscr{N}) \right) \\ \simeq & \int_{p_1} \left(R\Gamma_{[\Gamma_f]}(\mathscr{O}_{X\times Y})[d_Y^{\mathbb{C}}] \overset{D}{\otimes} Dp_2^*(j_Y^{-1}\mathscr{N}) \right) \simeq f^\dagger(j_Y^{-1}\mathscr{N}), \end{aligned}$$

again using the projection formula and lemma 4.10. □

The compatibility result we have been working for in this section so far is the following

Proposition 4.14. *Let $\mathscr{M} \in \mathrm{Hol}(\mathbf{X})$ and $\mathscr{N} \in \mathrm{Hol}(\mathbf{Y})$ be meromorphic connections on projective algebraic bordered spaces $\mathbf{X}$ resp. $\mathbf{Y}$. Let us denote by $j_{\mathbf{X}}$ resp. $j_{\mathbf{Y}}$ the bordered open embeddings $\mathbf{X} \to \check{X}$ resp. $\mathbf{Y} \to \check{Y}$ and by $j_X \colon X \to \check{X}$ resp. $j_Y \colon Y \to \check{Y}$ the non-bordered versions. Let $f \colon \mathbf{X} \to \mathbf{Y}$ be a morphism of bordered spaces such that $\int_f (j_X^{-1}\mathscr{M}) \in \mathrm{Hol}(\mathscr{D}_Y)$ and $f^\dagger(j_Y^{-1}\mathscr{N}) \in \mathrm{Hol}(\mathscr{D}_X)$. Then*

$$\begin{aligned} E(j_{\mathbf{Y}}^{\mathrm{an}})^{-1} DR^E_{\check{Y}^{\mathrm{an}}} \left(\left(\underline{\int_f^0} \mathscr{M} \right)^{\mathrm{an}} \right) \simeq & E f_*^{\mathrm{an}} E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} DR^E_{\check{X}^{\mathrm{an}}}(\mathscr{M}^{\mathrm{an}}), \\ E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} DR^E_{\check{X}^{\mathrm{an}}} \left(\left(\underline{f^{\dagger,0}} \mathscr{N} \right)^{\mathrm{an}} \right) \simeq & E(f^{\mathrm{an}})^! E(j_{\mathbf{Y}}^{\mathrm{an}})^{-1} DR^E_{\check{Y}^{\mathrm{an}}}(\mathscr{N}^{\mathrm{an}}). \end{aligned}$$

Before we prove this, we would like to state another auxiliary observation.

Lemma 4.15. *Let $\mathbf{X} = (X, \check{X})$ be an algebraic bordered space and let $j \colon X \to \check{X}$ denote the corresponding open embedding, whereas we write $j_{\mathbf{X}} \colon \mathbf{X} \to \check{X}$ for the bordered version, as above. For $\mathscr{M} \in \mathrm{Hol}(\check{X})$ we have*

$$E j_{\mathbf{X}}^{-1} DR^E_{\check{X}}(\mathscr{M}^{\mathrm{an}}) \simeq E j_{\mathbf{X}}^{-1} DR^E_{\check{X}}((D j_{!*} j^{-1} \mathscr{M})^{\mathrm{an}}).$$

Proof. By [DK16b, theorem 9.1.2 (iv)] and [DK16a, lemma 2.7.6] we have

$$\begin{aligned} E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} DR^E_{\check{X}^{\mathrm{an}}}((D j_{!*} j^{-1}\mathscr{M})^{\mathrm{an}}) \simeq & E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} E j_{\mathbf{X},*}^{\mathrm{an}} E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} DR^E_{\check{X}^{\mathrm{an}}}((D j_{!*} j^{-1}\mathscr{M})^{\mathrm{an}}) \\ \simeq & E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} R\mathscr{H}om(\pi^{-1}\mathbb{C}_{X^{\mathrm{an}}}, DR^E_{\check{X}^{\mathrm{an}}}((D j_{!*} j^{-1}\mathscr{M})^{\mathrm{an}}) \\ \simeq & E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} DR^E_{\check{X}^{\mathrm{an}}}(\mathscr{T}hom(\mathbb{C}_{X^{\mathrm{an}}}, \mathscr{O}_{\check{X}^{\mathrm{an}}}) \overset{D}{\otimes} (D j_{!*} j^{-1}\mathscr{M})^{\mathrm{an}}) \end{aligned}$$

$$
\begin{aligned}
&\simeq E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} DR^E_{\check{X}^{\mathrm{an}}}((j_*\mathscr{O}_X)^{\mathrm{an}} \overset{D}{\otimes} (Dj_{!*}j^{-1}\mathscr{M})^{\mathrm{an}})\\
&\simeq E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} DR^E_{\check{X}^{\mathrm{an}}}((j_*j^{-1}\mathscr{O}_{\check{X}} \overset{D}{\otimes} Dj_{!*}j^{-1}\mathscr{M})^{\mathrm{an}})\\
&\simeq E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} DR^E_{\check{X}^{\mathrm{an}}}((j_*j^{-1}\mathscr{M})^{\mathrm{an}})\\
&\simeq E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} DR^E_{\check{X}^{\mathrm{an}}}(j_*(\mathscr{O}_X \overset{D}{\otimes} j^{-1}\mathscr{M})^{\mathrm{an}})\\
&\simeq E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} DR^E_{\check{X}^{\mathrm{an}}}((j_*\mathscr{O}_X)^{\mathrm{an}} \overset{D}{\otimes} \mathscr{M}^{\mathrm{an}})\\
&\simeq E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} R\mathscr{I}hom(\pi^{-1}\mathbb{C}_{X^{\mathrm{an}}}, DR^E_{\check{X}^{\mathrm{an}}}(\mathscr{M}^{\mathrm{an}}))\\
&\simeq E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} DR^E_{\check{X}}(\mathscr{M}^{\mathrm{an}}).
\end{aligned}
$$

□

Proof of proposition 4.14. Recall that we denote by j_X resp. j_Y the open embeddings $X \to \check{X}$ resp. $Y \to \check{Y}$ and with $j_{\mathbf{X}}$ resp. $j_{\mathbf{Y}}$ the corresponding bordered versions. Again we would like to only prove the direct image case, the proceeding is completely analogous for the inverse images. With proposition 4.13 and lemma 4.15, we get

$$
\begin{aligned}
E(j_{\mathbf{Y}}^{\mathrm{an}})^{-1} DR^E_{\check{Y}^{\mathrm{an}}}\left(\left(\underline{\int_f^0}\mathscr{M}\right)^{\mathrm{an}}\right) &\simeq E(j_{\mathbf{Y}}^{\mathrm{an}})^{-1} DR^E_{\check{Y}^{\mathrm{an}}}\left(\left(Dj_{Y,!*}\int_f j_X^{-1}\mathscr{M}\right)^{\mathrm{an}}\right)\\
&\simeq E(j_{\mathbf{Y}}^{\mathrm{an}})^{-1} DR^E_{\check{Y}^{\mathrm{an}}}\left(\left(Dj_{Y,!*}j_Y^{-1}\underline{\int_f}\mathscr{M}\right)^{\mathrm{an}}\right)\\
&\simeq E(j_{\mathbf{Y}}^{\mathrm{an}})^{-1} DR^E_{\check{Y}^{\mathrm{an}}}\left(\left(\underline{\int_f}\mathscr{M}\right)^{\mathrm{an}}\right)\\
&\simeq E(j_{\mathbf{Y}}^{\mathrm{an}})^{-1} DR^E_{\check{Y}^{\mathrm{an}}}\left(\underline{\int_{f^{\mathrm{an}}}}\mathscr{M}^{\mathrm{an}}\right)\\
&\simeq Ef_*^{\mathrm{an}} E(j_{\mathbf{X}}^{\mathrm{an}})^{-1} DR^E_{\check{X}^{\mathrm{an}}}(\mathscr{M}^{\mathrm{an}}),
\end{aligned}
$$

where for the last line, we used the compatibility of $\underline{\int_{f^{\mathrm{an}}}}$ and $DR^E_{\mathbf{X}^{\mathrm{an}}}$ proven in [DK16a, proposition 4.15], cf. remark 4.9 – here, it is important that f is semi-proper (as $\mathbf{X}$ and $\mathbf{Y}$ are projective by hypothesis). □

By applying duality, we get analogous results for the case of using the enhanced solutions functor $\mathcal{S}ol^E$ instead of DR^E.

