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Gaussian and Non-Gaussian Stable Limit Laws in Wicksell's Corpuscle Problem

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Abstract. Suppose that a homogeneous system of spherical particles (d-spheres) with independent identically distributed radii is contained in some opaque d-dimensional body, and one is interested to estimate the common radius distribution. The only information one can get is by making a cross-section of that body with an s-flat ($1 \leq s \leq d-1$) and measuring the radii of the s-spheres and their midpoints. The analytical solution of (the hyper-stereological version of) Wicksell's corpuscle problem is used to construct an empirical radius distribution of the d-spheres. In this paper we study the asymptotic behaviour of this empirical radius distribution for $s = d-1$ and $s = d-2$ under the assumption that the intersection volume becomes unboundedly large and the point process of the midpoints of the d-spheres is Brillinger-mixing. Among others we generalize and extend some results obtained in [1] and [2] under the Poisson assumption for the special case $d = 3$, $s = 2$.

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This article contains detailed proofs of the Theorems 1 - 4 stated with concised proofs in a paper that appeared under the title "Limit Distributions of Some Stereological Estimators in Wicksell's Corpuscle Problem" in the journal "Image Analysis & Stereology" 26, No. 2, 63-71 (2007). An earlier draft without the below Theorems 5 and 6 and Theorems 1 - 4 under assumptions slightly different from those in the present version (and partly not complete) has been published in [6].

1. Introduction

Let $\Psi_d = \{[X_i, R_i] : i \geq 1\}$ be a stationary, independently marked point process in $\mathbb{R}^d$ with generic non-negative mark R_0 having the distribution function (briefly df) F_d. The intensity measure $\Lambda_d(\cdot)$ of Ψ_d is then given by $\Lambda_d(B \times (0,r]) = \lambda_d \nu_d(B) F_d(r)$, where ν_d denotes the d-volume and $\lambda_d = \mathsf{E}\#\{\Psi_d^* \cap [0,1]^d\}$ is the intensity of the corresponding stationary non-marked point process $\Psi_d^* = \{X_i : i \geq 1\}$, see Stoyan et al. [10] for details. To formulate appropriate mixing conditions on Ψ_d^* we need the higher-order cumulant measures $\gamma_k(\cdot)$ for any $k \geq 2$ defined on the Borel $\sigma-$field $\mathcal{B}(\mathbb{R}^{dk})$, see e.g. [3] for a precise definition. The stationarity of Ψ_d^* enables us to define an associated (signed) measure - the reduced kth-order cumulant measure - $\gamma_k^{(red)}(\cdot)$ on $\mathcal{B}(\mathbb{R}^{d(k-1)})$ by disintegration w.r.t. ν_d, i.e.

$$\gamma_k(\mathop{\times}_{i=1}^{k} B_i) = \lambda_d \int\limits_{B_k} \gamma_k^{(red)}(\mathop{\times}_{i=1}^{k-1}(B_i - x))\nu_d(\mathrm{d}x) \; .$$

Further, let $B_d(x,r)$ denote the closed sphere in $\mathbb{R}^d$ with radius $r > 0$ centered at x and ω_d stands for the d-volume of the unit sphere $B_d(o,1)$, i.e. $\omega_d = \pi^{d/2}/\Gamma(\frac{d}{2}+1)$.

Wicksell's corpuscle problem in its hyper-stereological version can be described as follows: The system of d-spheres $\Xi_d = \{B_d(X_i,R_i) : i \geq 1\}$ is intersected by the $s-$flat $H_s = \{x = (x_1,...,x_d) \in \mathbb{R}^d : x_{s+1} = \cdots = x_d = 0\}$ (which can be identified with $\mathbb{R}^s$).

We assume that the collection of non-empty s-spheres $\overline{\Xi}_s := \Xi_d \cap H_s = \{B_s(\overline{X}_i,\overline{R}_i) : i \geq 1\}$ in the linear subspace H_s can be observed (all radii and midpoints are visible, without considering overlappings and edge-effects) in an expanding sampling window $W_n^{(s)} := n\,W^{(s)}$, where $W^{(s)}$ is a fixed convex set in $\mathbb{R}^s$ with unit s-volume, i.e. $\nu_s(W^{(s)}) = 1$, and n runs through $\mathbb{N} = \{1,2,...\}$. Note that $B_s(\overline{X}_i,\overline{R}_i) \neq \emptyset$ iff $\overline{R}_i := (R_i^2 - \|\underline{X}_i\|_{d-s}^2)^{1/2} > 0$. Here and in what follows, write $\overline{x}$ (resp. $\underline{x}$) to indicate the projection of $x \in \mathbb{R}^d$ onto H_s) (resp. onto the orthogonal complement of H_s) ; $\|\cdot\|_{d-s}$ denotes the Euclidean norm in $\mathbb{R}^{d-s}$. The system of non-empty s-spheres $B_s(\overline{X}_i,\overline{R}_i)$ is completely described by the stationary marked point process $\overline{\Psi}_s = \{[\overline{X}_i,\overline{R}_i] : i \geq 1\}$ in $\mathbb{R}^s$ with intensity measure $\overline{\Lambda}_s(A \times (0,r]) = \overline{\lambda}_s\,\nu_s(A)\,\overline{F}_s(r)$, where $\overline{F}_s$ denotes the df of the typical radius $\overline{R}_0$.

In the next section we restate the well-known explicit expressions of the df $\overline{F}_s$ and the intensity $\overline{\lambda}_s$ in terms of F_d and λ_d together with the corresponding inversion formulae. After that we present our results on the asymptotic behaviour (as $n \to \infty$) of appropriate empirical counterparts of the radius df F_d which are obtained from a single observation of all s-spheres whose centers lie in $W_n^{(s)}$. In particular, we state asymptotic normality (Theorem 1) and weak consistency (Theorem 4) in the cases $s = d-1$ and $s = d-2$, respectively. Using the terminology of the limit theory for sums of independent identically distributed random variables we are in the situation of a non-normal domain of attraction of the Gaussian and the degenerate law, respectively, see Ibragimov and Linnik [7]. By $\underset{n\to\infty}{\Longrightarrow}$ and $\xrightarrow[n\to\infty]{\mathsf{P}}$ we designate weak convergence and convergence in probability P, respectively.

The Poisson framework as presupposed in [1] and [2] is replaced in the present paper by imposing a mixing condition on the point process Ψ_d^*. This special type of weak dependence between separated parts of the point field $\{X_i : i \geq 1\}$ requires the existence of moment measures of any order. It should be mentioned that similar asymptotic results under milder moment assumptions can be obtained for an absolutely regular point process Ψ_d^*, see [5], as well as for a Poisson cluster process Ψ_d^*, see [4].

However, it seems that the Poisson assumption can hardly be dropped in our Theorems 5 and 6 to derive α-stable limits (with $\alpha = 2/(d-s)$) for the fluctuation of the corresponding empirical df's of F_d when $d - s \geq 2$. In the final section we put together the essential steps of the proofs of our results.

2. RELATIONSHIPS BETWEEN THE RADIUS DF'S

By means of the Campbell theorem and the relation $\overline{R}_i^{\,2} = R_i^2 - \|\underline{X}_i\|_{d-s}^2 > 0$ the intensity measures $\overline{\Lambda}_s$ and Λ_d are connected by the identity

$$\overline{\Lambda}_s(A \times (a,b)) = \int\limits_{\mathbb{R}^d \times [0,\infty)} \mathbf{1}\Big(\big(\overline{x}, \sqrt{\max\{0, \rho^2 - \|\underline{x}\|_{d-s}^2\}}\big) \in (A \times (a,b))\Big)\Lambda_d(\mathrm{d}(x,\rho))$$

for any $A \in \mathcal{B}(\mathbb{R}^s)$ and $0 \le a < b \le \infty$ which leads (after putting $A = [0,1)^s$ and $a = r\,,\, b = \infty$) to the following Abel-type integral equation:

$$\overline{\lambda}_s\,(1 - \overline{F}_s(r)) = \lambda_d\, \omega_{d-s} \int\limits_r^\infty (\varrho^2 - r^2)^{(d-s)/2} \mathrm{d}F_d(\varrho)$$

$$= \lambda_d\,(d-s)\,\omega_{d-s} \int\limits_0^\infty (1 - F_d(\sqrt{r^2+\varrho^2}))\,\varrho^{d-s-1}\mathrm{d}\varrho\,.$$

Letting $r \to 0$, the previous formula yields

$$\overline{\lambda}_s = \lambda_d\,\omega_{d-s}\,\mathsf{E}R_0^{d-s}$$

provided that $\mathsf{E}R_0^{d-s} < \infty$, whence it follows that

$$1 - \overline{F}_s(r) = \frac{1}{\mathsf{E}R_0^{d-s}} \int\limits_r^\infty (\varrho^2 - r^2)^{(d-s)/2} \mathrm{d}F_d(\varrho)$$

and the probability density function $\overline{f}_s$ of $\overline{R}_0$ (which always exists !) takes the form

$$\overline{f}_s(r) = \frac{r\,(d-s)}{\mathsf{E}R_0^{d-s}} \int\limits_r^\infty (\varrho^2 - r^2)^{(d-s-2)/2} \mathrm{d}F_d(\varrho)\,.$$

Here and throughout, the integral $\int_r^\infty$ stretches over the interval (r,∞). To express the radius df F_d in terms of the radius df $\overline{F}_s$ for any $s \in \{1,\ldots,d-1\}$ one has to solve the above Abel-type integral equation by *unfolding*. For doing this we distinguish between the cases $d-s$ is even and $d-s$ is odd, respectively. Put $q = \lfloor (d-s-1)/2 \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x and $n!! = n(n-2)\cdot \ldots \cdot 4 \cdot 2$ or $3 \cdot 1$. Then the df F_d can be expressed in terms of the probability density function $\overline{f}_s$ by the following formulae:

$$1 - F_d(r) = (-1)^q \frac{\mathsf{E}R_0^{d-s}}{(d-s)!!} \begin{cases} \frac{1}{r} g_s(r) & ,\ d-s \text{ even} \\ \frac{2}{\pi} \int\limits_r^\infty g_s(\varrho)(\varrho^2 - r^2)^{-1/2}\mathrm{d}\varrho & ,\ d-s \text{ odd} \end{cases}$$

with

$$g_s(r) = \begin{cases} \overline{f}_s(r) & \text{if } d-s = 1\,,\,2 \\ \left(\frac{1}{r}\left(\frac{1}{r}\cdots\left(\frac{\overline{f}_s(r)}{r}\right)'\cdots\right)'\right)' & \text{if } d-s \ge 3\,, \end{cases}$$

where in the last line q derivatives occur.

