OLD AND NEW RESULTS IN REGULARITY THEORY FOR DIAGONAL ELLIPTIC SYSTEMS VIA BLOW UP TECHNIQUES

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Abstract. We consider quasilinear diagonal elliptic systems in bounded domains subject to Dirichlet, Neumann or mixed boundary conditions. The leading elliptic operator is assumed to have only measurable coefficients, and the nonlinearities (Hamiltonians) are allowed to be of quadratic (critical) growth in the gradient variable of the unknown. These systems appear in many applications, in particular in differential geometry and stochastic differential game theory. We impose on the Hamiltonians structural conditions developed between 1972–2002 and also a new condition (*sum coerciveness*) introduced in recent years (in the context of the pay off functional in stochastic game theory). We establish existence, Hölder continuity, Liouville properties, $W^{2,q}$ estimates, etc. for solutions, via a unified approach through the blow-up method. The main novelty of the paper is the introduction of a completely new technique, which in particular leads to smoothness of the solution also for dimensions $d \geq 3$.

1. Introduction

This paper is devoted to a generalization of indirect blow-up methods for proving existence and everywhere regularity for solutions to elliptic diagonal systems for which the right-hand side may be of critical growth with respect to the gradient of the unknown. It is well known that this type of system may admit discontinuous solutions, or even worse that solutions do not exist in general. Therefore, it is important to extend known or to find new structural conditions on the nonlinearity which allow to obtain regular solutions. In this paper, we introduce a new class of such assumptions, which in particular plays an important role in stochastic game theory and which leads to the existence of a regular solution (see however also a simple application to harmonic mappings below). As a byproduct of this newly developed method, we obtain also results about sequential compactness of solutions and Liouville-type theorems.

Outline of the assumptions. Here, we give the precise setting and the crucial conditions imposed on the system, which will enable us to prove the existence of solutions and to investigate some of their qualitative properties. Given a Lipschitz domain $\Omega \subset \mathbb{R}^d$ with $d \geq 2$, coefficients $a\colon \Omega \to \mathbb{R}^{d\times d}$, a nonlinearity $H\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}^N$ and a vector $\kappa \in \mathbb{R}^N$ with $N \in \mathbb{N}$, we want to find a (vector-valued) function $u\colon \Omega \to \mathbb{R}^N$ which solves the mixed boundary-value problem

$$-\sum_{i,j=1}^{d} D_i(a_{ij}(x)D_j u_\nu(x)) + \kappa_\nu u_\nu(x) = H_\nu(x, u(x), \nabla u(x)) \qquad \text{in } \Omega, \tag{1.1}$$

$$u_\nu(x) = 0 \qquad \text{on } \Gamma_D, \tag{1.2}$$

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$$\sum_{i,j=1}^{d} a_{ij}(x) D_j u_\nu(x) n_i(x) = 0 \qquad \text{on } \Gamma_N \tag{1.3}$$

for all $\nu = 1, \ldots, N$. Here, we abbreviated $D_k := \frac{\partial}{\partial x_k}$ and we assume that Γ_D and Γ_N are disjoint, relatively open parts of the boundary $\partial\Omega$ with $\overline{\Gamma}_D \cup \overline{\Gamma}_N = \partial\Omega$. In addition, we denoted by $n(x)$ the unit outward normal vector to $\partial\Omega$ at the point x. Let us emphasize already at this stage that we are in the setting of diagonal systems, meaning that the coefficients a have values in $\mathbb{R}^{d\times d}$ (and not in $\mathbb{R}^{(d\times N)\times(d\times N)}$), which allows interactions between the different component functions of u to happen only via the nonlinearity H.

To give a good meaning to the above problem, we need to prescribe some minimal assumptions on the data. Thus, in what follows we assume that δ is a positive constant and that K and K^* are nonnegative constants, and we formulate all the growth restrictions in terms of these constants. First, we assume that H is a Carathéodory mapping which fulfills for all $(u, z) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and almost all $x \in \Omega$

$$|H(x, u, z)| \leq K^* + K|z|^2. \tag{1.4}$$

For the matrix $(a_{ij})_{i,j=1}^d$ we require

$$\|a\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})} \leq K, \tag{1.5}$$

$$\sum_{i,j=1}^{d} a_{ij}(x) z_i z_j \geq \delta |z|^2 \tag{1.6}$$

for all $z \in \mathbb{R}^d$ and almost all $x \in \Omega$. Finally, for the vector $\kappa \in \mathbb{R}^N$ we assume

$$\kappa_\nu \geq \delta \qquad \text{for all } \nu = 1, \ldots, N. \tag{1.7}$$

These assumptions allow to introduce the notion of a weak solution to the mixed-boundary value problem (1.1)–(1.3) in a standard way, and our task is to establish its existence and, if possible, also its higher regularity. However, it is well known that a critical growth condition on H as in (1.4) ensures neither regularity nor existence of a weak solution. Therefore, further structured growth conditions on H need to be introduced which will allow to prove the desired results. We shall assume on the one hand estimates from above for each H_ν and on the other hand estimates from below for their sum. More precisely, we assume that there exists $q \in \mathbb{R}^N$ with $q_\nu \geq \delta > 0$ for all $\nu = 1, \ldots, N$ such that

$$H_\nu(x, u, z) \leq K^* + K|z_\nu||z|, \tag{1.8}$$

$$\sum_{\nu=1}^{N} q_\nu H_\nu(x, u, z) \geq -K^* - K\Big|\sum_{\nu=1}^{N} q_\nu z_\nu\Big|^2, \tag{1.9}$$

for all $(u, z) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and almost all $x \in \Omega$.

Applications in stochastic game theory. We would like to emphasize that the conditions (1.8)–(1.9) play an important role in the stochastic game theory, and the main novelty of the paper is that it covers such general cases. Indeed the conditions (1.8)–(1.9) naturally occur in game theory, see for example [5, 6, 7] for details, when one considers H_ν being independent of u and defined via point-wise Nash-points $v^* := (v_1^*, \ldots, v_N^*)$ of Lagrangians given as

$$L_\nu(x, v, z) := f_\nu(x, v) + z_\nu \cdot g(x, v),$$

where the so-called pay-off functions $f_\nu\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ may grow quadratically in v and where $g\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^d$, which represents the dynamics of the game, grows linearly in v with $|g(x, v)| \leq$

$K(1+|v|)$. Now, if v^* is a Nash point, one has (see [5, 6] for details)

$$\begin{aligned} H_\nu(x,z) := L_\nu(x,v^*,z) &\le L_\nu(x,v_1^*,\dots,v_{\nu-1}^*,0,v_{\nu+1}^*\dots,v_N^*,z) \\ &\le f_\nu(x,v_1^*,\dots,v_{\nu-1}^*,0,v_{\nu+1}^*,\dots,v_N^*) + K|z_\nu|(1+|v^*|). \end{aligned}$$

Under the assumption $f_\nu(x,v) \le K(1+|v_\nu||v|)$, which is for example valid for $f_\nu(v) = v_\nu \cdot Q_\nu v + K$, and under the regular solvability of the minimization problem for the Nash equilibrium, which yields $|v^*| \le K(1+|z|)$, one obtains (1.8). Concerning (1.9) in theoretical game applications, for the model case $q := (1,\dots,1)$, we first observe

$$\sum_{\nu=1}^N H_\nu(x,z) = \sum_{\nu=1}^N f_\nu(x,v^*) + \left(\sum_{\nu=1}^N z_\nu\right) \cdot g(x,v^*).$$

It is reasonable to assume the sum coerciveness for the functions f_ν, i.e.,

$$\sum_{\nu=1}^N f_\nu(x,v) \ge 2K_0|v|^2 - K.$$

Then, we conclude via Young's inequality

$$\sum_{\nu=1}^N H_\nu(x,z) \ge K_0|v^*|^2 - 2K - (K^2K_0^{-1}+K)\left|\sum_{\nu=1}^N z_\nu\right|^2,$$

which directly implies (1.9). A further generalization with applications to game theory is to require the sum coerciveness for groups I_γ of indices fulfilling $\cup_\gamma I_\gamma = \{1,\dots,N\}$ with a possible index dependence on g_γ, such that $g_\nu = g_\mu$ for $\nu,\mu \in I_\gamma$, and

$$\sum_{\nu\in I_\gamma} f_\nu(x,v) \ge 2K_0|v|^2 - K.$$

Applications for alternative proofs of known results. Another novelty of the paper is the applicability of the tools developed here for already solved problems, resulting in more elegant proofs. As one class of such problems one can consider the diagonal assumption

$$|H_\nu(x,u,z)| \le K^* + K|z_\nu||z| + K\sum_{\mu=1}^{\nu} |z_\mu|^2. \tag{1.10}$$

We would like to remark that, while in (1.8)–(1.9) we have considered only one sided bounds, in (1.10) a more restrictive assumption on the modulus of each H_ν is prescribed. However, the last term in (1.10) is not present in the corresponding one sided bound (1.8), and from this point of view the assumption (1.10) seems to be weaker, but does not cover the sum coerciveness case. The theory for this classical case (1.10) started in [2] and "ended" with [4]. The method introduced in this paper, besides some algebraic inequalities, is of a very different nature and the proof of the key uniform smallness of the Dirichlet integral (and consequently the VMO property which is the starting point for the further analysis) is not based on weighted estimates and the hole-filling technique but rather on the blow-up argument. It may be also of interest that the structure assumptions (1.8)–(1.9) or (1.10) naturally appear in other applications. For example one can use the theory developed in this paper for estimates for solutions of harmonic mappings in a neighborhood, where the components are positive (possibly after a transformation). The equation for harmonic mappings has the structure

$$-\triangle u_\nu = u_\nu|\nabla u|^2, \qquad \text{for } \nu = 1,\dots,N. \tag{1.11}$$

Then by a simple manipulation (and assuming that each $u_\nu > 0$ is positive), one can rewrite (1.11) as

$$-\Delta \ln u_\nu = |\nabla u|^2 + |\nabla \ln u_\nu|^2, \qquad \text{for } \nu = 1,\dots,N.$$

Consequently, introducing new variables

$$\begin{aligned} w_\nu &:= \ln u_\nu - \ln u_{\nu+1} \quad \text{for } \nu = 1, \dots, N-1, \\ w_N &:= \ln u_N, \end{aligned}$$

we compute

$$\begin{aligned} -\Delta w_\nu &= \nabla w_\nu \cdot \nabla \Big(w_\nu + 2 \sum_{\mu=\nu+1}^{N} w_\mu \Big) \qquad\qquad \text{for } \nu = 1, \dots, N-1, \\ -\Delta w_N &= \sum_{\mu=1}^{N} e^{2 \sum_{\nu=\mu}^{N} w_\nu} \Big| \nabla \Big(\sum_{\nu=\mu}^{N} w_\nu \Big) \Big|^2 + |\nabla w_N|^2. \end{aligned}$$

It is evident that the right-hand side satisfies the structural assumption (1.10) provided that $u_\nu \in L^\infty$ is positive for each $\nu = 1, \dots, N$. The existence of a $\mathcal{C}^{0,\alpha}$ regular (and consequently $\mathcal{C}^\infty$ regular) solution was obtained for example in [4].

Notions of weak solutions. We consider a slightly generalized notion of weak solutions to the problem (1.1)–(1.3) that seems to be suitable for systems with right-hand side H satisfying the condition (1.4) of critical growth and (1.8)–(1.9). Note that such a definition first appeared in [7]. However, for the sake of the clarity and to point out the slight differences in various definitions, we first recall the standard concept of the weak solution.

Definition 1.1 (Boundary value problem)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Assume that H satisfies* (1.4) *and that a satisfies* (1.5)*. We say that $u\colon \Omega \to \mathbb{R}^N$ is a* weak solution *to* (1.1)*–*(1.3) *if*

$$u \in L^\infty(\Omega; \mathbb{R}^N) \cap W^{1,2}_{\Gamma_D}(\Omega; \mathbb{R}^N),$$

and if the following identity holds

$$\int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij} D_j u_\nu D_i \varphi + \kappa_\nu u_\nu \varphi \Big) \, dx = \int_\Omega H_\nu(u, \nabla u) \varphi \, dx$$

for all $\nu = 1, \dots, N$ and all $\varphi \in L^\infty(\Omega) \cap W^{1,2}_{\Gamma_D}(\Omega)$.

Here, we have used the standard notation for the Lebesgue and the Sobolev spaces, with

$$W^{1,2}_{\Gamma_D}(\Omega) := \{ u \in W^{1,2}(\Omega) \colon u = 0 \text{ on } \Gamma_D \}.$$

denoting the subspace of functions vanishing on Γ_D (in the sense of traces). Obviously, in the case $\Gamma_D = \partial\Omega$ we have $W^{1,2}_{\Gamma_D} = W^{1,2}_0$ and the problem is reduced to the Dirichlet problem, while in the case $\Gamma_D = \emptyset$ we have $W^{1,2}_{\Gamma_D} = W^{1,2}$ and we consider the Neumann problem. This definition is standard and suitable in certain cases, for instance if one works under the additional assumption (1.10). However, in the sum coerciveness case, i.e., under the assumptions (1.8)–(1.9), it is not known how to obtain $W^{1,2}$ estimates up to the part of the boundary Γ_D, where we prescribe the Dirichlet data. Therefore, following [7, 1], we introduce a generalized concept of weak solutions that is on the one hand suitable for handling the assumptions (1.8)–(1.9) and on the other hand compatible with the standard Definition 1.1.

Definition 1.2 (Boundary value problem generalized)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Assume that H satisfies* (1.4) *and that a satisfies* (1.5)*. We say that $u\colon \Omega \to \mathbb{R}^N$ is a* generalized weak solution *to* (1.1)–(1.3) *if*

$$u \in L^\infty(\Omega;\mathbb{R}^N), \tag{1.12}$$

$$u \in W^{1,2}(\Omega_0;\mathbb{R}^N) \qquad \textit{for all open subsets } \Omega_0 \subset \Omega \textit{ with } \overline{\Gamma}_D \cap \overline{\Omega}_0 = \emptyset, \tag{1.13}$$

$$(u_\nu - \varepsilon)_+ \in W^{1,2}_{\Gamma_D}(\Omega) \qquad \textit{for all } \nu = 1,\dots,N \textit{ and } \varepsilon > 0, \tag{1.14}$$

$$\left(\sum_{\nu=1}^N q_\nu u_\nu\right)_- \in W^{1,2}_{\Gamma_D}(\Omega) \tag{1.15}$$

and if the following identity holds

$$\int_\Omega \left(\sum_{i,j=1}^d a_{ij} D_j u_\nu D_i \varphi + \kappa_\nu u_\nu \varphi\right) dx = \int_\Omega H_\nu(u,\nabla u)\varphi\, dx \tag{1.16}$$

for all $\nu = 1,\dots,N$ *and all* $\varphi \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ *vanishing in a neighborhood of* Γ_D*.*

Let us make some further comments on the relation between the Definitions 1.1 and 1.2. First, we note that in the case of only Neumann data ($\Gamma_D = \emptyset$), Definitions 1.1 and 1.2 are equivalent. Moreover, it is evident that if u is a solution in the sense of Definition 1.1, then it is a generalized weak solution in sense of Definition 1.2. On the contrary, if u is a generalized weak solution that in addition fulfills $u \in W^{1,2}(\Omega;\mathbb{R}^N)$, then (1.14) implies $u_\nu \le 0$ on Γ_D for all $\nu = 1,\dots,N$ and (1.15) implies $\sum_{\nu=1}^N q_\nu u_\nu \ge 0$ on Γ_D. Therefore, with $q_\nu > 0$ for all $\nu = 1,\dots,N$, we necessarily obtain $u \equiv 0$ on Γ_D, and consequently u is also a weak solution in sense of Definition 1.1. It is an important (and in our opinion solvable) problem to establish such $W^{1,2}$ estimates under the assumptions of Definition 1.2. One possibility could be to extend our interior blow-up argument up to the Dirichlet part of the boundary, which would lead in a first step to global $\mathcal{C}^{0,\alpha}$ estimates and in a second step to the desired $W^{1,2}$ estimates.

Statement of the main results. The first result, we present here, is the following statement on existence and (interior) regularity of generalized weak solutions.

Theorem 1.3 (Existence & Regularity)**.** *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz set. Assume that* H *satisfies* (1.4)*,* (1.8) *and* (1.9)*, the matrix* a *fulfills* (1.5)–(1.6) *and* κ *satisfies* (1.7)*. Then there exists a generalized weak solution to the mixed boundary value problem* (1.1)–(1.3)*. Moreover, for all open subsets* $\Omega_0 \subset \Omega$ *with* $\overline{\Gamma}_D \cap \overline{\Omega}_0 = \emptyset$ *there holds* $u \in \mathcal{C}^{0,\alpha}(\overline{\Omega}_0;\mathbb{R}^N)$ *with* α *depending only on* K*,* K^* *and* δ*. In addition, if* $a_{ij} \in \mathcal{C}^{0,1}(\overline{\Omega})$ *and* $\Omega \in \mathcal{C}^{1,1}$*, then* $u \in W^{2,q}(\Omega_0;\mathbb{R}^N)$ *for all* $q \in [1,\infty)$ *and all open subsets* $\Omega_0 \subset \Omega$ *with* $\overline{\Gamma}_D \cap \overline{\Omega}_0 = \emptyset$*.*

This theorem generalizes the results of [4, 5, 6, 7, 1] as far as the Hölder continuity is concerned in dimension $d \ge 3$. Furthermore, the structure of the assumptions on the right-hand side is slightly more general than those in the above mentioned papers. Moreover, the proof presented here seems to be much more straightforward and less technical than the one of the similar (but less general) result given in [7]. Indeed, in the present paper, the key step, i.e., the proof of the uniform smallness of the Dirichlet integral, is achieved via an indirect approach, which is based on the strategy of proof of the following new result of the paper.