4.2 Middle convolutions and enhanced Riemann–Hilbert correspondence

Consider the canonical projections

$$\mathbb{A}^1 \xleftarrow{p_1} \mathbb{A}^2 \xrightarrow{p_2} \mathbb{A}^1$$

and let us denote by (x, y) the coordinates on $\mathbb{A}^2$. Recall from [Mal91, appendix A.1] that for some $\mathscr{D}_{\mathbb{A}^1}$-module $\mathscr{M}$, the Fourier transform on $D^b_{\mathrm{hol}}(\mathscr{D}_{\mathbb{A}^1})$ may be defined as

$$\mathrm{FT}\,\mathscr{M} = \int_{p_2} (p_1^\bigstar \mathscr{M} \otimes e^{-xy})[1] = \int_{p_2} (p_1^\dagger \mathscr{M} \otimes e^{-xy})[-1],$$

with quasi-inverse $\mathrm{FT}^{-1}(\bullet) = \int_{p_2}(p_1^\bigstar(\bullet) \otimes e^{xy})[1]$. We know from [Mal91, appendix A.4] that we have

$$\mathrm{FT} \circ \mathbb{D}_{\mathbb{A}^1} \simeq \mathbb{D}_{\mathbb{A}^1} \circ \mathrm{FT}^{-1}.$$

As in section 2.4.1, we denote by $\mathcal{K}^\lambda \in \mathrm{Mod}_{\mathrm{rh}}(\mathbb{P}^1)$ the Kummer $\mathscr{D}$-module for some $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. If the context is clear, we will use the same label for the restriction $\mathcal{K}^\lambda|_{\mathbb{A}^1}$. Let j_0 be the open embedding $\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$. Now, in [Ari10], for some irreducible $\mathscr{M} \in \mathrm{Hol}(\mathbb{P}^1)$ with singularities containing $\infty \in \mathbb{P}^1$, writing $\mathscr{M}$ again for its restriction to $\mathbb{A}^1$, a middle convolution on $\mathbb{A}^1$ is defined by

$$\mathscr{M} *_{\mathrm{mid}} \mathcal{K}^\lambda := \mathrm{FT}^{-1}(Dj_{0,!*}(j_0^{-1}(\mathrm{FT}(\mathscr{M}) \overset{D}{\otimes} \mathcal{K}^{-\lambda}))). \tag{4.2}$$

We would like to refer to this construction as the Arinkin–Katz convolution. Let us set $\mathbf{A} = (\mathbb{A}^1, \mathbb{P}^1)$ and denote the corresponding open embedding by $j \colon \mathbb{A}^1 \to \mathbb{P}^1$ and its bordered version by $j_{\mathbf{A}} \colon \mathbf{A} \to \mathbb{P}^1$. As stated in [Ari10], this middle convolution $(\bullet) *_{\mathrm{mid}} \mathcal{K}^\lambda$ is an autoequivalence on the irreducible objects in $\mathrm{Hol}(\mathbf{A})$ with quasi-inverse $(\bullet) *_{\mathrm{mid}} \mathcal{K}^{-\lambda}$. For the sake of precision, let us formulate the details here. First, we would like to extend the above to

$$\mathscr{M} *_{\mathrm{mid}} \mathcal{K}^\lambda = Dj_{!*}\,\mathrm{FT}^{-1}(Dj_{0,!*}j_0^{-1}(\mathrm{FT}(j^{-1}\mathscr{M}) \overset{D}{\otimes} j^{-1}\mathcal{K}^{-\lambda})).$$

Then, for some irreducible $\mathscr{M} \in \mathrm{Hol}(\mathbf{A})$, we have

$$\begin{aligned}
&(\mathscr{M} *_{\mathrm{mid}} \mathcal{K}^\lambda) *_{\mathrm{mid}} \mathcal{K}^{-\lambda} \simeq \\
&\simeq Dj_{!*}\,\mathrm{FT}^{-1}\,Dj_{0,!*}j_0^{-1}(\mathrm{FT}(j^{-1}Dj_{!*}\,\mathrm{FT}^{-1}\,Dj_{0,!*}j_0^{-1}(\mathrm{FT}(j^{-1}\mathscr{M}) \overset{D}{\otimes} j^{-1}\mathcal{K}^{-\lambda})) \overset{D}{\otimes} j^{-1}\mathcal{K}^{\lambda}) \\
&\qquad \simeq Dj_{!*}\,\mathrm{FT}^{-1}\,Dj_{0,!*}j_0^{-1}(Dj_{0,!*}j_0^{-1}(\mathrm{FT}(j^{-1}\mathscr{M}) \overset{D}{\otimes} j^{-1}\mathcal{K}^{-\lambda}) \overset{D}{\otimes} j^{-1}\mathcal{K}^{\lambda}) \\
&\qquad \simeq Dj_{!*}\,\mathrm{FT}^{-1}\,Dj_{0,!*}(j_0^{-1}\,\mathrm{FT}(j^{-1}\mathscr{M}) \overset{D}{\otimes} j_0^{-1}j^{-1}\mathcal{K}^{-\lambda} \overset{D}{\otimes} j_0^{-1}j^{-1}\mathcal{K}^{\lambda}) \\
&\qquad\qquad \simeq Dj_{!*}\,\mathrm{FT}^{-1}\,Dj_{0,!*}j_0^{-1}\,\mathrm{FT}(j^{-1}\mathscr{M}) \simeq Dj_{!*}j^{-1}\mathscr{M} \simeq \mathscr{M},
\end{aligned}$$

where for the last line, we used that $\mathscr{M}$ is irreducible by hypothesis, which implies $j^{-1}\mathscr{M}$ and its Fourier transform $\mathrm{FT}(j^{-1}\mathscr{M})$ are irreducible as well (cf. [Ari10, section 2.2]), so $Dj_{0,!*}j_0^{-1}\,\mathrm{FT}(j^{-1}\mathscr{M}) \simeq \mathrm{FT}(j^{-1}\mathscr{M})$ and $Dj_{!*}j^{-1}\mathscr{M} \simeq \mathscr{M}$.

Remark 4.16. Before we formulate our conjecture, let us introduce one more piece of notation. Let X, Y be smooth algebraic complex varieties and $\mathrm{pr}_X \colon X \times Y \to X$,

$\mathrm{pr}_Y : X \times Y \to Y$ the usual projections. Then, one may define an external tensor product for (left) $\mathscr{D}$-modules by

$$\mathscr{M} \overset{D}{\boxtimes} \mathscr{N} := D\mathrm{pr}_X^* \mathscr{M} \overset{D}{\otimes} D\mathrm{pr}_Y^* \mathscr{N}.$$

By definition[7], the above satisfies $(\mathscr{M} \overset{D}{\boxtimes} \mathscr{N})^{\mathrm{an}} \simeq \mathscr{M}^{\mathrm{an}} \overset{D}{\boxtimes} \mathscr{N}^{\mathrm{an}}$ for the external product of analytic $\mathscr{D}$-modules of e. g. [Bjö93, section 2.4], in particular, we have

$$\begin{aligned} \mathbb{D}_{X\times Y}(\mathscr{M} \overset{D}{\boxtimes} \mathscr{N}) \simeq& \mathbb{D}_X \mathscr{M} \overset{D}{\boxtimes} \mathbb{D}_Y \mathscr{N} \\ \mathcal{S}ol^E_{(X\times Y)^{\mathrm{an}}}((\mathscr{M} \overset{D}{\boxtimes} \mathscr{N})^{\mathrm{an}}) \simeq& \mathcal{S}ol^E_X(\mathscr{M}^{\mathrm{an}}) \overset{+}{\boxtimes} \mathcal{S}ol^E_Y(\mathscr{N}^{\mathrm{an}}). \end{aligned}$$

[7] At this point we use again (as we already did before) that analytification commutes with the $\mathscr{D}$-module tensor product $(\bullet) \overset{D}{\otimes} (\bullet)$ – here, we would like to once precautionally reassure ourselves of this fact. By definition, for some smooth complex variety X and $\mathscr{M}, \mathscr{N} \in D^b(\mathscr{D}_X)$, the tensor product $\mathscr{M} \overset{D}{\otimes} \mathscr{N}$ is nothing but the $\mathscr{O}$-module $\mathscr{M} \overset{L}{\otimes}_{\mathscr{O}_X} \mathscr{N}$, equipped with the $\mathscr{D}$-module structure determined by

$$\theta(m \otimes n) = \theta(m) \otimes n + m \otimes \theta(n)$$

for a section θ of Θ_X. The definition is the same for the analytic case. On the level of $\mathscr{O}_{X^{\mathrm{an}}}$-modules, we have an isomorphism (let ι denote the continuous map $X^{\mathrm{an}} \to X$)

$$\begin{aligned} &\mathscr{M}^{\mathrm{an}} \overset{L}{\otimes}_{\mathscr{O}_{X^{\mathrm{an}}}} \mathscr{N}^{\mathrm{an}} = \\ &= (\mathscr{O}_{X^{\mathrm{an}}} \otimes_{\iota^{-1}\mathscr{O}_X} \iota^{-1}\mathscr{M}) \overset{L}{\otimes}_{\mathscr{O}_{X^{\mathrm{an}}}} (\mathscr{O}_{X^{\mathrm{an}}} \otimes_{\iota^{-1}\mathscr{O}_X} \iota^{-1}\mathscr{N}) \xrightarrow{\simeq} \mathscr{O}_{X^{\mathrm{an}}} \otimes_{\iota^{-1}\mathscr{O}_X} \iota^{-1}(\mathscr{M} \overset{L}{\otimes}_{\mathscr{O}_X} \mathscr{N}), \\ &\qquad (f \otimes m) \otimes (g \otimes n) \mapsto (fg) \otimes (m \otimes n). \end{aligned}$$

Furthermore, the $\mathscr{D}_{X^{\mathrm{an}}}$-structure on the $\mathscr{O}_{X^{\mathrm{an}}}$-module $\mathscr{M}^{\mathrm{an}} = \mathscr{O}_{X^{\mathrm{an}}} \otimes_{\iota^{-1}\mathscr{O}_X} \iota^{-1}\mathscr{M}$ is determined by

$$\partial_x(f \otimes m) = \partial_x(f) \otimes m + f \otimes \partial_x m,$$

for some $f \otimes m \in \mathscr{M}^{\mathrm{an}}$ and some local coordinate x on X. So the $\mathscr{D}_{X^{\mathrm{an}}}$-structure on the $\mathscr{O}_{X^{\mathrm{an}}}$-module $\mathscr{M}^{\mathrm{an}} \overset{L}{\otimes}_{\mathscr{O}_{X^{\mathrm{an}}}} \mathscr{N}^{\mathrm{an}}$ is determined, for some section $(f \otimes m) \otimes (g \otimes n)$, by

$$\begin{aligned} \partial_x((f \otimes m) \otimes (g \otimes n)) &= \partial_x(f \otimes m) \otimes (g \otimes n) + (f \otimes m) \otimes \partial_x(g \otimes n) \\ &= (\partial_x f \otimes m) \otimes (g \otimes n) + (f \otimes \partial_x m) \otimes (g \otimes n) + (f \otimes m) \otimes (\partial_x g \otimes n) + (f \otimes m) \otimes (g \otimes \partial_x n), \end{aligned}$$

which is mapped to

$$\partial_x(fg) \otimes (m \otimes n) + fg \otimes (\partial_x m \otimes n + m \otimes \partial_x n) \in \mathscr{O}_{X^{\mathrm{an}}} \otimes_{\iota^{-1}\mathscr{O}_X} \iota^{-1}(\mathscr{M} \overset{L}{\otimes}_{\mathscr{O}_X} \mathscr{N})$$

under the above isomorphism of $\mathscr{O}_{X^{\mathrm{an}}}$-modules. The latter one, on the other hand, by definition is nothing but the characterization of the $\mathscr{D}_{X^{\mathrm{an}}}$-action on the $\mathscr{O}_{X^{\mathrm{an}}}$-module

$$(\mathscr{M} \overset{D}{\otimes} \mathscr{N})^{\mathrm{an}} = \mathscr{O}_{X^{\mathrm{an}}} \otimes_{\iota^{-1}\mathscr{O}_X} \iota^{-1}(\mathscr{M} \overset{L}{\otimes}_{\mathscr{O}_X} \mathscr{N}).$$