However, the statistical solution of the above integral equation leads to an inverse estimation problem which is rather unstable from both the computational and statistical view point, see e.g. [1], [2], [8], [10], [11] for further details.

In the most important case $s = d-1$ it is rapidly verified by a straightforward application of Campbell's theorem, see e.g. Stoyan et al. [10], that

$$\widehat{U}_n(r) = \frac{1}{\pi n^{d-1}} \sum_{i\geq 1} \mathbf{1}(\overline{X}_i \in W_n^{(d-1)}) \, \frac{\mathbf{1}(\overline{R}_i > r)}{\sqrt{\overline{R}_i^{\,2} - r^2}}$$

is an unbiased estimation of $\lambda_d\,(1 - F_d(r))$. On the other hand, this calculation reveals that the variance of $\widehat{U}_n(r)$ does not exist (which has been first noticed in [1]).

We refer to the fact that, for any fixed $n \in \mathbb{N}$, the empirical process $\widehat{U}_n(r)$ regarded as a random function in $r \geq 0$ is by no means monotonically decreasing. It possesses downward jumps at the random points $r = \overline{R}_i$, however, between two such jumps $\widehat{U}_n(r)$ is strictly increasing. Such strange behaviour of this stereological estimator of $\lambda_d\,(1 - F_d(r))$ gave rise to consider several modified and smoothed versions of $\widehat{U}_n(r)$, see e.g. [2] for an isotonic estimation and its asymptotic analysis.

3. ASYMPTOTIC RESULTS

3.1 The Case $s = d-1$

We first put together some mixing-type conditions for the point process $\Psi_d^* = \{X_i : i \geq 1\}$ of the midpoints of the d-spheres.

Condition 1 Assume that Ψ_d^* is *Brillinger-mixing*, i.e. ,

$$\int_{(\mathbb{R}^d)^{k-1}} |\gamma_k^{(red)}(\mathrm{d}(x_1, ..., x_{k-1})| < \infty \quad \text{for} \quad k \geq 2\,.$$

Condition 2 Assume that the reduced second-order cumulant measure $\gamma_2^{(red)}(\cdot)$ satisfies

$$\int_{\mathbb{R}^{d-1}\times A} \left|\gamma_2^{(red)}(\mathrm{d}x)\right| \leq \text{const}\ \nu_1(A)$$

for any bounded Borel set $A \subset \mathbb{R}^1$.

Condition 3 Assume that the reduced second-order cumulant measure $\gamma_2^{(red)}(\cdot)$ has finite total variation, i.e. ,

$$\int_{\mathbb{R}^d} \left|\gamma_2^{(red)}(\mathrm{d}x)\right| < \infty\,.$$

Sufficient conditions for some classes of point processes to be Brillinger-mixing are discussed in [4]. For example, Poisson cluster processes are Brillinger-mixing iff the number of points in the typical cluster has moments of any order. Also, several types of dependently thinned Poisson processes such as Matérn's hard-core point processes possess this mixing property. If Ψ_d^* is additionally isotropic with *pair correlation function* $g(r)$, see Stoyan et al. [10], then Condition 2 is satisfied if

$$\sup_{a\geq 0} \int_0^{\infty} |\, g(\sqrt{r^2+a}) - 1 \,|\, r^{d-2}\, \mathrm{d}r < \infty \ .$$

This as well as Condition 3 are rather mild restrictions on the point process Ψ_d^*.

Theorem 1 *Let the Conditions 1 and 2 be satisfied. If*

$$\sigma^2(r) := \lambda_d \int_r^{\infty} (\varrho^2 - r^2)^{-1/2} \mathrm{d}F_d(\varrho) < \infty \tag{1}$$

for some fixed $r \geq 0$ *and* $\mathsf{E}R_0 < \infty$, *then*

$$\sqrt{\frac{\pi^2\, n^{d-1}}{\log n^{d-1}}} \Big(\widehat{U}_n(r) - \lambda_d (1 - F_d(r)) \Big) \underset{n\to\infty}{\Longrightarrow} N(0, \sigma^2(r)) \ ,$$

where $N(0,\sigma^2)$ *denotes a zero mean Gaussian random variable with variance* σ^2. *Furthermore, the relation* $\lambda_d\, \mathsf{E}R_0\, \overline{f}_{d-1}(r) = r\,\sigma^2(r)$ *shows that, for* $r > 0$, *condition* (1) *is equivalent to* $\overline{f}_{d-1}(r) < \infty$.

Remark 1 Provided that $F_d(0) = 0$, Theorem 1 (for $r = 0$) yields a central limit theorem for the unbiased estimator $\widehat{U}_n(0)$ of the intensity λ_d.

Note that, without assuming Brillinger mixing - merely under Condition 3 - $\widehat{U}_n(r)$ turns out to be weakly consistent for $\lambda_d\,(1 - F_d(r))$. Hence, we get that

$$\frac{\widehat{U}_n(r)}{\widehat{U}_n(0)} \quad \underset{n\to\infty}{\overset{\mathsf{P}}{\longrightarrow}} 1 - F_d(r) \qquad \text{for any} \quad r \geq 0\,.$$

It should be noted that, in case Ψ_d^* is a stationary ergodic point process, the latter relation holds $\mathsf{P}-$a.s..

Remark 2 For $r > 0$ the assumption (1) is satisfied if the df F_d is α-Hölder continuous for some $\alpha > 1/2$ in $[r, r+\delta]$, i.e. ,

$$F_d(\varrho) - F_d(r) \ \leq \ H_{\alpha,\delta}\, (\varrho - r)^{\alpha}$$

for $r \leq \varrho \leq r + \delta$ and some $\delta > 0$.

The multivariate extension of Theorem 1 (by employing the well-known method of Cramér - Wold) shows that the finite-dimensional distributions of the sequence of standardized empirical processes in Theorem 1 tend to those of a Gaussian ‘white noise’ process as $n \to \infty$.

Theorem 2 *Let the Conditions 1 and 2 and* (1) *for* $r \in \{r_1, \ldots, r_k\}$, $0 \leq r_1 < \cdots < r_k < \infty$, *be satisfied. Then*

$$\sqrt{\frac{\pi^2\, n^{d-1}}{\log n^{d-1}}} \Big(\frac{\widehat{U}_n(r_j) - \lambda_d (1 - F_d(r_j))}{\sqrt{\sigma^2(r_j)}} \Big)_{j=1}^{k} \underset{n\to\infty}{\Longrightarrow} N_k(\mathbf{o}, I_k)$$

where $N_k\big(\mathbf{o}, I_k\big)$ denotes a k-dimensional Gaussian random vector having zero mean components and a covariance matrix being equal to the unit matrix I_k.

As a simple application of Theorem 2 for $k = 2$, $r_1 = 0$, $r_2 = r$ (using the asymptotic independence of the components) and Slutski's theorem we obtain

Corollary 1 *Let the Conditions 1 and 2, $F_d(0) = 0$ and (1) for $r = 0$ and some $r > 0$ be satisfied. Then*

$$\sqrt{\frac{\pi^2\, n^{d-1}}{\log n^{d-1}}}\Big(\frac{\widehat{U}_n(r)}{\widehat{U}_n(0)} - (1 - F_d(r))\Big) \underset{n\to\infty}{\Longrightarrow} N(0, s^2(r)) \;,$$

where $s^2(r) := \big(\sigma^2(r) + \sigma^2(0)\,(1 - F_d(r))^2\big)/\lambda_d^2$.

There exists indeed a weakly consistent estimator of the asymptotic variance $\sigma^2(r)$ (although its expectation does not exist) which is given by the following 'overnormed' random sum

$$\widehat{\sigma}_n^2(r) := \frac{1}{n^{d-1}\log n^{d-1}} \sum_{i\geq 1} \mathbf{1}(\overline{X}_i \in W_n^{(d-1)})\; \frac{\mathbf{1}(\overline{R}_i > r)}{\overline{R}_i^{\,2} - r^2} \;.$$

Theorem 3 *Under Condition 3 and $\mathsf{E}R_0 < \infty$ it holds*

$$\widehat{\sigma}_n^2(r) \underset{n\to\infty}{\overset{\mathsf{P}}{\longrightarrow}} \sigma^2(r) \quad \textit{for each} \quad r \geq 0 \quad \textit{satisfying} \;\; (1) \quad .$$

Combining Theorem 1 with Theorem 3 together with Slutski's theorem provides

Corollary 2 *Let the Conditions 1 and 2, $\mathsf{E}R_0 < \infty$ and (1) for some fixed $r \geq 0$ be satisfied. Then*

$$\sqrt{\frac{\pi^2\, n^{d-1}}{\widehat{\sigma}_n^2(r)\, \log n^{d-1}}}\Big(\widehat{U}_n(r) - \lambda_d(1 - F_d(r))\Big) \underset{n\to\infty}{\Longrightarrow} N(0, 1) \;.$$

Remark 3 By means of Corollary 2 (applied to $r = 0$ provided $F_d(0) = 0$) we are able to construct an asymptotically exact confidence interval for the unknown intensity λ_d of the midpoints of d-spheres.