Theorem 1.4 (Liouville in $\mathbb{R}^d$)**.** *Assume that* H *satisfies* (1.4)*,* (1.8) *and* (1.9) *with* $K^* = 0$ *and that* a *satisfies* (1.5)–(1.6)*. If a function* $u \in L^\infty(\mathbb{R}^d;\mathbb{R}^N) \cap W^{1,2}_{loc}(\mathbb{R}^d;\mathbb{R}^N)$ *solves the system* (1.1) *with* $\kappa = 0$ *in the sense of distribution, then* u *is identically constant.*

Theorem 1.4 provides an efficient tool for studying further regularity properties of solutions to system (1.1). In fact, it is more in the spirit of the results obtained in [13, 17], where, for certain elliptic systems, the equivalence of the validity of the Liouville theorem and the higher regularity of their solutions was stated. For sake of completeness, we also formulate the Liouville-type theorem on Lipschitz cones.

Theorem 1.5 (Liouville in a cone)**.** *Let $\Omega \subsetneq \mathbb{R}^d$ be a Lipschitz cone, i.e., there holds $kx \in \Omega$ for all $k > 0$ and $x \in \Omega$ and $\Omega \cap B_1(0)$ is a Lipschitz domain. Assume that H satisfies* (1.4), (1.8) *and* (1.9) *with $K^* = 0$ and that a satisfies* (1.5)–(1.6)*. If $u \in L^\infty(\Omega;\mathbb{R}^N) \cap W^{1,2}_{loc}(\Omega;\mathbb{R}^N)$ is a function which solves the system* (1.1) *with $\kappa = 0$ in the sense of distribution and which, extended by zero outside Ω, satisfies $(u_\nu - \varepsilon)_+ \in W^{1,2}_{loc}(\mathbb{R}^d)$ for all $\nu = 1,\dots,N$ and $(\sum_{\nu=1}^N q_\nu u_\nu)_- \in W^{1,2}_{loc}(\mathbb{R}^d)$, then u is identically zero.*

It should be emphasized that this theorem works even for generalized weak solutions in sense of Definition 1.2 and is therefore stronger than the usual Liouville theorem (which automatically holds since any weak solution is also a generalized weak solution). Finally, we formulate two key results that will be used in the proof of Theorem 1.3. The first one deals with the sequential stability of generalized weak solutions.

Theorem 1.6 (Compactness)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $\{H^n, a^n, \kappa^n, q^n\}_{n=1}^\infty$ be a sequence of (K^*, K, δ)-*admissible representations *of systems of type* (1.1)*, i.e. the nonlinearities H^n satisfy* (1.4), (1.8)–(1.9)*, the matrices a^n satisfy* (1.5)–(1.6)*, with uniform constants K^*, K, δ and uniform condition $q^n_\nu \geq \delta$ for all $\nu \in \{1,\dots,N\}$, and the moduli of the vectors κ^n are uniformly bounded by K. In addition, we assume*

$$\kappa^n \to \kappa \qquad \text{in } \mathbb{R}^N, \tag{1.17}$$

$$q^n \to q \qquad \text{in } \mathbb{R}^N, \tag{1.18}$$

$$a^n \to a \qquad \text{in } L^1(\Omega;\mathbb{R}^{d\times d}), \tag{1.19}$$

$$H^n(x,\cdot) \to H(x,\cdot) \qquad \text{in } \mathcal{C}(S) \text{ for all compact sets } S \subset \mathbb{R}^d \times \mathbb{R}^{d\times N},\ \text{for a.e. } x \in \Omega, \tag{1.20}$$

$$H^n(\cdot,u,z) \to H(\cdot,u,z) \qquad \text{a.e. in } \Omega,\ \text{for all } (u,z) \in \mathbb{R}^d \times \mathbb{R}^{d\times N}. \tag{1.21}$$

Then for any sequence $\{u^n\}_{n=1}^\infty$ of generalized weak solutions corresponding to $\{H^n, a^n, \kappa^n\}$ that in addition are uniformly bounded with

$$\|u^n\|_\infty \leq M \tag{1.22}$$

there exists a subsequence (not relabeled) and a function $u \in L^\infty(\Omega;\mathbb{R}^N)$ such that

$$u^n \to u \qquad \text{strongly in } W^{1,2}(\Omega_0;\mathbb{R}^N) \qquad \text{for all open sets } \Omega_0 \subset \Omega \text{ with } \overline{\Gamma}_D \cap \overline{\Omega}_0 = \emptyset, \tag{1.23}$$

where u is a generalized weak solution to the system (1.1) *corresponding to $\{H, a, \kappa\}$.*

This compactness theorem further extends the result of [1], where less general structure of the right-hand sides H is treated. This theorem combined with the Liouville theorem can be used in an indirect approach to show uniform smallness of the Dirichlet integral. In fact, we here show a stronger result that provides such a uniform estimate for generalized solutions.

Theorem 1.7 (Uniform estimates)**.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and K^*, K and δ be given. Then, for any open set $\Omega_0 \subset \Omega$ with $\overline{\Gamma}_D \cap \overline{\Omega}_0 = \emptyset$ and any $\varepsilon > 0$, there exists $R_0 > 0$ such that for every $R \in (0, R_0)$, every $x_0 \in \Omega_0$, every (K^*, K, δ)-admissible representation $\{H, a, \kappa, q\}$ of a system of type* (1.1) *and every associated generalized weak solution u with L^∞-bound* (1.22)*, there holds*

$$\int_{B_R(x_0)\cap\Omega} \frac{|\nabla u(x)|^2}{R^{d-2}}\,dx \leq \varepsilon. \tag{1.24}$$

Moreover, there exists $\alpha > 0$ depending only on K^*, K and δ such that if u in addition belongs to $\mathcal{C}(\overline{\Omega}_0;\mathbb{R}^N)$, then it satisfies

$$\|u\|_{\mathcal{C}^{0,\alpha}(\overline{\Omega}_0)} \leq C, \tag{1.25}$$

where the constant C depends only on K, K^* and δ and is in particular independent of the modulus of continuity of u.

The part (1.24) of this theorem is the core of the paper. Not only it is the essential step in proving the existence of a continuous solution but it also provides uniform estimates for generalized solutions, which do not rely on the regularity of the matrix a (but only on K, K^* and δ). It is based on the blow-up technique, which is an efficient tool for obtaining a regularity criterion for non-linear partial differential equations via an indirect argument (negating the regularity criterion) and a scaling procedure. Such a tool is used for various kinds of elliptic problems (non-diagonal elliptic systems, Navier–Stokes equations, polyconvex variational methods, etc., see [9, 11, 12, 10, 19, 14, 16]). For systems treated here, we work with the criterion (1.24) that is supposed to hold uniformly with respect to x_0 and R, which finally leads to (1.25). The proof of (1.24) via blow-up technique is based on the contradiction argument and proper re-scaling of the equations. In case the matrix a is continuous (or has vanishing mean oscillations), one can directly combine the compactness result established in Theorem 1.6 with the Liouville theorem 1.4 to show (1.24). However, in our case, the situation is more complicated by the fact, that the matrix a has only measurable coefficients and that one has to work out the blow-up method for *sequences* of approximations, since irregular solutions of the basic system may occur.

In preceding papers in related situations, (1.24) has been frequently established by the hole-filling technique, see e.g. [1, 2, 4]. However, in the case of sum coercive Hamiltonians, those proofs (see [7]) for $d \geq 3$ are rather complicated and the structural assumptions on the Hamiltonians are much less general.

Finally, we shortly describe the structure of the paper. In Section 2, we prove several basic algebraic inequalities. Although their "simplicity" (they follow from the assumptions (1.8)–(1.9), see also [4]), they play the crucial role in all results of the paper, in particular for proving the Liouville and the compactness theorem. We then provide in Section 3 the proof of the standard version of the Liouville theorem for sub- or super-solution to general elliptic equations (Theorems 1.4 and 1.5). In Section 4, we prove Theorem 1.6 on the sequential compactness of the mixed boundary value problem (1.1)–(1.3). Section 5 is devoted to the proof of Theorem 1.7, and the main Theorem 1.3 is finally proved in Section 6.

Notation: In our paper we essentially use standard notations. In particular, given a set $S \subset \mathbb{R}^d$, we write ∂S for its topological boundary and $\overline{S}$ for its closure. Moreover, we denote by $B_R(x_0) := \{x \in \mathbb{R}^d \colon |x - x_0| < R\}$ the open ball with radius $R > 0$ and center x_0 in $\mathbb{R}^d$, and for $x_0 = 0$ we abbreviate $B_R := B_R(0)$. Similarly, we denote by $A_R(x_0) := \{x \in \mathbb{R}^d \colon R \leq |x - x_0| \leq 2R\}$ the annulus with center x_0 and radii R and $2R$, and we set $A_R := A_R(0)$. Finally, if $S \subset \mathbb{R}^d$ is a measurable set of positive, finite Lebesgue measure $|S| := \mathcal{L}^n(S)$, then we abbreviate by

$$(f)_S := \frac{1}{|S|} \int_S f(x)\, dx$$

the mean value of an integrable f over S, and by $f_+ := \max(0, f)$ and $f_- := \min(0, f)$ we indicate the positive and the negative part of f, respectively.

2. Auxiliary definitions, identities, inequalities and lemmas

This section is devoted to some preliminary and auxiliary results. First, we introduce auxiliary functions that will be used later for proving a priori estimates and the Liouville theorem. To this end, for $\nu = 1, \dots, N$, we assume that $\gamma_\nu \in \mathcal{C}^2(\mathbb{R})$ are given nonnegative, strictly convex nondecreasing functions and we define functions $\varphi_\nu \colon \mathbb{R}^N \to \mathbb{R}$ iteratively by setting

$$\begin{aligned} \varphi_N(u) &:= e^{\gamma_N(u_N)}, \\ \varphi_\nu(u) &:= e^{\gamma_\nu(u_\nu) + \varphi_{\nu+1}(u)} \qquad \text{for } \nu = 1, \dots, N-1, \end{aligned} \tag{2.1}$$

for all $u = (u_1, \ldots, u_N) \in \mathbb{R}^N$, and we use the abbreviations $D_{u_\mu}\varphi_\nu(u) := \partial\varphi_\nu(u)/\partial u_\mu$ for their partial derivatives for which we easily observe

$$D_{u_\mu}\varphi_\nu(u) = \begin{cases} 0 & \text{if } \mu < \nu, \\ \gamma_\mu'(u_\mu)\prod_{\alpha=\nu}^{\mu}\varphi_\alpha(u) & \text{if } \mu \geq \nu. \end{cases} \tag{2.2}$$

We can now formulate the result, which will help us to find a scalar quantity being a subsolution to an elliptic equation.

Lemma 2.1. *Assume that H satisfies* (1.4) *and* (1.8) *and that a satisfies* (1.5)–(1.6)*. Then for every $u \in L^\infty(\Omega;\mathbb{R}^N) \cap W^{1,2}(\Omega;\mathbb{R}^N)$ the following inequality holds almost everywhere in Ω*

$$\begin{aligned} &-K^*\sum_{\nu=1}^{N} D_{u_\nu}\varphi_1(u) + \delta\sum_{\nu=1}^{N} Z_\nu(u)\gamma_\nu''(u_\nu)|\nabla u_\nu|^2\prod_{\mu=1}^{\nu}\varphi_\mu(u) \\ &\quad\leq \sum_{\nu=1}^{N}\sum_{i,j=1}^{d} a_{ij}D_j u_\nu D_i D_{u_\nu}\varphi_1(u) - \sum_{\nu=1}^{N} D_{u_\nu}\varphi_1(u) H_\nu(u,\nabla u), \end{aligned} \tag{2.3}$$

where we have defined (with $\varphi_{N+1} := 1$)

$$Z_\nu(u) := 1 - \frac{K^2 N\prod_{\mu=\nu+1}^{N+1}\varphi_\mu(u)}{\delta^2\gamma_\nu''(u_\nu)} \qquad \text{for } \nu = 1, \ldots, N. \tag{2.4}$$

Proof. We start with the simple observations

$$\nabla\varphi_\nu(u) = \sum_{\mu=\nu}^{N}\nabla\gamma_\mu(u_\mu)\prod_{\alpha=\nu}^{\mu}\varphi_\alpha(u), \tag{2.5}$$

$$\nabla\gamma_\nu(u_\nu) = \nabla\ln(\varphi_\nu(u)) - \nabla\varphi_{\nu+1}(u), \tag{2.6}$$

which follow from definition (2.1) and formula (2.2) for all $\nu = 1, \ldots, N$. Moreover, in view of (2.2), the expression $D_{u_\nu}\varphi_1(u)$ appearing in (2.3) can be rewritten as $\gamma_\nu'(u_\nu)\prod_{\mu=1}^{\nu}\varphi_\mu(u)$, which is nonnegative almost everywhere in Ω since each γ_ν is nondecreasing and each φ_ν is nonnegative. Next, in order to verify the inequality (2.3), we find from this representation in a first step

$$\begin{aligned} &\sum_{\nu=1}^{N}\sum_{i,j=1}^{d} a_{ij}D_j u_\nu \cdot D_i D_{u_\nu}\varphi_1(u) \\ &\quad= \sum_{\nu=1}^{N}\sum_{i,j=1}^{d} a_{ij}\Big(\gamma_\nu''(u_\nu)D_j u_\nu D_i u_\nu\prod_{\mu=1}^{\nu}\varphi_\mu(u) + D_j(\gamma_\nu(u_\nu))D_i\prod_{\mu=1}^{\nu}\varphi_\mu(u)\Big). \end{aligned} \tag{2.7}$$

Employing $\varphi_\alpha(u)D_i(\ln\varphi_\alpha(u)) = D_i\varphi_\alpha(u)$ (for $i \in \{1,\dots,d\}$, $\alpha \in \{1,\dots,N\}$) and keeping in mind (2.5), we rewrite the last term in the previous identity as

$$
\begin{aligned}
&\sum_{\nu=1}^{N}\sum_{i,j=1}^{d} a_{ij}D_j(\gamma_\nu(u_\nu))D_i\Big(\prod_{\mu=1}^{\nu}\varphi_\mu(u)\Big)\\
&=\sum_{\nu=1}^{N}\sum_{\alpha=1}^{\nu}\sum_{i,j=1}^{d} a_{ij}D_j(\gamma_\nu(u_\nu))D_i(\ln\varphi_\alpha(u))\prod_{\mu=1}^{\nu}\varphi_\mu(u)\\
&=\sum_{\alpha=1}^{N}\sum_{\nu=\alpha}^{N}\sum_{i,j=1}^{d} a_{ij}D_j(\gamma_\nu(u_\nu))D_i(\ln\varphi_\alpha(u))\prod_{\mu=1}^{\nu}\varphi_\mu(u)\\
&=\sum_{\alpha=1}^{N}\sum_{i,j=1}^{d} a_{ij}\Big(\sum_{\nu=\alpha}^{N}D_j(\gamma_\nu(u_\nu))\prod_{\mu=\alpha}^{\nu}\varphi_\mu(u)\Big)D_i(\ln\varphi_\alpha(u))\prod_{\mu=1}^{\alpha-1}\varphi_\mu(u)\\
&=\sum_{\alpha=1}^{N}\sum_{i,j=1}^{d} a_{ij}D_j\varphi_\alpha(u)D_i(\ln\varphi_\alpha(u))\prod_{\mu=1}^{\alpha-1}\varphi_\mu(u)\\
&=\sum_{\nu=1}^{N}\sum_{i,j=1}^{d} a_{ij}D_i(\ln\varphi_\nu(u))D_j(\ln\varphi_\nu(u))\prod_{\mu=1}^{\nu}\varphi_\mu(u).
\end{aligned}
\tag{2.8}
$$

Thus, substituting (2.8) into (2.7) and employing (1.6) to bound the other term from below, we get the following estimate for the first term on the right-hand side of (2.3)

$$
\sum_{\nu=1}^{N}\sum_{i,j=1}^{d} a_{ij}D_ju_\nu\cdot D_iD_{u_\nu}\varphi_1(u) \geq \delta\sum_{\nu=1}^{N}(|\nabla\ln\varphi_\nu(u)|^2+\gamma_\nu''(u_\nu)|\nabla u_\nu|^2)\prod_{\mu=1}^{\nu}\varphi_\mu(u).
\tag{2.9}
$$