Consider two morphisms $f\colon X \to X'$ and $g\colon Y \to Y'$ of smooth complex varieties, then it is obvious from the definition that

$$D(f \times g)^*(\mathscr{M}' \overset{D}{\boxtimes} \mathscr{N}') \simeq Df^*\mathscr{M}' \overset{D}{\boxtimes} Dg^*\mathscr{N}',$$

where $\mathscr{M}'$ and $\mathscr{N}'$ are $\mathscr{D}_{X'}$- resp. $\mathscr{D}_{Y'}$-modules. Furthermore, we would like to prove that for $\mathscr{M}, \mathscr{N}$ as above, we have

$$\int_{f\times g} (\mathscr{M} \overset{D}{\boxtimes} \mathscr{N}) \simeq \int_f \mathscr{M} \overset{D}{\boxtimes} \int_g \mathscr{N},$$

(cf. [HTT08, proposition 1.5.30]). To do so, consider the diagram

$$\begin{tikzcd}
 & & X \times Y \arrow[dll, bend right, "p_1"'] \arrow[dl, "\tilde{g}"'] \arrow[dr, "\tilde{f}"] \arrow[drr, bend left, "p_2"] \arrow[dd, dotted, "f\times g"] & & \\
X \arrow[dr, "f"'] & X \times Y' \arrow[l, "\mathrm{pr}_1"'] \arrow[dr, "b"] & & X' \times Y \arrow[r, "\mathrm{pr}_2"] \arrow[dl, "a"'] & Y \arrow[dl, "g"] \\
 & X' & X' \times Y' \arrow[l, "\tilde{p}_1"'] \arrow[r, "\tilde{p}_2"] & Y' &
\end{tikzcd}$$

in which all three quadrangles are cartesian and where we are using the shorthands $\tilde{g} = \mathrm{Id}_X \times g$, $\tilde{f} = f \times \mathrm{Id}_Y$, $b = f \times \mathrm{Id}_{Y'}$, $a = \mathrm{Id}_{X'} \times g$. By repeatedly using base change ([HTT08, theorem 1.7.3]) and the projection formula ([HTT08, corollary 1.7.5]), we get (we will write $d_X := \dim(X)$ as usual)

$$\begin{aligned}
\int_f \mathscr{M} \overset{D}{\boxtimes} \int_g \mathscr{N} &= D\tilde{p}_1^* \int_f \mathscr{M} \overset{D}{\otimes} D\tilde{p}_2^* \int_g \mathscr{N} \\
&= \left(\tilde{p}_1^\dagger \int_f \mathscr{M} \overset{D}{\otimes} \tilde{p}_2^\dagger \int_g \mathscr{N}\right)[-d_{X'} - d_{Y'}] \\
&\simeq \left(\int_b \mathrm{pr}_1^\dagger \mathscr{M} \overset{D}{\otimes} \int_a \mathrm{pr}_2^\dagger \mathscr{N}\right)[-d_{X'} - d_{Y'}] \\
&\simeq \int_a \left(Da^* \int_b \mathrm{pr}_1^\dagger \mathscr{M} \overset{D}{\otimes} \mathrm{pr}_2^\dagger \mathscr{N}\right)[-d_{X'} - d_{Y'}] \\
&\simeq \int_a \left(a^\dagger \int_b \mathrm{pr}_1^\dagger \mathscr{M} \overset{D}{\otimes} \mathrm{pr}_2^\dagger \mathscr{N}\right)[-d_{X'} - d_{Y'} + d_{Y'} - d_Y] \\
&\simeq \int_a \left(\int_{\tilde{f}} \tilde{g}^\dagger \mathrm{pr}_1^\dagger \mathscr{M} \overset{D}{\otimes} \mathrm{pr}_2^\dagger \mathscr{N}\right)[-d_{X'} - d_Y] \\
&\simeq \int_a \int_{\tilde{f}} \left(p_1^\dagger \mathscr{M} \overset{D}{\otimes} D\tilde{f}^* \mathrm{pr}_2^\dagger \mathscr{N}\right)[-d_{X'} - d_Y] \\
&\simeq \int_{f\times g} (p_1^\dagger \mathscr{M} \overset{D}{\otimes} p_2^\dagger \mathscr{N})[-d_{X'} - d_Y + d_{X'} - d_X]
\end{aligned}$$

$$\simeq \int_{f\times g} (Dp_1^* \mathscr{M} \overset{D}{\otimes} Dp_2^* \mathscr{N}) = \int_{f\times g} \mathscr{M} \overset{D}{\boxtimes} \mathscr{N}.$$

With the help of the tools we collected so far, we will be able to reasonably substantiate our conjecture to the effect that our enhanced middle convolution construction is compatible to the one of [Ari08; Ari10] via the enhanced Riemann–Hilbert correspondence. Let us make this precise in form of the following

Conjecture 4.17. *Let* $i_{\mathbb{A}}\colon \mathbb{A}^1 \to \mathbb{P}^1$ *denote the open embedding and let* $\mathscr{M} \in \mathrm{Hol}(\mathbf{A})$ *be irreducible, such that*[8]

$$\int_\sigma (i_{\mathbb{A}}^{-1}\mathscr{M} \overset{D}{\boxtimes} i_{\mathbb{A}}^{-1}\mathcal{K}^\lambda) \in \mathrm{Hol}(\mathscr{D}_{\mathbb{A}^1}) \text{ and } \int_{\sigma!} (i_{\mathbb{A}}^{-1}\mathscr{M} \overset{D}{\boxtimes} i_{\mathbb{A}}^{-1}\mathcal{K}^\lambda) \in \mathrm{Hol}(\mathscr{D}_{\mathbb{A}^1}), \tag{4.3}$$

where σ *is the sum map*

$$\sigma\colon \mathbb{A}^2 \to \mathbb{A}^1, \quad (a,b) \mapsto a+b$$

of section 2, and set $K := E(j_{\mathbf{A}}^{\mathrm{an}})^{-1} \mathcal{S}ol_{\mathcal{P}}^E(\mathscr{M}^{\mathrm{an}})[1] \in {}^{1/2}E^0_{\mathbb{R}-c}(\mathbf{A}^{\mathrm{an}})$. *Then* $(K, L_\lambda^E[1])$ *has property* $\mathfrak{P}$ *(recall* $L_\lambda^E := \mathcal{S}ol_{\mathcal{P}}^E(\mathcal{K}^\lambda)$*, cf. section 2.4), we have*

$$K \overset{E}{*}_{\mathrm{mid}} L_\lambda^E[1] \simeq K \overset{E}{*}_{\mathrm{co-mid}} L_\lambda^E[1]$$

and furthermore

$$Ej_{\mathbf{A}}^{-1} \mathcal{S}ol_{\mathcal{P}}^E((\mathscr{M} *_{\mathrm{mid}} \mathcal{K}^\lambda)^{\mathrm{an}}) \simeq K \overset{E}{*}_{\mathrm{mid}} L_\lambda^E[1].$$

As announced in the introduction, we will be able to give a proof of conjecture 4.17 up to the verification of assumption 4.19 below. Consider the diagonal embedding

$$\Delta\colon \mathbb{A}^1 \to \mathbb{A}^2, \quad x \mapsto (x,x).$$

On the associated $\mathbb{C}$-vector spaces of closed points, this obviously is nothing but the transpose σ^T of the sum map

$$\sigma\colon \mathbb{A}^2 \to \mathbb{A}^1, \quad (x,y) \mapsto x+y.$$

This observation gives rise to the expectation (compare e. g. [KS16, proposition 5.6] for the enhanced ind-sheaf setting) that we should have a natural isomorphism

$$\mathrm{FT}\left(\int_\sigma \bullet\right) \simeq D(\sigma^T)^* \,\mathrm{FT}_{\mathbb{A}^2}(\bullet) = D\Delta^* \,\mathrm{FT}_{\mathbb{A}^2}(\bullet),$$

[8] Note that (4.3) is a necessary condition for (K, L_λ^E) having property $\mathfrak{P}$. The statement of the conjecture would then imply it is also sufficient.

where we write $\mathrm{FT}_{\mathbb{A}^2}$ for the Fourier transform on $\mathbb{A}^2$. In particular, for $\tilde{\mathscr{M}} := \mathscr{M}|_{\mathbb{A}^1}$ and $\mathscr{O}^\lambda := \mathcal{K}^\lambda|_{\mathbb{A}^1}$ as in the situation of conjecture 4.17, this would correspond to a canonical morphism

$$\mathrm{FT}\left(\int_\sigma \tilde{\mathscr{M}} \overset{D}{\boxtimes} \mathscr{O}^\lambda\right) \simeq D\Delta^* \,\mathrm{FT}_{\mathbb{A}^2}(\tilde{\mathscr{M}} \overset{D}{\boxtimes} \mathscr{O}^\lambda) \simeq D\Delta^*(\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathrm{FT}(\mathscr{O}^\lambda)), \tag{4.4}$$

where we are using that the Fourier transform acts component wise, i. e.

$$\mathrm{FT}_{\mathbb{A}^2}(\bullet \overset{D}{\boxtimes} \bullet) \simeq \mathrm{FT}(\bullet) \overset{D}{\boxtimes} \mathrm{FT}(\bullet),$$

cf. [Dai00, section 2.2]. Let us give a proof that this canonical morphism (4.4) exists.

