# Algebraic Neighbourhood Logic

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**Abstract**

We present an algebraic embedding of Neighbourhood Logic (NL) into the framework of semirings which yields various simplifications. For example, some of the NL axioms can be dropped, since they are theorems in our framework, and Galois connections produce properties for free. A further simplification is achieved, since the semiring methods used are easy and fairly standard. Moreover, this embedding allows us to reuse knowledge from Neighbourhood Logic in other areas of Computer Science. Since in its original axiomatisation the logic cannot handle intervals of infinite length and therefore not fully model and specify reactive and hybrid systems, using lazy semirings we introduce an extension of NL to intervals of finite and infinite length. Furthermore, we discuss connections between the (extended) logic and other application areas, like Allen's thirteen relations between intervals, the branching time temporal logic $\mathsf{CTL}^*$ and hybrid systems.

*Key words:* Neighbourhood Logic; semiring; lazy semiring; hybrid system; $\mathsf{CTL}^*$; Allen's interval relations

## 1 Introduction

Chop-based interval temporal logics like ITL [15] and IL [10] are useful for the specification and verification of safety properties of real-time systems, in particular, of hybrid systems (e.g. [46]). The Duration Calculus (DC) [45] is an extension of ITL in which real numbers model time and Boolean-valued functions over time model states and events of real-time systems. In particular, DC extends ITL by introducing notions of duration, measure and integration.

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However, all these approaches have natural limitations due to different aspects. First, it is clear that formulae of these logics cannot express unbounded liveness properties, since their truth value only depends on a given *finite* interval. Furthermore they consider only properties inside that interval and cannot be used for reasoning about properties "outside". Therefore, they are not able to express properties about the perpetual interaction of the system with its environment as in the case of hybrid or reactive systems. Second, notions from real analysis, such as limits, are not expressible in ITL.

In order to improve the expressiveness of ITL and DC, they were extended by *infinite intervals* [47,31,42] and *expanding* modalities [41,33,39,34] that are able to describe behaviour outside the interval under consideration. *Neighbourhood Logic* (NL) [43] is a first-order interval logic that uses expanding modalities. These atomic formulae relate time intervals to their (left and right) interval neighbours. It has been shown that the basic unary interval modalities of [16] and the three binary interval modalities (C, T and D) of [41] can be defined using the modalities of NL [14,5]. Hence NL subsumes those logics. It is also used for specifying liveness and fairness of computing systems and for defining notions of real analysis in terms of expanding modalities. Unfortunately NL, as an extension of ITL, is based on *finite* intervals and cannot handle infinite intervals. Therefore, although NL is able to reason about past and future behaviour via a universal modality, it cannot reason about unbounded infinite behaviour. Moreover, there is nearly no support from theorem provers for the above logics. The few existing approaches like [13] are special-purpose theorem provers, i.e. these automated theorem provers have to be developed and implemented for each single purpose.

Due to these deficiencies, we pursue two main aims with this paper.

First we present an algebraic embedding of NL into the algebraic framework of semirings, dealing mainly with the propositional aspects of NL. Then we extend NL from single intervals to sets of intervals, which also paves the way to an algebraic axiomatisation of NL. Moreover, we extend the approach to arbitrary idempotent semirings. This allows us to transfer the knowledge of NL to other areas of Computer Science and to reuse it there. Vice versa, we can transfer knowledge from semirings to NL. Because of work in [43] our extension is also an embedding of the logics of [16] and [41]. When deriving the algebraic version of NL we obtain further interesting results. For example, we show that the axioms K, M and another one can be dropped since they are theorems in the algebraic setting. Allen's thirteen relations between intervals [1,2] can also be embedded into semirings and are therefore related to NL. Since we are using first-order equationally based theory, we can use off-the-shelf theorem-provers like Prover9 [28] to support our calculations. The advantage of such general-purpose provers is that they are ready to use and there is no need to design and implement a new special-purpose theorem prover. Another advantage of the algebraic setting is that the methods used are fairly standard.

Second, since neither NL nor the algebraic version handle intervals of infinite length, we extend the algebraic approach to infinite time. Instead of using semirings, we relax them to lazy semirings. This allows us to handle infinite behaviour within the setting of NL. Moreover, due to the capability of handling infinite elements, NL and its algebraic counterpart can also be adapted to logics like $\mathsf{CTL}^*$ and hybrid systems. For the latter, semiring neighbours yield some safety and liveness properties. In both settings (full and lazy semirings) semiring neighbours are related by Galois connections, which give a lot of properties for free.

The paper is organised as follows: In Section 2 we recapitulate the necessary basics for our work. In particular we introduce NL and modal semirings. Afterwards we derive the algebraic embedding of NL in Section 3. There we also define semiring neighbours and boundaries in the setting of modal semirings and show some simplifications of NL which are enabled by the algebraic setting. In Section 4 we discuss properties beyond the original NL. More precisely we have a look at the chop operator and at Allen's 13 relations between intervals. In Section 5 we generalise the developed theory to the setting of lazy semirings; this starts the second part of the paper which deals with finite and infinite elements. After this, in Section 6, we present an extension of NL that can now handle intervals of finite and infinite length. Before summarising our results and presenting a short outlook on our future work in Section 8, we present applications of semiring neighbours beyond intervals. This can be done only since we abstract from concrete models and use (lazy) semirings instead. In particular, we sketch connections to temporal logics as $\mathsf{CTL}^*$ and to hybrid systems in Section 7.

## 2 Basic Definitions

### *2.1 Overview of Neighbourhood Logic*

Neighbourhood Logic (NL) is a logical formalism for reasoning about liveness and fairness properties in the framework of finite intervals, introduced by Zhou and Hansen in [43]. It is mainly based on Interval Temporal Logic (ITL) [15,16] and provides the possibility to "look" beyond the current interval. This additional feature allows expressing unbounded liveness properties, like "eventually there will be a time interval, where $\varphi$ holds" and "$\varphi$ will hold infinitely often in the future", which are not expressible in the setting of ITL. Similarly, NL was extended to obtain a real-time logic, called Duration Calculus (DC).

The vocabulary used consists of time-independent *global variable symbols* $x$, time-dependent *temporal variable symbols* $v$, time-independent *global function symbols* $f^n$, time-independent *temporal propositional letters* $X$ and time-independent *global relation symbols* $G^n$ (for certain arities $n \in \mathbb{N}$). There are two special global variables $\mathsf{true}$ and $\mathsf{false}$ and a special temporal variable $\ell$ which denotes the length of the interval under consideration.

The languages $\Theta$ of $\mathsf{NL}$ *terms* and $\Phi$ of $\mathsf{NL}$ *formulae* over that vocabulary are defined by the semi-formal grammar

$$\Theta \ ::= \ x \mid v \mid f^n(\underbrace{\Theta, \ldots, \Theta}_{n}) \ ,$$

$$\Phi \ ::= \ X \mid G^n(\underbrace{\Theta, \ldots, \Theta}_{n}) \mid \Phi \wedge \Phi \mid \neg\Phi \mid (\exists x)\Phi \mid \Diamond_l \Phi \mid \Diamond_r \Phi \ .$$

The logical connectives $\vee$, $\rightarrow$ are defined, as usual, by $\varphi \vee \psi \ =_{df} \ \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \rightarrow \psi \ =_{df} \ \neg\varphi \vee \psi$ and $(\forall x)\varphi \ =_{df} \ \neg\,(\exists x)\neg\varphi$. Furthermore, we define duals of the diamond operators $\Diamond_l$ and $\Diamond_r$ in the standard way by $\Box_l \varphi \ =_{df} \ \neg\Diamond_l\neg\varphi$ and $\Box_r\varphi \ =_{df} \ \neg\Diamond_r\neg\varphi$.

Now we briefly recapitulate the original semantics of $\mathsf{NL}$, which is based on the arithmetic of real numbers (see [43]). It is well known that $\mathsf{NL}$ allows an arbitrary cancellative commutative group as its time domain. However, since we want to derive an abstract algebraic version of $\mathsf{NL}$ later on, we focus on the intuition of $\mathsf{NL}$ which is more directly presented with real numbers. Moreover, non-expert readers will have less problems following arguments about real numbers than with an abstract time domain.

The basic objects of $\mathsf{NL}$ are real-valued intervals $[y, z]$ with $y \leq z$ and $y, z \in \mathbb{R}$. The meanings of $f^n$ and $G^n$ are straightforwardly given by functions $\underline{f}^n \in \mathbb{R}^n \rightarrow \mathbb{R}$ and $\underline{G}^n \in \mathbb{R}^n \rightarrow \{\mathsf{true}, \mathsf{false}\}$. The meanings of global variables are given by a *value assignment* $\mathcal{V}$, a function that assigns a real number $x^{\mathcal{V}}$ to each global variable $x$. The meanings of temporal variables and propositional letters are given by an *interpretation* $\mathcal{J}$, a function that associates a real-valued interval function $v^{\mathcal{J}}$ with each temporal variable $v$ and a truth-valued interval function $X^{\mathcal{J}}$ with each propositional letter X. For example, the interpretation of the temporal variable $\ell$ is

$$\ell^{\mathcal{J}}([y, z]) \ = \ z - y \ .$$

The semantics of terms and the logical symbols other than $\Diamond_l$ and $\Diamond_r$ is standard; for further details see Appendix A and [44,4]. For $\Diamond_l$ and $\Diamond_r$ one has

$$[y,z] \models_{\mathcal{J},\mathcal{V}} \Diamond_l \varphi \ \text{ iff } \ \exists \delta \geq 0 : [y-\delta, y] \models_{\mathcal{J},\mathcal{V}} \varphi$$
$$[y,z] \models_{\mathcal{J},\mathcal{V}} \Diamond_r \varphi \ \text{ iff } \ \exists \delta \geq 0 : [z, z+\delta] \models_{\mathcal{J},\mathcal{V}} \varphi$$

Intuitively, $\Diamond_l$ and $\Diamond_r$ allow reasoning about *left* and *right* neighbourhoods of a given interval.


where $u = y - \delta$ where $v = z + \delta$

By definition, $\Diamond_l \varphi$ will hold for an interval that has an interval on the *left* where $\varphi$ holds. Symmetrically, $\Diamond_r \varphi$ holds for an interval having a *right* neighbour interval where $\varphi$ holds.

One of the most interesting binary interval modalities is the *chop operator* $\frown$, interpreted as the operation of "chopping" an interval into two parts. Its semantics is given by

$$[y,z] \models_{\mathcal{J},\mathcal{V}} \varphi \frown \psi \ \text{ iff } \ \exists m : y \leq m \leq z \ \wedge \ [y,m] \models_{\mathcal{J},\mathcal{V}} \varphi \ \wedge \ [m,z] \models_{\mathcal{J},\mathcal{V}} \psi \ . \quad (1)$$

The chop operator cannot be derived from the basic unary modalities in a propositional logic like ITL [40,5], but it is expressible in NL and therefore ITL is subsumed [43]. More precisely,

$$\varphi \frown \psi \Leftrightarrow \exists x, y : ((\ell = x + y) \wedge \Diamond_l \Diamond_r ((\ell = x) \wedge \varphi \wedge \Diamond_r ((\ell = y) \wedge \psi))) \ .$$

There are various kinds of interval temporal logics in the literature, both propositional ([16,15,41]) and first-order ([10]). Most of these logics are subsumed by NL (e.g. [44]).

So far we have used real numbers as domain of time and values. It is known that it is impossible to have a complete axiomatisation of real numbers. One can develop different first order theories for real numbers, but none of them can be complete. However, Neighbourhood Logic is complete with respect to an abstract time domain [4].

Since NL is a logic based on *finite* intervals, an infinite behaviour can therefore only be approximated by finite prefixes of the respective infinite interval. In Sections 5 and 6 we will discuss a possibility to introduce infinite intervals into NL. An extension of ITL to discrete-time infinite intervals has already been given in [31].

Semirings have a wide range of applications in computer science: in the theory of formal languages and automata (regular expressions) (e.g. [27]), logic of programs (e.g. [26,17]) and many more [18,12].

A *semiring* (for clarity sometimes also called *full semiring*) is a quintuple $(S, +, \cdot, 0, 1)$ where $(S, +, 0)$ is a commutative monoid and $(S, \cdot, 1)$ is a monoid such that $\cdot$ distributes over $+$ and $\cdot$ is strict, i.e., $a \cdot 0 = 0 \cdot a = 0$ for all $a \in S$.

A semiring is called *idempotent* if $+$ is idempotent. On idempotent semirings the *natural order* $\leq$ on $S$ is given by $a \leq b \Leftrightarrow_{df} a + b = b$. It is the only order in which both operators are isotone. Moreover, 0 is the least element and $a + b$ is the join of $a$ and $b$. Hence every idempotent semiring is a semilattice. A semiring is *Boolean* if it is idempotent and its underlying semilattice is a Boolean algebra with meet operator $\sqcap$. In particular, every Boolean semiring has a complement operator, denoted by $\overline{\phantom{a}}$, and a greatest element $\top = \overline{0}$. An important proof principle in Boolean semirings is the *shunting rule*

$$a \sqcap b \leq c \Leftrightarrow a \leq \overline{b} + c \ .$$

Models for (Boolean) semirings include formal languages under concatenation as composition, relations under standard composition, sets of graph paths under path concatenation and sets of streams under concatenation. All these models are based on power set constructions and their addition is standard set union.