Next, in order to estimate also the second term on the right-hand side of (2.3), we take advantage of (1.8) and the nonnegativity of $D_{u_\nu}\varphi_1(u)$ to deduce

$$
\sum_{\nu=1}^{N} D_{u_\nu}\varphi_1(u)H_\nu(u,\nabla u) \leq K^*\sum_{\nu=1}^{N} D_{u_\nu}\varphi_1(u) + K\sum_{\nu=1}^{N} D_{u_\nu}\varphi_1(u)|\nabla u_\nu||\nabla u|.
\tag{2.10}
$$

Finally, to estimate the last term on the right-hand side of (2.10) we employ (2.5) and (2.6), and we get

$$\begin{aligned}
&\sum_{\nu=1}^{N} D_{u_\nu}\varphi_1(u)|\nabla u_\nu||\nabla u| = \sum_{\nu=1}^{N} |\nabla u||\nabla\gamma_\nu(u_\nu)| \prod_{\mu=1}^{\nu}\varphi_\mu(u)\\
&= |\nabla u||\nabla\gamma_N(u_N)| \prod_{\mu=1}^{N}\varphi_\mu(u) + \sum_{\nu=1}^{N-1} |\nabla u||\nabla\gamma_\nu(u_\nu)| \prod_{\mu=1}^{\nu}\varphi_\mu(u)\\
&= |\nabla u||\nabla(\ln\varphi_N(u))| \prod_{\mu=1}^{N}\varphi_\mu(u) + \sum_{\nu=1}^{N-1} |\nabla u||\nabla(\ln(\varphi_\nu(u)) - \varphi_{\nu+1}(u))| \prod_{\mu=1}^{\nu}\varphi_\mu(u)\\
&\le \sum_{\nu=1}^{N} |\nabla u||\nabla\ln(\varphi_\nu(u))| \prod_{\mu=1}^{\nu}\varphi_\mu(u) + \sum_{\nu=1}^{N-1} |\nabla u||\nabla\varphi_{\nu+1}(u)| \prod_{\mu=1}^{\nu}\varphi_\mu(u)\\
&= \sum_{\nu=1}^{N} |\nabla u||\nabla\ln(\varphi_\nu(u))| \prod_{\mu=1}^{\nu}\varphi_\mu(u) + \sum_{\nu=2}^{N} |\nabla u||\nabla\ln(\varphi_\nu(u))| \prod_{\mu=1}^{\nu}\varphi_\mu(u)\\
&\le 2\sum_{\nu=1}^{N} |\nabla u||\nabla\ln(\varphi_\nu(u))| \prod_{\mu=1}^{\nu}\varphi_\mu(u)\\
&\le \delta K^{-1}\sum_{\nu=1}^{N} |\nabla\ln(\varphi_\nu(u))|^2 \prod_{\mu=1}^{\nu}\varphi_\mu(u) + K\delta^{-1}\sum_{\nu=1}^{N}|\nabla u|^2\prod_{\mu=1}^{\nu}\varphi_\mu(u)\\
&\le \delta K^{-1}\sum_{\nu=1}^{N} |\nabla\ln(\varphi_\nu(u))|^2 \prod_{\mu=1}^{\nu}\varphi_\mu(u) + K\delta^{-1}N|\nabla u|^2\prod_{\mu=1}^{N}\varphi_\mu(u).
\end{aligned}$$

Consequently, using this in (2.10) and combining it with (2.9) we arrive at the assertion (2.3). □

Lemma 2.2. *Assume that H satisfies* (1.4) *and* (1.9) *and that a satisfies* (1.5)–(1.6)*. Let us define for arbitrary $u \in L^\infty(\Omega;\mathbb{R}^N) \cap W^{1,2}(\Omega;\mathbb{R}^N)$ and $\lambda > 0$*

$$w_\lambda(x) := e^{-\lambda\sum_{\nu=1}^{N} q_\nu u_\nu(x)}. \tag{2.11}$$

Then the following inequality holds almost everywhere in Ω

$$\begin{aligned}
-K^* w_\lambda + &\frac{\delta|\nabla w_\lambda|^2}{\lambda w_\lambda}\left(1 - \frac{K}{\delta\lambda}\right)\\
&\le -\sum_{i,j=1}^{d} a_{ij} D_j\left(\sum_{\nu=1}^{N} q_\nu u_\nu\right) D_i w_\lambda + w_\lambda \sum_{\nu=1}^{N} q_\nu H_\nu(u,\nabla u).
\end{aligned}\tag{2.12}$$

Proof. First, using (1.6) we obtain

$$-\sum_{i,j=1}^{d} a_{ij} D_j\left(\sum_{\nu=1}^{N} q_\nu u_\nu\right) D_i w_\lambda = \lambda^{-1} w_\lambda^{-1} \sum_{i,j=1}^{d} a_{ij} D_j w_\lambda D_i w_\lambda \ge \frac{\delta|\nabla w_\lambda|^2}{\lambda w_\lambda}.$$

Similarly, using (1.9) and the definition of w_λ, we get

$$\begin{aligned}
w_\lambda \sum_{\nu=1}^{N} q_\nu H_\nu(u,\nabla u) &\ge -K^* w_\lambda - K w_\lambda \left|\nabla \sum_{\nu=1}^{N} q_\nu u_\nu\right|^2\\
&= -K^* w_\lambda - K\lambda^{-2} w_\lambda^{-1}|\nabla w_\lambda|^2.
\end{aligned}$$

Combining these inequalities, we deduce (2.12). □

Furthermore, we need a version of Liouville theorem for subsolutions of a general linear elliptic equation on the whole $\mathbb{R}^d$.

Lemma 2.3 (Liouville for subsolutions on $\mathbb{R}^d$)**.** *Assume that $v \in L^\infty(\mathbb{R}^d) \cap W^{1,2}_{loc}(\mathbb{R}^d)$ solves*

$$\int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}(x) D_j v(x) D_i \psi(x)\, dx \le 0 \tag{2.13}$$

for all nonnegative $\psi \in \mathcal{D}(\mathbb{R}^d)$. Then for almost all $z \in \mathbb{R}^d$ there holds

$$v(z) \le \liminf_{R\to\infty} (v)_{A_R}. \tag{2.14}$$

In addition, we have

$$\int_{\mathbb{R}^d} \frac{|\nabla v(x)|^2}{|x|^{d-2}}\, dx \le C\|v\|_\infty^2 \tag{2.15}$$

for a constant C which depends only on d, δ and K.

The second Liouville-type theorem deals with subsolutions in a Lipschitz cone.

Lemma 2.4 (Liouville for subsolutions on a cone)**.** *Let $\Omega \subsetneq \mathbb{R}^d$ be a Lipschitz cone. Assume that $v \in L^\infty(\Omega) \cap W^{1,2}(\Omega \cap B_R)$ for all $R > 0$ is nonnegative with $v = 0$ on $\partial\Omega$ and that it solves*

$$\int_{\Omega} \sum_{i,j=1}^d a_{ij}(x) D_j v(x) D_i \psi(x)\, dx \le 0 \tag{2.16}$$

for all nonnegative $\psi \in \mathcal{D}(\mathbb{R}^d)$ vanishing on Γ_D. Then we have $v \equiv 0$ in Ω.

Proof of Lemma 2.3 and Lemma 2.4. We prove both lemmata simultaneously and only formally, because the proofs are almost the same and standard. First, in the case $\Omega \neq \mathbb{R}^d$, we may assume (by an extension argument) that the coefficients a_{ij} are given also outside Ω with (1.5)–(1.6). We then find the Green function to a_{ij} on $\mathbb{R}^d$, i.e., for $d \ge 3$ a function $G \in W^{1,1}_{loc}(\mathbb{R}^d)$ solving

$$\int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij} D_i G D_j \psi\, dx = \psi(0),$$

for all $\psi \in \mathcal{D}(\mathbb{R}^d)$ and satisfying $G(x) \to 0$ as $|x| \to \infty$. The existence of such G is established for bounded domains e.g. in [3, 15, 18] but it can be easily extended to the whole $\mathbb{R}^d$ by simple scaling arguments. Moreover, we know that G is positive with

$$\frac{1}{C|x|^{d-2}} \le G(x) \le \frac{C}{|x|^{d-2}} \qquad \text{for all } x \in \mathbb{R}^d \setminus \{0\}, \tag{2.17}$$

$$\int_{A_R} \frac{|\nabla G|^2}{G}\, dx \le C \qquad \text{for all } R > 0, \tag{2.18}$$

where C depends on a via the assumptions (1.5)–(1.6). Furthermore, in the case $d = 2$ we simply set $G \equiv 1$. Next, in what follows, we denote $\Omega := \mathbb{R}^d$ in case we deal with Lemma 2.3. Moreover, since v is bounded, we can always assume that it is nonnegative (otherwise we can add some constant). To conclude the preparations, we finally choose for any $R > 0$ a nonnegative function $\tau_R \in \mathcal{D}(B_{2R})$ with $\tau_R \equiv 1$ in B_R such that

$$R^2|\nabla^2 \tau_R| + R|\nabla \tau_R| \le C.$$

Proof of the estimate (2.15). For this purpose, we set $\psi := v\tau_R^2 G_\varrho$ in the inequalities (2.13) and (2.16), respectively, where G_ϱ is an approximation of G solving

$$\int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij} D_i G_\varrho D_j \psi\, dx = \frac{1}{|B_\varrho|} \int_{B_\varrho} \psi\, dx \tag{2.19}$$

for all $\psi \in \mathcal{D}(\mathbb{R}^d)$. Note that due to the zero trace (in case of Dirichlet data) and the nonnegativity of v such a setting is possible. Consequently, we obtain the inequality

$$\begin{aligned}&2\sum_{i,j=1}^d \int_\Omega a_{ij}D_jvD_iv\tau_R^2G_\varrho\,dx + \sum_{i,j=1}^d \int_\Omega a_{ij}D_j(v^2\tau_R^2)D_iG_\varrho\,dx \\ &\qquad\le \sum_{i,j=1}^d \int_\Omega a_{ij}v^2D_iG_\varrho D_j(\tau_R^2)\,dx - 2\sum_{i,j=1}^d \int_\Omega a_{ij}D_j(v^2)\tau_R D_i\tau_R G_\varrho\,dx.\end{aligned} \tag{2.20}$$

This inequality is now investigated in more detail. First, extending v outside Ω by zero and using (2.19), we see that the second term in (2.20) is nonnegative. Consequently, with the help of (1.5)–(1.6) we obtain

$$\delta\int_\Omega |\nabla v|^2\tau_R^2G_\varrho\,dx \le CR^{-1}\|v\|_\infty^2\int_{A_R}|\nabla G_\varrho|\,dx + CR^{-1}\|v\|_\infty\int_{A_R}|\nabla v|\tau_R G_\varrho\,dx$$

and then by Young's inequality

$$\delta\int_\Omega |\nabla v|^2\tau_R^2G_\varrho\,dx \le CR^{-1}\|v\|_\infty^2\int_{A_R}|\nabla G_\varrho|\,dx + CR^{-2}\|v\|_\infty^2\int_{A_R}G_\varrho\,dx.$$

Thus, letting $\varrho\to 0_+$, using the Fatou lemma and Young's inequality, we find

$$\begin{aligned}\int_\Omega |\nabla v|^2\tau_R^2G\,dx &\le CR^{-1}\|v\|_\infty^2\int_{A_R}|\nabla G|\,dx + CR^{-2}\|v\|_\infty^2\int_{A_R}G\,dx \\ &\le C\|v\|_\infty^2\int_{A_R}\frac{|\nabla G|^2}{G}\,dx + CR^{-2}\|v\|_\infty^2\int_{A_R}G\,dx \le C\|v\|_\infty^2,\end{aligned}$$

where the last inequality follows from (2.17) and (2.18). Hence, letting $R\to\infty$ and using (2.17), we gain (2.15) in both ($\mathbb{R}^d$ or cone) cases.

Proof of inequality (2.14). To this end, we set $\psi := (v-(v)_{A_R})_+\tau_R^2G_\varrho$ in (2.13), with $(v)_{A_R}$ defined as the mean values of v on A_R, which here is nonnegative since v is supposed to be nonnegative. Consequently, we obtain, analogously as (2.20), the identity

$$\begin{aligned}&2\sum_{i,j=1}^d \int_{\Omega_R^+} a_{ij}D_jvD_iv\tau_R^2G_\varrho\,dx + \sum_{i,j=1}^d \int_\Omega a_{ij}D_j((v-(v)_{A_R})_+^2\tau_R^2)D_iG_\varrho\,dx \\ &\le \sum_{i,j=1}^d \int_\Omega a_{ij}(v-(v)_{A_R})_+^2D_iG_\varrho D_j\tau_R^2\,dx - 2\sum_{i,j=1}^d \int_\Omega a_{ij}D_j(v-(v)_{A_R})_+^2\tau_R D_i\tau_R G_\varrho\,dx,\end{aligned}$$

where we have denoted by Ω_R^+ the subset of Ω where $v > (v)_{A_R}$. The second term is again nonnegative and can thus be neglected. Hence, letting $\varrho\to 0_+$ and using (1.6), we obtain

$$\begin{aligned}\int_{\Omega\cap B_R}|\nabla(v-(v)_{A_R})_+|^2G\,dx &\le CR^{-1}\int_{\Omega\cap A_R}(v-(v)_{A_R})^2|\nabla G|\,dx \\ &\quad + CR^{-1}\int_{\Omega\cap A_R}|v-(v)_{A_R}||\nabla(v-(v)_{A_R})|G\,dx.\end{aligned} \tag{2.21}$$

Next, using the Hölder inequality, the Poincaré inequality and (2.17)–(2.18), we find for the first term on the right-hand side of the previous inequality

$$\begin{aligned}\int_{\Omega\cap A_R}\frac{(v-(v)_{A_R})^2|\nabla G|}{R}\,dx &\le C\|v\|_\infty\left(\int_{A_R}\frac{|\nabla G|^2}{G}\,dx\right)^{\frac12}\left(\int_{\Omega\cap A_R}\frac{(v-(v)_{A_R})^2}{R^d}\,dx\right)^{\frac12} \\ &\le C\|v\|_\infty\left(\int_{\Omega\cap A_R}\frac{|\nabla v|^2}{R^{d-2}}\,dx\right)^{\frac12} \le C\|v\|_\infty\left(\int_{\Omega\cap A_R}\frac{|\nabla v|^2}{|x|^{d-2}}\,dx\right)^{\frac12}\end{aligned}$$

and for the second term via Jensen's inequality

$$\int_{\Omega\cap A_R} \frac{|v-(v)_{A_R}||\nabla(v-(v)_{A_R})|G}{R}\,dx \le C\|v\|_\infty \Big(\int_{\Omega\cap A_R} \frac{|\nabla v|^2}{|x|^{d-2}}\,dx\Big)^{\frac12}.$$

Hence, plugging these estimates into (2.21), we get

$$\int_{\Omega\cap B_R} |\nabla(v-(v)_{A_R})_+|^2 G\,dx \le C\|v\|_\infty \left(\int_{\Omega\cap A_R} \frac{|\nabla v|^2}{|x|^{d-2}}\,dx\right)^{\frac12}.$$

Finally, using (2.15) we end up with

$$\limsup_{R\to\infty} \int_{\Omega\cap B_R} |\nabla(v-(v)_{A_R})_+|^2 G\,dx = 0$$

and therefore, we conclude

$$\big(v - \liminf_{R\to\infty}(v)_{A_R}\big)_+ = const \qquad \text{a.e. in } \mathbb{R}^d. \tag{2.22}$$

Hence, if there is a set of positive measure where $v \le \liminf_{R\to\infty}(v)_{A_R}$, then we deduce from (2.22) that $v \le \liminf_{R\to\infty}(v)_{A_R}$ almost everywhere, and consequently (2.14) holds. If the opposite is true, i.e., if $v > \liminf_{R\to\infty}(v)_{A_R}$ holds almost everywhere, then (2.22) implies that

$$v - \liminf_{R\to\infty}(v)_{A_R} = const > 0 \qquad \text{a.e. in } \mathbb{R}^d.$$

After integration over A_r and division by $|A_r|$ for $r>0$, we have

$$(v)_{A_r} - \liminf_{R\to\infty}(v)_{A_R} = const > 0,$$

and considering the lim inf for $r\to\infty$ we arrive at a contradiction. This finishes the proof of (2.14) and thus of Lemma 2.3.