Proposition 4.18. *There is a canonical morphism*

$$\mathrm{FT}\left(\int_\sigma \tilde{\mathscr{M}} \overset{D}{\boxtimes} \mathscr{O}^\lambda\right) \simeq D\Delta^*(\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathrm{FT}(\mathscr{O}^\lambda)).$$

Proof. Consider the following diagram ([Kat90, section 12.2]), for which the outer rectangle is cartesian (here, as usual, $p_1, p_2 \colon \mathbb{A}^2 \to \mathbb{A}^1$ are meant to denote the canonical projections)

$$\begin{tikzcd}
\mathbb{A}^2 \times \mathbb{A}^1 \arrow[r, "(x,y,z)\mapsto(x,z,y,z)", "\delta"'] \arrow[d, "\sigma\times\mathrm{Id}"] & \mathbb{A}^2 \times \mathbb{A}^2 \arrow[dd, "p_2\times p_2"] \\
\mathbb{A}^1 \times \mathbb{A}^1 \arrow[d, "p_2"] & \\
\mathbb{A}^1 \arrow[r, "\Delta", "x\mapsto(x,x)"'] & \mathbb{A}^1 \times \mathbb{A}^1.
\end{tikzcd} \tag{4.5}$$

Let us start in the upper right corner, with the object (cf. [Kat90, section 12.2])

$$(p_1^\dagger \tilde{\mathscr{M}} \otimes e^{-xy}) \overset{D}{\boxtimes} (p_1^\dagger \mathscr{O}^\lambda \otimes e^{-xy})[-2].$$

Then, on the one hand, we have

$$\begin{aligned}
D\Delta^* \int_{p_2\times p_2} (p_1^\dagger \tilde{\mathscr{M}} \otimes e^{-xy}) \overset{D}{\boxtimes} (p_1^\dagger \mathscr{O}^\lambda \otimes e^{-xy})[-2] \simeq& \\
\simeq& D\Delta^*\left(\int_{p_2} (p_1^\dagger \tilde{\mathscr{M}} \otimes e^{-xy}) \overset{D}{\boxtimes} \int_{p_2} (p_1^\dagger \mathscr{O}^\lambda \otimes e^{-xy})\right)[-2] \\
\simeq& D\Delta^*\left(\int_{p_2} (p_1^\dagger \tilde{\mathscr{M}} \otimes e^{-xy})[-1] \overset{D}{\boxtimes} \int_{p_2} (p_1^\dagger \mathscr{O}^\lambda \otimes e^{-xy})[-1]\right) \\
\simeq& D\Delta^*(\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathrm{FT}(\mathscr{O}^\lambda))
\end{aligned}$$

$$\simeq \mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\otimes} \mathrm{FT}(\mathscr{O}^\lambda),$$

while on the other hand,

$$\begin{aligned}\int_{p_2}\int_{\sigma\times\mathrm{Id}} D\delta^*\left((p_1^\dagger\tilde{\mathscr{M}}\otimes e^{-xy}) \overset{D}{\boxtimes} (p_1^\dagger\mathscr{O}^\lambda\otimes e^{-xy})\right)[-2] &\simeq \\ \overset{(*)}{\simeq} \int_{p_2}\left(p_1^\dagger\int_\sigma(\tilde{\mathscr{M}}\overset{D}{\boxtimes}\mathscr{O}^\lambda)\otimes e^{-xy}\right)[-1]& \\ \simeq \mathrm{FT}\left(\int_\sigma(\tilde{\mathscr{M}}\overset{D}{\boxtimes}\mathscr{O}^\lambda)\right),&\end{aligned}$$

which shows, by using the base change theorem ([HTT08, theorem 1.7.3]), that there indeed is a canonical isomorphism

$$\mathrm{FT}\left(\int_\sigma(\tilde{\mathscr{M}}\overset{D}{\boxtimes}\mathscr{O}^\lambda)\right) \simeq D\Delta^*(\mathrm{FT}(\tilde{\mathscr{M}})\overset{D}{\boxtimes}\mathrm{FT}(\mathscr{O}^\lambda)) \simeq \mathrm{FT}(\tilde{\mathscr{M}})\overset{D}{\otimes}\mathrm{FT}(\mathscr{O}^\lambda). \tag{4.6}$$

Here, step $(*)$ seems to need some substantiation, which we would like to provide by the following

Claim. *With notations as above, we have*

$$\int_{\sigma\times\mathrm{Id}} D\delta^*\left((p_1^\dagger\tilde{\mathscr{M}}\otimes e^{-xy}) \overset{D}{\boxtimes} (p_1^\dagger\mathscr{O}^\lambda\otimes e^{-xy})\right) \simeq \left(p_1^\dagger\int_\sigma(\tilde{\mathscr{M}}\overset{D}{\boxtimes}\mathscr{O}^\lambda)\otimes e^{-xy}\right)[1].$$

Proof of claim. First, recall the definition of the Fourier kernel from [Dai00] – let x, y denote the coordinates on $\mathbb{A}^2$ and t the coordinate on $\mathbb{A}^1$. Then, consider the $\mathscr{D}_{\mathbb{A}^1}$-module $\tilde{\mathscr{L}}$ (notation as in [Dai00, section 2] which is defined by the connection

$$\nabla\colon \mathscr{O}_{\mathbb{A}^1}\to\Omega_{\mathbb{A}^1},\quad P\mapsto \mathrm{d}\,P - P\,\mathrm{d}\,t$$

and define the $\mathscr{D}_{\mathbb{A}^2}$-module

$$\mathscr{L} := Ds^*\tilde{\mathscr{L}}$$

for $s\colon \mathbb{A}^2\to\mathbb{A}^1$, $(x,y)\mapsto xy$ the inner product. We call $\mathscr{L}$ the Fourier kernel and denote it by e^{-xy}, i. e. for some $\mathscr{D}_{\mathbb{A}^1}$-module $\mathscr{M}$, we write

$$p_1^\dagger\mathscr{M}\otimes e^{-xy} := p_1^\dagger\mathscr{M}\overset{D}{\otimes}\mathscr{L}.$$

Now, let $q_1, q_2\colon \mathbb{A}^2\times\mathbb{A}^2\to\mathbb{A}^2$ and $r_1, r_2, r_3\colon \mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1\to\mathbb{A}^1$ denote the respective projections, then, obviously, $p_1\circ q_1\circ\delta = r_1$ and $p_1\circ q_2\circ\delta = r_2$, so that

$$D\delta^*\left((p_1^\dagger\tilde{\mathscr{M}}\otimes e^{-xy}) \overset{D}{\boxtimes} (p_1^\dagger\mathscr{O}^\lambda\otimes e^{-xy})\right)$$

$$\begin{aligned}
=&D\delta^*\left((Dp_1^*\tilde{\mathscr{M}}\overset{D}{\otimes}\mathscr{L})\overset{D}{\boxtimes}(Dp_1^*\mathscr{O}^\lambda\overset{D}{\otimes}\mathscr{L})\right)[2]\\
\simeq&Dr_1^*\tilde{\mathscr{M}}\overset{D}{\otimes}D\delta^*Dq_1^*\mathscr{L}\overset{D}{\otimes}Dr_2^*\mathscr{O}^\lambda\overset{D}{\otimes}D\delta^*Dq_2^*\mathscr{L}[2]\\
\simeq&Dr_{1,2}^*\left(\tilde{\mathscr{M}}\overset{D}{\boxtimes}\mathscr{O}^\lambda\right)\overset{D}{\otimes}\left(D\delta^*Dq_1^*\mathscr{L}\overset{D}{\otimes}D\delta^*Dq_2^*\mathscr{L}\right)[2],
\end{aligned}$$

where $r_{1,2}\colon \mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1\to\mathbb{A}^1\times\mathbb{A}^1$ is the projection on the first two factors. Let us now show that

$$\mathscr{L}' := D\delta^*Dq_1^*\mathscr{L}_{(w_1,w_2)}\overset{D}{\otimes}D\delta^*Dq_2^*\mathscr{L}_{(z_1,z_2)} = D(\sigma\times\mathrm{Id})^*\mathscr{L}_{(x_1,x_2)},$$

where we tagged the Fourier kernels with labels corresponding to the coordinates of the respective versions of $\mathbb{A}^2$. Both sides clearly are isomorphic to $\mathscr{O}_{\mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1}$ as $\mathscr{O}$-modules, so it is enough to compare the $\Theta_{\mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1}$-actions. Unrolling the definitions, let us note that the action of $\Theta_{\mathbb{A}^2}$ – for coordinates, say, (x,y) of $\mathbb{A}^2$ – on $\mathscr{L}$ is given by

$$\partial_x\cdot(g\otimes 1)=((\partial_x-y)g)\otimes 1,\quad \partial_y\cdot g\otimes 1=((\partial_y-x)g)\otimes 1$$

for a section $g\otimes 1$ of the $\mathscr{O}$-module $\mathscr{L}\simeq Ds^*\tilde{\mathscr{L}}=\mathscr{O}_{\mathbb{A}^2}\otimes_{s^{-1}\mathscr{O}_{\mathbb{A}^2}}s^{-1}\tilde{\mathscr{L}}\simeq\mathscr{O}_{\mathbb{A}^2}$, also cf. [Dai00, section 2.2]. The maps $q_1\circ\delta$, $q_2\circ\delta$ are given by

$$\begin{aligned}
q_1\circ\delta\colon \mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1\to\mathbb{A}^2,\quad (x,y,z)\mapsto(x,z),\\
q_2\circ\delta\colon \mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1\to\mathbb{A}^2,\quad (x,y,z)\mapsto(y,z).
\end{aligned}$$