Another power set construction yields a semiring of intervals (e.g. [19]) which we will use for the algebraic characterisation of $\mathsf{NL}$ in the next section.

Let $\mathbb{T}$ be a set of *time points* (e.g. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, ...) with a linear order $\preceq$. An interval $[y, z]$ over $\mathbb{T}$ is defined in the standard way as $[y, z] =_{df} \{t : y \preceq t \preceq z\}$ for all $y, z \in \mathbb{T}$ with $y \preceq z$; by $\mathbb{I}$ we denote the set of all intervals. Standard interval composition is given by $[y_1, z_1] \, ; [y_2, z_2] =_{df} [y_1, z_2]$ if $z_1 = y_2$ and undefined otherwise. Hence

$$\begin{aligned} & [y_1, z_1] \text{ is a left neighbour of } [y_2, z_2] \\ & \quad \Leftrightarrow [y_2, z_2] \text{ is a right neighbour of } [y_1, z_1] \\ & \quad \Leftrightarrow [y_1, z_1] \, ; [y_2, z_2] \text{ is defined} \\ & \quad \Leftrightarrow z_1 = y_2 \ . \end{aligned} \tag{2}$$

Now the structure

$$\mathrm{INT} =_{df} (\mathcal{P}(\mathbb{I}), \cup, ;, \emptyset, \mathbb{1}) \ ,$$

with pointwise lifted composition and $\mathbb{1}$ being the set of all one-point intervals forms a Boolean semiring. The natural order of this structure coincides with set inclusion.

To model assertions in semirings we use the idea of tests as introduced into Kleene algebras by Kozen [25]. In the semiring of relations and in INT a set of elements can be modelled as a subset of the identity; meets and joins of such partial identities coincide with their composition and addition. Generalising this, one defines a *test* in an idempotent semiring to be an element $p \leq 1$ that has a complement $q$ relative to 1, i.e., $p+q=1$ and $p \cdot q = 0 = q \cdot p$. The set of all tests of $S$ is denoted by $\mathsf{test}(S)$. It is not hard to show that $\mathsf{test}(S)$ is closed under $+$ and $\cdot$ and has 0 and 1 as its least and greatest elements. Moreover, the complement $\neg p$ of a test $p$ is uniquely determined by the definition and $\mathsf{test}(S)$ forms a Boolean algebra. If $S$ is Boolean itself, then $\mathsf{test}(S)$ coincides with the set of all elements below 1. For tests $p, q, r$ we have in arbitrary idempotent semirings the shunting rule $p \cdot q \leq r \Leftrightarrow p \leq \neg q + r$.

With the above definition of tests we deviate slightly from [25], where an arbitrary Boolean algebra of subidentities is allowed as $\mathsf{test}(S)$. The reason is that, as shown in Theorem 4.15 of [9], the axiomatisation of domain to be presented below forces every complemented subidentity to be in $\mathsf{test}(S)$ anyway.

We will consistently write $a, b, \ldots$ for arbitrary semiring elements and $p, q, \ldots$ for tests.

For ease of notation we introduce the operations $p \to q =_{df} \neg p + q$ and $p - q =_{df} p \cdot \neg q$ which obey their usual laws. The above shunting rule now reads $p \cdot q \leq r \Leftrightarrow p \leq q \to r$. In particular,

$$q \leq r \Leftrightarrow 1 \leq q \to r \ .$$

To prepare the connection to general modal logic we study unary functions on tests, later to be instantiated by box and diamond operators. Let $S$ be an idempotent semiring and $f : \mathsf{test}(S) \to \mathsf{test}(S)$ be a function. We distinguish the following properties of $f$:

$$f(1) = 1 \ , \qquad \text{(M)}$$
$$f(p \to q) \leq f(p) \to f(q) \ , \qquad \text{(K)}$$
$$f(p \cdot q) = f(p) \cdot f(q) \ . \qquad \text{(C)}$$

The labels M and K are taken from modal logic (see e.g. [24]), while C stands for conjunctivity. It is well known that C implies isotony of $f$. We have the following relations.

**Lemma 2.1**

(a) C $\Rightarrow$ K.
(b) C $\not\Rightarrow$ M.
(c) M $\wedge$ K $\Rightarrow$ C.

For the proof see Appendix B.

The *de Morgan dual* of $f$ is $f^\circ(p) =_{df} \neg f(\neg p)$. The duals of the above properties are

$$f^\circ(0) = 0 \ , \qquad (\mathrm{M}^\circ)$$
$$f^\circ(p) - f^\circ(q) \leq f^\circ(p - q) \ , \qquad (\mathrm{K}^\circ)$$
$$f^\circ(p + q) = f^\circ(p) + f^\circ(q) \ . \qquad (\mathrm{C}^\circ)$$

They are equivalent to the original ones by straightforward Boolean algebra. Hence we have

**Corollary 2.2**

(a) C° $\Rightarrow$ K°.
(b) C° $\not\Rightarrow$ M°.
(c) M° $\wedge$ K° $\Rightarrow$ C°.

We close with an auxiliary property that will be instrumental in the next section.

**Lemma 2.3** *In an idempotent semiring $S$ with greatest element $\top$ we have the following equivalences:*

(a) $a \cdot p \leq 0 \Leftrightarrow a \leq a \cdot \neg p \Leftrightarrow a \leq \top \cdot \neg p$
(b) $p \cdot a \leq 0 \Leftrightarrow a \leq \neg p \cdot a \Leftrightarrow a \leq \neg p \cdot \top$

*Proof.*

(a) The first equivalence is part of Lemma 3.4 of [9]. The first implication ($a \cdot p \leq 0 \Rightarrow a \leq a \cdot \neg p$) follows by identity, Boolean test algebra, distributivity and the assumption:

$$a = a \cdot 1 = a(p + \neg p) = a \cdot p + a \cdot \neg p \leq a \cdot \neg p \ .$$

The second implication follows directly by $a \leq \top$ and isotony. The third implication ($a \leq \top \cdot \neg \Rightarrow a \cdot p \leq 0$) is by isotony and Boolean test algebra:

$$a \leq \top \cdot \neg p \Rightarrow a \cdot p \leq \top \cdot \neg p \cdot p \Rightarrow a \cdot p \leq 0 \ .$$

(b) Similar to (a).

□

We introduce abstract domain and codomain operators that assign to a set of computations the tests that describe precisely its initial and final states. In combination with restriction, domain yields an abstract preimage operation and codomain an abstract image operation.

A *semiring with domain* [9] is a structure $(S, \ulcorner\,)$, where $S$ is an idempotent semiring and the *domain operator* $\ulcorner : S \to \mathsf{test}(S)$ satisfies

$$a \leq \ulcorner a \cdot a \quad (\mathrm{d1}), \qquad \ulcorner(p \cdot a) \leq p \quad (\mathrm{d2}), \qquad \ulcorner(a \cdot \ulcorner b) \leq \ulcorner(a \cdot b) \quad (\mathrm{d3}).$$

The domain of an element describes all its possible starting points.

In INT we have, for $A \subseteq \mathbb{I}$,

$$\ulcorner A \;=\; \{[x, x] : \exists\, y : [x, y] \in A\}\ . \tag{3}$$

We only give some properties which we need in the following sections; other properties can be found e.g. in [9]

First, the conjunction of (d1) and (d2) is equivalent to each of

$$\ulcorner a \leq p \Leftrightarrow a \leq p \cdot a \quad (\mathrm{llp}), \qquad \ulcorner a \leq p \Leftrightarrow \neg p \cdot a \leq 0 \quad (\mathrm{gla}).$$

(llp) says that $\ulcorner a$ is the least left preserver of $a$; (gla) that $\neg\ulcorner a$ is the greatest left annihilator of $a$. By Boolean algebra, (gla) is equivalent to

$$p \cdot \ulcorner a \leq 0 \Leftrightarrow p \cdot a \leq 0\ . \tag{4}$$

**Lemma 2.4** *(Lemma 4.11 of [9]) Let $S$ be a semiring with domain.*

(a) $\ulcorner$ *is isotone and universally disjunctive;*
*in particular* $\ulcorner 0 = 0$ *and* $\ulcorner(a + b) = \ulcorner a + \ulcorner b$.
(b) $\ulcorner a \leq 0 \Leftrightarrow a \leq 0$. (Full Strictness)
(c) $\ulcorner p = p$. (Stability)
(d) $\ulcorner(p \cdot a) = p \cdot \ulcorner a$. (Import/Export)
(e) $\ulcorner(a \cdot b) \leq \ulcorner a$.

A codomain operator $\urcorner$, which describes all possible ending states of an element, is easily defined as a domain operator in the opposite semiring (i.e., the one that swaps the order of composition). Due to the duality $a\urcorner$ is the least right preserver of $a$ and $\neg a\urcorner$ is the greatest right annihilator:

$$a\urcorner \leq p \Leftrightarrow a \leq a \cdot p \quad (\mathrm{lrp}), \qquad a\urcorner \leq p \Leftrightarrow a \cdot \neg p \leq 0 \quad (\mathrm{gra}).$$

Furthermore, the duality between domain and codomain immediately implies a Lemma dual to Lemma 2.4.

A *modal semiring* is a semiring with domain and codomain. In the remainder of this section we will only list the properties and proofs for domain; the ones for codomain are symmetric.

**Lemma 2.5** *If $S$ is Boolean then*

$$\neg\ulcorner a \le \ulcorner\overline{a},\ \textit{hence}\ \neg\ulcorner\overline{a} \le \ulcorner a, \quad \textit{and} \quad \overline{p\cdot\top} \;=\; \neg p\cdot\top\ .$$

*Proof.* By Boolean algebra and additivity of domain, $1 \;=\; \ulcorner\top \;=\; \ulcorner(a+\overline{a}) \;=\; \ulcorner a + \ulcorner\overline{a}$. Then the first two claims follow by shunting.

By Boolean algebra we only have to show that $\neg p\cdot\top + p\cdot\top \;=\; \top$ and $\neg p\cdot\top \sqcap p\cdot\top \;=\; 0$. The first equation follows by left-distributivity, the second one by Boolean algebra and the law (see [29])

$$p\cdot a \sqcap q\cdot a \;=\; p\cdot q\cdot a \tag{5}$$

that holds even in absence of a general meet operation. □

## 3 Embedding Neighbourhood Logic into Modal Semirings

In this section we present an embedding of Neighbourhood Logic into the interval semiring INT. This yields several advantages:

- It can be shown that some axioms of NL can be dropped since they are theorems in our setting.
- Using the algebra one can now use theorem provers to verify or falsify formulae and therefore has a computer-aided framework.
- The algebra gives a unifying framework in which the theory of NL can be reused in other areas, and vice versa, the theory of semirings can be applied to NL.

### *3.1 Towards an Algebraic Characterisation*

For the embedding we assume a fixed interpretation $\mathcal{J}$ and value assignment $\mathcal{V}$ and abbreviate $\models_{\mathcal{J},\mathcal{V}}$ by just $\models$. Given a formula $\varphi$ we define $\mathbb{I}_\varphi$ as the set of all intervals where $\varphi$ holds:

$$\mathbb{I}_\varphi \;=_{df}\; \{[y,z] : [y,z] \models \varphi\}\ .$$

The sets $\mathbb{I}_\varphi$ of intervals will be the elements of our algebraic structure.

Obviously, temporal and global variables as well as propositional letters can be used to construct such sets of intervals. For example, using the temporal variable $\ell$, we can characterise all intervals of length $x$ by $\mathbb{I}_{\ell=x}$. The embedding of other $\mathsf{NL}$ formulae is then straightforward by

$$\begin{aligned} [y,z] \models \varphi \vee \psi \ &\Leftrightarrow\ [y,z] \in \mathbb{I}_{\varphi\vee\psi} \ = \ \mathbb{I}_\varphi \cup \mathbb{I}_\psi \ , \\ [y,z] \models \neg\varphi \ &\Leftrightarrow\ [y,z] \in \overline{\mathbb{I}_\varphi} \ . \end{aligned}$$

We lift the validity assertion to sets of intervals by setting, for $A \subseteq \mathbb{I}$,

$$A \models \varphi \Leftrightarrow \forall [y,z] \in A : [y,z] \models \varphi \Leftrightarrow A \subseteq \mathbb{I}_\varphi \ .$$

But how to handle the neighbourhood modalities? For $\Diamond_l \varphi$ we get, using (2) and (3),

$$\begin{aligned} \{[y,z]\} \models \Diamond_l \varphi \ &\Leftrightarrow\ \exists\,[u_1,u_2] \in \mathbb{I}_\varphi : [u_1,u_2]\,;[y,z] \text{ is defined} \\ &\Leftrightarrow\ \exists\,[u_1,u_2] \in \mathbb{I}_\varphi : y = u_2 \\ &\Leftrightarrow\ {}^{\ulcorner}\{[y,z]\} \subseteq \mathbb{I}_\varphi{}^{\urcorner}, \end{aligned}$$

and therefore, for any set $A \subseteq \mathbb{I}$ of intervals, by disjunctivity of domain and codomain,

$$A \models \Diamond_l \varphi \Leftrightarrow {}^{\ulcorner}A \subseteq \mathbb{I}_\varphi{}^{\urcorner} \ , \tag{6}$$
$$A \models \Diamond_r \varphi \Leftrightarrow A^{\urcorner} \subseteq {}^{\ulcorner}\mathbb{I}_\varphi \ . \tag{7}$$

As a first result we note that at least one of the eight axioms postulated in [43] can be dropped, since it is a theorem in domain semirings.