In order to find an asymptotic confidence interval for $1 - F_d(r)$ we combine Corollary 1, Theorem 2 and Slutski's theorem and obtain

Corollary 3 *Assume that the Conditions 1 and 2, $\mathsf{E}R_0 < \infty$, $F_d(0) = 0$ and (1) for $r = 0$ and some fixed $r > 0$ are satisfied. Then*

$$\sqrt{\frac{\pi^2\, n^{d-1}}{\widehat{s}_n^2(r)\, \log n^{d-1}}}\Big(\frac{\widehat{U}_n(r)}{\widehat{U}_n(0)} - (1 - F_d(r))\Big) \underset{n\to\infty}{\Longrightarrow} N(0, 1) \;,$$

where $\widehat{s}_n^2(r) := \big(\widehat{\sigma}_n^2(r)\,\widehat{U}_n^2(0) + \widehat{\sigma}_n^2(0)\,\widehat{U}_n^2(r)\big)/\widehat{U}_n^4(0)$.

An immediate consequence of Theorem 2 and Slutski's theorem is

Corollary 4 *Let the assumptions of Theorem 2 and* $\mathsf{E}R_0 < \infty$ *be satisfied. Then*

$$\frac{\pi^2\, n^{d-1}}{\log n^{d-1}} \sum_{j=1}^{k} \frac{\Big(\widehat{U}_n(r_j) - \lambda_d(1 - F_d(r_j))\Big)^2}{\widehat{\sigma}_n^2(r_j)} \underset{n\to\infty}{\Longrightarrow} \chi_k^2\,,$$

where the random variable χ_k^2 *is* χ^2*-distributed with* k *degrees of freedom.*

The latter result can be used to test the goodness-of-fit of certain hypothesized radius df F_d (if λ_d is known).

3.2 The Case $s = d-2$

Define the empirical process

$$\widehat{V}_n(r) = \frac{1}{\pi\, n^{d-2}\, \log n^{d-2}} \sum_{i\geq 1} \mathbf{1}(\overline{X}_i \in W_n^{(d-2)})\, \frac{\mathbf{1}(\overline{R}_i > r)}{\overline{R}_i^{\,2} - r^2}$$

which has an infinite mean for any $r \geq 0$. Nevertheless, $\widehat{V}_n(r)$ is weakly consistent for $\lambda_d\,(1 - F_d(r))$ under slight additional assumptions.

Theorem 4 *Under Condition 3 and* $\mathsf{E}R_0^2 < \infty$ *it holds*

$$\widehat{V}_n(r) \overset{\mathsf{P}}{\underset{n\to\infty}{\longrightarrow}} \lambda_d(1 - F_d(r)) \qquad \text{for any} \quad r \geq 0\,,$$

and therefore, together with $F_d(0) = 0$,

$$\frac{\hat{V}_n(r)}{\hat{V}_n(0)} \overset{\mathsf{P}}{\underset{n\to\infty}{\longrightarrow}} 1 - F_d(r) \qquad \text{for any} \quad r \geq 0\,.$$

Theorem 5 *Let* $\Psi_d^* = \{X_i : i \geq 1\}$ *be a stationary Poisson process with intensity* λ_d. *If, in addition,*

$$\int_r^\infty |\,\log(\varrho^2 - r^2)\,|\,\mathrm{d}F_d(\varrho) < \infty \tag{2}$$

for some fixed $r \geq 0$ *with* $F_d(r) < 1$, *then*

$$\log n^{d-2} \Big(\frac{\widehat{V}_n(r)}{\lambda_d\,(1 - F_d(r))} - 1 \Big) - \log\Big(\pi\,\lambda_d\,(1 - F_d(r))\Big)$$

$$- \frac{\int_r^\infty \log(\varrho^2 - r^2)\,\mathrm{d}F_d(\varrho)}{1 - F_d(r)} - 1 + \gamma \underset{n\to\infty}{\Longrightarrow} S_1\,.$$

where $\gamma := \lim_{n\to\infty}(1+1/2+\cdots+1/n-\log n) \simeq 0.5772$ *denotes the Euler-Mascheroni constant and the random variable* S_1 *possesses a stable df with characteristic exponent* $\alpha = 1$ *and skewness parameter* $\beta = 1$ *having the characteristic function*

$$\mathsf{E}\exp\{i\,t\,S_1\} = \exp\Big\{-\frac{\pi}{2}\,|t|\, - i\,t\,\log|t|\Big\} \quad for \quad t\in\mathbb{R}^1\,.$$

Remark 4 Nolan [9] provides tables and numerical procedures for calculating the density of S_1 (and other stable densities). This gives at least in principle the possibility for testing the null hypothesis $H_0 : F_d = F_d^{(0)}\,,\ \lambda_d = \lambda_d^{(0)}$.

3.3 The Case $d - s > 2$

Of course, the previous cases are of particular interest in stereological practice for $d = 3, s = 2$, $d = 2, s = 1$ and $d = 3,\ s = 1$. To be complete we also investigate the asymptotic behaviour of a simple generalization of $\widehat{U}_n(r)$ resp. $\widehat{V}_n(r)$ to the case $d-s > 2$. The below result seems to be of interest for its own right (from the view point of pure asymptotics) and it gives insight how the instability increases when $d - s$ becomes greater than two.

Let $p := d - s$ and define

$$\widehat{Y}_n^{(p)}(r) = \frac{1}{n^{s\,p/2}}\sum_{i\geq 1}\mathbf{1}(\overline{X}_i\in W_n^{(s)})\,\frac{\mathbf{1}(\overline{R}_i > r)}{(\overline{R}_i^{\,2}-r^2)^{p/2}}$$

Theorem 6 *Let* $\Psi_d^* = \{X_i : i\geq 1\}$ *be a stationary Poisson process with intensity* λ_d *and* $\mathsf{E}R_0^{p-2} < \infty$*. Then, for any fixed* $r \geq 0$ *with* $F_d(r) < 1$*, it holds*

$$\frac{\widehat{Y}_n^{(p)}(r)}{\Big(c_p\,\lambda_d\,\int\limits_r^\infty(\varrho^2-r^2)^{(p-2)/2}\,\mathrm{d}F_d(\varrho)\Big)^{p/2}} \underset{n\to\infty}{\Longrightarrow} S_{2/p}\,,$$

where $c_p = \omega_p\,\frac{p}{2}\,\Gamma(1-\frac{2}{p})\,\cos(\frac{\pi}{p})$ *and the random variable* $S_{2/p}$ *possesses a stable df with characteristic exponent* $\alpha = 2/p \in (0,1)$ *and skewness parameter* $\beta = 1$ *having the characteristic function*

$$\mathsf{E}\exp\{i\,t\,S_{2/p}\} = \exp\Big\{-\,|t|^{2/p}\Big(1 - i\,\mathrm{sgn}(t)\,\tan\Big(\frac{\pi}{p}\Big)\Big)\Big\} \quad \text{for} \quad t\in\mathbb{R}^1\,.$$

4. PROOFS OF THE THEOREMS

First observe that the empirical processes $\widehat{U}_n(r)$, $\widehat{V}_n(r)$, $\widehat{\sigma}_n^2(r)$, and $\widehat{Y}_n^{(p)}(r)$ can be regarded as so-called shot-noise processes $\sum_{i\geq 1} f(\overline{X}_i, \underline{X}_i, R_i)$ with different 'response functions' $f\,|\,\mathbb{R}^s\times\mathbb{R}^{d-s}\times(0,\infty)\mapsto[0,\infty)$, see [3] and references therein. However, only $\widehat{U}_n(r)$ has a finite first moment.

In fact, applying Campbell's theorem gives $\mathsf{E}\widehat{U}_n(r) = \lambda_d\,(1 - F_d(r))$ and further $\mathsf{E}(\widehat{U}_n(r))^m < \infty$ for $1 < m < 2$, but $\mathsf{E}(\widehat{U}_n(r))^2 = \infty$. In order to prove Theorem 1 we have to replace the terms $(\overline{R}_i^{\,2} - r^2)^{-1/2}$ (which are responsible for the large fluctuations of the sum) by truncated terms. More precisely, for any $\varepsilon > 0$, we introduce the 'truncated' shot-noise process

$$\widehat{U}_{n,\varepsilon}(r) = \frac{1}{\pi n^{d-1}} \sum_{i\geq 1} \frac{\mathbf{1}(\overline{X}_i \in W_n^{(d-1)})}{\sqrt{\overline{R}_i^{\,2} - r^2}}\, \mathbf{1}\Big(\overline{R}_i^{\,2} - r^2 > \frac{\max\{\varepsilon, R_i^2 - r^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}}\Big)$$

and the nonnegative random integer

$$N_{n,\varepsilon}(r) = \sum_{i\geq 1} \mathbf{1}\big(\overline{X}_i \in W_n^{(d-1)}\big)\, \mathbf{1}\Big(0 < \overline{R}_i^{\,2} - r^2 \leq \frac{\max\{\varepsilon, R_i^2 - r^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}}\Big)\,.$$

First step: For any Borel set $B \subseteq \mathbb{R}^1$ we have the identity $\{\widehat{U}_{n,\varepsilon}(r) \in B\} \cap \{N_{n,\varepsilon}(r) = 0\} = \{\widehat{U}_n(r) \in B\} \cap \{N_{n,\varepsilon}(r) = 0\}$ and this in turn implies the estimate

$$\Big|\mathsf{P}(\widehat{U}_{n,\varepsilon}(r) \in B) - \mathsf{P}(\widehat{U}_n(r) \in B)\Big| \leq \mathsf{P}(N_{n,\varepsilon}(r) \geq 1) \leq \mathsf{E}N_{n,\varepsilon}(r) \tag{3}$$