So, for some section $g\otimes 1$ of the $\mathscr{O}$-module

$$\mathscr{L}_1 := D(q_1\circ\delta)^*\mathscr{L}=\mathscr{O}_{\mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1}\otimes_{(q_1\circ\delta)^{-1}\mathscr{O}_{\mathbb{A}^2}}(q_1\circ\delta)^{-1}\mathscr{L}\simeq\mathscr{O}_{\mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1},$$

the connection ∇_1 on $\mathscr{L}_1$ is given by

$$\begin{aligned}
\partial_x\cdot(g\otimes 1)=&(\partial_x g)\otimes 1+g\sum_{i=1}^{2}\partial_x(w_i\circ(q_1\circ\delta))\otimes\partial_{w_i}(1),\\
=&(\partial_x g-zg)\otimes 1\\
\partial_y\cdot(g\otimes 1)=&(\partial_y g)\otimes 1,\\
\partial_z\cdot(g\otimes 1)=&(\partial_z g-xg)\otimes 1
\end{aligned}$$

(cf. [HTT08, page 21] for the formula describing the connection on the inverse image that we used here). The very same way, we know that the connection ∇_2 on the $\mathscr{O}$-module

$$\mathscr{L}_2 := D(q_2\circ\delta)^*\mathscr{L}=\mathscr{O}_{\mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1}\otimes_{(q_2\circ\delta)^{-1}\mathscr{O}_{\mathbb{A}^2}}(q_2\circ\delta)^{-1}\mathscr{L}\simeq\mathscr{O}_{\mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1}$$

is determined by

$$\begin{aligned}\partial_x \cdot (g \otimes 1) =& (\partial_x g) \otimes 1,\\ \partial_y \cdot (g \otimes 1) =& (\partial_y g - zg) \otimes 1,\\ \partial_z \cdot (g \otimes 1) =& (\partial_z g - yg) \otimes 1.\end{aligned}$$

Therefore, we have that $\mathscr{L}' = D\delta^* Dq_1^* \mathscr{L}_{(w_1,w_2)} \overset{D}{\otimes} D\delta^* Dq_2^* \mathscr{L}_{(z_1,z_2)} = \mathscr{L}_1 \overset{D}{\otimes} \mathscr{L}_2$ is the $\mathscr{O}$-module $\mathscr{O}_{\mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1} \otimes_{\mathscr{O}} \mathscr{O}_{\mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1} \simeq \mathscr{O}_{\mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1}$, equipped with the connection ∇' determined by

$$\begin{aligned}\partial_x \cdot (g \otimes 1) =& \nabla_1(g)(\partial_x) \otimes 1 + g \otimes \nabla_2(1)(\partial_x) = (\partial_x g - zg) \otimes 1\\ =& \nabla_1(1)(\partial_x) \otimes g + 1 \otimes \nabla_2(g)(\partial_x) = \partial_x \cdot (1 \otimes g),\\ \partial_y \cdot (g \otimes 1) =& \nabla_1(g)(\partial_y) \otimes 1 + g \otimes \nabla_2(1)(\partial_y) = (\partial_y g) \otimes 1 + g \otimes (-z)\\ =& (\partial_y g - zg) \otimes 1 = \partial_y \cdot (1 \otimes g),\\ \partial_z(g \otimes 1) =& \nabla_1(g)(\partial_z) \otimes 1 + g \otimes \nabla_2(1)(\partial_z) = (\partial_z g - xg) \otimes 1 + g \otimes (-y)\\ =& (\partial_z g - (x+y)g) \otimes 1 = \partial_z \cdot (1 \otimes g).\end{aligned}$$

With regard to the above lines we might, in the sense of the Fourier kernel notation, denote this $\mathscr{D}$-module suggestively by $e^{-(x+y)z}$. As an $\mathscr{O}$-module,

$$\hat{\mathscr{L}} := D(\sigma \times \mathrm{Id})^* \mathscr{L} = \mathscr{O}_{\mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1} \otimes_{(\sigma\times\mathrm{Id})^{-1}\mathscr{O}_{\mathbb{A}^2}} (\sigma \times \mathrm{Id})^{-1} \mathscr{L} \simeq \mathscr{O}_{\mathbb{A}^1\times\mathbb{A}^1\times\mathbb{A}^1},$$

and $\sigma \times \mathrm{Id}$ is given by $(x, y, z) \mapsto (x + y, z)$. Recall we denote the coordinates of $\mathbb{A}^2$ by (x_1, x_2) here. Then, we get the connection $\hat{\nabla}$ on $\hat{\mathscr{L}}$, for some section $g \otimes 1$ of $\hat{\mathscr{L}}$, as

$$\begin{aligned}\partial_x \cdot (g \otimes 1) =& (\partial_x g) \otimes 1 + g \sum_{i=1}^{2} \partial_x (x_i \circ (\sigma \times \mathrm{Id}) \otimes \partial_{x_i}(1) = (\partial_x g - zg) \otimes 1,\\ \partial_y \cdot (g \otimes 1) =& (\partial_y g - zg) \otimes 1,\\ \partial_z \cdot (g \otimes 1) =& (\partial_z g - (x+y)g) \otimes 1,\end{aligned}$$

so we indeed have $\nabla' = \hat{\nabla}$, i. e. $\mathscr{L}' = \hat{\mathscr{L}}$. This allows us to complete the proof of the claim by observing that the morphism $\sigma \times \mathrm{Id}$ is defined via the cartesian diagram

$$\begin{tikzcd} \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1 \arrow[r, "\sigma\times\mathrm{Id}"] \arrow[d, "r_{1,2}"] & \mathbb{A}^1 \times \mathbb{A}^1 \arrow[d, "p_1"] \\ \mathbb{A}^1 \times \mathbb{A}^1 \arrow[r, "\sigma"] & \mathbb{A}^1, \end{tikzcd}$$

so, by base change, we have

$$\int_{\sigma\times\mathrm{Id}} \circ D r_{1,2}^*(\bullet) = \int_{\sigma\times\mathrm{Id}} \circ r_{1,2}^\dagger(\bullet)[-1] \simeq p_1^\dagger \circ \int_\sigma (\bullet)[-1].$$

□

Note that the proof would have worked the absolute same way when we replaced e^{-xy} with e^{xy}, the kernel of the inverse Fourier transform. So we get an induced canonical isomorphism

$$\begin{aligned}
\mathrm{FT}\left(\mathbb{D}_{\mathbb{A}^1}\int_\sigma \mathbb{D}_{\mathbb{A}^2}(\tilde{\mathscr{M}} \overset{D}{\boxtimes} \mathscr{O}^\lambda)\right) \simeq & \mathbb{D}_{\mathbb{A}^1}\,\mathrm{FT}^{-1}\left(\int_\sigma (\mathbb{D}_{\mathbb{A}^1}\tilde{\mathscr{M}} \overset{D}{\boxtimes} \mathbb{D}_{\mathbb{A}^1}\mathscr{O}^\lambda)\right)\\
\simeq & \mathbb{D}_{\mathbb{A}^1} D\Delta^*(\mathrm{FT}^{-1}(\mathbb{D}_{\mathbb{A}^1}\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathrm{FT}^{-1}(\mathbb{D}_{\mathbb{A}^1}\mathscr{O}^\lambda))\\
\simeq & \mathbb{D}_{\mathbb{A}^1} D\Delta^*(\mathbb{D}_{\mathbb{A}^1}\,\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathbb{D}_{\mathbb{A}^1}\,\mathrm{FT}(\mathscr{O}^\lambda))\\
\simeq & \mathbb{D}_{\mathbb{A}^1} D\Delta^*\mathbb{D}_{\mathbb{A}^2}(\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathrm{FT}(\mathscr{O}^\lambda)).
\end{aligned} \tag{4.7}$$

Recall the natural morphism $\mathbb{D}_{\mathbb{A}^1} D\Delta^*\mathbb{D}_{\mathbb{A}^2}(\bullet) \to D\Delta^*$, cf. [HTT08, theorem 2.7.1] and consider the following natural

Assumption 4.19. *The canonical isomorphisms* (4.4) *and* (4.7) *interchange the natural morphism* $\mathbb{D}_{\mathbb{A}^1}\int_\sigma \mathbb{D}_{\mathbb{A}^2} \to \int_\sigma$ *with the natural morphism* $\mathbb{D}_{\mathbb{A}^1} D\Delta^*\mathbb{D}_{\mathbb{A}^2} \to D\Delta^*$*, more precisely, the following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{FT}\left(\mathbb{D}_{\mathbb{A}^1}\int_\sigma \mathbb{D}_{\mathbb{A}^1}(\tilde{\mathscr{M}} \overset{D}{\boxtimes} \mathscr{O}^\lambda)\right) & \xrightarrow[(4.7)]{\simeq} & \mathbb{D}_{\mathbb{A}^1} D\Delta^*\mathbb{D}_{\mathbb{A}^1}\left(\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathrm{FT}(\mathscr{O}^\lambda)\right)\\
\downarrow & & \downarrow\\
\mathrm{FT}\left(\int_\sigma (\tilde{\mathscr{M}} \overset{D}{\boxtimes} \mathscr{O}^\lambda)\right) & \xrightarrow[(4.4)]{\simeq} & D\Delta^*\left(\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathrm{FT}(\mathscr{O}^\lambda)\right).
\end{array}$$

Although assumption 4.19 seems highly plausible, finding a proof has turned out to be surprisingly intricate. It appears that the main part of the difficulties arises from the fact that the construction of the natural morphism $\mathbb{D}\int_\sigma \mathbb{D} \to \int_\sigma$ relies on a factorization of σ as an open embedding, followed by a proper morphism, e. g. – in notation from section 2 – as $\sigma = q_2 \circ (u \circ \alpha)$, with $q_2 \colon \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^2$ the second projection, $\alpha \colon \mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, x+y)$ and $u \colon \mathbb{A}^2 \to \mathbb{P}^1 \times \mathbb{A}^2$ the open embedding. Then, writing $j := u \circ \alpha$, a canonical morphism $\mathbb{D}\int_j \mathbb{D} \to \int_j$ may easily be found, e. g. using the adjunction $\int_{j!} \dashv j^\dagger$ and the fact that $j^\dagger \int_j \simeq \mathrm{Id}$, cf. [HTT08, theorem 3.2.16]. The morphism $\mathbb{D}\int_\sigma \mathbb{D} \to \int_\sigma$ is then obtained by applying $\mathbb{D}\int_{q_2} \mathbb{D} \simeq \int_{q_2}$. However, this two stage process is highly incompatible to any functorial properties of the Fourier transform.