**Theorem 3.1** *On single intervals, $\Diamond(\varphi \vee \psi) \Leftrightarrow \Diamond\varphi \vee \Diamond\psi$, where $\Diamond$ is $\Diamond_l$ or $\Diamond_r$. Hence Axiom 4 of [43], which postulates the distributivity of $\Diamond$ over disjunction, is now a conclusion.*

*Proof.* Using Equation (6), that ${}^{\ulcorner}\{[y,z]\} = \{[y,y]\}$ is a singleton set and additivity of codomain, we get

$$\begin{aligned} [y,z] \models \Diamond_l\varphi \vee \Diamond_l\psi \ &\Leftrightarrow\ {}^{\ulcorner}\{[y,z]\} \subseteq \mathbb{I}_\varphi{}^{\urcorner} \ \vee\ {}^{\ulcorner}\{[y,z]\} \subseteq \mathbb{I}_\psi{}^{\urcorner} \\ &\Leftrightarrow\ {}^{\ulcorner}\{[y,z]\} \subseteq \mathbb{I}_\varphi{}^{\urcorner} \cup \mathbb{I}_\psi{}^{\urcorner} \\ &\Leftrightarrow\ {}^{\ulcorner}\{[y,z]\} \subseteq (\mathbb{I}_\varphi \cup \mathbb{I}_\varphi)^{\urcorner} \ = \ (\mathbb{I}_{\varphi\vee\psi})^{\urcorner} \\ &\Leftrightarrow\ [y,z] \models \Diamond_l(\varphi \vee \psi) \ . \end{aligned}$$

The proof of distributivity of $\Diamond_r$ is similar. □

More precisely, the corresponding logical part of Theorem 3.1 splits into two parts. The first one, $\Diamond(\varphi \vee \psi) \Rightarrow \Diamond\varphi \vee \Diamond\psi$ is a consequence of the axioms M and K for modal logic, as stated in Lemma 2.1 and Corollary 2.2. The second, $\Diamond\varphi \vee \Diamond\psi \Rightarrow \Diamond(\varphi \vee \psi)$ is already a theorem in $\mathsf{NL}$(e.g. Theorem NL3 in [44]).

Now we will discuss the box operators $\Box_l\varphi$ and $\Box_r\varphi$ of Zhou and Hansen in the setting of modal semirings. The meaning of $\Box_l\varphi$ and $\Box_r\varphi$ is the following:

$$[y,z] \models \Box_l\varphi \Leftrightarrow i \models \varphi \text{ for all left neighbour intervals } i \text{ of } [y,z]\ ,$$
$$[y,z] \models \Box_r\varphi \Leftrightarrow i \models \varphi \text{ for all right neighbour intervals } i \text{ of } [y,z]\ .$$

Again we start with the pointwise characterisation of the boxes in INT.

Since $\ulcorner\{[y,z]\} = \{[y,y]\}$ is a singleton set,

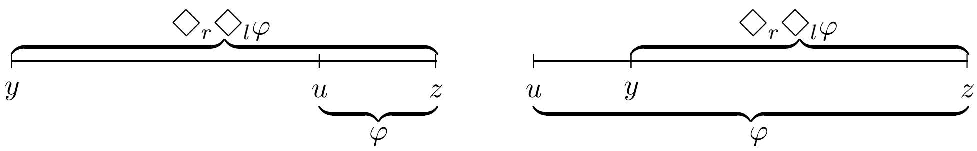
$$\begin{aligned} [y,z] \models \Box_l\varphi &\Leftrightarrow [y,z] \models \neg\Diamond_l\neg\varphi \\ &\Leftrightarrow \ulcorner\{[y,z]\} \not\subseteq (\mathbb{I}_{\neg\varphi})^{\urcorner} \\ &\Leftrightarrow \ulcorner\{[y,z]\} \subseteq \neg(\mathbb{I}_{\neg\varphi})^{\urcorner} \\ &\Leftrightarrow (\mathbb{I}_{\neg\varphi})^{\urcorner}\,;\,\ulcorner\{[y,z]\} \subseteq \emptyset\ . \end{aligned}$$

Note that the symbol $\neg$ is overloaded and used in two different contexts; on the one hand it is the logical negation of $\mathsf{NL}$ and on the other hand it denotes the complement of tests.

Since $\mathbb{I}_{\neg\varphi}$ characterises the set of all intervals where $\varphi$ does not hold, it is the same as $\overline{\mathbb{I}_\varphi}$ using the complement function of Boolean semirings. Using the same generalisation as above we get, for $A \subseteq \mathbb{I}$,

$$A \models \Box_l\varphi \Leftrightarrow (\mathbb{I}_{\neg\varphi})^{\urcorner}\,;\,\ulcorner A \subseteq \emptyset\ , \tag{8}$$
$$A \models \Box_r\varphi \Leftrightarrow A^{\urcorner}\,;\,\ulcorner(\mathbb{I}_{\neg\varphi}) \subseteq \emptyset\ . \tag{9}$$

In [44] the authors introduce additional neighbourhood modalities for $\mathsf{NL}$ which are given by composing the basic modalities $\Diamond_l$ and $\Diamond_r$. In the remaining section we show that they are again diamonds closely related to $\Diamond_l$ and $\Diamond_r$. First we want to illustrate the meaning of $\Diamond_r\Diamond_l\varphi$.




with $u = z - \delta$. Therefore in this case, $[u,z]$ is a postfix of $[y,z]$, or $[y,z]$ is a postfix of $[u,z]$. These nested diamond operators are closely related to the modal operators $E, \overline{E}, B$ and $\overline{B}$ of the logic defined in [16]. For details of the relationship e.g. [44].

In contrast to the simple neighbourhood operators where some starting points have to be equal to some end points of sets of intervals, here only end points occur. The end points of $\Diamond_r \Diamond_l \varphi$ have to form a subset of the ones of $\varphi$. Using the (co)domain interpretation of (6) and (7),

$$\begin{aligned}
[y,z] \models \Diamond_r \Diamond_l \varphi \;&\Leftrightarrow\; \{[y,z]\}^{\urcorner} \subseteq {}^{\ulcorner}\!\left(\mathbb{I}_{\Diamond_l \varphi}\right) \\
&\Leftrightarrow\; \{[y,z]\}^{\urcorner} \subseteq {}^{\ulcorner}\{[u,v] : {}^{\ulcorner}\{[u,v]\} \subseteq \mathbb{I}_\varphi^{\urcorner}\} \\
&\Leftrightarrow\; \{[y,z]\}^{\urcorner} \subseteq \{[u,u] : [u,u] \in \mathbb{I}_\varphi^{\urcorner}\} \\
&\Leftrightarrow\; \{[y,z]\}^{\urcorner} \subseteq \mathbb{I}_\varphi^{\urcorner} \,.
\end{aligned}$$

We can derive a similar expression for $\Diamond_l \Diamond_r \varphi$ as ${}^{\ulcorner}\{[y,z]\} \subseteq {}^{\ulcorner}\mathbb{I}_\varphi$. We see in our setting the characterisation of $\Diamond_r \Diamond_l \varphi$ and $\Diamond_l \Diamond_r \varphi$ is no more complicated than that of the single neighbourhood modalities. The four neighbourhood operators ($\Diamond_l$, $\Diamond_r$, $\Diamond_l \Diamond_r$, $\Diamond_r \Diamond_l$) represent all combinations for comparing domain and codomain and therefore motivate the generalised definition in the next section.

### *3.2 Semiring Neighbours*

Starting with the expressions for the neighbourhoods derived in the previous section and motivated by NL we now give definitions that work on general modal semirings. We simply have to replace sets of intervals by semiring elements, $\emptyset$ by 0 and inclusion $\subseteq$ by the natural order $\leq$. In the remainder some proofs are presented only for one of a series of similar cases.

**Definition 3.2** Let $S$ be a modal semiring. Then

| | | | | |
|---|---|---|---|---|
| (a) | $a$ is a *left neighbour* of $b$ | (or $a \leq {\Diamond\!\!\!\!\!\!\;{\scriptstyle n}}_l\, b$ for short) | iff | $a^{\urcorner} \leq {}^{\ulcorner}b$, |
| (b) | $a$ is a *right neighbour* of $b$ | (or $a \leq {\Diamond\!\!\!\!\!\!\;{\scriptstyle n}}_r\, b$ for short) | iff | ${}^{\ulcorner}a \leq b^{\urcorner}$, |
| (c) | $a$ is a *left boundary* of $b$ | (or $a \leq {\Diamond\!\!\!\!\!\!\;{\scriptstyle b}}_l\, b$ for short) | iff | ${}^{\ulcorner}a \leq {}^{\ulcorner}b$, |
| (d) | $a$ is a *right boundary* of $b$ | (or $a \leq {\Diamond\!\!\!\!\!\!\;{\scriptstyle b}}_r\, b$ for short) | iff | $a^{\urcorner} \leq b^{\urcorner}$. |

We will see below that the use of $\leq$ is justified. Note that the $b$ inside the diamond stands for boundary and is not related to the argument $b$, which is an arbitrary element. By *semiring neighbours* we mean both, left/right neighbours and boundaries.

Now we take a closer look at the definition and its interpretation in INT. It is straightforward to see the connection between semiring neighbours and the modalities of NL. As an example take the equivalence

$$i \models \Diamond_l \varphi \quad \text{iff} \quad \{i\} \leq {\Diamond\!\!\!\!\!\!\;{\scriptstyle n}}_r\, \mathbb{I}_\varphi \,,$$

for any interval $i$. The change in the direction is caused by different points of view. The original interpretation of $i \models \Diamond_l \varphi$ was that $i$ *has* a left neighbour interval where $\varphi$ holds. Our reading of $i \le \Diamond^n_r \mathbb{I}_\varphi$ is that $i$ *is* a right neighbour of some interval in $\mathbb{I}_\varphi$.

In our opinion the latter notation is more intuitive, since, looking at the figures of Section 2.1 $\Diamond_l \varphi$ is on the right hand side of the interval where $\varphi$ holds.

Starting from the definitions of semiring neighbours we calculate an explicit form of these operators if the existence of a greatest element $\top$ is guaranteed. Such an element exists in nearly all semirings that occur in applications. In particular, all semirings which are built via a power set construction have a greatest element, namely the set of all elements. For example in INT the greatest element is the set of all intervals.

**Lemma 3.3** *If $\top$ exists, neighbours and boundaries can be expressed as*

$$\Diamond^n_l b = \top \cdot \ulcorner b\ , \quad \Diamond^n_r b = b^\urcorner \cdot \top\ , \quad \Diamond^b_l b = \ulcorner b \cdot \top\ , \quad \Diamond^b_r b = \top \cdot b^\urcorner\ .$$

*Consequently,* $(\Diamond^n_l b)^\urcorner = \ulcorner b$, $\ulcorner(\Diamond^n_r b) = b^\urcorner$, $\ulcorner(\Diamond^b_l b) = \ulcorner b$ *and* $(\Diamond^b_r b)^\urcorner = b^\urcorner$.

*Proof.* By definition, (lrp) and Lemma 2.3.(b) we get

$$a \le \Diamond^n_l b \Leftrightarrow a^\urcorner \le \ulcorner b \Leftrightarrow a \le a \cdot \ulcorner b \Leftrightarrow a \le \top \cdot \ulcorner b\ .$$

The second part follows from Lemma 2.4(d) and $\ulcorner\top = 1$. $\square$

As a direct consequence of the explicit expressions and the equation $\ulcorner(p\cdot\top) = p$, we have the following cancellation properties for nested neighbours.

**Corollary 3.4**

(a) $\Diamond^n_l \Diamond^n_r b = \Diamond^b_r b$ *and* $\Diamond^n_r \Diamond^n_l b = \Diamond^b_l b$,
(b) $\Diamond^b_l \Diamond^n_r b = \Diamond^n_r b$ *and* $\Diamond^b_r \Diamond^n_l b = \Diamond^n_l b$,
(c) $\Diamond^b_l \Diamond^b_l b = \Diamond^b_l b$ *and* $\Diamond^b_r \Diamond^b_r b = \Diamond^b_r b$,
(d) $\Diamond^n_l \Diamond^b_l b = \Diamond^n_l b$ *and* $\Diamond^n_r \Diamond^b_r b = \Diamond^n_r b$.