For brevity put $\alpha_n = (\,\varepsilon\, n^{d-1}\, \log n^{d-1}\,)^{-1}$. By applying Campbell's theorem and using that the assumptions (1) and $\mathsf{E}R_0 < \infty$ are satisfied we may write

$$\begin{aligned}
\mathsf{E}N_{n,\varepsilon}(r) &= \mathsf{E}\sum_{i\geq 1} \mathbf{1}(\overline{X}_i \in W_n^{(d-1)})\, \mathbf{1}\Big(0 < \overline{R}_i^{\,2} - r^2 \leq \frac{\max\{\varepsilon, R_i^2 - r^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}}\Big) \\
&= \lambda_d\, n^{d-1} \int\limits_r^{\infty} \int\limits_{-\sqrt{\varrho^2 - r^2}}^{\sqrt{\varrho^2 - r^2}} \mathbf{1}\Big(\varrho^2 - x^2 - r^2 \leq \frac{\max\{\varepsilon, \varrho^2 - r^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}}\Big)\, \mathrm{d}x\, \mathrm{d}F_d(\varrho) \\
&= 2\,\lambda_d\, n^{d-1} \int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \Big(\sqrt{\varrho^2 - r^2} - \sqrt{(\varrho^2 - r^2)(1 - \alpha_n/\varepsilon)}\Big)\, \mathrm{d}F_d(\varrho) \\
&+ 2\,\lambda_d\, n^{d-1} \int\limits_r^{\sqrt{r^2+\varepsilon}} \Big(\sqrt{\varrho^2 - r^2} - \sqrt{\max\{0, \varrho^2 - r^2 - \alpha_n\}}\Big)\, \mathrm{d}F_d(\varrho) \\
&\leq 2\,\lambda_d\, n^{d-1} \alpha_n \left(\frac{1}{\varepsilon} \int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \sqrt{\varrho^2 - r^2}\, \mathrm{d}F_d(\varrho) + \int\limits_r^{\sqrt{r^2+\varepsilon}} \frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2 - r^2}}\right)
\end{aligned}$$

Since, by our assumptions, both integrals in the last line exist, it follows that

$$\mathsf{E}N_{n,\varepsilon}(r) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for any} \quad \varepsilon > 0\,. \tag{4}$$

Second step: Once more using assumption (1) we are able to verify that

$$\sqrt{\frac{\pi^2\, n^{d-1}}{\log n^{d-1}}}\Big(\mathsf{E}\widehat{U}_{n,\varepsilon}(r) - \lambda_d\,(1-F_d(r))\Big) \underset{n\to\infty}{\longrightarrow} 0\,. \tag{5}$$

For this we again employ Campbell's theorem combined with $\int_0^1 (1-w^2)^{-1/2}\mathrm{d}w = \pi/2$ which leads to

$$\begin{aligned}\mathsf{E}\widehat{U}_{n,\varepsilon}(r) &= \frac{2\,\lambda_d}{\pi}\left(\int\limits_{\sqrt{r^2+\varepsilon}}^{\infty}\ \int\limits_{0}^{\sqrt{(\varrho^2-r^2)(1-\alpha_n/\varepsilon)}} \frac{\mathrm{d}x\,\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2-x^2}} + \int\limits_{\sqrt{r^2+\alpha_n}}^{\sqrt{r^2+\varepsilon}}\ \int\limits_{0}^{\sqrt{\varrho^2-r^2-\alpha_n}} \frac{\mathrm{d}x\,\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2-x^2}}\right)\\ &= \lambda_d\,(1-F_d(r)) - \lambda_d\,(F_d(\sqrt{r^2+\alpha_n}) - F_d(r)) - \frac{2\,\lambda_d}{\pi}\int\limits_{\sqrt{r^2+\alpha_n}}^{\sqrt{r^2+\varepsilon}}\ \int\limits_{\sqrt{1-\frac{\alpha_n}{\varrho^2-r^2}}}^{1} \frac{\mathrm{d}x\,\mathrm{d}F_d(\varrho)}{\sqrt{1-x^2}}\,.\end{aligned}$$

By obvious rearrangements we get that

$$F_d(\sqrt{r^2+\alpha_n}) - F_d(r) \;\le\; \sqrt{\alpha_n}\int\limits_{r}^{\sqrt{r^2+\alpha_n}} \frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2}}$$

and

$$\int\limits_{\sqrt{1-\frac{\alpha_n}{\varrho^2-r^2}}}^{1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} \;\le\; \int\limits_{0}^{1-\sqrt{1-\frac{\alpha_n}{\varrho^2-r^2}}} \frac{\mathrm{d}x}{\sqrt{x}} \;=\; 2\sqrt{1-\sqrt{1-\frac{\alpha_n}{\varrho^2-r^2}}} \;\le\; 2\sqrt{\frac{\alpha_n}{\varrho^2-r^2}}\,.$$

Hence,

$$|\,\mathsf{E}\widehat{U}_{n,\varepsilon}(r) - \lambda_d\,(1-F_d(r))\,| \le \lambda_d\left(1+\frac{4}{\pi}\right)\sqrt{\alpha_n}\int\limits_{r}^{\sqrt{r^2+\varepsilon}} \frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2}}\,,$$

which, together with assumption (1), implies (5). Combining (4) and (5) shows that the sequences $\sqrt{n^{d-1}/\log n^{d-1}}\,\big(\widehat{U}_{n,\varepsilon}(r) - \mathsf{E}\widehat{U}_{n,\varepsilon}(r)\big)$ and $\sqrt{n^{d-1}/\log n^{d-1}}\,\big(U_n(r) - \lambda_d\,(1-F_d(r))\big)$ posseses the same limit distribution with mean zero. We shall verify that this limit distribution is Gaussian with variance $\sigma^2(r)/\pi^2$.

Third step: By virtue of Condition 2 we may verify that

$$\overline{\lim_{n\to\infty}}\left|\frac{\pi^2\,n^{d-1}}{\log n^{d-1}}\,\mathsf{Var}\big(\widehat{U}_{n,\varepsilon}(r)\big) - \sigma^2(r)\right| \le \lambda_d \int\limits_{r}^{\sqrt{r^2+\varepsilon}} \frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2}} \quad \text{for any} \quad \varepsilon > 0\,. \tag{6}$$

To derive this estimate we rewrite the variance of $\widehat{U}_{n,\varepsilon}(r)$ by applying Campbell's formula and remembering the definition of the second-order reduced factorial cumulant measure $\gamma_2^{(red)}(\cdot)$

$$\mathsf{Var}\big(\widehat{U}_{n,\varepsilon}(r)\big) = \frac{2\,\lambda_d}{\pi^2\,n^{d-1}} \left(\int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \int\limits_0^{\sqrt{(\varrho^2-r^2)(1-\alpha_n/\varepsilon)}} \frac{\mathrm{d}x\,\mathrm{d}F_d(\varrho)}{\varrho^2-r^2-x^2} + \int\limits_{\sqrt{r^2+\alpha_n}}^{\sqrt{r^2+\varepsilon}} \int\limits_0^{\sqrt{\varrho^2-r^2-\alpha_n}} \frac{\mathrm{d}x\,\mathrm{d}F_d(\varrho)}{\varrho^2-r^2-x^2} \right)$$

$$+ \quad \frac{\lambda_d}{\pi^2\,n^{2(d-1)}} \int\limits_0^{\infty}\int\limits_0^{\infty}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \frac{\mathbf{1}(\overline{x}\in W_n^{(d-1)})}{\sqrt{\varrho^2-r^2-\underline{x}^2}}\,\mathbf{1}\Big(\varrho^2-r^2-\underline{x}^2 > \frac{\max\{\varepsilon,\varrho^2-r^2\}}{\varepsilon^2 n^{d-1}\log n^{d-1}}\Big)$$

$$\times \quad \frac{\mathbf{1}(\overline{y}+\overline{x}\in W_n^{(d-1)})}{\sqrt{\tau^2-r^2-(\underline{y}+\underline{x})^2}}\,\mathbf{1}\Big(\tau^2-r^2-(\underline{y}+\underline{x})^2 > \frac{\max\{\varepsilon,\tau^2-r^2\}}{\varepsilon^2 n^{d-1}\log n^{d-1}}\Big)$$

$$\times \quad \gamma_2^{(red)}(\mathrm{d}(\overline{y},\underline{y}))\,\mathrm{d}(\overline{x},\underline{x})\,\mathrm{d}F_d(\varrho)\,\mathrm{d}F_d(\tau) = T_1^{(n)} + T_2^{(n)} + T_3^{(n)}\,.$$

The first term is easy to treat and yields the following limit :

$$\frac{\pi^2\,n^{d-1}}{\log n^{d-1}}\,T_1^{(n)} = \frac{2\,\lambda_d}{\log n^{d-1}} \int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2}} \int\limits_0^{\sqrt{1-\alpha_n/\varepsilon}} \frac{\mathrm{d}x}{1-x^2}$$

$$= \quad \frac{\lambda_d}{\log n^{d-1}} \int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2}} \Big(-\log(1-\sqrt{1-\alpha_n/\varepsilon}) + \log(1+\sqrt{1-\alpha_n/\varepsilon})\Big)$$

$$= \quad \frac{\lambda_d}{\log n^{d-1}} \int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2}} \Big(\log\Big(\varepsilon^2\,n^{d-1}\,\log n^{d-1}\Big) + 2\,\log(1+\sqrt{1-\alpha_n/\varepsilon})\Big)$$

$$\underset{n\to\infty}{\longrightarrow} \quad \lambda_d \int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2}}\,.$$