Theorem 4.20. *If assumption 4.19 holds, then conjecture 4.17 is true.*

Proof. The basic idea for the proof is to use proposition 4.14 to get an affine $\mathscr{D}$-module counterpart to the enhanced convolutions, and to then check the statement on $\mathrm{Hol}(\mathscr{D}_{\mathbb{A}^1})$.

First note that, in the situation of proposition 4.14, i.e. $f\colon \mathbf{X}\to\mathbf{Y}$ a morphism of bordered spaces, $\mathscr{M}\in\mathrm{Hol}(\mathbf{X})$, applying $\mathrm{D}^E_{\mathbf{X}^{\mathrm{an}}}$ gives us

$$E(j_Y^{\mathrm{an}})^{-1}\mathcal{S}ol^E_{\check{Y}^{\mathrm{an}}}\left(\left(\underline{\int_f}^0 \mathscr{M}\right)^{\mathrm{an}}\right)\simeq Ef^{\mathrm{an}}_{!!}E(j_X^{\mathrm{an}})^{-1}\mathcal{S}ol^E_{\check{X}^{\mathrm{an}}}(\mathscr{M}^{\mathrm{an}})[d_X^{\mathbb{C}}-d_Y^{\mathbb{C}}]$$

and, similarly, using $\mathcal{S}ol^E_{\check{Y}}(\mathbb{D}^E_{\check{Y}}\bullet)\simeq DR^E_{\check{Y}}(\bullet)[-d^{\mathbb{C}}_{\check{Y}}]$ and $DR^E_{\check{X}}(\mathbb{D}_{\check{X}}\bullet)\simeq\mathcal{S}ol^E_{\check{X}}(\bullet)[d^{\mathbb{C}}_X]$ yields

$$E(j_Y^{\mathrm{an}})^{-1}\mathcal{S}ol^E_{\check{Y}^{\mathrm{an}}}\left(\left(\underline{\int_{f!}}^0 \mathscr{M}\right)^{\mathrm{an}}\right)\simeq Ef^{\mathrm{an}}_{*}E(j_X^{\mathrm{an}})^{-1}\mathcal{S}ol^E_{\check{X}^{\mathrm{an}}}(\mathscr{M}^{\mathrm{an}})[d_X^{\mathbb{C}}-d_Y^{\mathbb{C}}].$$

Note that $\overset{D}{\boxtimes}$ induces an operation

$$(\bullet)\overset{D}{\boxtimes}(\bullet)\colon\ \mathrm{Hol}(\mathbf{A})\times\mathrm{Hol}(\mathbf{A})\to\mathrm{Hol}(\mathbf{A}),$$

so we may apply proposition 4.14 and, denoting the analytified bordered open embeddings with $j_{\mathbf{A}^{\mathrm{an}}}\colon \mathbf{A}^{\mathrm{an}}\to\mathcal{P}$ resp. $j_{(\mathbf{A}^{\mathrm{an}})^2}\colon(\mathbf{A}^{\mathrm{an}})^2\to\mathcal{P}\times\mathcal{P}$, get

$$\begin{aligned}Ej^{-1}_{\mathbf{A}^{\mathrm{an}}}\mathcal{S}ol^E_{\mathcal{P}}\left(\left(\underline{\int_\sigma}^0(\mathscr{M}\overset{D}{\boxtimes}\mathcal{K}^\lambda)\right)^{\mathrm{an}}\right)[1]&\simeq E\sigma^{\mathrm{an}}_{!!}Ej^{-1}_{(\mathbf{A}^{\mathrm{an}})^2}\mathcal{S}ol^E_{\mathcal{P}\times\mathcal{P}}((\mathscr{M}\overset{D}{\boxtimes}\mathcal{K}^\lambda)^{\mathrm{an}})[2]\\&\simeq E\sigma^{\mathrm{an}}_{!!}Ej^{-1}_{(\mathbf{A}^{\mathrm{an}})^2}(\mathcal{S}ol^E_{\mathcal{P}}(\mathscr{M}^{\mathrm{an}})[1]\overset{+}{\boxtimes}\mathcal{S}ol^E_{\mathcal{P}}((\mathcal{K}^\lambda)^{\mathrm{an}})[1])\\&\simeq E\sigma^{\mathrm{an}}_{!!}(K\overset{+}{\boxtimes}L^E_\lambda[1])\end{aligned}$$

and, analogously,

$$\begin{aligned}Ej^{-1}_{\mathbf{A}^{\mathrm{an}}}\mathcal{S}ol^E_{\mathcal{P}}\left(\left(\underline{\int_{\sigma!}}^0(\mathscr{M}\overset{D}{\boxtimes}\mathcal{K}^\lambda)\right)^{\mathrm{an}}\right)[1]&\simeq E\sigma^{\mathrm{an}}_{*}Ej^{-1}_{(\mathbf{A}^{\mathrm{an}})^2}\mathcal{S}ol^E_{\mathcal{P}\times\mathcal{P}}((\mathscr{M}\overset{D}{\boxtimes}\mathcal{K}^\lambda)^{\mathrm{an}})[2]\\&\simeq E\sigma^{\mathrm{an}}_{*}Ej^{-1}_{(\mathbf{A}^{\mathrm{an}})^2}(\mathcal{S}ol^E_{\mathcal{P}}(\mathscr{M}^{\mathrm{an}})[1]\overset{+}{\boxtimes}\mathcal{S}ol^E_{\mathcal{P}}((\mathcal{K}^\lambda)^{\mathrm{an}})[1])\\&\simeq E\sigma^{\mathrm{an}}_{*}(K\overset{+}{\boxtimes}L^E_\lambda[1]).\end{aligned}$$

In particular, by our hypothesis, $\underline{\int_\sigma}^0(\mathscr{M}\overset{D}{\boxtimes}\mathcal{K}^\lambda)\in\mathrm{Hol}(\mathbf{A})$ and $\underline{\int_{\sigma!}}^0(\mathscr{M}\overset{D}{\boxtimes}\mathcal{K}^\lambda)\in\mathrm{Hol}(\mathbf{A})$, which means, as $\mathcal{S}ol^E_{\mathcal{P}}(\bullet)[1]$ and $(\bullet)^{\mathrm{an}}$ are exact (and thus in particular commute with images), that the pair $(K,L^E_\lambda[1])$ has property $\mathfrak{P}$ and

$$K\overset{E}{*}_{\mathrm{mid}}L^E_\lambda[1]\simeq K\overset{E}{*}_{\mathrm{co-mid}}L^E_\lambda[1],$$

as the standard t-structure of $D^b_{\mathrm{hol}}(\mathscr{D}_{\mathcal{P}})$ is 1-indexed (cf. corollary 3.8), which proves the first part of conjecture 4.17.

From here, what remains to show is that the operation

$$\mathscr{M} \overset{D}{*}_{\mathrm{mid}} \mathcal{K}^{\lambda} := \mathrm{Im}\left(\int^{0}_{\sigma !} (\mathscr{M} \overset{D}{\boxtimes} \mathcal{K}^{\lambda}) \to \int^{0}_{\sigma} (\mathscr{M} \overset{D}{\boxtimes} \mathcal{K}^{\lambda}) \right) \tag{4.8}$$

agrees with $\mathscr{M} *_{\mathrm{mid}} \mathcal{K}^{\lambda}$ on $\mathbb{A}^1$. With regard to (4.8), note that $\mathrm{Hol}(\mathbf{A}) \subset \mathrm{Hol}(\mathscr{D}_{\mathcal{P}})$ is closed with respect to taking images, by

Proposition 4.21. *Let X be a smooth complex variety and $j\colon U \to X$ an affine open embedding such that $Z := X \setminus U$ is a smooth variety. Then, $Dj_{!*}\colon \mathrm{Hol}(\mathscr{D}_U) \to \mathrm{Hol}(\mathscr{D}_X)$ preserves injectivity and surjectivity, i. e.*

i) if $0 \to \mathscr{M} \to \mathscr{N}$ is exact in $\mathrm{Hol}(\mathscr{D}_U)$, then

$$0 \longrightarrow Dj_{!*}\mathscr{M} \longrightarrow Dj_{!*}\mathscr{N}$$

is exact in $\mathrm{Hol}(\mathscr{D}_X)$,

ii) if $\mathscr{M} \to \mathscr{N} \to 0$ is exact in $\mathrm{Hol}(\mathscr{D}_U)$, then

$$Dj_{!*}\mathscr{M} \longrightarrow Dj_{!*}\mathscr{N} \longrightarrow 0$$

is exact in $\mathrm{Hol}(\mathscr{D}_X)$.