Proof of Lemma 2.4. We let $\varrho\to 0_+$ in (2.20) and neglecting the second nonnegative term, using the Young inequality for terms on the right-hand side, using the estimate (2.17) and the assumptions (1.5)–(1.6), we deduce

$$\begin{aligned}\int_\Omega |\nabla v|^2\tau_R^2 G\,dx \le{}& C\|v\|_\infty \left(\int_{\Omega\cap A_R} \frac{|v|^2}{R^d}\,dx\right)^{\frac12} \left(\int_{A_R} \frac{|\nabla G|^2}{G}\,dx\right)^{\frac12}\\ &+ C\|v\|_\infty \left(\int_{\Omega\cap A_R} |\nabla v|^2 G\,dx\right)^{\frac12} \left(\int_{A_R} GR^{-2}\,dx\right)^{\frac12},\end{aligned}$$

which reduces with (2.17)–(2.18) to

$$\int_\Omega |\nabla v|^2\tau_R^2 G\,dx \le C\|v\|_\infty \left(\int_{\Omega\cap A_R} \frac{|v|^2}{R^d}\,dx\right)^{\frac12} + C\|v\|_\infty \left(\int_{\Omega\cap A_R} |\nabla v|^2 G\,dx\right)^{\frac12}. \tag{2.23}$$

Next, since Ω is a Lipschitz cone and v is zero on $\partial\Omega$ we can use the Poincaré inequality for the first term on the right-hand side of (2.23) to deduce

$$\int_{\Omega\cap A_R} \frac{|v|^2}{R^d}\,dx \le C\int_{\Omega\cap A_R} \frac{|\nabla v|^2}{R^{d-2}}\,dx \le C\int_{\Omega\cap A_R} |\nabla v|^2 G\,dx,$$

where we used once again (2.17). Hence, (2.23) reduces to

$$\int_\Omega |\nabla v|^2\tau_R^2 G\,dx \le C\left(\int_{\Omega\cap A_R} |\nabla v|^2 G\,dx\right)^{\frac12}.$$

Finally, letting $R\to\infty$ and using (2.15), we find

$$\int_\Omega |\nabla v|^2 G\,dx = 0,$$

and consequently v is identically constant. Since it is zero on the boundary, we see that v is identically zero. $\square$

3. Proofs of Theorems 1.4 and 1.5

First, we focus on the proof of Theorem 1.4.

Proof of Theorem 1.4. We start by denoting $M := \|u\|_\infty$, which according to the assumption on u is finite. Then we introduce

$$\bar{u}_\nu := \lim_{R\to\infty} (u_\nu)_{A_R}.$$

Note here that the limit may not exist but surely we can find an increasing sequence of radii R_k such that the above definition is meaningful. Our goal is to show that for almost all $x \in \mathbb{R}^d$ we have

$$u_\nu(x) \le \bar{u}_\nu \qquad \text{for } \nu = 1, \dots, N, \tag{3.1}$$

$$\sum_{\nu=1}^N q_\nu u_\nu(x) \ge \sum_{\nu=1}^N q_\nu \bar{u}_\nu. \tag{3.2}$$

Once, this is achieved, we can use the positivity of each q_ν to deduce

$$q_\mu \bar{u}_\mu \ge q_\mu u_\mu(x) = \sum_{\nu=1}^N q_\nu u_\nu(x) - \sum_{\nu=1,\nu\neq\mu}^N q_\nu u_\nu(x) \ge \sum_{\nu=1}^N q_\nu \bar{u}_\nu - \sum_{\nu=1,\nu\neq\mu}^N q_\nu \bar{u}_\nu = q_\mu \bar{u}_\mu$$

for arbitrary $\mu = 1, \dots, N$, and Theorem 1.4 follows. Thus, it is sufficient to prove (3.1)–(3.2).

Proof of the lower bound (3.2). We will essentially take advantage of the estimates stated in Lemma 2.2. Hence, let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be an arbitrary nonnegative function. We test the ν-th equation in (1.1) by $w_\lambda \varphi$, where w_λ is defined in (2.11), multiply the result by q_ν and sum with respect to $\nu = 1, \dots, N$ to observe

$$\sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j\Big(\sum_{\nu=1}^N q_\nu u_\nu\Big) D_i(w_\lambda\varphi)\,dx = \int_{\mathbb{R}^d} w_\lambda \varphi \sum_{\nu=1}^N q_\nu H_\nu(u,\nabla u)\,dx. \tag{3.3}$$

Next, we use the definition of w_λ and rewrite the first term as

$$\begin{aligned}
&\sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j\Big(\sum_{\nu=1}^N q_\nu u_\nu\Big) D_i(w_\lambda\varphi)\,dx \\
&\quad = \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j\Big(\sum_{\nu=1}^N q_\nu u_\nu\Big) D_i w_\lambda \varphi\,dx + \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} w_\lambda D_j\Big(\sum_{\nu=1}^N q_\nu u_\nu\Big) D_i\varphi\,dx \\
&\quad = \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j\Big(\sum_{\nu=1}^N q_\nu u_\nu\Big) D_i w_\lambda \varphi\,dx - \lambda^{-1} \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j w_\lambda D_i\varphi\,dx.
\end{aligned}$$

Employing this identity in (3.3) we arrive at

$$\begin{aligned}
&\sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j w_\lambda D_i \varphi\,dx \\
&\quad = -\lambda \int_{\mathbb{R}^d} \varphi\Big(w_\lambda \sum_{\nu=1}^N q_\nu H_\nu(u,\nabla u) - \sum_{i,j=1}^d a_{ij} D_j\Big(\sum_{\nu=1}^N q_\nu u_\nu\Big) D_i w_\lambda\Big)\,dx.
\end{aligned}$$

Thus, recalling $K^*=0$ by assumption, using (2.12) and the nonnegativity of φ, we find

$$\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}D_j w_\lambda D_i\varphi\,dx+\int_{\mathbb{R}^d}\frac{\delta|\nabla w_\lambda|^2\varphi}{\lambda w_\lambda}\left(1-\frac{K}{\delta\lambda}\right)dx\le 0.$$

Hence, setting $\lambda:=\frac{K}{\delta}$ we deduce

$$\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}D_j w_\lambda D_i\varphi\,dx\le 0.$$

Therefore, w_λ is a subsolution to $-\sum_{i,j=1}^d D_i(a_{ij}D_jv)=0$ and we may apply Lemma 2.3 to conclude that for almost every $z\in\mathbb{R}^d$

$$w_\lambda(z)\le\liminf_{R\to\infty}(w_\lambda)_{A_R}. \tag{3.4}$$

In order to show that (3.4) leads indeed to (3.2), we first use (2.15) and the L^∞ bound for u to observe

$$\int_{\mathbb{R}^d}\frac{|\nabla\sum_{\nu=1}^N q_\nu u_\nu|^2}{|x|^{d-2}}\,dx=\lambda^{-2}\int_{\mathbb{R}^d}\frac{|\nabla w_\lambda|^2}{w_\lambda^2|x|^{d-2}}\,dx\le C\int_{\mathbb{R}^d}\frac{|\nabla w_\lambda|^2}{|x|^{d-2}}\,dx\le C(M,\lambda,q). \tag{3.5}$$

Furthermore, using the definition $w_\lambda:=e^{-\lambda\sum_{\nu=1}^N q_\nu u_\nu}$ combined with the algebraic inequality $|e^x-e^y|\le e^{|x|+|y|}|x-y|$, we get the estimate

$$\begin{aligned}
&\left|\frac{1}{|A_R|}\int_{A_R}\left(w_\lambda-e^{-\lambda(\sum_{\nu=1}^N q_\nu u_\nu)_{A_R}}\right)dx\right|\\
&\quad\le\lambda e^{2NM\lambda\max\{|q_\nu|\}}\frac{1}{|A_R|}\int_{A_R}\left|\sum_{\nu=1}^N q_\nu u_\nu-\left(\sum_{\nu=1}^N q_\nu u_\nu\right)_{A_R}\right|dx\\
&\quad\le C\int_{A_R}\frac{|\nabla\sum_{\nu=1}^N q_\nu u_\nu|}{R^{d-1}}\,dx\\
&\quad\le C(d,N,M,\lambda,q)\left(\int_{A_R}\frac{|\nabla\sum_{\nu=1}^N q_\nu u_\nu|^2}{|x|^{d-2}}\,dx\right)^{\frac12}.
\end{aligned}$$

Consequently, using the estimate (3.5) we have

$$\lim_{R\to\infty}\left|\frac{1}{|A_R|}\int_{A_R}\left(w_\lambda-e^{-\lambda(\sum_{\nu=1}^N q_\nu u_\nu)_{A_R}}\right)dx\right|=0,$$

and via the triangle inequality we deduce from (3.4)

$$w_\lambda(x)\le\liminf_{R\to\infty}e^{-\lambda(\sum_{\nu=1}^N q_\nu u_\nu)_{A_R}}\le e^{-\lambda\sum_{\nu=1}^N q_\nu\bar u_\nu}.$$

Thus, applying the logarithm on both sides, we gain

$$-\lambda\sum_{\nu=1}^N q_\nu u_\nu(x)\le-\lambda\sum_{\nu=1}^N q_\nu\bar u_\nu$$

and (3.2) follows directly.

Proof of the upper bound (3.1). We proceed similarly as before. Let γ_ν be increasing, convex, nonnegative function and define $\varphi_\nu(u)$ by (2.1). Then we test the ν-th equation in (1.1) by

$D_{u_\nu}\varphi_1(u)\psi$ where $\psi \in \mathcal{D}(\mathbb{R}^d)$ is an arbitrary, nonnegative function. Summing the result with respect to $\nu = 1, \dots, N$, we gain

$$\begin{aligned}
&\sum_{\nu=1}^{N}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij} D_j u_\nu D_{u_\nu}\varphi_1(u) D_i\psi \, dx \\
&= \sum_{\nu=1}^{N}\int_{\mathbb{R}^d}\Big(D_{u_\nu}\varphi_1(u) H_\nu(u,\nabla u) - \sum_{i,j=1}^{d} a_{ij} D_j u_\nu D_i D_{u_\nu}\varphi_1(u)\Big)\psi \, dx.
\end{aligned} \tag{3.6}$$

Due to the nonnegativity of ψ we can estimate the term on the right-hand side of (3.6) via (2.3) (note $K^* = 0$) and rewrite the left-hand side of (3.6) via the identity $\sum_{\nu=1}^N D_j u_\nu D_{u_\nu}\varphi_1(u) = D_j\varphi_1(u)$. In this way, we conclude

$$\int_{\mathbb{R}^d}\Big(\sum_{i,j=1}^{d} a_{ij} D_j\varphi_1(u) D_i\psi + \delta\sum_{\nu=1}^{N} Z_\nu(u)\gamma_\nu''(u_\nu)|\nabla u_\nu|^2 \prod_{\mu=1}^{\nu}\varphi_\mu(u)\psi\Big)\, dx \le 0, \tag{3.7}$$

with $Z_\nu(u)$ defined as in (2.4). We now want to apply Lemma 2.3 again. For this purpose, we need to choose the functions γ_ν suitably to guarantee the nonnegativity (or even the uniform positivity) of $Z_\nu(u)$. For the choice

$$\gamma_\nu(u_\nu) := e^{u_\nu + M + c_\nu}$$

with nonnegative constants c_ν the definition of M gives

$$\gamma_\nu''(u_\nu) \ge e^{c_\nu}. \tag{3.8}$$

Moreover, from the definition of the functions φ_ν in (2.1) we also observe

$$\begin{aligned}
&\varphi_N(u) \le e^{e^{2M + c_N}} =: \varphi_N(u_{\max}), \\
&\varphi_\nu(u) \le e^{e^{2M + c_\nu} + \varphi_{\nu+1}(u_{\max})} =: \varphi_\nu(u_{\max}) \quad \text{for } \nu = 1, \dots, N-1.
\end{aligned} \tag{3.9}$$

Finally, using the definition (2.4) of $Z_\nu(u)$ and the estimates (3.8)–(3.9), we gain

$$Z_\nu(u) \ge 1 - \frac{K^2 N \prod_{\mu=\nu+1}^{N+1}\varphi_\mu(u_{\max})}{\delta^2 e^{c_\nu}}. \tag{3.10}$$

It is important to notice that the constant $\varphi_\nu(u_{\max})$ depends on M and $c_\nu, \dots, c_N$, but not on $c_1, \dots, c_{\nu-1}$. This gives us a possibility to choose all constants in such a way that

$$Z_\nu(u) \ge \frac{1}{2}, \qquad \text{for all } \nu = 1, \dots, N. \tag{3.11}$$

Indeed, if we define c_ν iteratively as (note here that $c_2, \dots, c_N$ depend only on N, δ, K and M)

$$\begin{aligned}
&c_N := \ln\Big(\frac{2K^2 N}{\delta^2}\Big), \\
&c_\nu := c_{\nu+1} + \ln\varphi_{\nu+1}(u_{\max}) \quad \text{for } \nu = 2, \dots, N-1, \\
&c_1 \ge c_2 + \ln\varphi_2(u_{\max})
\end{aligned} \tag{3.12}$$

then a direct computation leads to (3.11). Therefore, we see from (3.7) that $\varphi_1(u)$ is a subsolution to $-\sum_{i,j=1}^d D_i(a_{ij}D_j v) = 0$ and we may apply Lemma 2.3 to conclude

$$\varphi_1(u(x)) \le \liminf_{R\to\infty}(\varphi_1(u))_{A_R} \qquad \text{a.e. in } \mathbb{R}^d. \tag{3.13}$$

In addition, we can also mimic the proof of Lemma 2.3 to show an estimate of the form

$$\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^{d-2}}\, dx \le C(c_1, d, N, \delta, K, M). \tag{3.14}$$

Indeed, setting $\psi := G\tau_R \geq 0$ in (3.7), using (3.8), (3.11) and $\varphi_\nu \geq 1$ for all ν, we get

$$\begin{aligned}
\int_{\mathbb{R}^d} |\nabla u|^2 G\tau_R \, dx &\leq -C \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij} D_j \varphi_1(u) D_i(G\tau_R)\, dx \\
&\leq C \int_{A_R} \left(|\nabla \varphi_1(u)| G R^{-1} + |\varphi_1(u)||\nabla G| R^{-1} \right) dx \\
&\quad - C \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij} D_j(\varphi_1(u)\tau_R) D_i G\, dx \\
&\leq C \left[\left(\int_{A_R} |\nabla \varphi_1(u)|^2 G\, dx \right)^{\frac12} + \left(\int_{A_R} \frac{|\nabla G|^2}{G}\, dx \right)^{\frac12} \right] \left(\int_{A_R} \frac{|G|}{R^2}\, dx \right)^{\frac12} \\
&\leq C(c_1, d, N, \delta, K, M),
\end{aligned}$$

where the third inequality follows from the the nonnegativity of φ_1 and the definition of G, while for the last inequality we have used (2.15), (2.17) and (2.18). Hence, letting $R \to \infty$, we deduce (3.14). As the next step we show that (3.13) and (3.14) imply

$$\varphi_1(u(x)) \leq \liminf_{R\to\infty} \varphi_1\left((u)_{A_R}\right) = \varphi_1(\bar{u}) \qquad \text{a.e. in } \mathbb{R}^d. \tag{3.15}$$

Indeed, since φ_1 is Lipschitz continuous and u is bounded, we observe

$$\begin{aligned}
\left| (\varphi_1(u))_{A_R} - \varphi_1\left((u)_{A_R}\right) \right| &\leq \frac{1}{|A_R|} \int_{A_R} \left| \varphi_1(u(x)) - \varphi_1\left((u)_{A_R}\right) \right| dx \\
&\leq \frac{C(c_1, N, \delta, M)}{|A_R|} \int_{A_R} \left| u(x) - (u)_{A_R} \right| dx \\
&\leq C(c_1, d, N, \delta, M) \left(\int_{A_R} \frac{|\nabla u(x)|^2}{|x|^{d-2}}\, dx \right)^{\frac12},
\end{aligned}$$

where for the last inequality we have used Poincaré's and Hölder's inequality. Hence, taking into account (3.14) we see that the right-hand side vanishes in the limit $R \to \infty$. Consequently, using the triangle inequality in (3.13) we gain (3.15). Note here that (3.15) is valid for all possible choice of c_1 from (3.12), while the constant $c_2, \dots, c_N$ are already fixed. Thus, using definitions of φ_1 and γ_1, it is a straightforward to see that (3.15) implies

$$e^{u_1(x)+M+c_1} + \varphi_2(u(x)) \leq e^{\bar{u}_1+M+c_1} + \varphi_2(\bar{u}) \qquad \text{for a.e. } x \in \mathbb{R}^d.$$

Finally, we let $c_1 \to \infty$ (which is possible in (3.12)) to conclude that

$$e^{u_1(x)} \leq e^{\bar{u}_1},$$

and (3.1) for $\nu = 1$ directly follows. However, since our assumption are completely independent of the order of the unknowns we can repeat the same procedure step by step to obtain the same result also for $\nu = 2, \dots, N$. Hence, the proof of Theorem 1.4 is complete. □

We continue with the proof of Theorem 1.5 for the Liouville property in a cone.