Analogously,

$$\frac{\pi^2\,n^{d-1}}{\log n^{d-1}}\,T_2^{(n)} = \frac{2\,\lambda_d}{\log n^{d-1}} \int\limits_{\sqrt{r^2+\alpha_n}}^{\sqrt{r^2+\varepsilon}} \int\limits_0^{\sqrt{1-\frac{\alpha_n}{\varrho^2-r^2}}} \frac{\mathrm{d}x}{1-x^2}\,\frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2}}$$

$$= \frac{\lambda_d}{\log n^{d-1}} \int\limits_{\sqrt{r^2+\alpha_n}}^{\sqrt{r^2+\varepsilon}} \left(\log\Big(\frac{\varrho^2-r^2}{\alpha_n}\Big) + 2\,\log\Big(1+\sqrt{1-\frac{\alpha_n}{\varrho^2-r^2}}\Big)\right) \frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2}}$$

leading to

$$\overline{\lim_{n\to\infty}} \frac{\pi^2\, n^{d-1}}{\log n^{d-1}}\, T_2^{(n)} \;\le\; \lambda_d \int\limits_r^{\sqrt{r^2+\varepsilon}} \frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2 - r^2}} \quad \text{for any} \quad \varepsilon > 0\,.$$

Condition 2 and $\int_0^1 (1-w^2)^{-1/2}\mathrm{d}w = \pi/2$ guarantee that the third summand can be bounded by

$$\begin{aligned}
\frac{\pi^2\, n^{d-1}}{\log n^{d-1}}\, T_3^{(n)} \quad &\le \quad \frac{\lambda_d\,\mathrm{const}}{\log n^{d-1}} \left(2 \int\limits_0^\infty \int\limits_0^\infty \frac{1}{\sqrt{\varrho^2 - r^2 - \underline{x}^2}}\, \mathbf{1}\Big((\varrho^2 - r^2)(1-\alpha_n/\varepsilon) > \underline{x}^2 \Big)\, \mathrm{d}\underline{x}\, \mathrm{d}F_d(\varrho) \right)^2 \\
&\le \quad \frac{\lambda_d\, \pi^2\, \mathrm{const}}{\log n^{d-1}} \;\;,
\end{aligned}$$

which together with the above relations immediately confirm (6).

In the final step we make use of Condition 1 and derive bounds of the cumulants of order $m \ge 3$ (abbreviated by the symbol Cum_m) of $(\pi^2\, n^{d-1}/\log n^{d-1})^{1/2}\, \hat{U}_{n,\varepsilon}(r)$, which are uniformly bounded in n and tend to zero as ε does so.

More precisely, using the representation formula for cumulants of general shot-noise processes obtained in Heinrich and Schmidt [3] we get

$$\begin{aligned}
&\mathrm{Cum}_m\{\widehat{U}_{n,\varepsilon}(r)\} = \\
&= \sum_{p=1}^m \frac{1}{p\,!} \sum_{\substack{m_1+\cdots+m_p=m \\ m_j\ge 1\,,\, j=1,\ldots,p}} \frac{m!}{m_1!\cdots m_p!} \int\limits_{(\mathbb{R}^d)^p} \prod_{j=1}^p \int_0^\infty \Big(f_{n,\varepsilon}(x_j,\varrho) \Big)^{m_j} \mathrm{d}F(\varrho)\, \gamma_p\big(\mathrm{d}(x_1,\ldots,x_p)\big) \\
&= \lambda_d \int\limits_{\mathbb{R}^d} \int_0^\infty \Big(f_{n,\varepsilon}(\,x,\varrho\,) \Big)^m \mathrm{d}F(\varrho)\,\mathrm{d}x + \lambda_d \sum_{p=2}^m \frac{1}{p\,!} \sum_{\substack{m_1+\cdots+m_p=m \\ J=\min\{j:m_j\ge 2\}}} \frac{m!}{m_1!\cdots m_p!} \\
&\times \int\limits_{\mathbb{R}^d} \int\limits_{(\mathbb{R}^d)^{p-1}} \prod_{\substack{j=1 \\ j\ne J}}^p \int_0^\infty \Big(f_{n,\varepsilon}(x_j + x_J,\varrho) \Big)^{m_j} \mathrm{d}F(\varrho) \gamma_p^{(red)}\big(\mathrm{d}(x_j; j\ne J)\big) \\
&\times \int_0^\infty \Big(f_{n,\varepsilon}(x_J,\varrho) \Big)^{m_J} \mathrm{d}F(\varrho)\,\mathrm{d}x_J\;,
\end{aligned}$$

where the 'response function' $f_{n,\varepsilon}\,|\,\mathbb{R}^{d-1}\times\mathbb{R}^1\times[0,\infty)\mapsto\mathbb{R}^1$ of the truncated shot-noise process $\widehat{U}_{n,\varepsilon}(r)$ is given by

$$f_{n,\varepsilon}(x,\varrho) = \frac{1}{\pi\, n^{d-1}} \frac{\mathbf{1}(\overline{x} \in W_n^{(d-1)})}{\sqrt{\varrho^2 - r^2 - \underline{x}^2}}\, \mathbf{1}\Big(\varrho^2 - r^2 - \underline{x}^2 > \frac{\max\{\varepsilon, \varrho^2 - r^2\}}{\varepsilon^2\, n^{d-1} \log n^{d-1}} \Big) \quad \text{for} \quad x = (\overline{x}, \underline{x})\,.$$

Since $0 \le f_{n,\varepsilon}(x,\varrho) \le \sqrt{\varepsilon \, \log n^{d-1}/\pi^2 \, n^{d-1}}$ uniformly in $x \in \mathbb{R}^d$ and $\varrho \in (0,\infty)$, we arrive at the estimate

$$| \,\mathrm{Cum}_m\{\widehat{U}_{n,\varepsilon}(r)\}\, | \quad \le \quad \lambda_d \, C_m \Big(\frac{\varepsilon \, \log n^{d-1}}{\pi^2 \, n^{d-1}} \Big)^{(m-2)/2} \int\limits_{\mathbb{R}^d} \int\limits_0^\infty f^2_{n,\varepsilon}(x,\varrho) \, \mathrm{d}F(\varrho) \, \mathrm{d}x \quad \text{for} \quad m \ge 3 \,,$$

where the constant C_m depends on the total variations of the signed measures $\gamma_p^{(red)}(\cdot)$ for $p = 2, ..., m$ in the following way

$$C_m = 1 + \sum_{p=2}^m \frac{1}{p\,!} \sum_{\substack{m_1+\cdots+m_p=m \\ m_j \ge 1, j=1,\ldots,p}} \frac{m!}{m_1! \cdots m_p!} \int\limits_{(\mathbb{R}^d)^{p-1}} | \, \gamma_p^{(red)}(\mathrm{d}(x_1,...,x_{p-1}) \, | \, .$$

Making use of the abbreviation α_n intoduced at the beginning of the proof we find that

$$\begin{aligned}
\int\limits_{\mathbb{R}^d} \int\limits_0^\infty f^2_{n,\varepsilon}(x,\varrho) \, \mathrm{d}F(\varrho) \, \mathrm{d}x \quad &\le \quad \frac{2}{\pi^2 \, n^{d-1}} \int\limits_0^\infty \int\limits_0^\infty \frac{\mathbf{1}\big(\varrho^2 - r^2 - y^2 > \alpha_n \, (\varrho^2 - r^2)/\varepsilon\big)}{\varrho^2 - r^2 - y^2} \, \mathrm{d}F(\varrho) \, \mathrm{d}y \\
&= \quad \frac{2}{\pi^2 \, n^{d-1}} \int\limits_r^\infty \frac{\mathrm{d}F(\varrho)}{\sqrt{\varrho^2 - r^2}} \int\limits_0^{\sqrt{1-\alpha_n/\varepsilon}} \frac{\mathrm{d}y}{1-y^2} \\
&= \quad \frac{1}{\pi^2 \, n^{d-1}} \Big(\log(\varepsilon/\alpha_n) + 2 \, \log(1 + \sqrt{1-\alpha_n/\varepsilon}) \Big) \int\limits_r^\infty \frac{\mathrm{d}F(\varrho)}{\sqrt{\varrho^2 - r^2}} \\
&\le \quad \frac{\log n^{d-1}}{\pi^2 \, n^{d-1}} \Big(1 + \frac{\log\big(4 \, \varepsilon^2 \, \log n^{d-1}\big)}{\log n^{d-1}} \Big) \int\limits_r^\infty \frac{\mathrm{d}F(\varrho)}{\sqrt{\varrho^2 - r^2}} \, .
\end{aligned}$$

Thus, summarizing the above steps yields the estimate

$$\overline{\lim_{n\to\infty}} \Big(\frac{\pi^2 \, n^{d-1}}{\log n^{d-1}} \Big)^{m/2} \Big| \mathrm{Cum}_m\{\widehat{U}_{n,\varepsilon}(r)\} \Big| \le \varepsilon^{(m-2)/2} \, C_m \, \sigma^2(r) \quad \text{for any} \quad \varepsilon > 0 \;\; \text{and} \;\; m \ge 3 \, .$$

This last step confirms the asymptotic normality of the truncated shot-noise process $\widehat{U}_{n,\varepsilon}(r)$ by applying the classical 'method of moments'.