Proof. The idea for the proof is the very same as in [HTT08, proposition 8.2.7 and corollaries 8.2.8, 8.2.9] for the case of perverse sheaves. Let us denote by i the closed embedding $Z \to X$. If, for any $\mathscr{M} \in \mathrm{Hol}(\mathscr{D}_U)$, A is a subobject of $\int_j \mathscr{M} = j_*\mathscr{M}$ with $\mathrm{Supp}(A) \subset Z$, then $A = \int_i H^0 i^{\dagger} A$ by Kashiwara's equivalence ([HTT08, theorem 1.6.1]). Applying $H^0 i^{\dagger}$ to the exact sequence

$$0 \longrightarrow A \longrightarrow j_*\mathscr{M},$$

we get $H^0 i^{\dagger} A \simeq 0$, so $A \simeq 0$. Analogously, if $j_!\mathscr{M} \twoheadrightarrow B$ is a quotient (here, as usual, $j_! = \int_{j!} = \mathbb{D}_X j_* \mathbb{D}_U$) with $\mathrm{Supp}(B) \subset Z$, then, again, $B \simeq \int_i H^0 i^{\dagger} B$ by Kashiwara's equivalence, so let us apply $H^0 i^{\star}$ to the exact sequence

$$j_!\mathscr{M} \longrightarrow B \longrightarrow 0.$$

This yields $0 \simeq H^0 i^{\star} \int_i H^0 i^{\dagger} B \simeq H^0 i^{\dagger} B$, where we used $\int_i \simeq \int_{i!}$ as i is proper as a closed embedding, cf. [HTT08, theorem 3.2.16], so $B = 0$. In particular, let $A \subset Dj_{!*}\mathscr{M}$ be a subobject and $Dj_{!*}\mathscr{M} \twoheadrightarrow B$ a quotient such that $\mathrm{Supp}(A), \mathrm{Supp}(B) \subset Z$. Then the diagram

$$\begin{tikzcd}
 & B & \\
j_!\mathscr{M} \arrow[r, two heads] \arrow[ur, dashed, two heads] & Dj_{!*}\mathscr{M} \arrow[u, two heads] \arrow[r, hook] & j_*\mathscr{M} \\
 & A \arrow[u, hook] \arrow[ur, dashed, hook] &
\end{tikzcd}$$

shows that $A \simeq 0 \simeq B$. So, for i), let A be the kernel

$$0 \longrightarrow A \longrightarrow Dj_{!*}\mathscr{M} \longrightarrow Dj_{!*}\mathscr{N}.$$

As $0 \to \mathscr{M} \to \mathscr{N}$ is exact by hypothesis (and $j^{-1}Dj_{!*} \simeq \mathrm{Id}$), we know that $j^{-1}A \simeq 0$, i.e. $\mathrm{Supp}(A) \subset Z$ and thus $A \simeq 0$ by the above. For ii), we analogously get $B \simeq 0$ for the cokernel

$$Dj_{!*}\mathscr{M} \longrightarrow Dj_{!*}\mathscr{N} \longrightarrow B \longrightarrow 0.$$

□

As we already know that the Arinkin–Katz convolution is an autoequivalence, with $((\bullet) *_{\mathrm{mid}} \mathcal{K}^{\lambda})^{-1} \simeq (\bullet) *_{\mathrm{mid}} \mathcal{K}^{-\lambda}$, it actually is enough to prove that $(\bullet) \overset{D}{*}_{\mathrm{mid}} \mathcal{K}^{\lambda}$ is a right-sided quasi-inverse to $(\bullet) *_{\mathrm{mid}} \mathcal{K}^{-\lambda}$, i.e. there is a natural isomorphism

$$(i_{\mathbb{A}}^{-1}\mathscr{M} \overset{D}{*}_{\mathrm{mid}} i_{\mathbb{A}}^{-1}\mathcal{K}^{\lambda}) *_{\mathrm{mid}} i_{\mathbb{A}}^{-1}\mathcal{K}^{-\lambda} \simeq i_{\mathbb{A}}^{-1}\mathscr{M},$$

where $i_{\mathbb{A}} \colon \mathbb{A}^1 \to \mathbb{P}^1$, as above. As the Fourier transform is exact and thus compatible with images, as well as $i_{\mathbb{A}}^{-1}$, we are lead to determining the image of

$$\begin{aligned} &\mathrm{FT}\left(i_{\mathbb{A}}^{-1}\underline{\int_{\sigma!}}^{0}(\mathscr{M} \overset{D}{\boxtimes} \mathcal{K}^{\lambda})\right) \simeq \\ &\simeq \mathrm{FT}\left(\mathbb{D}_{\mathbb{A}^1}\int_{\sigma}\mathbb{D}_{\mathbb{A}^2}(i_{\mathbf{A}}^{-1}\mathscr{M} \overset{D}{\boxtimes} i_{\mathbb{A}}^{-1}\mathcal{K}^{\lambda})\right) \longrightarrow \mathrm{FT}\left(\int_{\sigma}(i_{\mathbb{A}}^{-1}\mathscr{M} \overset{D}{\boxtimes} i_{\mathbb{A}}^{-1}\mathcal{K}^{\lambda})\right) \simeq \\ &\qquad\qquad\simeq \mathrm{FT}\left(i_{\mathbb{A}}^{-1}\underline{\int_{\sigma}}^{0}(\mathscr{M} \overset{D}{\boxtimes} \mathcal{K}^{\lambda})\right) \end{aligned} \tag{4.9}$$

For the following, let us again denote $\tilde{\mathscr{M}} := i_{\mathbb{A}}^{-1}\mathscr{M}$ and $\mathscr{O}^{\lambda} := i_{\mathbb{A}}^{-1}\mathcal{K}^{\lambda}$ for the sake of notational compactness. Now, assumption 4.19 would give us a commutative square

$$\begin{array}{ccc} \mathrm{FT}\left(\mathbb{D}_{\mathbb{A}^1}\int_{\sigma}\mathbb{D}_{\mathbb{A}^1}(\tilde{\mathscr{M}} \overset{D}{\boxtimes} \mathscr{O}^{\lambda})\right) & \xrightarrow[(4.7)]{\simeq} & \mathbb{D}_{\mathbb{A}^1}\Delta^*\mathbb{D}_{\mathbb{A}^1}\left(\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathrm{FT}(\mathscr{O}^{\lambda})\right) \\ \downarrow & & \downarrow \\ \mathrm{FT}\left(\int_{\sigma}(\tilde{\mathscr{M}} \overset{D}{\boxtimes} \mathscr{O}^{\lambda})\right) & \xrightarrow[(4.4)]{\simeq} & \Delta^*\left(\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathrm{FT}(\mathscr{O}^{\lambda})\right), \end{array}$$

and state that (4.9) corresponds to the canonical morphism

$$\mathbb{D}_{\mathbb{A}^1}D\Delta^*\mathbb{D}_{\mathbb{A}^2}(\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathrm{FT}(\mathscr{O}^{\lambda})) \longrightarrow D\Delta^*(\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathrm{FT}(\mathscr{O}^{\lambda})). \tag{4.10}$$

For the subsequent calculations, we will use the following

Lemma 4.22. *We have* $\mathrm{FT}(\mathscr{O}^\lambda) \simeq \mathscr{O}^{-\lambda}$.

Proof. Recall that $\mathscr{O}^\lambda$ is the regular $\mathscr{D}_{\mathbb{A}^1}$-module $j_{0,*}\widetilde{\mathcal{K}^\lambda}$, for the integrable connection

$$\widetilde{\mathcal{K}^\lambda} = \left(\mathscr{O}_{\mathbb{A}^1\setminus\{0\}}, \mathrm{d} + \frac{\lambda}{z}\,\mathrm{d}\,z\right),$$

where z denotes the affine coordinate on $\mathbb{A}^1$, cf. section 2.4.1. In particular, we have $\mathscr{O}^\lambda \simeq \mathscr{D}_{\mathbb{A}^1}/\mathscr{D}_{\mathbb{A}^1}P$ for $P = z\partial_z + \lambda$. So, by definition of the algebraic Fourier transform, $\mathrm{FT}(\mathscr{O}^\lambda) \simeq \mathscr{D}_{\mathbb{A}^1}/\mathscr{D}_{\mathbb{A}^1}P'$ with

$$P' = -\partial_z z + \lambda = -z\partial_z + (\lambda - 1),$$

cf. [Dai00, section 2], i. e. $\mathrm{FT}(\mathscr{O}^\lambda) \simeq \mathscr{O}^{1-\lambda}$. So we will finish the proof with showing that $\mathscr{O}^{1-\lambda} \simeq \mathscr{O}^{-\lambda}$. To do so, note that we have

$$\mathscr{O}^\mu \simeq j_{0,*}j_0^{-1}\mathscr{O}^\mu$$

for any $\mu \in \mathbb{C}\setminus\mathbb{Z}$ and $j_0\colon \mathbb{A}^1\setminus\{0\} \to \mathbb{A}^1$ the open embedding, cf. lemma 1.63. So it is certainly enough to verify $j_0^{-1}\mathscr{O}^{1-\lambda} \simeq j_0^{-1}\mathscr{O}^{-\lambda}$. By definition, these are the connections

$$\begin{aligned}
j_0^{-1}\mathscr{O}^{1-\lambda} &= \left(\mathscr{O}_{\mathbb{A}^1\setminus\{0\}}, \nabla\right), \quad && \nabla\colon \mathscr{O}_{\mathbb{A}^1\setminus\{0\}} \to \Omega_{\mathbb{A}^1\setminus\{0\}} \\
& && \qquad f \mapsto \mathrm{d}\,f + \frac{(1-\lambda)f}{z}\,\mathrm{d}\,z \\
j_0^{-1}\mathscr{O}^{-\lambda} &= \left(\mathscr{O}_{\mathbb{A}^1\setminus\{0\}}, \nabla'\right), \quad && \nabla'\colon \mathscr{O}_{\mathbb{A}^1\setminus\{0\}} \to \Omega_{\mathbb{A}^1\setminus\{0\}} \\
& && \qquad f \mapsto \mathrm{d}\,f - \frac{\lambda f}{z}\,\mathrm{d}\,z
\end{aligned}$$

and we find that $j_0^{-1}\mathscr{O}^{1-\lambda} \xrightarrow{z\cdot} j_0^{-1}\mathscr{O}^{-\lambda}$ is an isomorphism of $\mathscr{D}_{\mathbb{A}^1\setminus\{0\}}$-modules. To prove this, it is enough to assert the compatibility with the action of ∂_z. For some $f \in \mathscr{O}_{\mathbb{A}^1\setminus\{0\}}$, we have

$$\begin{aligned}
\nabla'(zf)(\partial_z) = \left(\partial_z - \frac{\lambda}{z}\right)(zf) = \partial_z(zf) - \lambda f = z\partial_z f + f - \lambda f = \\
= z\left(\partial_z + \frac{1-\lambda}{z}\right) f = z\nabla(f)(\partial_z).
\end{aligned}$$