Proof of Theorem 1.5. Since the proof is almost the same as the one of Theorem 1.4, we only point out the key differences. First, we can deduce an inequality similar to (3.7), which by using (3.11) (and with the same choice of functions φ_ν) reduces to

$$\int_\Omega \Big(\sum_{i,j=1}^d a_{ij} D_j \varphi_1(u) D_i \psi + \frac{\delta}{2} |\nabla u|^2 \psi \Big)\, dx \leq 0, \tag{3.16}$$

for all nonnegative functions $\psi \in \mathcal{D}(\Omega)$. Next, we would like to apply Lemma 2.4 and repeat the above procedure, but we need to proceed slightly differently because of the boundary condition for $\varphi_1(u)$. Denoting by $\tau_h \in \mathcal{D}(\Omega)$ a nonnegative function with $\tau_h \equiv 1$ on the set $\{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) \geq h\}$ and fulfilling $|\nabla \tau_h| \leq 2h^{-1}$ (which is possible since Ω is Lipschitz), we set $\psi := (u_\nu - \varepsilon)_+^2 \tau_R^2 \tau_h^2$ in (3.16) to get via Young's inequality

$$\int_\Omega (u_\nu - \varepsilon)_+^2 \tau_R^2 \tau_h^2 |\nabla u|^2 \, dx \leq \frac{1}{2} \int_\Omega (u_\nu - \varepsilon)_+^2 \tau_R^2 \tau_h^2 |\nabla u|^2 \, dx$$
$$+ C \int_\Omega \Big(|\nabla (u - \varepsilon)_+|^2 \tau_R^2 \tau_h^2 + (u_\nu - \varepsilon)_+^2 |\nabla \tau_R|^2 \tau_h^2 + (u_\nu - \varepsilon)_+^2 \tau_R^2 |\nabla \tau_h|^2 \Big) \, dx.$$

Hence, using the facts that $(u_\nu - \varepsilon)_+ \in W^{1,2}(\Omega \cap B_R)$ for all balls B_R and that $(u_\nu - \varepsilon)_+$ vanishes on $\partial\Omega$, we let $h \to 0_+$ and observe

$$\int_\Omega (u_\nu - \varepsilon)_+^2 |\nabla u|^2 \tau_R^2 \, dx \leq C(c_1, N, \delta, K, M, R, \varepsilon). \tag{3.17}$$

Since (3.17) is valid for all $\nu = 1, \dots, N$, we see that for U_ε defined as

$$U_\varepsilon := \{x \in \Omega\colon u_\nu \geq 2\varepsilon \text{ for some } \nu \in \{1, \dots, N\}\} \tag{3.18}$$

we deduce

$$\int_{U_\varepsilon} |\nabla u|^2 \tau_R^2 \, dx \leq C(c_1, N, \delta, K, M, R, \varepsilon). \tag{3.19}$$

We next claim that for all $\eta > 0$ and all choices of c_1 (note that $c_2, \dots, c_N$ were fixed in order to verify (3.11)) we can find $\varepsilon > 0$ such that

$$\operatorname{supp} \, (\varphi_1(u) - \varphi_1(0) - \eta)_+ \subset U_\varepsilon. \tag{3.20}$$

Indeed, if this is not the case, there is some $x_0 \in \Omega$ such that $u_\nu(x_0) < 2\varepsilon$ for all $\nu = 1, \dots, N$ and

$$\eta < \varphi_1(u(x_0)) - \varphi_1(0).$$

But since by definition φ_1 is increasing in any component and Lipschitz regular, we infer

$$\eta < \varphi_1(u(x_0)) - \varphi_1(0) \leq \varphi_1(2\varepsilon, \dots, 2\varepsilon) - \varphi_1(0) \leq C(c_1, N, \delta, M)\varepsilon.$$

Consequently, we may find $\varepsilon > 0$ sufficiently small such that we directly obtain a contradiction, and therefore (3.20) holds. Then with the help of (3.19), we verify $(\varphi_1(u) - \varphi_1(0) - \eta)_+ \in W^{1,2}(\Omega \cap B_R)$ with $(\varphi_1(u) - \varphi_1(0) - \eta)_+ = 0$ on $\partial\Omega$. Thus, setting for any nonnegative function $\tilde{\psi} \in \mathcal{D}(\Omega)$ and positive α

$$\psi := \tilde{\psi} \frac{(\varphi_1(u) - \varphi_1(0) - \eta)_+}{\alpha + (\varphi_1(u) - \varphi_1(0) - \eta)_+}$$

in (3.16), we deduce

$$\int_\Omega \sum_{i,j=1}^d a_{ij} D_j (\varphi_1(u) - \varphi_1(0) - \eta)_+ D_i \tilde{\psi} \frac{(\varphi_1(u) - \varphi_1(0) - \eta)_+}{\alpha + (\varphi_1(u) - \varphi_1(0) - \eta)_+} \, dx$$
$$+ \frac{\delta}{2} \int_\Omega |\nabla u|^2 \tilde{\psi} \frac{(\varphi_1(u) - \varphi_1(0) - \eta)_+}{\alpha + (\varphi_1(u) - \varphi_1(0) - \eta)_+} \, dx$$
$$\leq -\alpha \int_\Omega \sum_{i,j=1}^d a_{ij} D_j (\varphi_1(u) - \varphi_1(0) - \eta)_+ \frac{\tilde{\psi} D_i (\varphi_1(u) - \varphi_1(0) - \eta)_+}{(\alpha + (\varphi_1(u) - \varphi_1(0) - \eta)_+)^2} \, dx \leq 0.$$

Hence, letting $\alpha \to 0_+$, we find

$$\int_\Omega \sum_{i,j=1}^d a_{ij} D_j (\varphi_1(u) - \varphi_1(0) - \eta)_+ D_i \tilde{\psi} \, dx \leq 0.$$

Taking advantage of Lemma 2.4 for the function $\varphi_1(u) - \varphi_1(0) - \eta)_+$, we thus obtain

$$(\varphi_1(u) - \varphi_1(0) - \eta)_+ \equiv 0,$$

which leads to

$$\varphi_1(u) \le \varphi_1(0) + \eta \qquad \text{a.e. in } \Omega.$$

Thus, repeating the procedure at the end of the previous proof, i.e., letting first $\eta \to 0_+$ and then $c_1 \to \infty$, we gain $u_1 \le 0$ a.e. in Ω. The rest of the proof is then the same as above and we skip it for the sake of brevity. □

4. Proof of Theorem 1.6

This section is inspired by the method introduced in [1]. In fact, using the auxiliary inequalities for the Hamiltonian H provided in Section 2, we can directly mimic the procedure developed in [1].

Initial integral identities and inequalities. Using the notation from the previous section, the same definitions of the functions φ_ν and applying the same procedure, we gain for all $\psi \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ vanishing in a neighborhood of Γ_D (see Definition 1.2 of a generalized weak solution) the identity

$$\begin{aligned}\sum_{i,j=1}^{d} \int_\Omega a_{ij}^n D_j \varphi_1(u^n) D_i \psi \, dx + \sum_{\nu=1}^{N} \int_\Omega \kappa_\nu^n u_\nu^n D_{u_\nu} \varphi_1(u^n) \psi \, dx \\ = \sum_{\nu=1}^{N} \int_\Omega \Big(D_{u_\nu} \varphi_1(u^n) H_\nu^n(u^n, \nabla u^n) - \sum_{i,j=1}^{d} a_{ij}^n D_j u_\nu^n D_i D_{u_\nu} \varphi_1(u^n) \Big) \psi \, dx\end{aligned} \tag{4.1}$$

(compare (3.6)). Therefore, using (2.3) and assuming $\psi \ge 0$, we observe

$$\begin{aligned}\int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij}^n D_j \varphi_1(u^n) D_i \psi + \delta \sum_{\nu=1}^{N} Z_\nu(u^n) \gamma_\nu''(u_\nu^n) |\nabla u_\nu^n|^2 \prod_{\mu=1}^{\nu} \varphi_\mu(u^n) \psi \Big) \, dx \\ \le \sum_{\nu=1}^{N} \int_\Omega (K^* - \kappa_\nu^n u_\nu^n) D_{u_\nu} \varphi_1(u^n) \psi \, dx.\end{aligned} \tag{4.2}$$

Hence, similarly as before, we set

$$\gamma_\nu(u_\nu^n) := e^{u_\nu^n + M + c_\nu},$$

and keeping the notation from (3.9) and defining c_ν in the same way as in (3.12), we see that (4.2) reduces to

$$\int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij}^n D_j \varphi_1(u^n) D_i \psi + \frac{\delta}{2} |\nabla u^n|^2 \psi \Big) \, dx \le C(c_1, N, \delta, K, K^*, M) \int_\Omega \psi \, dx.$$

Multiplying the ν-th equation in (1.16) by q_ν^n, summing the result over $\nu = 1, \dots, N$ and setting $\varphi := w_\lambda^n \psi$ (see (2.11) for the definition of w_λ) with an arbitrary nonnegative function $\psi \in L^\infty \cap W^{1,2}(\Omega)$ vanishing in a neighborhood of Γ_D, we deduce

$$\begin{aligned}-\lambda^{-1} \int_\Omega \sum_{i,j=1}^{d} a_{ij}^n D_j w_\lambda^n D_i \psi \, dx + \int_\Omega \sum_{\nu=1}^{N} q_\nu^n \kappa_\nu^n u_\nu^n w_\lambda^n \psi \, dx \\ = \int_\Omega \sum_{\nu=1}^{N} q_\nu^n H_\nu^n(u^n, \nabla u^n) w_\lambda^n \psi \, dx - \int_\Omega \sum_{i,j=1}^{d} a_{ij}^n D_j \Big(\sum_{\nu=1}^{N} q_\nu u_\nu^n \Big) D_i w_\lambda^n \psi \, dx.\end{aligned} \tag{4.3}$$

Similarly as above, using (2.12), we obtain the inequality

$$
\begin{aligned}
&\lambda^{-1}\int_\Omega \sum_{i,j=1}^d a_{ij}^n D_j w_\lambda^n D_i\psi\,dx + \frac{\delta}{\lambda}\left(1-\frac{K}{\delta\lambda}\right)\int_\Omega \frac{|\nabla w_\lambda^n|^2}{w_\lambda}\psi\,dx \\
&\qquad \le K^*\int_\Omega w_\lambda^n\psi\,dx + \int_\Omega \sum_{\nu=1}^N q_\nu^n \kappa_\nu^n u_\nu^n w_\lambda^n \psi\,dx.
\end{aligned} \tag{4.4}
$$

Derivation of a priori estimates. By an approximation argument, we may choose $\psi := d_\eta^2$ in (4.2), with $d_\eta(x) := \min\{1, \eta^{-1}\operatorname{dist}(x,\Gamma_D)\}$ for $\eta > 0$ (and hence there holds $d_\eta \equiv 1$ outside of the η-neighborhood of Γ_D), to observe the local estimate

$$
\int_\Omega |\nabla u^n|^2 d_\eta^2\,dx \le C(c_1, N, \delta, K, K^*, M, \eta). \tag{4.5}
$$

Next, choosing $\psi := (\varphi_1(u^n) - \varphi_1(0))_+$ in (4.2), we obtain

$$
\int_\Omega \Big(|\nabla\varphi_1(u^n)|^2\chi_{\{x;\,\varphi_1(u^n)\ge\varphi_1(0)\}} + |\nabla u^n|^2(\varphi_1(u^n)-\varphi_1(0))_+\Big)\,dx \le C(c_1,N,\delta,K,K^*,M), \tag{4.6}
$$

and choosing $\psi := \lambda(w_\lambda^n - 1)_+$ in (4.4) (note that both functions ψ have zero trace on Γ_D), we get for $\lambda \ge \frac{K}{\delta}$

$$
\int_\Omega |\nabla w_\lambda^n|^2 \chi_{\{x;\,w_\lambda^n\ge 1\}}\,dx \le C(\delta, K, K^*, M, q^n, \lambda). \tag{4.7}
$$

Thus, using the definition (2.11) of w_λ^n, we see that (4.7) implies the uniform bound

$$
\left\|\Big(\sum_{\nu=1}^N q_\nu^n u_\nu^n\Big)_-\right\|_{1,2} \le C.
$$

Finally, we derive a uniform estimate also for the positive parts of u^n from (4.6). To this end, we let $c_1 \to \infty$ and see that for any $\varepsilon > 0$ we can find $c_1 \gg 1$ with

$$
\varphi_1(u^n) - \varphi_1(0) \ge 1 \qquad \text{on the set } \{x\in\Omega;\, u_1^n \ge 2\varepsilon\}.
$$

Consequently, (4.6) gives

$$
\int_{\{u_1^n\ge 2\varepsilon\}} |\nabla u^n|^2\,dx \le C(N,\delta,K,K^*,M,\varepsilon).
$$

Using the fact that all estimates do not depend on the order of unknowns, we finally get on U_ε^n (defined analogously to (3.18)) the uniform bound

$$
\int_{U_\varepsilon^n} |\nabla u^n|^2\,dx \le C(N,\delta,K,K^*,M,\varepsilon)
$$

and in particular

$$
\int_\Omega |\nabla(u_\nu^n - \varepsilon)_+|^2\,dx \le C(N,\delta,K,K^*,M,\varepsilon) \qquad \text{for all } \nu = 1,\dots,N.
$$

Preliminary convergence results for u^n. First, it follows from the previous a priori estimates that there exists a subsequence (not relabeled) and a function $u \in L^\infty(\Omega;\mathbb{R}^N)$ such that for any $\eta > 0$ there hold

$$
u^n d_\eta \rightharpoonup u d_\eta \qquad \text{weakly in } W^{1,2}(\Omega;\mathbb{R}^N), \tag{4.8}
$$

$$
u^n \rightharpoonup^* u \qquad \text{weakly}^* \text{ in } L^\infty(\Omega;\mathbb{R}^N), \tag{4.9}
$$

$$
u^n \to u \qquad \text{strongly in } L^2(\Omega;\mathbb{R}^N) \text{ and pointwise a.e. in } \Omega \tag{4.10}
$$

and in addition (using also (1.18))

$$(\varphi_1(u^n)-\varphi_1(0))_+ \rightharpoonup (\varphi_1(u)-\varphi_1(0))_+ \qquad \text{weakly in } W^{1,2}_{\Gamma_D}(\Omega), \tag{4.11}$$

$$\Big(\sum_{\nu=1}^N q_\nu^n u_\nu^n\Big)_- \rightharpoonup \Big(\sum_{\nu=1}^N q_\nu u_\nu\Big)_- \qquad \text{weakly in } W^{1,2}_{\Gamma_D}(\Omega), \tag{4.12}$$

$$H^n(u^n,\nabla u^n)d_\eta^2\,dx \rightharpoonup^* \overline{H}d_\eta^2 \qquad \text{weakly}^* \text{ in } \mathcal{M}(\overline{\Omega};\mathbb{R}^N). \tag{4.13}$$

Hence, having these convergence results and (1.19), we can let $n\to\infty$ in the weak formulation (1.16) for u^n and $\psi\in W^{1,2}(\Omega)\cap\mathcal{C}(\overline{\Omega})$ vanishing in a neighborhood of Γ_D. This gives (with an appropriate choice of η in dependency of ψ)

$$\int_\Omega \Big(\sum_{i,j=1}^d a_{ij}(x)D_ju_\nu(x)D_i\psi(x)+\kappa_\nu u_\nu(x)\psi(x)\Big)\,dx = \langle \overline{H}_\nu,\psi\rangle, \quad \nu=1,\dots,N \tag{4.14}$$

(and with $\langle\cdot,\cdot\rangle$ denoting the duality pairing). Postponing the strong convergence (1.23) of the sequence u^n in $W^{1,2}(\Omega_0,\mathbb{R}^N)$ to the end of the proof, we now continue by establishing

$$\nabla u^n \to \nabla u \text{ a.e. in } \Omega. \tag{4.15}$$

Indeed, defining T_ε as a standard cut-off function via

$$T_\varepsilon(s) := \min(\varepsilon,|s|)\operatorname{sign} s$$

and testing the weak formulation for u^n by $T_\varepsilon(u^n-u)d_\eta^2$ (which again is possible by approximation, due to the bound (4.5)), we observe the identity

$$\int_\Omega \Big(\sum_{i,j=1}^d a_{ij}^n D_ju_\nu^n D_i(T_\varepsilon(u_\nu^n-u_\nu)d_\eta^2)+\kappa_\nu^n u_\nu^n T_\varepsilon(u_\nu^n-u_\nu)d_\eta^2\Big)\,dx = \int_\Omega H_\nu^n(u^n,\nabla u^n)T_\varepsilon(u_\nu^n-u_\nu)d_\eta^2\,dx$$

for $\nu=1,\dots,N$. Thus, using (4.13) we get

$$\limsup_{n\to\infty}\int_\Omega \sum_{i,j=1}^d a_{ij}^n D_ju_\nu^n D_i(T_\varepsilon(u_\nu^n-u_\nu)d_\eta^2)\,dx \le C\varepsilon$$

for $\nu=1,\dots,N$ and a constant C depending only on $\delta,K,K^*,M,c_1,N,\eta$ and Ω. Consequently, via the identity (4.14) for (approximations of) the same test function combined with the strong convergence (1.19) of the matrices a^n and the convergences (4.8) and (4.10) of the generalized solutions u^n, we deduce

$$\limsup_{n\to\infty}\int_\Omega \sum_{i,j=1}^d a_{ij}^n D_j(u_\nu^n-u_\nu)D_i(T_\varepsilon(u_\nu^n-u_\nu))d_\eta^2\,dx \le C\varepsilon,$$

which gives by the ellipticity condition (1.6) for a^n

$$\limsup_{n\to\infty}\int_\Omega |\nabla(T_\varepsilon(u_\nu^n-u_\nu))|^2 d_\eta^2\,dx \le C\varepsilon$$

for $\nu = 1, \dots, N$. Next, Hölder inequality, (4.5) and (4.10) imply

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega |\nabla(u^n_\nu - u_\nu)| d_\eta \, dx &\le C \limsup_{n\to\infty} \int_\Omega |\nabla T_\varepsilon(u^n_\nu - u_\nu)| d_\eta \, dx \\ &\quad + C \limsup_{n\to\infty} \big|\{x \in \Omega;\ |u^n_\nu(x) - u_\nu(x)| > \varepsilon\}\big|^{\frac12} \\ &\le C \limsup_{n\to\infty} \Big(\int_\Omega |\nabla T_\varepsilon(u^n_\nu - u_\nu)|^2 d^2_\eta \, dx \Big)^{\frac12} \le C\varepsilon^{\frac12}.\end{aligned}$$

Since ε is arbitrary, this leads to strong $L^1(\Omega)$ and thus (after passage to a subsequence) to pointwise convergence in Ω of $\nabla u^n d_\eta$ to $\nabla u d_\eta$, which directly yields the claim (4.15).