This corollary shows that Axiom 6 of [43], which postulates that left and right neighbourhoods of an interval always end and start at the same point, is also a theorem in our setting.

To define boxes similar to $\Box_l$ and $\Box_r$ in the general setting we assume that the underlying semiring $S$ is Boolean.

**Definition 3.5**

(a) $a$ is a *perfect left neighbour* of $b$ (or $a \le \Box^n_l b$) iff $a^\urcorner \cdot \ulcorner\overline{b} \le 0$,
(b) $a$ is a *perfect right neighbour* of $b$ (or $a \le \Box^n_r b$) iff $\overline{b}^\urcorner \cdot \ulcorner a \le 0$,
(c) $a$ is a *perfect left boundary* of $b$ (or $a \le \Box^b_l b$) iff $\ulcorner a \cdot \ulcorner\overline{b} \le 0$,
(d) $a$ is a *perfect right boundary* of $b$ (or $a \le \Box^b_r b$) iff $a^\urcorner \cdot \overline{b}^\urcorner \le 0$.

(a) and (b) correspond to the box-operators of NL. By (c) and (d) we have an additional extension of NL. These two definitions provide “box operators” for the nested neighbourhood modalities $\Diamond_l\Diamond_r$ and $\Diamond_r\Diamond_l$, which are not defined in the semantics of NL in [44].

To justify the definitions above we have

**Lemma 3.6** *Each perfect neighbour (boundary) is a neighbour (boundary):*

$$\boxed{n}_l b \le \langle n\rangle_l b\ , \qquad \boxed{n}_r b \le \langle n\rangle_r b\ , \qquad \boxed{b}_l b \le \langle b\rangle_l b\ , \qquad \boxed{b}_r b \le \langle b\rangle_r b\ .$$

For boundaries this corresponds to the fact in modal logic that $\Box\varphi \to \Diamond\varphi$ iff the underlying relation is total, since every interval is a boundary of itself.

*Proof.* By definition, shunting, Lemma 2.5 and definition again:

$$a \le \boxed{n}_l b \Leftrightarrow a^{\urcorner} \le \neg{}^{\ulcorner}\overline{b} \Rightarrow a^{\urcorner} \le {}^{\ulcorner}b \Leftrightarrow a \le \langle n\rangle_l b\ . \qquad \Box$$

From the definitions we immediately get the box exchange rule

$$a \le \boxed{n}_l b \Leftrightarrow \overline{b} \le \boxed{n}_r \overline{a}\ . \tag{bexc}$$

Like neighbours/boundaries we can characterise the box operators in an explicit form.

**Lemma 3.7** *Perfect neighbours and boundaries can be expressed as*

$$\boxed{n}_l b = \top\cdot\neg{}^{\ulcorner}\overline{b}\ , \quad \boxed{n}_r b = \neg\overline{b}^{\urcorner}\cdot\top\ , \quad \boxed{b}_l b = \neg{}^{\ulcorner}\overline{b}\cdot\top\ , \quad \boxed{b}_r b = \top\cdot\neg\overline{b}^{\urcorner}\ .$$

*Consequently,* $(\boxed{n}_l b)^{\urcorner} = \neg{}^{\ulcorner}\overline{b}$, ${}^{\ulcorner}(\boxed{n}_r b) = \neg\overline{b}^{\urcorner}$, ${}^{\ulcorner}(\boxed{b}_l b) = \neg{}^{\ulcorner}\overline{b}$ *and* $(\boxed{b}_r b)^{\urcorner} = \neg\overline{b}^{\urcorner}$.

*Proof.* By definition, shunting, (lrp) and Lemma 2.3.(a):

$$a \le \boxed{n}_l b \Leftrightarrow a^{\urcorner}\cdot{}^{\ulcorner}\overline{b} \le 0 \Leftrightarrow a^{\urcorner} \le \neg{}^{\ulcorner}\overline{b} \Leftrightarrow a \le a\cdot\neg{}^{\ulcorner}\overline{b} \Leftrightarrow a \le \top\cdot\neg{}^{\ulcorner}\overline{b}\ . \qquad \Box$$

In the remainder of this section we show some properties of (perfect) neighbours and boundaries and compare them to properties of NL. To reduce calculations we introduce $\diamondcdot$ and $\boxdot$ as parameterised versions that can be instantiated by either $\langle n\rangle_l$, $\langle n\rangle_r$, $\langle b\rangle_l$ or $\langle b\rangle_r$ and $\boxed{n}_l$, $\boxed{n}_r$, $\boxed{b}_l$ or $\boxed{b}_r$, respectively. The instantiation must be consistent for all occurrences of $\diamondcdot$ and $\boxdot$. The following proofs are only done for one instance of $\diamondcdot$ or $\boxdot$; for all other instances they are similar. If the “direction” of $\diamondcdot$ or $\boxdot$ is important we use formulae like $\diamondcdot_l$ and $\boxdot_r$ where only one degree of freedom remains.

The above explicit forms (Lemma 3.3 and 3.7) show immediately that boxes and diamonds are connected via the de Morgan dualities

$$\boxdot a = \overline{\diamondcdot\overline{a}} \qquad \text{and} \qquad \diamondcdot a = \overline{\boxdot\overline{a}}\ ;$$

hence they form proper modal operators. Additionally, we show that diamonds and boxes are lower and upper adjoints of Galois connections:

**Lemma 3.8** *Diamonds and boxes form the following Galois connections.*

$$\Diamond_l a \leq b \Leftrightarrow a \leq \boxdot_r b\ , \qquad \Diamond_r a \leq b \Leftrightarrow a \leq \boxdot_l b\ .$$

*Proof.* By de Morgan duality, Boolean algebra, definition of $\boxed{n}_l$, Boolean algebra again and definition of $\boxed{n}_r$

$$\overset{n}{\Diamond}_l a \leq b \Leftrightarrow \overline{\boxed{n}_l \overline{a}} \leq b \Leftrightarrow \overline{b} \leq \boxed{n}_l \overline{a} \Leftrightarrow \overline{b}^{\urcorner} \cdot {}^{\ulcorner}a \leq 0 \Leftrightarrow a \leq \boxed{n}_r b\ . \qquad \square$$

### *3.3 Simplifications of Neighbourhood Logic*

We have already seen that at least two axioms of NL can be dropped since they are theorems in our setting. Since Galois connections are useful as theorem generators and dualities as theorem transformers (e.g. [3]) we get many properties of (perfect) neighbours and (perfect) boundaries for free and can simplify NL even more.

For example we get directly by the Galois connection

**Corollary 3.9**

(a) $\Diamond$ *and* $\boxdot$ *are isotone.*
(b) $\Diamond$ *is disjunctive and* $\boxdot$ *is conjunctive.*
(c) *The cancellation laws* $\Diamond_l \boxdot_r a \leq a \leq \boxdot_r \Diamond_l a$ *and* $\Diamond_r \boxdot_l a \leq a \leq \boxdot_l \Diamond_r a$ *hold.*

All these properties are standard implications of the Galois connection (e.g. Lemma 7.26 and Proposition 7.31 of [7]). Therefore no proofs are needed and the corresponding properties for NL come for free.

Since 0 is the least element with respect to $\leq$ and domain as well as codomain are strict, 0 is a neighbour and boundary of each element. Furthermore, special neighbours and boundaries are summarised in

**Lemma 3.10**

(a) $\Diamond 1 = \Diamond \top = \boxdot \top = \top, \quad \Diamond 0 = \boxdot 0 = 0.$
(b) $\Diamond a \leq 0 \Leftrightarrow a \leq 0.$
(c) *By isotony,* ${}^{\ulcorner}a \leq \Diamond_l a$ *and* $a^{\urcorner} \leq \Diamond_r a$*. Additionally, a is a left (right) boundary of itself, i.e.,* $a \leq \overset{b}{\Diamond}_l a$ *and* $a \leq \overset{b}{\Diamond}_r a$*.*
(d) *By the Galois connections and (a) we get* $\top \leq \boxdot y \Leftrightarrow \top \leq y$*.*

Part (a) shows that our box operators satisfy the modal axiom M. Part (c) cannot be transferred from $\Diamond$ to $\boxdot$, i.e., $x \leq [b]x, \ulcorner x \leq [n]_l x, \ldots$ do not hold, since in general $\ulcorner\overline{a} \neq \neg\ulcorner a$.

To summarise, nearly all theorems of NL given in [37,43,44] hold in the generalisation. Most of them already follow from the Galois connections and the lemmas above. The few which cannot be proved in the generalised setting need special properties of the time domain. An example is the density of $\mathbb{R}$. This property implies that each proper interval $[x, y]$ where $x \neq y$ can be split into two proper subintervals. A translation table between [44] and our approach is given in Appendix A of [20].

Hence the Galois connections and the algebraic equations provide simplifications of NL. Next to that, algebraic structures like modal semirings are first-order and therefore automated theorem provers for first-order and equational logic like Prover9 [28] can be used. A procedure to prove theorems fully automatically is given in [23]. Some of the properties above are already shown there. However, a full investigation whether all above properties of NL can be shown automatically has yet to be carried out. We expect only minor problems when doing this, except whenever specific knowledge is needed. For example the fact that every interval of length $\ell = 2x$ can be split into two intervals of length $\ell = x$ needs particular knowledge about addition. Mace4 [28], which is provided together with Prover9, searches for finite counterexamples. Therefore it can be used to disprove false conjectures about NL.

With Corollary 3.4 we have already given cancellation laws for semiring neighbours. Using the explicit forms, we can show many more cancellation laws like

$$\langle b \rangle_l [b]_l = [b]_l b \quad \text{and} \quad [b]_l \langle b \rangle_l = \langle b \rangle_l b \, . \tag{10}$$

In fact there are altogether 32 such laws, which are summarised in [20].

Within the calculations the combination $\boxdot\,\boxdot\, b = \overline{\Diamond\Diamond\overline{b}}$ turns out to be very useful. Furthermore, the "inner" operator dominates the "outer" one; i.e., in those cases, where $\boxdot\Diamond$ or $\Diamond\boxdot$ fulfils one of the cancellation laws, the expression is the same as $\Diamond\Diamond$ and $\boxdot\boxdot$, respectively.

Further simplifications and properties are discussed in [20]. There it is also shown how to handle the nested modality $[n]_l [n]_l [n]_r [n]_r a$, which is used in [44] for a deduction theorem. In INT this complex formula becomes either the greatest element $\top$ or, if $a = 0$, represents the empty set.

## 4 Beyond Neighbourhood Logic

So far, we have discussed semiring neighbours and boundaries, their properties and their connection to NL. This section gives a short overview over further interval operators in our algebraic treatment.

### *4.1 The Chop Operator and Interval Logics*

As mentioned in the introduction, $\varphi^\frown\psi$ holds on $[y,z]$ iff there is an $m$ with $y \leq m \leq z$ such that $\varphi$ holds on $[y,m]$ and $\psi$ holds on $[m,z]$. Equation (1) shows the connection to NL. In the setting of the semiring INT this becomes

$$\begin{aligned} i \models \varphi^\frown\psi \;&\Leftrightarrow\; \exists\, j,k : i \,=\, j \,;k \;\wedge\; j \in \mathbb{I}_\varphi \;\wedge\; k \in \mathbb{I}_\psi \\ &\Leftrightarrow\; i \in \mathbb{I}_\varphi \,; \mathbb{I}_\psi \;,, \end{aligned}$$

where ; is the lifted interval composition defined in Section 2.2. Hence, for $A \subseteq \mathbb{I}$,

$$A \models \varphi^\frown\psi \;\Leftrightarrow\; A \,\subseteq\, \mathbb{I}_\varphi \,; \mathbb{I}_\psi \;,$$

so that in a general semiring we can simply identify $a^\frown b$ with $a \cdot b$. This interpretation of chop is much easier than (1) and coincides well with the standard definitions in semirings. All the explicit treatment of the interval lengths in (1) can be skipped, since they are encoded in the concatenation of intervals, abstractly in the equation $a^\frown b \,=\, a \cdot b$.

It has been shown elsewhere that ITL is subsumed by NL and that all modalities of Halpern and Shoham [16] and the one of Venema [41] can be expressed in NL. Due to our algebraisation of NL we therefore provide also algebraic versions of these logics. It is straightforward to derive algebraic expressions using the representation of [43] of these operators in NL.

### *4.2 Allen's Relations Between Intervals*

Another temporal-based interval logic, introduced in [1,2], describes all possible relationships between intervals $i$ and $k$ over a linear time. Over a partially ordered time a few more exist, but these are not discussed in [1].