□

This being said, let us continue the proof of theorem 4.20. We would now like to prove that (4.10) is actually an isomorphism on $\mathbb{A}^1\setminus\{0\}$, due to the fact that $\mathscr{O}^\lambda$ is an integrable connection on $\mathbb{A}^1\setminus\{0\}$, which we want to denote by $\mathscr{L}^\lambda := \widetilde{\mathcal{K}^\lambda} = j_0^{-1}\mathscr{O}^\lambda$ for the rest of this section.

Claim. *If $\mathscr{M}$ is some holonomic $\mathscr{D}$-module on $X := \mathbb{A}^1 \setminus \{0\}$ and $\mathscr{L}$ is an integrable connection (on X), then $\Delta_X = \Delta|_X : X \to X \times X$ is non-characteristic for $\mathscr{M} \overset{D}{\boxtimes} \mathscr{L}$, i. e. the canonical morphism*

$$\mathbb{D}_X D\Delta_X^*(\mathscr{M} \overset{D}{\boxtimes} \mathscr{L}) \to D\Delta_X^* \mathbb{D}_{X\times X}(\mathscr{M} \overset{D}{\boxtimes} \mathscr{L})$$

is an isomorphism

$$\begin{aligned}\mathbb{D}_X(\mathscr{M} \overset{D}{\otimes} \mathscr{L}) \simeq& \mathbb{D}_X D\Delta_X^*(\mathscr{M} \overset{D}{\boxtimes} \mathscr{L}) \underset{(*)}{\overset{\simeq}{\to}} D\Delta_X^*(\mathbb{D}_X\mathscr{M} \overset{D}{\boxtimes} \mathbb{D}_X\mathscr{L}) \\ \simeq& \mathbb{D}_X\mathscr{M} \overset{D}{\otimes} \mathbb{D}_X\mathscr{L} \simeq \mathbb{D}_X\mathscr{M} \overset{D}{\otimes} \mathscr{L}^\vee.\end{aligned} \tag{4.11}$$

Proof of claim. By the very definition of $\overset{D}{\boxtimes}$, we have

$$\mathscr{M} \overset{D}{\otimes} \mathscr{L} \simeq D\Delta_X^*(\mathscr{M} \overset{D}{\boxtimes} \mathscr{L}).$$

Let us convince ourselves that Δ_X is non-characteristic for $\mathscr{M} \overset{D}{\boxtimes} \mathscr{L}$, which then proves the remaining step $(*)$ of (4.11), cf. [HTT08, thorem 2.7.1]. We know that (cf. e. g. [Bjö93, remark 2.7.5, theorem 2.7.16] – note also that the construction of the characteristic variety is compatible with analytification)

$$\mathrm{CV}(\mathscr{M} \overset{D}{\boxtimes} \mathscr{L}) = \mathrm{CV}(\mathscr{M}) \times \mathrm{CV}(\mathscr{L}) \subset T^*X \times T^*X \simeq T^*(X \times X),$$

where we have $\mathrm{CV}(\mathscr{L}) = T_X^*X$. Labeling the coordinates of $T^*(X \times X)$ with (x, y, ξ, ν), we have

$$T_X^*(X \times X) = \{(x, x, \xi, -\xi)\} \subset T^*(X \times X),$$

so that indeed

$$\Delta_{X,\pi}^{-1}(\mathrm{CV}(\mathscr{M} \overset{D}{\boxtimes} \mathscr{L})) \cap T_X^*(X \times X) \subset X \times_{X\times X} T_{X\times X}^* X \times X,$$

which proves that Δ_X is non-characteristic for $\mathscr{M} \overset{D}{\boxtimes} \mathscr{L}$ and thus finishes the proof of the claim (recall $\Delta_{X,\pi}$ is the projection $X \times_{X\times X} T^*(X \times X) \to T^*(X \times X)$).

Using the claim for $\mathscr{M} = j_0^{-1}\mathbb{D}_{\mathbb{A}^1}\,\mathrm{FT}(\tilde{\mathscr{M}})$ and $\mathscr{L} = j_0^{-1}\mathbb{D}_{\mathbb{A}^1}\,\mathrm{FT}(\mathscr{O}^\lambda) \simeq \mathscr{L}^\lambda$, we have that

$$\begin{aligned}& j_0^{-1}\,\mathrm{Im}\left(\mathbb{D}_{\mathbb{A}^1} D\Delta^* \mathbb{D}_{\mathbb{A}^2}(\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathrm{FT}(\mathscr{O}^\lambda)) \to D\Delta^*(\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathrm{FT}(\mathscr{O}^\lambda))\right) \\ \simeq\, & \mathrm{Im}\left(\mathbb{D}_X D\Delta_X^*(j_0^{-1}\mathbb{D}_{\mathbb{A}^1}\,\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} j_0^{-1}\mathbb{D}_{\mathbb{A}^1}\,\mathrm{FT}(\mathscr{O}^\lambda)) \to D\Delta_X^*(j_0^{-1}\,\mathrm{FT}(\tilde{\mathscr{M}}) \overset{D}{\boxtimes} \mathscr{L}^{-\lambda})\right)\end{aligned}$$

$$\begin{aligned}
&\simeq \operatorname{Im}\left(D\Delta_X^*\mathbb{D}_{X\times X}\left(\mathbb{D}_X j_0^{-1}\operatorname{FT}(\tilde{\mathscr{M}})\overset{D}{\boxtimes}\mathbb{D}_X\mathscr{L}^{-\lambda}\right)\to D\Delta_X^*\left(j_0^{-1}\operatorname{FT}(\tilde{\mathscr{M}})\overset{D}{\boxtimes}\mathscr{L}^{-\lambda}\right)\right)\\
&\simeq \operatorname{Im}\left(D\Delta_X^*\left(j_0^{-1}\operatorname{FT}(\tilde{\mathscr{M}})\overset{D}{\boxtimes}\mathscr{L}^{-\lambda}\right)\xrightarrow{\sim} D\Delta_X^*\left(j_0^{-1}\operatorname{FT}(\tilde{\mathscr{M}})\overset{D}{\boxtimes}\mathscr{L}^{-\lambda}\right)\right)\\
&\simeq \operatorname{Im}\left(j_0^{-1}\operatorname{FT}(\tilde{\mathscr{M}})\overset{D}{\otimes}\mathscr{L}^{-\lambda}\xrightarrow{\sim} j_0^{-1}\operatorname{FT}(\tilde{\mathscr{M}})\overset{D}{\otimes}\mathscr{L}^{-\lambda}\right)\\
&\simeq j_0^{-1}\operatorname{FT}(\tilde{\mathscr{M}})\overset{D}{\otimes}\mathscr{L}^{-\lambda}.
\end{aligned}$$

Note that, even without using assumption 4.19, we would at this point have shown that there is an isomorphism

$$j_0^{-1}\operatorname{FT}\left(\mathbb{D}_{\mathbb{A}^1}\int_\sigma \mathbb{D}_{\mathbb{A}^2}(\tilde{\mathscr{M}}\overset{D}{\boxtimes}\mathscr{O}^\lambda)\right)\simeq j_0^{-1}\operatorname{FT}\left(\int_\sigma(\tilde{\mathscr{M}}\overset{D}{\boxtimes}\mathscr{O}^\lambda)\right),$$

but we could not know if it is really induced by (4.4). Putting it all together, we have shown (under assumption 4.19) that

$$\begin{aligned}
i_{\mathbb{A}}^{-1}(\mathscr{M}\overset{D}{*}_{\mathrm{mid}}\mathcal{K}^\lambda) *_{\mathrm{mid}} i_{\mathbb{A}}^{-1}\mathcal{K}^{-\lambda} &\simeq \operatorname{FT}^{-1}(Dj_{0,!*}((j_0^{-1}\operatorname{FT}(\tilde{\mathscr{M}})\overset{D}{\otimes}\mathscr{L}^{-\lambda})\overset{D}{\otimes}\mathscr{L}^{\lambda}))\\
&\simeq \operatorname{FT}^{-1}(Dj_{0,!*}j_0^{-1}\operatorname{FT}(\tilde{\mathscr{M}}))\\
&\simeq \operatorname{FT}^{-1}(\operatorname{FT}(\tilde{\mathscr{M}}))\simeq\tilde{\mathscr{M}}=i_{\mathbb{A}}^{-1}\mathscr{M},
\end{aligned}$$

where, for the last line, we used that $\mathscr{M}$ (so in particular $\tilde{\mathscr{M}}=i_{\mathbb{A}}^{-1}\mathscr{M}$) is irreducible by hypothesis, thus so is its Fourier transform (cf. e. g. [Ari10, section 2.2]), which implies

$$Dj_{0,!*}j_0^{-1}\operatorname{FT}(\tilde{\mathscr{M}})\simeq\operatorname{FT}(\tilde{\mathscr{M}}).$$

□

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