Identification of $\overline{H}$. The next step is to show

$$\overline{H} = H(u(x), \nabla u(x)) \qquad \text{in } \overline{\Omega} \setminus \overline{\Gamma}_D, \tag{4.16}$$

and consequently, $\overline{H} d^2_\eta = H(u(x), \nabla u(x)) d^2_\eta \in L^1(\Omega)$ for all $\eta > 0$. To this end, we first want to let $n \to \infty$ in (4.1) for nonnegative functions $\psi \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ vanishing in a neighborhood of Γ_D. It is a direct consequence of (1.19) and (4.8)–(4.10) that for the left-hand side of (4.1) there holds

$$\begin{aligned}\lim_{n\to\infty} \sum_{i,j=1}^d \int_\Omega a^n_{ij} D_j \varphi_1(u^n) D_i \psi \, dx + \sum_{\nu=1}^N \int_\Omega \kappa^n_\nu u^n_\nu D_{u_\nu} \varphi_1(u^n) \psi \, dx \\ = \sum_{i,j=1}^d \int_\Omega a_{ij} D_j \varphi_1(u) D_i \psi \, dx + \sum_{\nu=1}^N \int_\Omega \kappa_\nu u_\nu D_{u_\nu} \varphi_1(u) \psi \, dx.\end{aligned} \tag{4.17}$$

Applying (2.3) with the choices (3.12) for the constants c_ν, we see that the integrand on the right-hand side of (4.1) is bounded from above. Consequently, we can use the Fatou lemma, (4.10), (4.15) and (1.20)–(1.21) to deduce

$$\begin{aligned}\limsup_{n\to\infty} \sum_{\nu=1}^N \int_\Omega \Big(D_{u_\nu} \varphi_1(u^n) H^n_\nu(u^n, \nabla u^n) - \sum_{i,j=1}^d a^n_{ij} D_j u^n_\nu D_i D_{u_\nu} \varphi_1(u^n) \Big) \psi \, dx \\ \le \sum_{\nu=1}^N \int_\Omega \Big(D_{u_\nu} \varphi_1(u) H_\nu(u, \nabla u) - \sum_{i,j=1}^d a_{ij} D_j u_\nu D_i D_{u_\nu} \varphi_1(u) \Big) \psi \, dx.\end{aligned} \tag{4.18}$$

Therefore, applying the chain rule and combining (4.1) with (4.17) and (4.18), we get

$$\begin{aligned}&\sum_{\nu=1}^N \int_\Omega \Big(\sum_{i,j=1}^d a_{ij} D_j u_\nu D_i \big(D_{u_\nu} \varphi_1(u) \psi \big) + \kappa_\nu u_\nu D_{u_\nu} \varphi_1(u) \psi \Big) \, dx \\ &= \sum_{i,j=1}^d \int_\Omega a_{ij} \Big(D_j \varphi_1(u) D_i \psi + \sum_{\nu=1}^N D_j u_\nu D_i D_{u_\nu} \varphi_1(u) \Big) \psi \Big) \, dx + \sum_{\nu=1}^N \int_\Omega \kappa_\nu u_\nu D_{u_\nu} \varphi_1(u) \psi \, dx \\ &\le \sum_{\nu=1}^N \int_\Omega D_{u_\nu} \varphi_1(u) H_\nu(u, \nabla u) \psi \, dx\end{aligned}$$

for all nonnegative $\psi \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ vanishing in a neighborhood of Γ_D. Finally, for an arbitrary nonnegative function $\tilde\psi \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ vanishing in a neighborhood of Γ_D, we set

$$\psi := \big(D_{u_1} \varphi_1(u) \big)^{-1} \tilde\psi$$

in the previous inequality to arrive at

$$
\begin{aligned}
&\int_\Omega \Big(\sum_{i,j=1}^d a_{ij} D_j u_1 D_i \tilde\psi + \kappa_1 u_1 \tilde\psi \Big)\, dx \le \int_\Omega H_1(u,\nabla u)\tilde\psi\, dx \\
&+ \sum_{\nu=2}^N \int_\Omega D_{u_\nu}\varphi_1(u) H_\nu(u,\nabla u)\big(D_{u_1}\varphi_1(u)\big)^{-1}\tilde\psi\, dx \\
&- \sum_{\nu=2}^N \int_\Omega \sum_{i,j=1}^d a_{ij} D_j u_\nu D_i\Big(D_{u_\nu}\varphi_1(u)\big(D_{u_1}\varphi_1(u)\big)^{-1}\tilde\psi\Big)\, dx \\
&- \sum_{\nu=2}^N \int_\Omega \kappa_\nu u_\nu D_{u_\nu}\varphi_1(u)\big(D_{u_1}\varphi_1(u)\big)^{-1}\tilde\psi\, dx.
\end{aligned}
$$

Using the definition (2.1) of φ_1 (see also (2.2)) we have

$$
D_{u_\nu}\varphi_1(u) = \begin{cases} \gamma_1'(u_1)\varphi_1(u) & \text{for } \nu = 1, \\ \varphi_1(u) D_{u_\nu}\varphi_2(u) & \text{for } \nu > 1, \end{cases}
$$

and consequently, the previous inequality can be rewritten as

$$
\begin{aligned}
&\int_\Omega \Big(\sum_{i,j=1}^d a_{ij} D_j u_1 D_i \tilde\psi + \kappa_1 u_1 \tilde\psi \Big)\, dx \le \int_\Omega H_1(u,\nabla u)\tilde\psi\, dx \\
&+ \sum_{\nu=2}^N \int_\Omega D_{u_\nu}\varphi_2(u) H_\nu(u,\nabla u)\left(\gamma_1'(u_1)\right)^{-1}\tilde\psi\, dx \\
&- \sum_{\nu=2}^N \int_\Omega \sum_{i,j=1}^d a_{ij} D_j u_\nu D_i\Big(D_{u_\nu}\varphi_2(u)\left(\gamma_1'(u_1)\right)^{-1}\tilde\psi\Big)\, dx \\
&- \sum_{\nu=2}^N \int_\Omega \kappa_\nu u_\nu D_{u_\nu}\varphi_2(u)\left(\gamma_1'(u_1)\right)^{-1}\tilde\psi\, dx.
\end{aligned}
$$

Hence, using (4.8) combined with (4.5), (1.22) and the definition of γ_ν, we obtain

$$
\int_\Omega \Big(\sum_{i,j=1}^d a_{ij} D_j u_1 D_i \tilde\psi + \kappa_1 u_1 \tilde\psi \Big)\, dx \le \int_\Omega H_1(u,\nabla u)\tilde\psi\, dx + e^{-c_1} C(N,\delta,K,K^*,M,\tilde\psi).
$$

Consequently, letting $c_1 \to \infty$ and observing that the roles of $u_1, \dots, u_N$ can be interchanged, we arrive at

$$
\int_\Omega \Big(\sum_{i,j=1}^d a_{ij} D_j u_\nu D_i \tilde\psi + \kappa_\nu u_\nu \tilde\psi \Big)\, dx \le \int_\Omega H_\nu(u,\nabla u)\tilde\psi\, dx \tag{4.19}
$$

for all $\nu = 1, \dots, N$.

Next, in the same spirit, we let $n \to \infty$ in (4.3) to observe (for sufficiently large λ, cp. (2.12))

$$
\begin{aligned}
&-\lambda^{-1} \int_\Omega \sum_{i,j=1}^d a_{ij} D_j w_\lambda D_i \psi\, dx + \int_\Omega \sum_{\nu=1}^N q_\nu \kappa_\nu u_\nu w_\lambda \psi\, dx \\
&\ge \int_\Omega \sum_{\nu=1}^N q_\nu H_\nu(u,\nabla u) w_\lambda \psi\, dx - \int_\Omega \sum_{i,j=1}^d a_{ij} D_j \Big(\sum_{\nu=1}^N q_\nu u_\nu\Big) D_i w_\lambda \psi\, dx,
\end{aligned} \tag{4.20}
$$

where $w_\lambda(x) := e^{-\lambda \sum_{\nu=1}^N q_\nu u_\nu(x)}$ is defined as before in (2.11). Thus, setting

$$\psi := e^{\lambda \sum_{\nu=1}^N q_\nu u_\nu} \tilde{\psi},$$

in (4.20) with an arbitrary nonnegative function $\tilde{\psi} \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ vanishing in a neighborhood of Γ_D, we get

$$\sum_{\nu=1}^N \int_\Omega \Big(q_\nu \sum_{i,j=1}^d a_{ij} D_j u_\nu D_i \tilde{\psi} + q_\nu \kappa_\nu u_\nu \tilde{\psi} \Big)\, dx \geq \int_\Omega \sum_{\nu=1}^N q_\nu H_\nu(u, \nabla u) \tilde{\psi}\, dx. \tag{4.21}$$

Hence, combining (4.21) with (4.19) for $\nu = 2, \dots, N$ and using the nonnegativity of q_ν, we observe

$$\begin{aligned}
\int_\Omega \sum_{\nu=1}^N q_\nu H_\nu(u, \nabla u) \tilde{\psi}\, dx &\leq \sum_{\nu=1}^N \int_\Omega q_\nu \Big(\sum_{i,j=1}^d a_{ij} D_j u_\nu D_i \tilde{\psi} + \kappa_\nu u_\nu \tilde{\psi} \Big)\, dx \\
&= \int_\Omega q_1 \Big(\sum_{i,j=1}^d a_{ij} D_j u_1 D_i \tilde{\psi} + \kappa_1 u_1 \tilde{\psi} \Big)\, dx \\
&\quad + \sum_{\nu=2}^N \int_\Omega q_\nu \Big(\sum_{i,j=1}^d a_{ij} D_j u_\nu D_i \tilde{\psi} + \kappa_\nu u_\nu \tilde{\psi} \Big)\, dx \\
&\leq \int_\Omega q_1 \Big(\sum_{i,j=1}^d a_{ij} D_j u_1 D_i \tilde{\psi} + \kappa_1 u_1 \tilde{\psi} \Big)\, dx + \int_\Omega \sum_{\nu=2}^N q_\nu H_\nu(u, \nabla u) \tilde{\psi}\, dx,
\end{aligned}$$

which leads (via the positivity of q_1) to

$$\int_\Omega H_1(u, \nabla u) \tilde{\psi}\, dx \leq \int_\Omega \Big(\sum_{i,j=1}^d a_{ij} D_j u_1 D_i \tilde{\psi} + \kappa_1 u_1 \tilde{\psi} \Big)\, dx.$$

Consequently, it directly follows from (4.19) for $\nu = 1$ that in fact equality holds, i.e. we have

$$\int_\Omega H_1(u, \nabla u) \tilde{\psi}\, dx = \int_\Omega \Big(\sum_{i,j=1}^d a_{ij} D_j u_1 D_i \tilde{\psi} + \kappa_1 u_1 \tilde{\psi} \Big) dx \tag{4.22}$$

for all nonnegative $\tilde{\psi} \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ vanishing near Γ_D. Since any $\tilde{\psi}$ can be decomposed into the positive and negative part, it is evident that (4.22) holds for all $\tilde{\psi}$. Finally, this can be repeated for any $\nu > 1$ to deduce the identity (1.16), and thus, the identification of $\overline{H}$ is complete.

Strong convergence of ∇u. To finish the proof, we still need to show (1.23). To this end, exactly as for the derivation of (4.1) (now with u instead of u^n), we deduce from the weak formulation (1.16), which is obtained from (4.14) combined with (4.16), the identity

$$\begin{aligned}
&\sum_{i,j=1}^d \int_\Omega a_{ij} D_j \varphi_1(u) D_i \psi\, dx + \sum_{\nu=1}^N \int_\Omega \kappa_\nu u_\nu D_{u_\nu} \varphi_1(u) \psi\, dx \\
&= \sum_{\nu=1}^N \int_\Omega \Big(D_{u_\nu} \varphi_1(u) H_\nu(u, \nabla u) - \sum_{i,j=1}^d a_{ij} D_j u_\nu D_i D_{u_\nu} \varphi_1(u) \Big) \psi\, dx
\end{aligned}$$

for all $\psi \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ vanishing in a neighborhood of Γ_D, and by approximation also for $\psi = d_\eta^2$ for any $\eta > 0$. Hence, using (4.8)–(4.10) we can let $n \to \infty$ in (4.1) to conclude

$$\lim_{n\to\infty} \sum_{\nu=1}^{N} \int_\Omega \Big(D_{u_\nu}\varphi_1(u^n) H_\nu^n(u^n, \nabla u^n) - \sum_{i,j=1}^{d} a_{ij}^n D_j u_\nu^n D_i D_{u_\nu}\varphi_1(u^n) \Big) \psi \, dx$$
$$= \sum_{\nu=1}^{N} \int_\Omega \Big(D_{u_\nu}\varphi_1(u) H_\nu(u, \nabla u) - \sum_{i,j=1}^{d} a_{ij} D_j u_\nu D_i D_{u_\nu}\varphi_1(u) \Big) \psi \, dx.$$

In view of the choices (3.12) of the constants c_ν, the latter integrand is bounded from above by $\sum_{\nu=1}^N K^* D_{u_\nu}\varphi_1(u^n)$, and hence we obtain from (4.15) the strong convergence

$$\sum_{\nu=1}^{N} \Big(D_{u_\nu}\varphi_1(u^n) H_\nu^n(u^n, \nabla u^n) - \sum_{i,j=1}^{d} a_{ij}^n D_j u_\nu^n D_i D_{u_\nu}\varphi_1(u^n) - K^* D_{u_\nu}\varphi_1(u^n) \Big) d_\eta^2$$
$$\to \sum_{\nu=1}^{N} \Big(D_{u_\nu}\varphi_1(u) H_\nu(u, \nabla u) - \sum_{i,j=1}^{d} a_{ij} D_j u_\nu D_i D_{u_\nu}\varphi_1(u) - K^* D_{u_\nu}\varphi_1(u) \Big) d_\eta^2$$

in $L^1(\Omega)$ for all $\eta > 0$. Consequently, using (2.3), (4.15) (combined with the choice (3.12) of the constants c_ν) and a variant of Lebesgue's dominated convergence theorem, we easily deduce the strong convergence $\nabla u^n d_\eta \to \nabla u d_\eta$ in $L^2(\Omega; \mathbb{R}^{d\times N})$ for all $\eta > 0$ and hence in particular (1.23). This concludes the proof of Theorem 1.6.

5. Proof of Theorem 1.7

In this section we prove the main theorem of the paper on uniform smallness of the Dirichlet integral, i.e., Theorem 1.7. First, we give the reduction to interior estimates via a reflection method near the Neumann boundary. Then we focus on the key estimate (1.24). Note that in case of smoother data, we could directly use the indirect approach and Theorems 1.4–1.6 to get the desired result. In case of general data such a simple argument is not possible. However, we can still follow a similar strategy of proof as for the Liouville theorem to recover the estimate (1.24). Finally, we will establish the uniform regularity improvement from continuity to Hölder continuity.