The proof of Theorem 3 is quite similar to that of Theorem 4. For this reason we present a detailed proof only in case of Theorem 4 and outline the essential proving steps

Let $\delta > 0$ be arbitrarily small, but fixed and $\varepsilon > 0$ be chosen small enough (in fact, $\varepsilon = \varepsilon_n$ can be thought of as a positive sufficiently slowly decreasing null sequence). Define in analogy to $\widehat{U}_{n,\varepsilon}(r)$ the truncated shot-noise process

$$\widehat{V}_{n,\varepsilon}(r) \;=\; \frac{1}{\pi n^{d-2}\log n^{d-2}} \sum_{i\geq 1} \frac{\mathbf{1}(\overline{X}_i \in W_n^{(d-2)})}{\overline{R}_i^{\,2} - r^2} \,\mathbf{1}\Big(\overline{R}_i^{\,2} - r^2 > \frac{\max\{\varepsilon, R_i^2 - r^2\}}{\varepsilon^2 n^{d-2}\log n^{d-2}}\Big)$$

and let $M_{n,\varepsilon}(r)$ denote the above random integer $N_{n,\varepsilon}(r)$ with $d-2$ instead of $d-1$.

Using the analog to the 'truncation inequality' (3) and Chebychev's inequality we get

$$\begin{aligned}
\mathsf{P}(\big|\widehat{V}_n(r) - \lambda_d(1-F_d(r))\big| \geq \delta) \;&\leq\; \mathsf{P}(M_{n,\varepsilon}(r) \geq 1) + \mathsf{P}(\,\big|\widehat{V}_{n,\varepsilon}(r) - \lambda_d(1-F_d(r))\,\big| \geq \delta\,) \\
&\leq\; \mathsf{E}M_{n,\varepsilon}(r) + \frac{\mathsf{Var}\big(\widehat{V}_{n,\varepsilon}(r)\big)}{\delta^2} + \frac{\big(\mathsf{E}\widehat{V}_{n,\varepsilon}(r) - \lambda_d(1-F_d(r))\big)^2}{\delta^2}\,.
\end{aligned}$$

The following relations can be proved for any $\varepsilon > 0$:

$$\mathsf{E}M_{n,\varepsilon}(r)) \underset{n\to\infty}{\longrightarrow} 0 \qquad (\;\text{since } \mathsf{E}R_0^2 < \infty\;)\,, \tag{7}$$

$$\overline{\lim_{n\to\infty}}\,\big|\,\mathsf{E}\widehat{V}_{n,\varepsilon}(r) - \lambda_d\,(1-F_d(r))\,\big| \leq\; \lambda_d\,\big(\,F_d(\sqrt{r^2+\varepsilon}) - F_d(r)\,\big) \tag{8}$$

and

$$\overline{\lim_{n\to\infty}}\,\mathsf{Var}\big(\widehat{V}_{n,\varepsilon}(r)\big) \;\leq\; \varepsilon\,\frac{\lambda_d}{\pi}\int_{\mathbb{R}^d} |\,\gamma_2^{(red)}(\mathrm{d}x)\,|\,. \tag{9}$$

Next we present detailed proofs of these three relations. Using the abbreviation $\beta_n = (\varepsilon\, n^{d-2}\, \log n^{d-2})^{-1}$ we may write the expectation $\mathsf{E}M_{n,\varepsilon}(r)$ as follows :

$$\begin{aligned}
\mathsf{E}M_{n,\varepsilon}(r) &= \mathsf{E}\sum_{i\geq 1}\mathbf{1}(\overline{X}_i \in W_n^{(d-2)})\,\mathbf{1}\Big(0 < \overline{R}_i^{\,2} - r^2 \leq \frac{\max\{\varepsilon, R_i^2 - r^2\}}{\varepsilon^2 n^{d-2}\log n^{d-2}}\Big) \\
&=\; \lambda_d\, n^{d-2}\int_0^\infty\int_{\mathbb{R}^2}\mathbf{1}\Big(\varrho^2 - \|\underline{x}\|_2^2 - r^2 \leq \frac{\max\{\varepsilon, \varrho^2 - r^2\}}{\varepsilon^2 n^{d-2}\log n^{d-2}}\Big)\,\mathrm{d}\underline{x}\,\mathrm{d}F_d(\varrho) \\
&=\; 2\,\pi\,\lambda_d\, n^{d-2}\int_0^\infty\int_0^\infty\mathbf{1}\Big(\varrho^2 - r^2 - x^2 \leq \frac{\max\{\varepsilon, \varrho^2 - r^2\}}{\varepsilon^2 n^{d-2}\log n^{d-2}}\Big)\,x\,\mathrm{d}x\,\mathrm{d}F_d(\varrho) \\
&=\; \pi\,\lambda_d\, n^{d-2}\,\Big(\,\frac{\beta_n}{\varepsilon}\int_{\sqrt{r^2+\varepsilon}}^\infty (\,\varrho^2 - r^2\,)\,\mathrm{d}F_d(\varrho) \,+\, \int_r^{\sqrt{r^2+\varepsilon}} \min\{\varrho^2 - r^2, \beta_n\}\,\mathrm{d}F_d(\varrho)\Big)\,.
\end{aligned}$$

Thus, by $n^{d-2}\,\beta_n \underset{n\to\infty}{\longrightarrow} 0$, (7) is shown.

Furthermore, by introducing planar polar coordinates we are led to

$$
\begin{aligned}
\mathsf{E}\widehat{V}_{n,\varepsilon}(r) &= \frac{\lambda_d}{\pi\,\log n^{d-2}} \int\limits_0^\infty \int\limits_{\mathbb{R}^2} \frac{\mathbf{1}\big(\varrho^2 - r^2 - \|\underline{x}\|_2^2 > \frac{\max\{\varepsilon,\varrho^2-r^2\}}{\varepsilon^2 n^{d-2}\log n^{d-2}}\big)}{\varrho^2 - r^2 - \|\underline{x}\|_2^2}\,\mathrm{d}\underline{x}\,\mathrm{d}F_d(\varrho) \\
&= \frac{\lambda_d}{\log n^{d-2}} \left(\int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \int\limits_0^{(\varrho^2-r^2)(1-\beta_n/\varepsilon)} \frac{\mathrm{d}x\,\mathrm{d}F_d(\varrho)}{\varrho^2 - r^2 - x} + \int\limits_{\sqrt{r^2+\beta_n}}^{\sqrt{r^2+\varepsilon}} \int\limits_0^{\varrho^2-r^2-\beta_n} \frac{\mathrm{d}x\,\mathrm{d}F_d(\varrho)}{\varrho^2 - r^2 - x} \right) \\
&= \frac{\lambda_d\,\big(1 - F_d(\sqrt{r^2+\varepsilon})\big)}{\log n^{d-2}}\,\log\Big(\frac{\varepsilon}{\beta_n}\Big) + \frac{\lambda_d}{\log n^{d-2}} \int\limits_{\sqrt{r^2+\beta_n}}^{\sqrt{r^2+\varepsilon}} \log\Big(\frac{\varrho^2-r^2}{\beta_n}\Big)\,\mathrm{d}F_d(\varrho)\,,
\end{aligned}
$$

whence, in view of $\log(\varepsilon/\beta_n)/\log n^{d-2} \underset{n\to\infty}{\longrightarrow} 1$ for all $\varepsilon > 0$, relation (8) follows. In like manner, we may express the variance of the truncated shot-noise process $\widehat{V}_{n,\varepsilon}(r)$:

$$
\begin{aligned}
\mathsf{Var}\big(\widehat{V}_{n,\varepsilon}(r)\big) &= \frac{\lambda_d\,\pi\,n^{d-2}}{(\pi\,n^{d-2}\log n^{d-2})^2} \\
&\times \left(\int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \int\limits_0^{(\varrho^2-r^2)(1-\beta_n/\varepsilon)} \frac{\mathrm{d}x\,\mathrm{d}F_d(\varrho)}{(\varrho^2 - r^2 - x)^2} + \int\limits_{\sqrt{r^2+\beta_n}}^{\sqrt{r^2+\varepsilon}} \int\limits_0^{\varrho^2-r^2-\beta_n} \frac{\mathrm{d}x\,\mathrm{d}F_d(\varrho)}{(\varrho^2 - r^2 - x)^2} \right) \\
&+ \frac{\lambda_d}{(\pi\,n^{d-2}\,\log n^{d-2})^2} \int\limits_0^\infty \int\limits_0^\infty \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \frac{\mathbf{1}(\overline{x} \in W_n^{(d-2)})}{\varrho^2 - r^2 - \|\underline{x}\|_2^2}\,\mathbf{1}\Big(\varrho^2 - r^2 - \|\underline{x}\|_2^2 > \frac{\max\{\varepsilon, \varrho^2 - r^2\}}{\varepsilon^2 n^{d-2}\log n^{d-2}}\Big) \\
&\times \frac{\mathbf{1}(\overline{y} + \overline{x} \in W_n^{(d-2)})}{\tau^2 - r^2 - \|\underline{y} + \underline{x}\|_2^2}\,\mathbf{1}\Big(\tau^2 - r^2 - \|\underline{y} + \underline{x}\|_2^2 > \frac{\max\{\varepsilon, \tau^2 - r^2\}}{\varepsilon^2 n^{d-2}\log n^{d-2}}\Big) \\
&\times \gamma_2^{(red)}(\mathrm{d}(\overline{y},\underline{y}))\,\mathrm{d}(\overline{x},\underline{x})\,\mathrm{d}F_d(\varrho)\,\mathrm{d}F_d(\tau) = T_4^{(n)} + T_5^{(n)} + T_6^{(n)}\,.
\end{aligned}
$$