The essential relationships are shown in Figure 1. Considering also the converses of these relations, there are in total thirteen ways in which an ordered pair of intervals can be related. For example $k$ after $i$ is the converse of $i$ before $k$ when considering only single intervals. Obviously, equal can be expressed

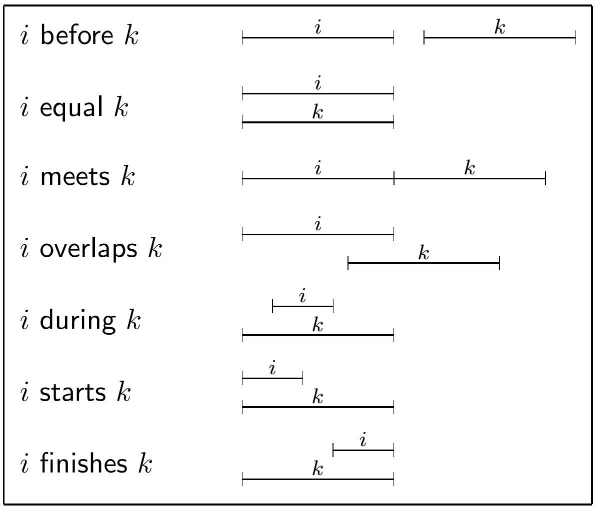



Fig. 1. Possible relationships

by $=$ in the semiring setting. Further, starts and finishes coincide with boundaries. More precisely, $i$ starts $j \Leftrightarrow i \leq \langle b \rangle_l j$ and $i$ finishes $j \Leftrightarrow i \leq \langle b \rangle_r j$. Moreover, meets can be expressed easily by neighbours:

$$i \text{ meets } j \Leftrightarrow i \leq \langle n \rangle_l j \Leftrightarrow j \leq \langle n \rangle_r i \ . \tag{11}$$

For sets of intervals we have to distinguish left and right neighbours. Therefore meets splits into two relationships and we have to be careful which one we mean in each case.

For the description of the further relations we need the following simple fact. For arbitrary functions $f, g$ such that $f$ is isotone,

$$a \leq f(g(b)) \Leftrightarrow \exists\, c : a \leq f(c) \wedge c \leq g(b) \ . \tag{12}$$

For ($\Rightarrow$) choose $c = g(b)$; ($\Leftarrow$) follows by isotony of $f$ and transitivity of $\leq$.

Now we treat the before relation, for which we slightly deviate from Allen. He considers only intervals of positive length, and "before" means "properly before". Since we have also included "improper" one-point intervals, it seems more natural to define before to include meets as a special case. Then we can simply set, even when $i$ and $j$ stand for sets of intervals,

$$\begin{aligned} i \text{ before } k &\Leftrightarrow \exists\, j : i \text{ meets } j \wedge j \text{ meets } k \\ &\Leftrightarrow \exists\, j : i \leq \langle n \rangle_l j \wedge j \leq \langle n \rangle_l k \\ &\Leftrightarrow i \leq \langle n \rangle_l \langle n \rangle_l k \ , \end{aligned}$$

using the first representation of meets from (11) together with (12). Similarly, $k$ after $i$ becomes $k \leq \langle n \rangle_r \langle n \rangle_r i$. Therefore in our setting before and after are converses of each other on single intervals, but for general $a, b$ we have $a$ before $b \not\Leftrightarrow b$ after $a$.

If one wants to model the relation "properly before" one can, in a Boolean semiring, use a modified neighbour operator

$$\Diamond^{n\prime}_{l} b =_{df} \top \cdot \overline{1} \cdot \ulcorner b$$

and set

$$a \ \mathsf{pbefore}\ b \Leftrightarrow_{df} a \leq \Diamond^{n\prime}_{l} b \ .$$

In contrast to the above expression for before this also works for a time domain which is not densely ordered.

Now let us take a closer look at the more complex relations during and overlaps. Both can be characterised using the previous ones.

Using again (12) as well as the duality expressed in (11), we calculate

$$\begin{aligned} i \ \mathsf{during}\ k &\Leftrightarrow \exists\, j_1, j_2 : i \ \mathsf{meets}\ j_1 \ \wedge\ j_1 \ \mathsf{finishes}\ k \ \wedge\ j_2 \ \mathsf{meets}\ i \ \wedge\ j_2 \ \mathsf{starts}\ k \\ &\Leftrightarrow \exists\, j_1, j_2 : i \leq \Diamond^{n}_{l} j_1 \ \wedge\ j_1 \leq \Diamond^{b}_{r} k \ \wedge\ i \leq \Diamond^{n}_{r} j_2 \ \wedge\ j_2 \leq \Diamond^{b}_{l} k \\ &\Leftrightarrow i \leq \Diamond^{n}_{l} \Diamond^{b}_{r} k \ \wedge\ i \leq \Diamond^{n}_{r} \Diamond^{b}_{l} k \ . \end{aligned}$$

In INT, also a meet-operator is available; hence $i$ during $k$ is the same as $i \leq \Diamond^{n}_{l} \Diamond^{b}_{r} k \sqcap \Diamond^{n}_{r} \Diamond^{b}_{l} k$. overlaps can also be expressed by the other relationships and therefore embedded in our setting. By similar reasoning as above,

$$\begin{aligned} i \ \mathsf{overlaps}\ k &\Leftrightarrow \exists\, j_1, j_2 : i \ \mathsf{meets}\ j_1 \ \wedge\ j_1 \ \mathsf{finishes}\ k \ \wedge\ k \ \mathsf{meets}\ j_2 \ \wedge\ j_2 \ \mathsf{starts}\ i \\ &\Leftrightarrow i \leq \Diamond^{n}_{l} \Diamond^{b}_{r} k \ \wedge\ k \leq \Diamond^{n}_{r} \Diamond^{b}_{l} i \end{aligned}$$

Hence we obtain closed algebraic expressions for all of Allen's relations except overlaps. Furthermore it seems that the modalities $\Diamond^{n}_{l} \Diamond^{b}_{r}$ and $\Diamond^{n}_{r} \Diamond^{b}_{l}$ are important, since they occur frequently; for example they were used in the derivation of during and overlaps.

### *4.3 Further Applications*

Recapitulating, we have shown that modal semirings not only provide an algebraic framework for NL but also for many other temporal interval-based logics. But the algebraic structure can be exploited even more. On the one hand it equips us with easy, straightforward calculation rules which can be supported by theorem provers. On the other hand, since there are many other areas in computer science where modal semirings play an important role (e.g. [8,27,18]), the knowledge of NL can be shifted and reused in these areas. Vice versa, one can also apply knowledge from other areas to NL. In

Section 7 we will show how to apply semiring neighbours and therefore Neighbourhood Logic to other subjects. Some interpretations of neighbours in other settings are already given in [20].

## 5 Adding Infinity

A severe limitation of NL and also the presented algebraic setting is that they cannot handle *infinite* intervals, which is necessary to describe infinite behaviour. For example, including infinite intervals allows expressing properties about perpetual interaction of a system with its environment as in the case of hybrid or reactive systems. Therefore we will now present another extension of NL in which this can be modelled.

### *5.1 Lazy Semirings*

We will now assume that the time domain $\mathbb{T}$ includes a special value $\infty$ which is required to be the greatest element of $\mathbb{T}$; e.g., $\mathbb{T} = \mathbb{R} \cup \{\infty\}$. Then we consider two types of intervals. As before, we restrict ourselves to the real numbers to make the discussion more understandable. Obviously $\mathbb{R}$ can be replaced by an abstract time domain as in NL.

- A *finite* interval has the form $[y, z]$ with $y, z \neq \infty$.
- An *infinite* interval has the form $[y, \infty[$ with $y \neq \infty$.

(An interval like $[\infty, z]$ would not be meaningful.) We denote the set of all such intervals by $\mathbb{I}^\infty$.

Now we extend interval composition to $\mathbb{I}^\infty$. For two finite intervals their composition is defined as before. The missing cases are $[y_1, z_1] \,;\, [y_2, \infty[ \,=_{df}\, [y_1, \infty[$ if $z_1 = y_2$ and undefined otherwise, and, $[y, \infty[ \,;\, i =_{df} [y, \infty[$ for any interval $i \in \mathbb{I}^\infty$. Intuitively, the latter case describes the situation that the second interval is never reached, since the first one is already infinite.

We can split each set of intervals into its finite and infinite part. For $A \subseteq \mathbb{I}^\infty$ we set $\mathsf{fin}\, A =_{df} \{i : i \in A,\ i \text{ is finite}\}$ and $\mathsf{inf}\, A =_{df} \{i : i \in A,\ i \text{ is infinite}\}$. The composition of two sets of intervals now becomes

$$A \,;\, B =_{df} \mathsf{inf}\, A \cup \{i \,;\, j : i \in \mathsf{fin}\, A, j \in B\} \;.$$

This implies that $A ; \emptyset = \mathsf{inf}\, A$. Therefore $\emptyset$ is not a right annihilator anymore and $\mathrm{INT}^\infty =_{df} (\mathcal{P}(\mathbb{I}^\infty), \cup, ;, \emptyset, \mathbb{1})$ does not form a full semiring.

Algebraically we relax the definition of a semiring. A *lazy semiring* (also called a *left semiring*) is a quintuple $(S, +, \cdot, 0, 1)$ where $(S, +, 0)$ is a commutative monoid and $(S, \cdot, 1)$ is a monoid such that $\cdot$ is left-distributive over $+$ and *left-strict*, i.e., $0 \cdot a = 0$. A lazy semiring in which $\cdot$ is also right-distributive and right-strict forms a full semiring. Therefore every full semiring is also a lazy semiring. A lazy semiring structure forms the core of process algebra frameworks.

The lazy semiring is *idempotent* if $+$ is idempotent and $\cdot$ is right-isotone, i.e., $b \leq c \Rightarrow a \cdot b \leq a \cdot c$. Left-isotony of $\cdot$ follows from its left-distributivity.

The definitions of *Boolean* lazy semirings and of *tests* are identical to the ones for semirings (cf. [29]).

It is straightforward to check that the structure

$$\mathrm{INT}^\infty = (\mathcal{P}(\mathbb{I}^\infty), \cup, ;, \emptyset, \mathbb{1})$$

forms a Boolean lazy semiring which is even right-distributive. More examples for idempotent lazy semirings are given in [29] and in the following sections. In particular, we will present a lazy semiring describing hybrid systems and another one for temporal logics like $\mathsf{CTL}^*$.

Note that $A \subseteq \mathbb{I}^\infty$ consists of infinite intervals only, i.e., $A = \mathsf{inf}\, A$, iff $A;B = A$ for all $B \subseteq \mathbb{I}^\infty$. We call such an interval set *infinite*, too. Contrarily, $A$ consists of finite intervals only, i.e., $A = \mathsf{fin}\, A$, iff $A ; \emptyset = \emptyset$. We call such an interval set *finite*, too.

We now generalise these notions from $\mathrm{INT}^\infty$ to an arbitrary idempotent lazy semiring $S$. An element $a \in S$ is called *infinite* if it is a left zero, i.e., $a \cdot b = a$ for all $b \in S$, which is equivalent to $a \cdot 0 = a$. By this property, $a \cdot 0$ may be considered as the *infinite part* of $a$, i.e., the part consisting just of infinite computations (if any). We assume the existence of a largest infinite element $\mathsf{N}$, i.e.,

$$a \leq \mathsf{N} \Leftrightarrow_{df} a \cdot 0 = a \ .$$

Dually, we call an element *a finite* if its infinite part is trivial, i.e., if $a \cdot 0 = 0$. We also assume that there is a largest finite element $\mathsf{F}$, i.e.,

$$a \leq \mathsf{F} \Leftrightarrow_{df} a \cdot 0 = 0 \ .$$

This implies, in particular, $1 \leq \mathsf{F}$.

Finally, we assume that every element can be split into its finite and infinite parts: $a = \mathsf{fin}\, a + \mathsf{inf}\, a$, where $\mathsf{fin}\, a =_{df} a \sqcap \mathsf{F}$ and $\mathsf{inf}\, a =_{df} a \sqcap \mathsf{N}$ (but no general meet operation is assumed to exist). In particular, $\top = \mathsf{N} + \mathsf{F}$.

In general $\mathsf{N}$ and $\mathsf{F}$ need not exist (cf. [29]); but if the underlying semiring is Boolean they do, viz. $\mathsf{N} = \top \cdot 0$ and $\mathsf{F} = \overline{\mathsf{N}}$. Since all semirings presented above satisfy that assumption, we will freely use these equations.

**Lemma 5.1** *Assume an idempotent lazy semiring with* $\mathsf{F}$ *and* $\mathsf{N}$.

(a) *If* $a \leq \mathsf{F}$ *then* $a \cdot p \sqcap a \cdot q = a \cdot p \cdot q$.
(b) *If* $S$ *is Boolean and right-distributive then* $\overline{\top \cdot p} = \mathsf{F} \cdot \neg p$.

Note that (a), contrary to (5), needs the premise $a \leq \mathsf{F}$.

### *5.2 Modal Lazy Semirings*

As we have seen, domain and codomain abstractly characterise, in the form of tests, the sets of initial and final states of a set of computations. In contrast to the domain and codomain operators of full semirings, the operators for lazy semirings are not symmetric. Therefore we recapitulate their definitions [29] and establish some properties we need later.