Reflection near the Neumann boundary Γ_N. Before starting the proof of (1.24), we focus on the behavior of the solution near the Neumann part of the boundary Γ_N in order to avoid the difficulties with the localization in what follows. Since Ω is Lipschitz, for any relatively open set $\Gamma_0 \subset\subset \Gamma_N$ there exist $\alpha, \beta > 0$, $m \in \mathbb{N}$, m coordinate systems, m functions $b_k \in \mathcal{C}^{0,1}([-\alpha, \alpha]^{d-1})$ with $k = 1, \dots, m$ and open sets $\{V_k\}_{k=1}^m$ in $\mathbb{R}^d$ which cover Γ_0, i.e. $\Gamma_0 \subset \bigcup_{k=1}^m V_k$, and such that (after a possible change of coordinates):

$$\begin{aligned}
V_k^+ &:= \{x = (x', x_d) \in \mathbb{R}^d; |x'| < \alpha \text{ and } b_k(x') < x_d < b_k(x') + \beta\} \subset \Omega\,,\\
V_k^- &:= \{x = (x', x_d) \in \mathbb{R}^d; |x'| < \alpha \text{ and } b_k(x') - \beta < x_d < b_k(x')\} \subset \mathbb{R}^d \setminus \overline{\Omega}\,,\\
V_k^= &:= \{x = (x', x_d) \in \mathbb{R}^d; |x'| < \alpha \text{ and } b_k(x') = x_d\} \subset \Gamma_0\,,\\
V_k &:= V_k^+ \cup V_k^- \cup V_k^=\,,
\end{aligned}$$

hold for each $k = 1, \dots m$. Our goal is to show that for any V_k^+ we can extend a weak solution of (1.1) to V_k^- such that it is a solution of the system under consideration on the whole set V_k. Hence, we fix some $k \in \{1, \dots, m\}$ and omit writing this index in what follows. Let us first introduce the (surjective) Lipschitz continuous mapping $T : V^- \to V^+$ by

$$(T(x))_i := \begin{cases} x_i & i = 1, \dots, d-1,\\ -x_d + 2b(x') & i = d. \end{cases}$$

Then, it directly follows from the definition that

$$(A(x'))_{ij} := D_j(T(x))_i = \begin{cases} = \delta_{ij} & i \neq d, \\ = 2D_j b(x') & i = d,\, j \neq d, \\ = -1 & i = j = d. \end{cases}$$

Note that the Jacobian of T is identically equal to one and the same also holds for the Lipschitz continuous inverse T^{-1}. In addition, since b is Lipschitz, the matrix A is bounded. Also, it is evident that for all $x_0 \in V^=$ we have $\lim_{x\to x_0} T(x) = x_0$. Finally, let us assume that u is a generalized weak solution in the sense of Definition 1.2. Then necessarily, for all $\varphi \in L^\infty(V) \cap W_0^{1,2}(V)$ the following holds

$$\int_{V^+} \left(\sum_{i,j=1}^d a_{ij} D_j u_\nu D_i \varphi + \kappa_\nu u_\nu \varphi \right) dx = \int_{V^+} H_\nu(u, \nabla u) \varphi \, dx. \tag{5.1}$$

Then we find $\tilde{a}$, $\tilde{u}$ and $\tilde{H}$ such that they are equal to a, u, H in V^+ and for $x \in V^-$ are defined as:

$$\begin{aligned} \tilde{u}(x) &:= u(T(x)), \\ \tilde{a}_{ij}(x) &:= \sum_{k,\ell=1}^d A^{-1}_{ik}(x') A^{-1}_{j\ell}(x') a_{k\ell}(T(x)), \\ \tilde{H}(x,u,z) &:= H(T(x), u, (A^{-1})^T(x') z). \end{aligned}$$

Then due to the properties of T, it is evident that $\tilde{u} \in L^\infty(V; \mathbb{R}^N) \cap W^{1,2}(V; \mathbb{R}^N)$. In addition, $\tilde{H} \colon V \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to \mathbb{R}^N$ remains Carathéodory and due to the fact that A is bounded, the properties (1.4), (1.8) and (1.9) are still valid. Moreover, in view of (1.5) we also have $\tilde{a}_{ij} \in L^\infty(V)$. In order to check also the ellipticity condition (1.6), we notice

$$\sum_{i,j=1}^d \tilde{a}_{ij}(x) z_i z_j \geq \delta |A^{-1}(x') z|^2$$

for almost all $x \in V^-$. Since the determinant of A is identically -1 (and thus A is regular) and A^{-1} is uniformly bounded, we conclude that for some $\tilde{\delta} > 0$

$$\sum_{i,j=1}^d \tilde{a}_{ij}(x) z_i z_j \geq \tilde{\delta} |z|^2$$

for almost all $x \in V$. Consequently, (1.6) holds with $\tilde{\delta}$ instead of δ. Finally, for $\varphi \in L^\infty(V) \cap W_0^{1,2}(V)$ we get via the weak formulation (5.1)

$$\begin{aligned} &\int_V \left(\sum_{i,j=1}^d \tilde{a}_{ij}(x) D_j \tilde{u}_\nu D_i \varphi + \kappa_\nu \tilde{u}_\nu \varphi - \tilde{H}_\nu(\tilde{u}, \nabla \tilde{u}) \varphi \right) dx \\ &= \int_{V^-} \left(\sum_{i,j=1}^d \tilde{a}_{ij} D_j \tilde{u}_\nu D_i \varphi + \kappa_\nu \tilde{u}_\nu \varphi - \tilde{H}_\nu(\tilde{u}, \nabla \tilde{u}) \varphi \right) dx. \end{aligned}$$

Next, we can find $\psi \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ such that for almost all $x \in V^{-1}$ there holds $\varphi(x) = \psi(T(x))$. Therefore, by using also the definition of $\tilde{a}$, $\tilde{u}$ and $\tilde{H}$ and thanks to the properties of

T, the above relation reduces to

$$
\begin{aligned}
&\int_V \left(\sum_{i,j=1}^{d} \tilde{a}_{ij}(x) D_j \tilde{u}_\nu D_i \varphi + \kappa_\nu \tilde{u}_\nu \varphi - \tilde{H}_\nu(\tilde{u}, \nabla \tilde{u}) \varphi \right) dx \\
&= \int_{V^-} \left(\sum_{i,j,k,\ell=1}^{d} A^{-1}_{ik}(x') A^{-1}_{j\ell}(x') a_{k\ell}(T(x)) D_{x_j} u_\nu(T(x)) D_{x_i} \psi(T(x)) + \kappa_\nu u_\nu(T(x)) \psi(T(x)) \right) dx \\
&\quad - \int_{V^-} H_\nu(T(x), u(T(x)), (A^{-1})^T(x') \nabla_x u(T(x))) \psi(T(x))\, dx \\
&= \int_{V^-} \left(\sum_{i,j=1}^{d} a_{ij}(T(x)) D_{(T(x))_j} u_\nu(T(x)) D_{(T(x))_i} \psi(T(x)) + \kappa_\nu u_\nu(T(x)) \psi(T(x)) \right) dx \\
&\quad - \int_{V^-} H_\nu(T(x), u(T(x)), \nabla_{T(x)} u(T(x))) \psi(T(x))\, dx \\
&= \int_{V^+} \left(\sum_{i,j=1}^{d} a_{ij} D_j u_\nu D_i \psi + \kappa_\nu u_\nu \psi \right) dx - \int_{V^+} H_\nu(u, \nabla u) \psi \, dx \overset{(5.1)}{=} 0.
\end{aligned}
$$

Consequently, we see that that $\tilde{u}$ is a weak solution in V (without prescribed boundary values on ∂V).

Proof of the uniform smallness (1.24) *of the Dirichlet integral.* Since we have already observed that we can extend the solution by reflection outside Ω in a neighborhood of Γ_N, we can restrict ourselves to the treatment of only interior estimates. Consequently, we now focus on the proof (1.24) only on the interior of Ω. We proceed by contradiction, hence, we assume that there exist an open set $\Omega_0 \subset \Omega$, a number $\varepsilon > 0$, a decreasing sequence of numbers $R_n \searrow 0$ in $(0,1)$, a sequence of points $\{x_0^n\}$ in Ω_0 and a sequence of (K^*, K, δ)-admissible representations $\{H^n, a^n, \kappa^n, q^n\}$ of systems of the type (1.1) with an associated sequence of generalized weak solutions $\{u^n\}$ such that $B_{R_n}(x_0^n) \subset \Omega$ and

$$
\int_{B_{R_n}(x_0^n)} \frac{|\nabla u^n(x)|^2}{R_n^{d-2}}\, dx > \varepsilon \tag{5.2}
$$

hold. Then we introduce the scaled function v^n as

$$
v^n(x) := \begin{cases} u^n(x_0^n + R_n x) & \text{for all } x \in \Omega^n := \{x \in \mathbb{R}^d; (x_0^n + R_n x) \in \Omega\}, \\ 0 & \text{otherwise,} \end{cases}
$$

for which the condition (5.2) can be rewritten as

$$
\int_{B_1(0)} |\nabla v^n(x)|^2\, dx > \varepsilon. \tag{5.3}
$$

Next, we deduce from the weak formulation (1.16) satisfied by the functions u^n the corresponding (rescaled) identity for v^n. To this end, for an arbitrary function $\psi_R \in W^{1,2}_{\Gamma_D}(\Omega) \cap L^\infty(\Omega)$ vanishing in a neighborhood of Γ_D, we introduce the scaled function ψ as

$$
\psi(x) := \psi_R(x_0^n + R_n x) \Leftrightarrow \psi_R(x) := \psi\left(R_n^{-1}(x - x_0^n)\right)
$$

which belongs to $W^{1,2}_{\Gamma_D^n}(\Omega^n) \cap L^\infty(\Omega^n)$ by definition of Ω^n and with

$$
\Gamma_D^n := \left\{x \in \mathbb{R}^d;\ (x_0^n + R_n x) \in \Gamma_D\right\}.
$$

Thus, setting $\varphi := \psi_R$ in the system of type (1.16) satisfied by u^n and using the substitution theorem, we find

$$\int_{\Omega^n} \sum_{i,j=1}^d a_{ij}^{R_n}(x) D_j v_\nu^n(x) D_i \psi(x)\, dx + R_n^2 \int_{\Omega^n} \kappa_\nu^n v_\nu^n(x)\psi(x)\, dx = \int_{\Omega^n} H_\nu^{R_n}(x, v^n(x), \nabla v^n(x))\psi(x)\, dx$$

for $\nu = 1, \dots, N$, where

$$a_{ij}^{R_n}(x) := a_{ij}^n(x_0^n + R_n x),$$
$$H_\nu^{R_n}(x,u,z) := R_n^2 H_\nu^n(x_0^n + R_n x, u, R_n^{-1} z).$$

It is evident from the definition that a^{R_n} still satisfies (1.5)–(1.6) and that H^{R_n} satisfies (1.4) and also

$$H_\nu^{R_n}(x,w,z) \le K_n^* + K|z_\nu||z|,$$
$$\sum_{\nu=1}^N q_\nu^n H_\nu^{R_n}(x,w,z) \ge -K_n^* - K\left|\sum_{\nu=1}^N q_\nu^n z_\nu\right|^2,$$

where $K_n^* := K^* R_n^2$. Our goal is to show

$$v^n \to 0 \qquad \text{in } W^{1,2}_{loc}(\mathbb{R}^d; \mathbb{R}^N) \tag{5.4}$$

(note that for each positive L we have $B_L(0) \subset \Omega^n$ for n sufficiently large), which directly contradicts (5.3) and thus finishes the proof of (1.24).

Thus, we now focus on the proof of (5.4). Since we assume that the functions u^n are uniformly bounded by M (i.e., (1.22)), we can use the same procedure as in Section 4 to derive the a priori estimates and consequently to find a subsequence (not relabeled) such that

$$x_0^n \to x_0 \qquad \text{in } \overline{\Omega}_0, \tag{5.5}$$

$$v^n \rightharpoonup v \qquad \text{weakly in } W^{1,2}_{loc}(\mathbb{R}^d, \mathbb{R}^N), \tag{5.6}$$

$$v^n \rightharpoonup^* v \qquad \text{weakly}^* \text{ in } L^\infty(\mathbb{R}^d, \mathbb{R}^N), \tag{5.7}$$

$$|\nabla v^n| \rightharpoonup \overline{|\nabla v|} \qquad \text{weakly in } L^2_{loc}(\mathbb{R}^d). \tag{5.8}$$

At this step we mimic and combine the proofs presented in Sections 3–4, with the essential difference that now, for the Liouville-type arguments, we have to deal with nonlinearities H^n with constants K_n^* for the lower-order growth which vanish only in the limit $n \to \infty$. First, for $d = 2$ we set $G_y^n := 1$, and for $d \ge 3$ we denote by G_y^n the Green function corresponding to a^{R_n} centered in y, i.e., a function solving

$$-\sum_{i,j=1}^d D_j(a_{ij}^{R_n} D_i G_y^n) = \delta_y$$

and vanishing for $|x| \to \infty$. Moreover, we use the notation $G^{n,\rho}$ for its ρ-approximation, i.e., the solution to

$$-\sum_{i,j=1}^d D_j(a_{ij}^{R_n} D_i G_y^{n,\rho}) = \frac{\chi_{B_\rho(y)}}{|B_\rho(y)|}$$

which vanishes for $|x| \to \infty$. We also recall the uniform growth and weighted integrability properties of the Green functions of the form (2.17) and (2.18) (now with $x - y$ instead of x) which follow from the uniform boundedness and ellipticity of the matrices a^{R_n}, while the approximations $G^{n,\rho}$ satisfy the analogous inequalities

$$\frac{1}{C|x-y|^{d-2}} \leq G_y^{n,\rho}(x) \leq \frac{C}{|x-y|^{d-2}} \quad \text{and} \quad \int_{A_R(y)} \frac{|\nabla G_y^{n,\rho}(x)|^2}{R^{d-2}}\, dx \leq C \tag{5.9}$$

for all $x \in \mathbb{R}^d$ with $|x-y| > 2\rho$ and all $R > 2\rho$, with a uniform constant C depending only on d, K and δ. Secondly, applying the same procedure as in Section 3, we deduce (compare (3.7) and (4.2)) that for all $\Omega' \subset\subset \Omega^n$, the choice (2.1) for the functions φ_ν and all nonnegative functions $\psi \in W_0^{1,2}(\Omega') \cap L^\infty(\Omega')$ there holds

$$\begin{aligned} &\int_{\Omega^n} \Big(\sum_{i,j=1}^d a_{ij}^{R_n} D_j\varphi_1(v^n) D_i\psi + \delta \sum_{\nu=1}^N Z_\nu(v^n)\gamma_\nu''(v_\nu^n)|\nabla v_\nu^n|^2 \prod_{\mu=1}^\nu \varphi_\mu(v^n)\psi \Big)\, dx \\ &\qquad \leq \sum_{\nu=1}^N \int_{\Omega^n} R_n^2(K^* - \kappa_\nu^n v_\nu^n) D_{u_\nu}\varphi_1(v^n)\psi\, dx. \end{aligned} \tag{5.10}$$

Hence, similarly as before, setting $\gamma_\nu(v_\nu^n) := e^{v_\nu^n + M + c_\nu}$, keeping the notation from (3.9) and defining c_ν in the same way as in (3.12), we see that (5.10) leads to

$$\int_{\Omega^n} \Big(\sum_{i,j=1}^d a_{ij}^{R_n} D_j\varphi_1(v^n) D_i\psi + \frac{\delta}{2}|\nabla v^n|^2\psi \Big)\, dx \leq C(c_1, N, \delta, K, K^*, M) \int_{\Omega^n} R_n^2 \psi\, dx. \tag{5.11}$$

Next, for arbitrary $y \in \mathbb{R}^d$ and $R > 0$ we denote by $\tau_{R;y}$ a nonnegative function in $\mathcal{D}(B_{2R})$ with $\tau_{R;y} \equiv 1$ in $B_R(y)$ and $|\nabla \tau_{R;y}| \leq CR^{-1}$. Then, we choose $\psi := G_y^{n,\rho}\tau_{R;y}^2$ in (5.11) (we assume from here that $d \geq 3$, since the proof for $d = 2$ is easier), which is possible since the domains Ω^n exhaust $\mathbb{R}^d$, and we obtain in this way the identity

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{\delta}{2}|\nabla v^n|^2 G_y^{n,\rho}\tau_{R;y}^2\, dx &\leq C \int_{\mathbb{R}^d} R_n^2 G_y^{n,\rho}\tau_{R;y}^2\, dx \\ &\qquad - \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}^{R_n} D_j\varphi_1(v^n) D_i(G_y^{n,\rho}\tau_{R;y}^2)\, dx, \end{aligned} \tag{5.12}$$

with C still depending only on c_1, N, δ, K, K^* and M. Using the definition of $G^{n,\rho}$, the estimate (5.9) and the Hölder inequality, we can evaluate and estimate the last term for $R \geq \rho$ as

$$\begin{aligned}
&\int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}^{R_n} D_j\varphi_1(v^n) D_i(G_y^{n,\rho}\tau_{R;y}^2)\,dx = \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}^{R_n} D_j\varphi_1(v^n) D_i G_y^{n,\rho}\tau_{R;y}^2\,dx \\
&\quad + 2\int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}^{R_n} D_j\varphi_1(v^n)\tau_{R;y} D_i\tau_{R;y} G_y^{n,\rho}\,dx \\
&= \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}^{R_n} D_j(\tau_{R;y}^2\varphi_1(v^n)) D_i G_y^{n,\rho}\,dx - \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}^{R_n} D_j(\tau_{R;y}^2(\varphi_1(v^n))_{A_R(y)}) D_i G_y^{n,\rho}\,dx \\
&\quad - \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}^{R_n} D_j(\tau_{R;y}^2) D_i G_y^{n,\rho}(\varphi_1(v^n) - (\varphi_1(v^n))_{A_R(y)})\,dx \\
&\quad + 2\int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}^{R_n} D_j\varphi_1(v^n)\tau_{R;y} D_i\tau_{R;y} G_y^{n,\rho}\,dx \\
&\geq (\varphi_1(v^n))_{B_\rho(y)} - (\varphi_1(v^n))_{A_R(y)} - C\Big(\int_{A_R(y)} \frac{|\varphi_1(v^n) - (\varphi_1(v^n))_{A_R(y)}|^2}{R^d}\,dx\Big)^{\frac{1}{2}} \\
&\quad - C\int_{A_R(y)} \frac{|\nabla\varphi_1(v^n)|\tau_{R;y}}{R^{d-1}}\,dx,
\end{aligned}$$

with $C = C(c_1, d, N, \delta, K, K^*, M)$. Consequently, plugging this estimate into (5.12), we obtain

$$\begin{aligned}
&(\varphi_1(v^n))_{B_\rho(y)} + \int_{\mathbb{R}^d} \frac{\delta}{2}|\nabla v^n|^2 G_y^{n,\rho}\tau_{R;y}^2\,dx \\
&\leq (\varphi_1(v^n))_{A_R(y)} + CR_n^2\int_{\mathbb{R}^d} G_y^{n,\rho}\tau_{R;y}^2\,dx + C\int_{A_R(y)} \frac{|\nabla\varphi_1(v^n)|\tau_{R;y}}{R^{d-1}}\,dx \\
&\quad + C\Big(\int_{A_R(y)} \frac{|\varphi_1(v^n) - (\varphi_1(v^n))_{A_R(y)}|^2}{R^d}\,dx\Big)^{\frac{1}{2}}.
\end{aligned} \tag{5.13}$$