Some obvious rearrangements show that

$$T_4^{(n)} + T_5^{(n)} = \frac{\lambda_d}{\pi\, n^{d-2}\,(\log n^{d-2})^2} \left(\int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \frac{\mathrm{d}F_d(\varrho)}{\varrho^2 - r^2} \int\limits_{\beta_n/\varepsilon}^{1} \frac{\mathrm{d}x}{x^2} + \int\limits_{\sqrt{r^2+\beta_n}}^{\sqrt{r^2+\varepsilon}} \int\limits_{\frac{\beta_n}{\varrho^2-r^2}}^{1} \frac{\mathrm{d}x}{x^2} \frac{\mathrm{d}F_d(\varrho)}{\varrho^2 - r^2} \right)$$

$$= \frac{\lambda_d}{\pi\, n^{d-2}\,(\log n^{d-2})^2} \left(\int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \frac{\mathrm{d}F_d(\varrho)}{\varrho^2 - r^2} \Big(\frac{\varepsilon}{\beta_n} - 1 \Big) + \int\limits_{\sqrt{r^2+\beta_n}}^{\sqrt{r^2+\varepsilon}} \Big(\frac{\varrho^2 - r^2}{\beta_n} - 1 \Big) \frac{\mathrm{d}F_d(\varrho)}{\varrho^2 - r^2} \right)$$

$$\le \frac{\lambda_d}{\pi\, n^{d-2}\,(\log n^{d-2})^2\, \beta_n} \underset{n\to\infty}{\longrightarrow} 0$$

and

$$|\, T_6^{(n)} \,| \le \frac{\lambda_d\, \varepsilon}{\pi\, \log n^{d-2}} \int\limits_{\mathbb{R}^d} |\gamma_2^{(red)}(\mathrm{d}y)\,| \int\limits_0^{\infty} \int\limits_0^{\infty} \frac{\mathbf{1}\Big(\varrho^2 - r^2 - x > \frac{\max\{\varepsilon, \varrho^2 - r^2\}}{\varepsilon^2 n^{d-2} \log n^{d-2}} \Big)}{\varrho^2 - r^2 - x} \, \mathrm{d}x \, \mathrm{d}F_d(\varrho)$$

$$= \frac{\lambda_d\, \varepsilon}{\pi\, \log n^{d-2}} \int\limits_{\mathbb{R}^d} |\gamma_2^{(red)}(\mathrm{d}y)\,|$$

$$\times \left(\big(1 - F_d(\sqrt{r^2 + \varepsilon}) \big) \, \log \Big(\frac{\varepsilon}{\beta_n} \Big) + \int\limits_{\sqrt{r^2+\beta_n}}^{\sqrt{r^2+\varepsilon}} \log \Big(\frac{\varrho^2 - r^2}{\beta_n} \Big) \, \mathrm{d}F_d(\varrho) \right)$$

$$\le \frac{\lambda_d\, \varepsilon}{\pi\, \log n^{d-2}} \, \log \Big(\frac{\varepsilon}{\beta_n} \Big) .$$

These estimates imply immediately (9).

Combining the relations (7), (8), (9), Condition 3, and the right-continuity of the df F_d completes the proof of Theorem 4.

As announced above we outline some calculations needed to prove Theorem 3. First we introduce a truncated version of the estimator $\widehat{\sigma}_n^2(r)$ and calculate its mean by using Campbell's formula. Let

$$\widehat{\sigma}_{n,\varepsilon}^2(r) := \frac{1}{n^{d-1} \log n^{d-1}} \sum_{i \ge 1} \frac{\mathbf{1}(\overline{X}_i \in W_n^{(d-1)})}{\overline{R}_i^{\,2} - r^2} \, \mathbf{1}\Big(\overline{R}_i^{\,2} - r^2 > \frac{\max\{\varepsilon, R_i^2 - r^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}} \Big)$$

so that the expectation $\mathsf{E}\widehat{\sigma}_{n,\varepsilon}^2(r)$ exists for any $\varepsilon > 0$. More precisely,

$$\mathsf{E}\widehat{\sigma}^2_{n,\varepsilon}(r) \;=\; \frac{2\,\lambda_d}{\log n^{d-1}} \int\limits_0^\infty \int\limits_0^\infty \frac{\mathbf{1}\Big(\varrho^2 - r^2 - x^2 > \frac{\max\{\varepsilon,\varrho^2-r^2\}}{\varepsilon^2 n^{d-1}\log n^{d-1}}\Big)}{\varrho^2 - r^2 - x^2}\,\mathrm{d}x\,\mathrm{d}F_d(\varrho)$$

$$= \frac{2\,\lambda_d}{\log n^{d-1}} \left(\int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \int\limits_0^{\sqrt{(\varrho^2-r^2)(1-\alpha_n/\varepsilon)}} \frac{\mathrm{d}x\,\mathrm{d}F_d(\varrho)}{\varrho^2-r^2-x^2} + \int\limits_{\sqrt{r^2+\alpha_n}}^{\sqrt{r^2+\varepsilon}} \int\limits_0^{\sqrt{\varrho^2-r^2-\alpha_n}} \frac{\mathrm{d}x\,\mathrm{d}F_d(\varrho)}{\varrho^2-r^2-x^2} \right)$$

$$= \frac{\lambda_d}{\log n^{d-1}} \int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2}} \left(\log\Big(\frac{\varepsilon}{\alpha_n}\Big) + 2\,\log\Big(1+\sqrt{1-\frac{\alpha_n}{\varepsilon}}\,\Big)\right)$$

$$+ \frac{\lambda_d}{\log n^{d-1}} \int\limits_{\sqrt{r^2+\alpha_n}}^{\sqrt{r^2+\varepsilon}} \left(\log\Big(\frac{\varrho^2-r^2}{\alpha_n}\Big) + 2\,\log\Big(1+\sqrt{1-\frac{\alpha_n}{\varrho^2-r^2}}\,\Big)\right) \frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2}}\,,$$

whence, together with $\log(\varepsilon/\alpha_n)/\log n^{d-1} \underset{n\to\infty}{\longrightarrow} 1$ and $\lim_{n\to\infty}\int_r^{\sqrt{r^2+\alpha_n}} (\varrho^2-r^2)^{-1/2}\,\mathrm{d}F_d(\varrho) = 0\,,$ it follows that

$$\overline{\lim_{n\to\infty}}\, \mathsf{E}\widehat{\sigma}^2_{n,\varepsilon}(r) \le \sigma^2(r) \quad\text{and}\quad \overline{\lim_{n\to\infty}}\, \Big|\, \mathsf{E}\widehat{\sigma}^2_{n,\varepsilon}(r) - \sigma^2(r)\,\Big| \le\; \lambda_d \int\limits_r^{\sqrt{r^2+\varepsilon}} \frac{\mathrm{d}F_d(\varrho)}{\sqrt{\varrho^2-r^2}} \tag{10}$$

for all $\varepsilon > 0$. It remains to show that

$$\overline{\lim_{n\to\infty}}\, \mathsf{Var}\big(\widehat{\sigma}^2_{n,\varepsilon}(r)\big) \le\; \varepsilon\,\sigma^2(r) \int\limits_{\mathbb{R}^d} |\,\gamma_2^{(red)}(\mathrm{d}x)\,|\,. \tag{11}$$

In analogy to the computation of $\mathsf{Var}\big(\widehat{U}_{n,\varepsilon}(r)\big)$ and $\mathsf{Var}\big(\widehat{V}_{n,\varepsilon}(r)\big)$ we have

$$\mathsf{Var}\big(\widehat{\sigma}^2_{n,\varepsilon}(r)\big) \;=\; \frac{2\,\lambda_d\, n^{d-1}}{(n^{d-1}\log n^{d-1})^2}$$

$$\times \left(\int\limits_{\sqrt{r^2+\varepsilon}}^{\infty} \int\limits_0^{\sqrt{(\varrho^2-r^2)(1-\alpha_n/\varepsilon)}} \frac{\mathrm{d}x\,\mathrm{d}F_d(\varrho)}{(\varrho^2-r^2-x^2)^2} + \int\limits_{\sqrt{r^2+\alpha_n}}^{\sqrt{r^2+\varepsilon}} \int\limits_0^{\sqrt{\varrho^2-r^2-\alpha_n}} \frac{\mathrm{d}x\,\mathrm{d}F_d(\varrho)}{(\varrho^2-r^2-x^2)^2} \right)$$

$$+ \quad \frac{\lambda_d}{(n^{d-1}\,\log n^{d-1})^2} \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}(\overline{x} \in W_n^{(d-1)})}{\varrho^2 - r^2 - \underline{x}^2}\, \mathbf{1}\Big(\varrho^2 - r^2 - \underline{x}^2 > \frac{\max\{\varepsilon, \varrho^2 - r^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}}\Big)$$

$$\times \quad \frac{\mathbf{1}(\overline{y} + \overline{x} \in W_n^{(d-1)})}{\tau^2 - r^2 - (\underline{y} + \underline{x})^2}\, \mathbf{1}\Big(\tau^2 - r^2 - (\underline{y} + \underline{x})^2 > \frac{\max\{\varepsilon, \tau^2 - r^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}}\Big)$$

$$\times \quad \gamma_2^{(red)}(\mathrm{d}(\overline{y}, \underline{y}))\, \mathrm{d}(\overline{x}, \underline{x})\, \mathrm{d}F_d(\varrho)\, \mathrm{d}F_d(\tau) \le \frac{2\,\lambda_d}{n^{d-1}\,(\log n^{d-1})^2}$$

$$\times \quad \Bigg(\int_{\sqrt{r^2+\varepsilon}}^\infty \frac{\mathrm{d}F_d(\varrho)}{(\varrho^2 - r^2)^{3/2}} \int_0^{\sqrt{1-\alpha_n/\varepsilon}} \frac{\mathrm{d}x}{(1-x^2)^2} + \int_{\sqrt{r^2+\alpha_n}}^{\sqrt{r^2+\varepsilon}} \int_0^{\sqrt{1-\frac{\beta_n}{\varrho^2-r^2}}} \frac{\mathrm{d}x}{(1-x^2)^2}\, \frac{\mathrm{d}F_d(\varrho)}{(\varrho^2 - r^2)^{3/2}} \Bigg)$$

$$+ \quad \frac{2\,\lambda_d\,\varepsilon}{\log n^{d-1}} \int_{\mathbb{R}^d} |\gamma_2^{(red)}(\mathrm{d}y)| \int_0^\infty \int_0^\infty \frac{\mathbf{1}\Big(\varrho^2 - r^2 - x^2 > \frac{\max\{\varepsilon, \varrho^2 - r^2\}}{\varepsilon^2 n^{d-1} \log n^{d-1}}\Big)}{\varrho^2 - r^2 - x^2}\, \mathrm{d}x\, \mathrm{d}F_d(\varrho)$$

$$\le \quad \frac{4\,\sigma^2(r)}{n^{d-1}\,(\log n^{d-1})^2\,\alpha_n} + \varepsilon \int_{R^d} |\gamma_2^{(red)}(\mathrm{d}y)|\; \mathsf{E}\widehat{\sigma}_{n,\varepsilon}(r)\,.$$