Since the domain describes all possible starting states of an element, it is easy to see that "laziness" of the underlying semiring doesn't matter. Therefore the axioms for domain are the same as in full semirings and most properties of [9] can also be proved in lazy semirings with domain. For example the Equations (llp) and (gla) (cf. Section 2.4) and Lemma 2.4 still hold.

But due to the absence of right-distributivity and right-strictness, a codomain operator can no longer be defined as a domain operator in the opposite semiring; we need an additional axiom.

A *lazy semiring with codomain* is a structure $(S, ^\urcorner)$, where $S$ is an idempotent lazy semiring and the *codomain operator* $^\urcorner : S \to \mathsf{test}(S)$ satisfies for all $a, b \in S$ and $p \in \mathsf{test}(S)$

$$a \leq a \cdot a^\urcorner \quad \text{(lcd1)}, \qquad (a \cdot p)^\urcorner \leq p \quad \text{(lcd2)},$$
$$(a^\urcorner \cdot b)^\urcorner \leq (a \cdot b)^\urcorner \quad \text{(lcd3)}, \qquad (a + b)^\urcorner \geq a^\urcorner + b^\urcorner \quad \text{(lcd4)}.$$

(lcd4) is equivalent to postulating isotony of the codomain operator. As for domain, the conjunction of (lcd1) and (lcd2) together with (lcd4) is equivalent to

$$a^\urcorner \leq p \Leftrightarrow a \leq a \cdot p \;, \tag{lrp}$$

i.e., $a^\urcorner$ is the least right preserver of $a$. However, due to lack of right-strictness $\neg a^\urcorner$ need not be the greatest right annihilator; we only have the weaker equivalence

$$a^{\urcorner} \le p \Leftrightarrow a \cdot \neg p \le a \cdot 0 \ . \qquad \text{(wgra)}$$

With one exception, codomain satisfies the dual of Lemma 2.4:

**Lemma 5.2** *(Lemma 7.7, Lemma 7.8 and its comment in [29]) Let S be a $^{\urcorner}$-lazy semiring.*

(a) $^{\urcorner}$ *is universally disjunctive;*
*in particular* $0^{\urcorner} = 0$ *and* $(a+b)^{\urcorner} = a^{\urcorner} + b^{\urcorner}$.
(b) $a^{\urcorner} \le 0 \Leftrightarrow a \le \mathsf{N}$.
(c) $p^{\urcorner} = p$. (Stability)
(d) $(a \cdot p)^{\urcorner} = a^{\urcorner} \cdot p$. (Import/Export)
(e) $(a \cdot b)^{\urcorner} \le b^{\urcorner}$.

Lemma 2.4.(b) and Lemma 5.2.(b) show the asymmetry between domain and codomain.

A *modal lazy semiring* is a lazy semiring with domain and codomain. The following lemma has some important consequences for the next sections and illustrates again the asymmetry in lazy semirings; it is the counterpart of Lemma 2.3 for lazy semirings.

**Lemma 5.3** *In a modal lazy semiring with a greatest element* $\top$ *we have*

(a) $\neg p \cdot a \le 0 \Leftrightarrow {}^{\ulcorner}a \le p \Leftrightarrow a \le p \cdot a \Leftrightarrow a \le p \cdot \top$.
(b) $a \cdot \neg p \le a \cdot 0 \Leftrightarrow a^{\urcorner} \le p \Leftrightarrow a \le a \cdot p \Leftrightarrow a \le \top \cdot p$.
(c) $a \le \mathsf{F} \Leftrightarrow (a \le a \cdot p \Leftrightarrow a \cdot \neg p \le 0) \Leftrightarrow (a \le \top \cdot p \Leftrightarrow a \cdot \neg p \le 0)$.
*Therefore, in general,* $a \le a \cdot p \not\Rightarrow a \cdot \neg p \le 0$ *and* $a \le \top \cdot p \not\Rightarrow a \cdot \neg p \le 0$.

Property (c) says that we do not have a law for codomain symmetric to (a). Further properties of (co)domain in the setting of lazy semirings can be found in [9,29].

In the lazy semiring of intervals $\mathrm{INT}^{\infty}$, (co)domain can be defined as

$$\begin{aligned} {}^{\ulcorner}A &=_{df} \{[y,y] \mid [y,z] \in A \text{ or } [y,\infty[ \in A\} \ , \\ A^{\urcorner} &=_{df} \{[z,z] \mid [y,z] \in A, z \ne \infty\} \ . \end{aligned}$$

With these definitions we will derive a new version $\mathsf{NL}^{\infty}$ of $\mathsf{NL}$ that handles intervals with infinite durations.

# 6 Neighbourhood Logic with Infinite Duration

## *6.1 Lazy Semiring Neighbours*

The above definitions of semiring neighbours required full semirings as the underlying algebraic structure. In this section we use the same axiomatisation to define neighbours and boundaries in lazy semirings. Since the domain and codomain operators are not symmetric, we will also discuss some properties and consequences those arise out right-distributivity and right-strictness.

The definitions of (perfect) neighbours and boundaries remain unchanged. But their properties are slightly different in the lazy setting. Most of the properties given in Section 2.2 use Lemma 2.3 in their proofs. Unfortunately, by Lemma 5.3.(b) and (c), we do not have that symmetry and we have to check them again. Since most of the interesting properties depend on a greatest element $\top$, we assume its existence in the remainder.

First, the explicit representations for (perfect) semiring neighbours (Lemmas 3.3 and 3.7), the cancellation properties for nested neighbours (Lemma 3.4 and Appendix of [20]), Lemma 3.6, and the exchange rule for boxes (Equation (bexc)) also hold for lazy semirings.

Furthermore, we still have the de Morgan dualities between right neighbours and between left boundaries and hence the following Galois connections.

$$\langle n\rangle_r a \le b \Leftrightarrow a \le \boxed{n}_l b \quad \text{and} \quad \langle b\rangle_l a \le b \Leftrightarrow a \le \boxed{b}_r b\ . \tag{13}$$

As a consequence we get, even in the setting of lazy semirings, that $\langle n\rangle_r$, $\langle b\rangle_l$, $\boxed{n}_l$, $\boxed{b}_r$ are isotone, $\langle n\rangle_r$, $\langle b\rangle_l$ are disjunctive, $\boxed{n}_l$, $\boxed{b}_r$ are conjunctive and $\langle n\rangle_r \boxed{n}_l a \le a \le \boxed{n}_l \langle n\rangle_r a$ and $\langle b\rangle_l \boxed{b}_r a \le a \le \boxed{b}_r \langle b\rangle_l a$.

Between left neighbours and between right boundaries only weak, inequational forms of de Morgan dualities hold.

**Lemma 6.1** *Let $S$ be a right-distributive idempotent semiring with $\top$.*

(a) $\overline{\langle n\rangle_l \overline{b}} \le \boxed{n}_l b$ *and* $\overline{\boxed{n}_l \overline{b}} \le \langle n\rangle_l b$ ,
(b) $\overline{\langle b\rangle_r \overline{b}} \le \boxed{b}_r b$ *and* $\overline{\boxed{b}_r \overline{b}} \le \langle b\rangle_r b$ .

*Proof.*

(a) By Lemma 3.3, Lemma 2.5 (5.1), isotony and Lemma 3.7, $\overline{\langle n\rangle_l b} = \overline{\top \cdot {}^{\ulcorner}b} = \mathsf{F} \cdot \neg{}^{\ulcorner}b \le \top \cdot \neg{}^{\ulcorner}b = \boxed{n}_l \overline{b}$. The inequation $\overline{\boxed{n}_l \overline{b}} \le \langle n\rangle_l b$ then follows by shunting. $\square$

The converse inequations do not hold due to Lemmas 5.1(a) and 5.3(c), since in most cases $\top \not\leq \mathsf{F}$ (if there is at least one infinite element $\neq 0$). Hence $\overline{\langle n\rangle_l \overline{\top}} = \overline{\top \cdot \ulcorner 0} = \overline{\top \cdot 0} = \overline{\mathsf{N}} = \mathsf{F}$ and $[n]_l \top = \top \cdot \neg\ulcorner 0 = \top$. Similarly, the above Galois connections are not valid for left neighbours and right boundaries, but one implication can still be proved.

**Lemma 6.2** *For $S$ as in Lemma 6.1,*

$$\langle n\rangle_l a \leq b \Rightarrow a \leq [n]_r b \ , \qquad \langle b\rangle_r a \leq b \Rightarrow a \leq [b]_r b \ .$$

*Proof.* By Lemma 6.1.(a), Boolean algebra and the exchange rule (bexc)

$$\langle n\rangle_l a \leq b \Rightarrow \overline{[n]_l \overline{a}} \leq b \Leftrightarrow \overline{b} \leq [n]_l \overline{a} \Leftrightarrow a \leq [n]_r b \ . \qquad \square$$

By lack of Galois connections, we do not have a full analogue to Corollary 3.9.

**Lemma 6.3** *For $S$ as in Lemma 6.1,*

(a) $\langle n\rangle_l$, $\langle b\rangle_r$, $[n]_r$ *and* $[b]_l$ *are isotone.*
(b) *If $S$ is right-distributive, then* $\langle n\rangle_l$, $\langle b\rangle_r$ *are disjunctive and* $[n]_r$, $[b]_l$ *are conjunctive.*

*Proof.*

(a) The claim follows directly by the explicit representation of (perfect) neighbours and boundaries (Lemma 3.3 and Lemma 3.7).
(b) By Lemma 3.3, additivity of domain and right-distributivity we get $\langle n\rangle_l (a+b) = \top \cdot \ulcorner(a+b) = \top \cdot (\ulcorner a + \ulcorner b) = \top \cdot \ulcorner a + \top \cdot \ulcorner b = \langle n\rangle_l a + \langle n\rangle_l b$. $\square$

Until now, we have seen that most of the properties of semiring neighbours hold in full semirings as well as in lazy semirings. At some points, we need additional assumptions like right-distributivity. Many more properties, like $\overline{b}^{\urcorner} \leq \langle b\rangle_r b$, can be shown. Most proofs use the explicit forms for lazy semiring neighbours or the Galois connections. Some of them can be found in [21].

However, since lazy semirings reflect some aspects of infinity, we get some useful properties, which are different from the properties of full semirings. Some are summarised in the following lemma.

**Lemma 6.4**

(a) $\langle n\rangle_l \mathsf{F} = \langle n\rangle_r \mathsf{F} = \langle b\rangle_l \mathsf{F} = \langle b\rangle_r \mathsf{F} = \top$.
(b) $b \leq \mathsf{N} \Leftrightarrow \langle n\rangle_r b \leq 0 \Leftrightarrow \langle b\rangle_r b \leq \mathsf{N}$.
(c) $[n]_l \mathsf{N} = [b]_r \mathsf{N} = \mathsf{N}$ *and* $[n]_r \mathsf{N} = [b]_l \mathsf{N} = 0$.
(d) $\overline{b} \leq \mathsf{N} \Leftrightarrow \mathsf{F} \leq b \Leftrightarrow [n]_r b = \top \Leftrightarrow [b]_r b = \top$.

*Proof.* First we note that by straightforward calculations using Lemmas 2.4

and 5.2 and isotony we get

$$\top \cdot p \leq \top \cdot q \Leftrightarrow p \leq q \Leftrightarrow p \cdot \top \leq q \cdot \top \ . \tag{14}$$

Furthermore, by Lemmas 2.4(c) and 5.2(c) we get ${}^{\ulcorner}\mathsf{F} = \mathsf{F}^{\urcorner} = 1$.

(a) Directly by Lemma 3.3 and $\mathsf{F}^{\urcorner} = 1$:

$$\langle n \rangle_l \mathsf{F} = \top \cdot {}^{\ulcorner}\mathsf{F} = \top \cdot 1 = \top.$$

(b) By Lemma 5.2, (14), left-strictness and definition of $\langle n \rangle_l$

$$b \leq \mathsf{N} \Leftrightarrow b^{\urcorner} \leq 0 \Leftrightarrow b^{\urcorner} \cdot \top \leq 0 \cdot \top \Leftrightarrow \langle n \rangle_r b \leq 0 \ .$$

(c) By Lemma 3.7 and ${}^{\ulcorner}\mathsf{F} = 1$ we get

$$[n]_l \mathsf{N} = \top \cdot \neg {}^{\ulcorner}\mathsf{N} = \top \cdot \neg {}^{\ulcorner}\mathsf{F} = \top \cdot 0 = \mathsf{N} \ .$$

(d) Similar to (b). □

Note that (a) implies $\langle n \rangle_l \top = \langle n \rangle_r \top = \langle b \rangle_l \top = \langle b \rangle_r \top = \top$ using isotony. (c) shows again that the inequations of Lemma 6.1 cannot be strengthened to equations.