Hence, taking advantage of the uniform L^∞ bound for v^n and (5.9), we get in the limit $\rho \to 0_+$

$$\int_{\mathbb{R}^d} \frac{|\nabla v^n|^2\tau_{R;y}^2}{|x-y|^{d-2}}\,dx \leq CR^2R_n^2 + C + C\int_{A_R(y)} \frac{|\nabla\varphi_1(v^n)|\tau_{R;y}}{|x-y|^{d-1}}\,dx.$$

Then, using the point-wise estimate $|\nabla\varphi_1(v^n)| \leq C|\nabla v^n|$ and Young's inequality, we gain

$$\int_{\mathbb{R}^d} \frac{|\nabla v^n|^2\tau_{R;y}^2}{|x-y|^{d-2}}\,dx \leq CR^2R_n^2 + C.$$

Therefore, letting $n \to \infty$, we observe from weak lower semicontinuity

$$\int_{\mathbb{R}^d} \frac{|\nabla v|^2\tau_{R;y}^2}{|x-y|^{d-2}}\,dx \leq \int_{\mathbb{R}^d} \frac{(\overline{|\nabla v|})^2\tau_{R;y}^2}{|x-y|^{d-2}}\,dx \leq C,$$

which directly implies (by letting $R \to \infty$) that for all $y \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x-y|^{d-2}}\,dx \leq \int_{\mathbb{R}^d} \frac{(\overline{|\nabla v|})^2}{|x-y|^{d-2}}\,dx \leq C, \tag{5.14}$$

where C still depends only on $c_1, d, N, \delta, K, K^*$ and M. Next, going back to (5.13), letting $n \to \infty$, using the convergence results (5.6)–(5.8) and the compact embedding, we gain

$$\begin{aligned} &(\varphi_1(v))_{B_\rho(y)} + \limsup_{n\to\infty} \int_{\mathbb{R}^d} \frac{\delta}{2} |\nabla v^n|^2 G_y^{n,\rho} \tau_{R;y}^2 \, dx \\ &\quad \le (\varphi_1(v))_{A_R(y)} + C \int_{A_R(y)} \frac{\overline{|\nabla v|}}{R^{d-1}} \, dx + C\Big(\int_{A_R(y)} \frac{|\varphi_1(v) - (\varphi_1(v))_{A_R(y)}|^2}{R^d} \, dx \Big)^{\frac{1}{2}}. \end{aligned} \tag{5.15}$$

Finally, estimating the last term on the right-hand side via Poincaré's and the Hölder's inequality and dropping the second term on the left-hand side, we find

$$(\varphi_1(v))_{B_\rho(y)} \le (\varphi_1(v))_{A_R(y)} + C\Big(\int_{A_R(y)} \frac{|\nabla v|^2 + (\overline{|\nabla v|})^2}{|x-y|^{d-2}} \, dx \Big)^{\frac{1}{2}}.$$

Therefore, using (5.14) and letting $R \to \infty$, we deduce

$$(\varphi_1(v))_{B_\rho(y)} \le \liminf_{R\to\infty} (\varphi_1(v))_{A_R(y)} = \liminf_{R\to\infty} (\varphi_1(v))_{A_R}, \tag{5.16}$$

where the second relation follows from the Poincaré inequality and the estimate (5.14) (see also Section 3). Since (5.16) holds for all y and all ρ, we have

$$\varphi_1(v(x)) \le \liminf_{R\to\infty} (\varphi_1(v))_{A_R}$$

for almost all $x \in R^d$. Hence, we are exactly in the same position as in (3.13) in Section 3, and by the same procedure we then deduce

$$v = const \qquad \text{in } \mathbb{R}^d.$$

Since v is constant, the inequality (5.15) implies

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} \frac{\delta}{2} |\nabla v^n|^2 G_y^{n,\rho} \tau_{R;y}^2 \, dx \le C \int_{A_R(y)} \frac{\overline{|\nabla v|} \tau_{R;y}}{R^{d-1}} \, dx,$$

and letting $R \to \infty$, we obtain from (5.14)

$$\limsup_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^d} |\nabla v^n|^2 G_y^{n,\rho} \tau_{R;y}^2 \, dx \le 0.$$

However, this implies in particular

$$\limsup_{n\to\infty} \int_{B_1(0)} |\nabla v^n|^2 \, dx = 0,$$

which contradicts (5.3), and thus the proof of (1.24) is complete.

Proof of the Hölder estimate (1.25). We now start from a continuous solution u. First, following [8], we can show the existence of positive number α depending only on δ, K and K^* such that u is actually α-Hölder continuous, but the constant C in the corresponding estimate (1.25) depends on the modulus of continuity of u. In addition, one can show that for any open $\Omega_0 \subset \Omega$ and all $x_0 \in \Omega_0$ we have

$$\int_{B_R(x_0)} \frac{|\nabla u|^2}{R^{d-2+2\alpha}} \, dx \le C \tag{5.17}$$

with C depending in particular on this modulus of continuity. Hence, in what follows, we show that (1.25) and (5.17) hold with a constant which does not depend on the modulus of continuity of u. To prove this, we denote

$$\begin{aligned} S(x_0,\varrho_0,\alpha) &:= \sup_{\varrho\in(0,\varrho_0)} \int_{B_\varrho(x_0)} \frac{|\nabla u(x)|^2}{\varrho^{d-2+2\alpha}}\,dx,\\ \mathcal{S}(\varrho_0,\alpha) &:= \sup_{x_0\in\Omega_0} S(x_0,\varrho_0,\alpha),\\ |u|_{\alpha,\varrho_0} &:= \sup_{x,y\in\Omega_0;\,0<|x-y|<\varrho_0} \frac{|u(x)-u(y)|}{|x-y|^\alpha}. \end{aligned}$$

Note that the Morrey embedding gives

$$|u|^2_{\alpha,\varrho_0} \le C(d,\Omega_0)\mathcal{S}(\varrho_0,\alpha). \tag{5.18}$$

Since we have the uniform smallness (1.24), we can continue by the Campanato comparison method. To this end, we assume from now on $\varrho, R < \varrho_0 \le 1$ with ϱ_0 so small such that for all $x_0 \in \Omega_0$ we have $B_{4\varrho_0}(x_0) \subset \Omega$. Hence, for arbitrary x_0 and R, we find N auxiliary functions $h_\nu \in u_\nu + W^{1,2}_0(B_R(x_0))$ solving

$$\sum_{i,j=1}^d D_i(a_{ij}D_jh_\nu) = 0 \quad \text{in } B_R(x_0) \tag{5.19}$$

for all $\nu = 1,\dots,N$. From the standard theory for linear equations with bounded and elliptic coefficients (1.5)–(1.6), we know that h is locally Hölder continuous with

$$\int_{B_\varrho(x_0)} |\nabla h|^2\,dx \le C\left(\frac{\varrho}{R}\right)^{d-2+2\alpha_0}\int_{B_R(x_0)} |\nabla u|^2\,dx \qquad \text{for all } \varrho\le R, \tag{5.20}$$

$$\inf_{y\in B_R(x_0)} u_\nu(y) \le h_\nu(x) \le \sup_{y\in B_R(x_0)} u_\nu(y) \qquad \text{for all } x\in B_R(x_0), \tag{5.21}$$

for all $\nu = 1,\dots,N$, where α_0 and C depend only on d, δ and K. Next, using (1.5)–(1.6) and (5.20), we get that

$$\begin{aligned} &\frac{\delta}{2}\int_{B_\varrho(x_0)} |\nabla u|^2\,dx\\ &\le \sum_{\nu=1}^N\sum_{i,j=1}^d \int_{B_\varrho(x_0)} a_{ij}D_j(u_\nu-h_\nu)D_i(u_\nu-h_\nu)\,dx + C\int_{B_\varrho(x_0)}|\nabla h|^2\,dx\\ &\le \sum_{\nu=1}^N\sum_{i,j=1}^d \int_{B_R(x_0)} a_{ij}D_j(u_\nu-h_\nu)D_i(u_\nu-h_\nu)\,dx + C\left(\frac{\varrho}{R}\right)^{d-2+2\alpha_0}\int_{B_R(x_0)}|\nabla u|^2\,dx. \end{aligned} \tag{5.22}$$

We now estimate the first term on the right-hand side by subtracting (5.19) from (1.1), multiplying the result by $u-h$ and integrating over $B_R(x_0)$ (note that the boundary term vanishes). Using also (1.22), (1.4) and (5.21), we then deduce

$$\begin{aligned} &\sum_{\nu=1}^N\sum_{i,j=1}^d \int_{B_R(x_0)} a_{ij}D_j(u_\nu-h_\nu)D_i(u_\nu-h_\nu)\,dx\\ &= \sum_{\nu=1}^N \int_{B_R(x_0)} \kappa_\nu u_\nu(h_\nu-u_\nu) + H_\nu(u,\nabla u)(u_\nu-h_\nu)\,dx\\ &\le C(d,K,K^*,M)R^d + C(d,\delta,K)R^\alpha |u|_{\alpha,R}\int_{B_R(x_0)}|\nabla u|^2\,dx \end{aligned}$$

for any $\alpha \in (0, \alpha_0)$. Thus, substituting this into (5.22) we gain

$$\begin{aligned}\int_{B_\varrho(x_0)} &|\nabla u|^2\,dx \\ &\le CR^d + C\left(\frac{\varrho}{R}\right)^{d-2+2\alpha_0}\int_{B_R(x_0)}|\nabla u|^2 + CR^\alpha|u|_{\alpha,R}\int_{B_R(x_0)}|\nabla u|^2\,dx,\end{aligned} \tag{5.23}$$

with $C = C(d,\delta,K,K^*,M)$. In order to finish the proof we fix some $\alpha < \alpha_0$ and determine $\varrho_1 < \varrho_0$ such that

$$C\Big(\frac{\varrho_1}{\varrho_0}\Big)^{2\alpha_0-2\alpha} = \frac{1}{2}. \tag{5.24}$$

We now distinguish two cases. First, we assume that the supremum in the definition of $S(x_0,\varrho_0,\alpha)$ is attained for some radius $\varrho \in [\varrho_1,\varrho_0]$, which then implies

$$S(x_0,\varrho_0,\alpha) = \int_{B_\varrho(x_0)}\frac{|\nabla u|^2}{\varrho^{d-2+2\alpha}}\,dx \le C(\alpha,\alpha_0,\Omega_0)\varrho_0^{2-d-2\alpha}. \tag{5.25}$$

In the opposite case we set $R := \varrho\frac{\varrho_0}{\varrho_1} < \varrho_0$ in (5.23) and after division by $\varrho^{d-2+2\alpha}$ we get

$$\begin{aligned}\int_{B_\varrho(x_0)}\frac{|\nabla u|^2}{\varrho^{d-2+2\alpha}}\,dx &\le CR^{2-2\alpha} + \frac{1}{2}\int_{B_R(x_0)}\frac{|\nabla u|^2}{R^{d-2+2\alpha}}\,dx \\ &\quad + C|u|_{\alpha,\varrho_0}\Big(\int_{B_R(x_0)}\frac{|\nabla u|^2}{R^{d-2+2\alpha}}\,dx\Big)^{\frac12}\Big(\int_{B_R(x_0)}\frac{|\nabla u|^2}{R^{d-2}}\,dx\Big)^{\frac12}.\end{aligned}$$

Hence, taking the supremum over ϱ on the both sides, we gain (note $\varrho_0 \le 1$)

$$\begin{aligned}S(x_0,\varrho_0,\alpha) &\le C + \frac{1}{2}S(x_0,\varrho_0,\alpha) \\ &\quad + C|u|_{\alpha,\varrho_0}\left(S(x_0,\varrho_0,\alpha)\right)^{\frac12}\left(\sup_{R\in(0,\varrho_0)}\int_{B_R(x_0)}\frac{|\nabla u|^2}{R^{d-2}}\,dx\right)^{\frac12}.\end{aligned} \tag{5.26}$$

Consequently, combining the estimates (5.25) and (5.26) in the two cases and using Young's inequality, we deduce

$$S(x_0,\varrho_0,\alpha) \le C + C\varrho_0^{2-d-2\alpha} + C|u|^2_{\alpha,\varrho_0}\sup_{R\in(0,\varrho_0)}\int_{B_R(x_0)}\frac{|\nabla u|^2}{R^{d-2}}\,dx.$$

Finally, taking the supremum over all $x_0 \in \Omega_0$ and invoking (5.18), we observe

$$\mathcal{S}(\varrho_0,\alpha) \le C + C\varrho_0^{2-d-2\alpha} + C\mathcal{S}(\varrho_0,\alpha)\sup_{x_0\in\Omega_0}\sup_{R\in(0,\varrho_0)}\int_{B_R(x_0)}\frac{|\nabla u|^2}{R^{d-2}}\,dx. \tag{5.27}$$

with $C = C(d,\delta,K,K^*,M,\alpha,\alpha_0,\Omega_0)$. Now we are in the position to take advantage of the uniform smallness (1.24). Indeed, for ϱ_0 small enough we have

$$C\sup_{x_0\in\Omega_0}\sup_{R\in(0,\varrho_0)}\int_{B_R(x_0)}\frac{|\nabla u|^2}{R^{d-2}}\,dx \le \frac{1}{2}.$$

Consequently, this choice of ϱ_0 combined with (5.27) gives

$$\mathcal{S}(\varrho_0,\alpha) \le C + C\varrho_0^{2-d-2\alpha},$$

and by using (5.18) once again the estimate (1.25) follows. Hence the proof is complete.

6. Proof of Theorem 1.3

In this final subsection we prove Theorem 1.3. For this purpose, we consider a sequence of regularized Hamiltonians of the form

$$H^n(x,u,z) := \frac{H(x,u,z)}{1+n^{-1}|H(x,u,z)|},$$

which obviously still fulfill (1.4), (1.8) and (1.9). Moreover, for the mixed boundary value problem (1.1)–(1.3) with these regularized Hamiltonians it is not difficult to find a solution which is in addition continuous. Then, relying on Theorem 1.6 and Theorem 1.7, we let $n \to \infty$ and observe that the limit of the solutions to the regularized problems is a generalized weak solution to the original problem which in addition fulfills (1.25). Moreover, the $W^{2,q}$ theory then follows by a standard procedure and we do not provide the proof here. The only necessary assumption to check is the L^∞ bound, i.e., (1.22). We provide here only the formal proof. First we assume that u_ν attains its maximum in $x_0 \in \Omega$. Then surely $\nabla u_\nu(x_0) = 0$ holds and we deduce from (1.1) and (1.8) that (note that a_{ij} is positively definite)

$$\kappa_\nu u_\nu(x_0) \le H(x_0, u(x_0), \nabla u(x_0)) \le K^* + K|\nabla u_\nu(x_0)||\nabla u(x_0)| = K^*.$$

Hence, since the maximum is attained in x_0, we have

$$u_\nu \le \frac{K^*}{\kappa_\nu} \le \frac{K^*}{\delta} \qquad \text{in } \Omega. \tag{6.1}$$

Next, if the maximum is attained at a point in Γ_D and hence, by the Dirichlet condition, equals zero, then (6.1) holds trivially. Finally, if the maximum is attained at some point x_0 in Γ_N, then for any tangential vector τ at x_0 there holds

$$\nabla u_\nu(x_0) \cdot \tau = 0,$$

which also implies

$$\nabla u_\nu(x_0) = (\nabla u_\nu(x_0) \cdot n(x_0)) n(x_0)$$

(where $n(x_0)$ denotes the unit outward normal vector at x_0). Consequently, using (1.5)–(1.6) and the Neumann boundary condition (1.3) we get

$$\begin{aligned} 0 &= \left| \sum_{i,j=1}^d a_{ij}(x_0) D_j u_\nu(x_0) n_i(x_0) \right| \\ &= |\nabla u_\nu(x_0) \cdot n(x_0)| \sum_{i,j=1}^d a_{ij}(x_0) n_j(x_0) n_i(x_0) \ge \delta |\nabla u_\nu(x_0) \cdot n(x_0)|. \end{aligned}$$

Therefore, also the normal component of the gradient is zero and consequently, we have $\nabla u_\nu(x_0) = 0$. Thus, we can use the same procedure as for the interior point to obtain the claim (6.1).

In a similar manner we can also show

$$\sum_{\nu=1}^N q_\nu u_\nu \ge -K. \tag{6.2}$$

Finally, since q_ν are strictly positive, we see that (6.1) and (6.2) directly imply (1.22) and the proof is complete.

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