Thus, (11) is an immediate consequence of (10). To accomplish the proof of Theorem 4 we remember that, in analogy to the proof of Theorem 3 and in view of (3), we have

$$\mathsf{P}\big(\big|\widehat{\sigma}_n^2(r) - \sigma^2(r)\big| \ge \delta\big) \le\ \mathsf{P}(N_{n,\varepsilon}(r) \ge 1) + \frac{\mathsf{Var}\big(\widehat{\sigma}_{n,\varepsilon}^2(r)\big)}{\delta^2} + \frac{\big(\mathsf{E}\widehat{\sigma}_{n,\varepsilon}^2(r) - \sigma^2(r)\big)^2}{\delta^2}$$

for arbitrarily small, but fixed $\delta > 0$. Due to (4), (10), and (11) the right-hand side of the latter inequality tends to zero as $\varepsilon = \varepsilon_n \downarrow 0$.

The proofs of the Theorems 5 and 6 rely on the exponential shape of the generating functional of the stationary, independently marked Poisson process Ψ_d, which is as follows:

$$\mathsf{E}\prod_{i\ge 1} v(X_i, R_i) = \exp\Big\{\lambda_d \int_{\mathbb{R}^d} \int_0^\infty \big(v(x,\rho) - 1\big)\, \mathrm{d}F_d(\rho)\, \mathrm{d}x\Big\}$$

for any Borel-measurable, complex-valued function $v(\cdot)$ on $R^d \times [0,\infty)$ satisfying $\int_{\mathbb{R}^d} \int_0^\infty |\, v(x,\rho) - 1\,| \mathrm{d}F_d(\rho)\, \mathrm{d}x < \infty$, see e.g. Stoyan et al. [10].

Choosing

$$v(x,\rho) = \exp\Big\{\frac{i\,t\,\mathbf{1}(\overline{x} \in W_n^{(d-2)})\,\mathbf{1}(\rho^2 - \|\underline{x}\|_2^2 > r^2)}{\pi\,n^{d-2}\,(\rho^2 - \|\underline{x}\|_2^2 - r^2)}\Big\}$$

yields the following expression for the logarithm of the characteristic function $\mathsf{E}\exp\{it\,\log n^{d-2}\,\widehat{V}_n(r)\}$:

$$\lambda_d \, n^{d-2} \int\limits_{\mathbb{R}^2} \int\limits_{\sqrt{\rho^2+\|x\|_2^2}}^{\infty} \Big(\exp\Big\{ \frac{i\,t}{\rho^2 - \|x\|_2^2 - r^2} \Big\} - 1 \Big) \, \mathrm{d}F_d(\rho) \, \mathrm{d}x$$

$$\lambda_d \, \pi \, n^{d-2} \int\limits_r^{\infty} \int\limits_0^{\rho^2 - r^2} \Big(\exp\Big\{ \frac{i\,t}{\pi\, n^{d-2}\, y} \Big\} - 1 \Big) \, \mathrm{d}y \, \mathrm{d}F_d(\rho)$$

$$= \lambda_d \int\limits_r^{\infty} \int\limits_{\left(\pi\, n^{d-2}\, (\rho^2 - r^2)\right)^{-1}}^{\infty} \frac{\exp\{i\,t\,z\} - 1}{z^2} \, \mathrm{d}z \, \mathrm{d}F_d(\rho) \, . \tag{12}$$

The inner integral in (12) can be approximated by elementary functions with explicit remainder term in the following way :

$$\int\limits_A^{\infty} \frac{\exp\{i\,t\,z\} - 1}{z^2} \, \mathrm{d}z = -\frac{\pi}{2}\,|\,t| - i\,t\,\log|\,t| + i\,t\,\big(1 - \gamma - \log A\big) + \frac{A\,t^2}{2}\,(1 + A\,|\,t|)\,\theta \, ,$$

where $A = \big(a_n\,(\rho^2 - r^2)\big)^{-1}$, $a_n = \pi\, n^{d-2}$, and θ denotes some complex number satisfying $|\theta| \le 1$. Next, splitting the outer integral in (12) into two integrals over $(r_n(\varepsilon), \infty)$ and $(r, r_n(\varepsilon)]$ with $r_n(\varepsilon) = \sqrt{r^2 + (\varepsilon\, a_n)^{-1}}$, we arrive at

$$\log \mathsf{E} \exp\{i\,t\,\log n^{d-2}\,\widehat{V}_n(r)\} = \lambda_d\,(1 - F_d(r_n(\varepsilon)))\,\Big(-\frac{\pi}{2}\,|t| - i\,t\,\log|t| + i\,t\,\big(1 - \gamma + \log a_n\big) \Big)$$

$$+ i\,t\,\lambda_d \int\limits_{r_n(\varepsilon)}^{\infty} \log(\rho^2 - r^2)\,\mathrm{d}F_d(\rho) + \frac{\varepsilon\,\lambda_d}{2}\,t^2\,(1 + \varepsilon|t|)\,\theta \; + \; 2\,\lambda_d\,\widetilde{\theta}\,a_n \int\limits_r^{r_n(\varepsilon)} (\rho^2 - r^2)\,\mathrm{d}F_d(\rho)$$

with some complex number $\widetilde{\theta}$ satisfying $|\widetilde{\theta}| \le 1$. Since, in view of (2), the last term in the previous line vanishes as $n \to \infty$ for any $\varepsilon > 0$ and also

$$\begin{aligned} &\log n^{d-2}\,\big(F_d(r_n(\varepsilon)) - F_d(r)\big) \\ \le \; &\frac{\log n^{d-2}}{\log\big(\varepsilon\, a_n\big)} \int\limits_r^{r_n(\varepsilon)} |\,\log(\rho^2 - r^2)|\,\mathrm{d}F_d(\rho) \underset{n\to\infty}{\longrightarrow} 0 \, , \end{aligned}$$

it follows from the foregoing equation (after replacing t by $t/\lambda_d\,(1 - F_d(r))$ and some further rearrangements) that

$$\log \mathsf{E} \exp\Big\{ i\,t\,\log n^{d-2}\,\Big(\frac{\widehat{V}_n(r)}{\lambda_d\,(1 - F_d(r))} - 1 \Big) \Big\}$$

$$\underset{n\to\infty}{\longrightarrow} \log \mathsf{E}\exp\{i\,t\,S_1\} + i\,t\,\frac{\int_r^\infty \log(\rho^2-r^2)\,\mathrm{d}F_d(\rho)}{1-F_d(r)} + i\,t\,\log\Big(\pi\lambda_d\,\big(1-F_d(r)\big)\Big) + i\,t\,\big(1-\gamma\big)$$

which is nothing else but the assertion of Theorem 5.

To prove Theorem 6 we make use of the subsequent representation of $L_n^{(p)}(t) := \log \mathsf{E}\exp\{i\,t\,\widehat{Y}_n^{(p)}(r)\}$ which can be derived in analogy to (12) by using the generating functional of the Poisson process $\Psi_d^* = \{X_i : i \geq 1\}$:

$$L_n^{(p)}(t) = \lambda_d\,\omega_p \int_r^\infty \int_{\left(n^s\,(\rho^2-r^2)\right)^{-p/2}}^\infty \frac{\exp\{i\,t\,z\}-1}{z^{1+2/p}}\,\big(\rho^2-r^2-z^{-2/p}\,n^{-s}\big)^{-1+p/2}\,\mathrm{d}z\,\mathrm{d}F_d(\rho)\,.$$

The following formula goes back to L. Euler and can be found in any 'Table of Integrals' for $0<\alpha<1$:

$$\int_0^\infty \frac{\exp\{i\,t\,z\}-1}{z^{1+\alpha}}\,\mathrm{d}z = \frac{\Gamma(1-\alpha)}{\alpha}\,\cos\Big(\frac{\alpha\,\pi}{2}\Big)\;|\,t|^\alpha\,\Big(-1+i\,\mathrm{sgn}(t)\,\tan\Big(\frac{\pi\,\alpha}{2}\Big)\Big)\,, \tag{13}$$

where $\Gamma(1-\alpha) = \int_0^\infty e^{-x}\,x^{-\alpha}\,\mathrm{d}x$.

Therefore, applying (13) for $\alpha = \frac{2}{p}$ we obtain after a simple calculation that

$$L_n^{(p)}(t) \underset{n\to\infty}{\longrightarrow} c_p\,\lambda_d\,I_p(r)\;\log \mathsf{E}\exp\{i\,t\,S_{2/p}\} \;=\; \log \mathsf{E}\exp\{i\,t\,(c_p\,\lambda_d\,I_p(r))^{p/2}\,S_{2/p}\}\,,$$

where $I_p(r) = \int_r^\infty (\varrho^2-r^2)^{(p-2)/2}\,\mathrm{d}F_d(\varrho)$ and c_p is as defined in Theorem 6. Thus, replacing t by $t/(c_p\,\lambda_d\,I_p(r))^{p/2}$ completes the proof of Theorem 6.

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