### *6.2 Infinite Neighbourhood Logic*

We have shown that $\mathrm{INT}^{\infty}$ forms a lazy semiring. Further we have defined semiring neighbours within the setting of lazy semirings. Thereby we have defined a new version $\mathsf{NL}^{\infty}$ of $\mathsf{NL}$ which handles intervals with infinite durations. We already mentioned that $\mathsf{NL}$ subsumes logics like the one of Halpern and Shoham [16], the binary interval modalities of Venema ([41]) and $\mathsf{ITL}$([15]). Using the same arguments, $\mathsf{NL}^{\infty}$ subsumes extensions of those logics. In particular it covers the logics presented in [39,42,34] and [47].

Since $\mathrm{INT}^{\infty}$ is right-distributive, all Lemmas and Corollaries of the preceding section hold in this model.

Due to the algebraic structure, there is nothing more to do to get an extended Neighbourhood Logic. Nevertheless we will give a short example.

We want to discuss the neighbourhood modalities $\Diamond_l$ and $\Diamond_r$ in the infinite setting of $\mathsf{NL}^{\infty}$. By splitting into finite and infinite parts, Lemma 6.3.(b) and Lemma 6.4.(b) we get

$$\langle n \rangle_r b = \langle n \rangle_r(\mathsf{fin}\, b + \mathsf{inf}\, b) = \langle n \rangle_r \mathsf{fin}\, b + \langle n \rangle_r \mathsf{inf}\, b = \langle n \rangle_r \mathsf{fin}\, b.$$

Therefore

$$i \models \Diamond_l \varphi \;\Leftrightarrow\; \{i\} \leq \langle\!\langle n \rangle\!\rangle_r \mathbb{I}_\varphi \;\Leftrightarrow\; \{i\} \leq \langle\!\langle n \rangle\!\rangle_r \mathsf{fin}\,(\mathbb{I}_\varphi)$$

Informally, $\Diamond_l \varphi$ holds on $[y,z]$ ($[y,\infty[$) iff there exists a finite interval $[u,y]$ where $\varphi$ holds; this behaviour fits well with our intuition, since an infinite interval has no neighbour at the "right hand side".

In contrast to this and due to the asymmetry of domain and codomain, $\Diamond_r$ behaves differently. $\Diamond_r \varphi$ holds on $[y,z]$ iff there exists any interval (finite or infinite) where $\varphi$ holds. This can be shown by straightforward calculations and is similar to the original $\mathsf{NL}$. For an infinite interval we calculate

$$\begin{aligned}
[y,\infty[\; \models \Diamond_r \varphi \;&\Leftrightarrow\; \{[y,\infty[\} \leq \langle\!\langle n \rangle\!\rangle_l \mathbb{I}_\varphi \\
&\Leftrightarrow\; (\{[y,\infty[\})^\urcorner \leq \langle\!\langle n \rangle\!\rangle_l \mathbb{I}_\varphi \\
&\Leftrightarrow\; 0 \leq \langle\!\langle n \rangle\!\rangle_l \mathbb{I}_\varphi \\
&\Leftrightarrow\; \mathsf{true}\ .
\end{aligned}$$

The third step holds due to Lemma 5.2.(b). Informally, $\Diamond_r \varphi$ holds *always* on $[y,\infty[$. Since the set of end points of $[y,\infty[$ is empty, every property holds there vacuously.

The chop operator in $\mathsf{NL}^\infty$ and lazy semirings has now to guarantee the reachability of the second part of the "chopped" interval. Therefore we set, for any $A \subseteq \mathbb{I}^\infty$,

$$A \models \varphi {}^\frown \psi \;\Leftrightarrow\; A \leq \mathsf{fin}\,(\mathbb{I}_\varphi)\,;\mathbb{I}_\psi$$

Similar to the above calculations every property of $\mathsf{NL}$ and $\mathsf{NL}^\infty$ can be determined and interpreted in an algebraic setting and therefore allows simple, elegant and convenient calculations. Therefore we won't discuss further properties of $\mathsf{NL}^\infty$. Instead we will present a few further applications for semiring neighbours beyond Neighbourhood Logic.

## 7 More Applications for Semiring Neighbours

### *7.1 The Lazy Semiring of Streams and Trajectories*

The applications presented in this paper will be in the area of temporal logic and hybrid systems. For this we introduce another important Boolean lazy semiring. It is based on trajectories (e.g. [38]) that reflect the values of the system variables over time and was introduced in [22].

Let $D$ be a set of *durations* (e.g. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, ...). The elements of $D$ will represent interval lengths (see also the special variable $\ell$ of the duration calculus). We assume a cancellative addition $+$ on $D$ and an element $0 \in D$ such that $(D, +, 0)$ is a commutative monoid and the relation $x \preceq y \Leftrightarrow_{df} \exists z . x + z = y$ is a linear order on $D$. Then 0 is the least element and $+$ is isotone with respect to $\preceq$. Moreover, 0 is indivisible, i.e., $x + y = 0 \Leftrightarrow x = y = 0$. Similarly to $\mathbb{T}$, the set $D$ may include the special value $\infty$. It is required to be an annihilator with respect to $+$ and hence the greatest element of $D$ (and cancellativity of $+$ is restricted to elements in $D - \{\infty\}$).

For $d \in D$ we define the interval $\mathsf{intv}\, d$ of admissible times as

$$\mathsf{intv}\, d =_{df} \begin{cases} [0, d] & \text{if } d \neq \infty \\ [0, d[ & \text{otherwise} . \end{cases}$$

Compared to the algebra of intervals, $\mathsf{intv}\, d$ is just a subset of $\mathbb{I}^\infty$. Similarly, we call an interval of the form $[0, d]$ *finite* and one like $[0, d[$ *infinite*.

Let now be $V$ a set of *values*. A *trajectory* $t$ is a pair $(d, g)$, where $d \in D$ and $g : \mathsf{intv}\, d \to V$ is a function. Then $d$ is the *duration* of the trajectory. This view models *oblivious* systems in which the evolution of a trajectory is independent of the history before the starting time. The model is more abstract than that of general intervals, since we cannot distinguish different starting times; they are all set to 0.

The set of all trajectories is denoted by TRA. Composition of trajectories $(d_1, g_1)$ and $(d_2, g_2)$ is defined by

$$(d_1, g_1) \cdot (d_2, g_2) =_{df} \begin{cases} (d_1 + d_2, g) & \text{if } d_1 \neq \infty \wedge g_1(d_1) = g_2(0) \\ (d_1, g_1) & \text{if } d_1 = \infty \\ \text{undefined} & \text{otherwise} \end{cases}$$

with $g(x) = g_1(x)$ for all $x \in [0, d_1]$ and $g(x + d_1) = g_2(x)$ for all $x \in \mathsf{intv}\, d_2$. For a value $v \in V$, let $\underline{v} =_{df} (0, g)$ with $g(0) = v$ be the corresponding

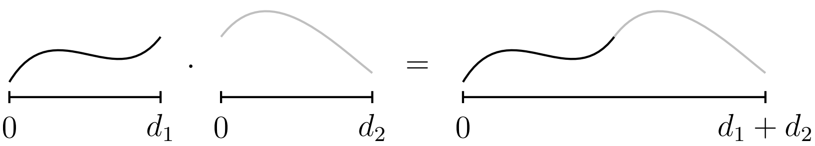



Fig. 2. Composition of two finite trajectories

zero-length trajectory having value $v$ at its only time point 0. Moreover, set $I =_{df} \{\underline{v} \mid v \in V\}$.

A *process* is a set of trajectories. Again, similar to $\mathrm{INT}^\infty$, the *infinite part* of process $A$ are its processes over infinite intervals, i.e., $\mathsf{inf}\, A =_{df} \{(d, g) \in$

$A \mid d = \infty\}$, while the *finite part* of $A$ are its processes over finite intervals, i.e., $\mathsf{fin}\, A =_{df} A - \mathsf{inf}\, A$. Furthermore composition is lifted as

$$A \cdot B =_{df} \mathsf{inf}\, A \cup \{a \cdot b \mid a \in \mathsf{fin}\, A, b \in B\} .$$

With this, we obtain the lazy Boolean semiring

$$\mathrm{PRO} =_{df} (\mathcal{P}(\mathrm{TRA}), \cup, \cdot, \emptyset, I) ,$$

with test set $\mathsf{test}(\mathrm{PRO}) =_{df} \mathcal{P}(I)$.

For a discrete infinite set $D$, e.g. $D = \mathbb{N}$, trajectories are isomorphic to nonempty finite or infinite words over the value set $V$.

If $V$ consists of states of computations, then the elements of PRO can be viewed as sets of computation streams; therefore we also write $\mathrm{STR}(V)$ instead of PRO in this case.

### *7.2 Semiring Neighbours and* $\mathsf{CTL}^*$

The branching time temporal logic $\mathsf{CTL}^*$ (e.g. [11]) is a well-known tool for analysing and describing parallel as well as reactive and hybrid systems. In $\mathsf{CTL}^*$ one distinguishes state formulae and path formulae, the former denoting sets of states, the latter sets of computation traces. We want to show how neighbourhood concepts figure in the semantics of $\mathsf{CTL}^*$.

The language $\Psi$ of $\mathsf{CTL}^*$ *formulae* over a set $\Phi$ of atomic propositions is defined by the grammar

$$\Psi \ ::= \ \bot \mid \Phi \mid \Psi \to \Psi \mid \mathsf{X}\,\Psi \mid \Psi\,\mathsf{U}\,\Psi \mid \mathsf{E}\Psi ,$$

where $\mathsf{X}$ and $\mathsf{U}$ are the next-time and until operators and $\mathsf{E}$ is the existential quantifier on paths. As usual, $\neg\varphi =_{df} \varphi \to \bot$, $\varphi \wedge \psi =_{df} \neg(\varphi \to \neg\psi)$, $\varphi \vee \psi =_{df} \neg\varphi \to \psi$, $\mathsf{A}\varphi =_{df} \neg\,\mathsf{E}\neg\varphi, \mathsf{true} =_{df} \neg\bot, \mathsf{F}\varphi =_{df} \mathsf{true}\,\mathsf{U}\,\varphi, \mathsf{G}\varphi =_{df} \neg\mathsf{F}\neg\varphi$.

In [30] a connection between $\mathsf{CTL}^*$ and Boolean modal left quantales is presented. A *left quantale* [32] is an idempotent and right-distributive semiring that is a complete lattice under the natural order. In particular, all the lemmas of the previous sections can still be used.

More concretely, one can use the left quantale $\mathrm{STR}(A)$ (cf. Section 7.1) of sets of finite and infinite streams over a set $A$ of states as a model. The tests in that semiring are sets of singleton streams (finite streams consisting of only one state each) and hence are isomorphic to sets of states.

For an arbitrary Boolean modal quantale $S$, the concrete standard semantics for $\mathsf{CTL}^*$ is generalised to a function $[\![\_]\!] : \Psi \to S$ as follows, where $[\![\varphi]\!]$ abstractly represents the set of paths satisfying formula $\varphi$. Atomic formulae from $\Phi$ are represented by tests $p$. Moreover, one fixes an element $\mathsf{n}$ ($\mathsf{n}$ standing for "next") as representing the transition system underlying the logic. This leads to the following semantic clauses:

$$
\begin{aligned}
[\![\bot]\!] &= 0 \;, \\
[\![p]\!] &= p \cdot \top \;, \\
[\![\varphi \to \psi]\!] &= \overline{[\![\varphi]\!]} + [\![\psi]\!] \;, \\
[\![\mathsf{X}\,\varphi]\!] &= \mathsf{n} \cdot [\![\varphi]\!] \;, \\
[\![\varphi\,\mathsf{U}\,\psi]\!] &= \bigsqcup_{j \geq 0} (\mathsf{n}^j \cdot [\![\psi]\!] \sqcap \bigsqcap_{k<j} \mathsf{n}^k \cdot [\![\varphi]\!]) \;, \\
[\![\mathsf{E}\varphi]\!] &= {}^{\ulcorner}[\![\varphi]\!] \cdot \top \;.
\end{aligned}
$$

The expression $p \cdot \top$ for $[\![p]\!]$ describes the set of all paths that start with a state in $p$; the same idea is used to define the semantics of $\mathsf{E}\varphi$. For further details and explanations we have to refer to [30].

Using these definitions, it is straightforward to check that $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] + [\![\psi]\!]$, $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \sqcap [\![\psi]\!]$ and $[\![\neg\varphi]\!] = \overline{[\![\varphi]\!]}$.

By simple calculations one gets the following result.

**Lemma 7.1** *(Theorem 6.1 of [30]) Let $\varphi$ be a state formula of $\mathsf{CTL}^*$. Then*

$$[\![\mathsf{A}\varphi]\!] = \neg{}^{\ulcorner}(\overline{[\![\varphi]\!]}) \cdot \top \;.$$

Hence we see that $[\![\mathsf{E}\varphi]\!]$ corresponds to a left boundary and $[\![\mathsf{A}\varphi]\!]$ to a perfect left boundary, i.e.,

$$[\![\mathsf{E}\varphi]\!] = \Diamond\!\!\!\!\;b_l\,[\![\varphi]\!] \quad \text{and} \quad [\![\mathsf{A}\varphi]\!] = \boxed{b}_l\,[\![\varphi]\!] \;.$$

Hence we have built a bridge between $\mathsf{CTL}^*$ and $\mathsf{NL}$ via semiring neighbours and therefore can transfer knowledge between both logics. For example, from the cancellation laws for (perfect) semiring neighbours and equations (10) we obtain immediately

$$[\![\mathsf{EE}\varphi]\!] = [\![\mathsf{E}\varphi]\!] \;, \quad [\![\mathsf{AA}\varphi]\!] = [\![\mathsf{A}\varphi]\!] \;, \quad [\![\mathsf{EA}\varphi]\!] = [\![\mathsf{A}\varphi]\!] \;, \quad [\![\mathsf{AE}\varphi]\!] = [\![\mathsf{E}\varphi]\!] \;.$$

Of course, there are many more dual lemmas which we do not discuss here. The other two boundaries as well as all variants of (perfect) neighbours do not occur in $\mathsf{CTL}^*$ itself. However, the extension $\mathsf{PCTL}^*$ (e.g. [35,36]) of $\mathsf{CTL}^*$ provides operators for describing behaviour in the past. Therefore right boundaries occur in that setting.

In the next section we will build a bridge between semiring neighbours and hybrid systems. In particular, we also build a bridge between $\mathsf{NL}$, $\mathsf{CTL}^*$ and hybrid systems.

### *7.3 Semiring Neighbours and Hybrid Systems*

Hybrid systems are dynamical heterogeneous systems characterised by the interaction of discrete and continuous dynamics. In [22] we use the lazy semiring PRO of processes taken from Section 7.1 for the description of hybrid systems.

**Hybrid systems and $\mathsf{NL}$.** In PRO the left/right neighbours describe a kind of composability, i.e., for processes $A$, $B$,

$$A \leq \overset{n}{\Diamond}_l B \quad \text{iff} \quad \forall\, t \in A : \exists\, u \in B : t \cdot u \text{ is defined}, \tag{15}$$

$$A \leq \overset{n}{\Diamond}_r B \quad \text{iff} \quad \forall\, t \in A : \exists\, u \in \mathsf{fin}\,(B) : u \cdot t \text{ is defined}. \tag{16}$$

Both $\overset{n}{\Diamond}_r$ and $\overset{n}{\Diamond}_l$ guarantee the existence of a composable element. Especially, $\overset{n}{\Diamond}_r B = B^{\rceil} \cdot \top \neq 0$ guarantees that for every trajectory $t \in B$ there exists a trajectory $u$ that can continue $t$. Therefore $\overset{n}{\Diamond}_r B$ is a form of *liveness assertion*. In particular, the process $\overset{n}{\Diamond}_r B$ contains all trajectories that are composable with the "running" one. If $\overset{n}{\Diamond}_r B = \emptyset$, we know that the system will terminate if all trajectories of the running process have finite durations. Note that in (15) the composition $t \cdot u$ is defined if either $f(d_1) = g(0)$ (assuming $a = (d_1, f)$ and $b = (d_2, g)$) or $a$ has infinite duration, i.e., $d = \infty$.

The next paragraph will show that left and right boundaries of lazy semirings are closely connected to temporal logics for hybrid systems. But, by Lemma 3.4, they are also useful as operators that simplify nestings of semiring neighbours.

The situation for right/left perfect neighbours is more complicated. As shown in [20], $\overset{n}{\Box}_r B$ is the set of those trajectories which can be reached only from $B$, not from $\overline{B}$. Hence it describes a situation of guaranteed non-reachability from $\overline{B}$. The situation with $\overset{n}{\Box}_l$ is similar for finite processes, because of the symmetry between left and right perfect neighbours.

**Hybrid systems and $\mathsf{CTL}^*$.** Above we have shown how lazy semiring neighbours are characterised in PRO. Although a next-time operator is not meaningful in continuous time models, the other operators of $\mathsf{CTL}^*$ still make sense.

Since PRO is a Boolean modal left quantale, we simply reuse the above semantic equations (except those for $\mathsf{X}$ and $\mathsf{U}$) and obtain a semantics of a fragment of $\mathsf{CTL}^*$ for hybrid systems. In particular, the existential quantifier $\mathsf{E}$ is a left boundary also in hybrid systems. The operators $\mathsf{F}$, $\mathsf{G}$ and $\mathsf{U}$ can be realised as

$$[\![\mathsf{F}\varphi]\!] =_{df} \mathsf{F}\cdot[\![\varphi]\!]\ , \quad \mathsf{G}\varphi =_{df} \neg\mathsf{F}\neg\varphi\ , \quad [\![\varphi\,\mathsf{U}\,\psi]\!] =_{df} (\mathsf{fin}\,[\![\mathsf{G}\varphi]\!])\cdot[\![\psi]\!]\ .$$

Note that in the first of these equations the $\mathsf{F}$ on the left is the $\mathsf{CTL}^*$ operator "finally", while on the right it is the largest finite element. A straightforward calculation shows that $\mathsf{F}\varphi = \mathsf{true}\,\mathsf{U}\,\varphi$ is still valid.

Of course all other kinds of left and right (perfect) neighbours and boundaries have their own interpretation in PRO and in (the extended) $\mathsf{CTL}^*$, respectively. A detailed discussion of all these interpretations is part of our future work (cf. Section 8).

## 8 Conclusion and Outlook

In this paper we have discussed different algebraic views of Neighbourhood Logic. In particular, we have given an embedding of Neighbourhood Logic into semirings and its connections to other interval logics in the algebraic setting. Furthermore we have extended Neighbourhood Logic by infinite intervals and shown other applications for Neighbourhood Logics beyond intervals.

The abstract algebraic structure of semirings has wide-spread applications in Computer Science, for example to formal languages and graph algorithms, and one of their advantages is a simple first-order based meta-calculus. It is also at the right level of abstraction for deriving an algebraic version of Neighbourhood Logic. In this setting we have shown that at least two axioms of this logic can be dropped and that the neighbourhood modalities can be expressed in a much more general framework, namely modal semirings. There arbitrary sets of intervals can be handled and, due to the first-order setting, we are able to use theorem provers to verify or falsify formulae.

These modalities of Neighbourhood Logic satisfy Galois connections which yield several properties for free. We have also shown that the algebraic setting can be used for characterising further interval operations. In particular, we have given a common framework for Neighbourhood Logic and Allen's thirteen interval relations.

In the second part of this paper we have generalised semiring neighbours to lazy semirings which are suitable for modelling non-strict systems with infinite behaviour. As a result, our *Algebraic Neighbourhood Logic* now uniformly handles intervals of finite or infinite length. With that extended logic we can now express unlimited processes and treat infinite elements. During the development of lazy semiring neighbours it turned out that they are not only useful and necessary for this particular logic, but also in other areas of computer science; in particular, we have sketched connections to temporal logics like $\mathsf{CTL}^*$ and to hybrid systems.

Since the developed algebraic theory of neighbours is based on (lazy and full) semirings, it is obvious that one can use it also in the framework of (lazy) Kleene algebra and (lazy) omega algebra (see e.g. [6,29]) with operators for finite and infinite iteration. Hence Neighbourhood Logic can be extended by iteration.

Some of our further aims in this area are, on the one hand, to find more applications for neighbours and boundaries in both settings (full and lazy semirings) and, on the other hand, using a concrete example of a hybrid system and to algebraically prove safety and liveness properties using neighbourhood concepts, in particular, using at least partial automation.

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## A Original Semantics of NL

Based on the definitions in Section 2.1, the semantics $\theta^{\mathcal{J},\mathcal{V}}$ of a term $\theta$ w.r.t. interpretation $\mathcal{J}$ and value assignment $\mathcal{V}$ can be given inductively as an interval function:

$$\begin{aligned} x^{\mathcal{J},\mathcal{V}}([y,z]) &=_{df} \mathcal{V}(x), \\ v^{\mathcal{J},\mathcal{V}}([y,z]) &=_{df} v^{\mathcal{J}}([y,z]), \\ f^n(\theta_1 \dots \theta_n)^{\mathcal{J},\mathcal{V}}([y,z]) &=_{df} \underline{f}^n(c_1 \dots c_n)\ , \end{aligned}$$

where $c_i = \theta_i^{\mathcal{J},\mathcal{V}}([y,z]), i = 1 \dots n$. Further on, one inductively defines when a formula $\varphi$ *holds* for an interpretation $\mathcal{J}$, a value assignment $\mathcal{V}$ and an interval $[y,z]$, in signs $[y,z] \models_{\mathcal{J},\mathcal{V}} \varphi$, (e.g., [44]):

$$\begin{array}{lll} [y,z] \models_{\mathcal{J},\mathcal{V}} X & \text{iff} & X^{\mathcal{J}}([y,z]) = \mathsf{true} \\ [y,z] \models_{\mathcal{J},\mathcal{V}} G^n(\theta_1,\dots,\theta_n) & \text{iff} & \underline{G}^n(c_1,\dots,c_n) = \mathsf{true}, \\ & & \text{where } c_i = \theta_i^{\mathcal{J},\mathcal{V}}([y,z]),\ i = 1 \dots n \\ [y,z] \models_{\mathcal{J},\mathcal{V}} \neg\varphi & \text{iff} & [y,z] \not\models_{\mathcal{J},\mathcal{V}} \varphi, \\ [y,z] \models_{\mathcal{J},\mathcal{V}} \varphi \vee \psi & \text{iff} & [y,z] \models_{\mathcal{J},\mathcal{V}} \varphi \text{ or } [y,z] \models_{\mathcal{J},\mathcal{V}} \psi, \\ [y,z] \models_{\mathcal{J},\mathcal{V}} (\exists x)\varphi & \text{iff} & [y,z] \models_{\mathcal{J},\mathcal{V}'} \varphi \text{ for some } \mathcal{V}' \text{ that agrees with } \mathcal{V} \\ & & \text{for all global variables } u \neq x \\ [y,z] \models_{\mathcal{J},\mathcal{V}} \Diamond_l\varphi & \text{iff} & \exists\delta \geq 0 : [y-\delta, y] \models_{\mathcal{J},\mathcal{V}} \varphi \\ [y,z] \models_{\mathcal{J},\mathcal{V}} \Diamond_r\varphi & \text{iff} & \exists\delta \geq 0 : [z, z+\delta] \models_{\mathcal{J},\mathcal{V}} \varphi \end{array}$$

## B Proof of Lemma 2.1

(a)
$$\begin{array}{ll} & f(p \to q) \leq f(p) \to f(q) \\ \Leftrightarrow & \quad \{\!\![\ \text{shunting}\ ]\!\!\} \\ & f(p \to q) \cdot f(p) \leq f(q) \\ \Leftrightarrow & \quad \{\!\![\ \text{Axiom C}\ ]\!\!\} \\ & f((p \to q) \cdot p) \leq f(q) \\ \Leftarrow & \quad \{\!\![\ \text{isotony (implied by C)}\ ]\!\!\} \\ & (p \to q) \cdot p \leq q \\ \Leftrightarrow & \quad \{\!\![\ \text{Boolean algebra}\ ]\!\!\} \\ & \mathsf{true} \end{array}$$

(b) Set $f(p) =_{df} 0$ for all $p$.

(c) ($\leq$) First, by $q \leq 1$, isotony and shunting,

$$\mathsf{true} \Leftrightarrow p \cdot q \leq p \Leftrightarrow 1 \leq p \cdot q \to p\ ,$$

i.e., $1 = p \cdot q \to p$. Now,

$$\begin{array}{ll} & 1 \\ = & \quad \{\!\![\ \text{Axiom M}\ ]\!\!\} \\ & f(1) \end{array}$$

$=$ $\quad \{\![$ Axiom M $]\!\}$
$\quad f(p \cdot q \to p)$
$\leq$ $\quad \{\![$ Axiom K $]\!\}$
$\quad f(p \cdot q) \to f(p)$

and shunting yields $f(p \cdot q) \leq f(p)$. Likewise, $f(p \cdot q) \leq f(p)$, showing the claim. Moreover, by this, $f$ is isotone.

$(\geq)$

$\quad f(p) \cdot f(q) \leq f(p \cdot q)$
$\Leftrightarrow$ $\quad \{\![$ shunting $]\!\}$
$\quad f(p) \leq f(q) \to f(p \cdot q)$
$\Leftarrow$ $\quad \{\![$ Axiom K $]\!\}$
$\quad f(p) \leq f(q \to p \cdot q)$
$\Leftrightarrow$ $\quad \{\![$ shunting $]\!\}$
$\quad 1 \leq f(p) \to f(q \to p \cdot q)$
$\Leftarrow$ $\quad \{\![$ Axiom K $]\!\}$
$\quad 1 \leq f(p \to q \to p \cdot q)$
$\Leftarrow$ $\quad \{\![$ Axiom M $]\!\}$
$\quad 1 \leq p \to q \to p \cdot q$
$\Leftrightarrow$ $\quad \{\![$ Boolean algebra $]\!\}$
$\quad \mathsf{true}$

□