Regular and irregular solutions for a class of elliptic systems in the critical dimension

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August 13, 2012

Abstract

We study regularity properties of weak solutions in the Sobolev space $W_0^{1,n}$ to inhomogeneous elliptic systems under a natural growth condition and on bounded Lipschitz domains in $\mathbb{R}^n$, i. e. we investigate weak solutions in the limiting situation of the Sobolev embedding. Several counterexamples of irregular solutions are constructed in cases, where additional structure conditions might have led to regularity. Among others we present both bounded irregular and unbounded weak solutions to elliptic systems obeying a one-sided condition, and we further construct unbounded extremals of two-dimensional variational problems. These counterexamples do not exclude the existence of a regular solution. In fact, we establish the existence of regular solutions – under standard assumptions on the principal part and the aforementioned one-sided condition on the inhomogeneity. This extends previous works for $n = 2$ to more general cases, including arbitrary dimensions. Moreover, this result is achieved by a simplified proof invoking modern techniques.

MSC (2010): 35J47, 35B65 (primary); 35A01 (secondary)

Keywords: Quasi-linear elliptic systems; regularity of weak solutions; one-sided condition; counterexamples

1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain. We study the existence and regularity of vector-valued weak solutions $u\colon \Omega \to \mathbb{R}^N$ of the following inhomogeneous system in divergence form

$$-\sum_{i=1}^{n} D_i a_i(x, u, Du) + a_0(x, u, Du) = 0 \qquad \text{in } \Omega\,, \tag{1.1}$$

subject to zero-Dirichlet boundary conditions. We are interested in systems where the vector field $a = (a_1, \ldots, a_n)$ is of $(p-1)$-growth and where the inhomogeneity a_0 satisfies a natural (also called critical) growth condition, i. e.

$$|a_i(x, u, z)| \leq K(1 + |z|^{p-1})\,, \qquad i = 1, \ldots n\,, \tag{1.2}$$

$$|a_0(x, u, z)| \leq K_0(1 + |z|^p) \tag{1.3}$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$ and $z \in \mathbb{R}^{Nn}$, and clearly we suppose that the functions a_i $(i = 0, 1, \ldots, n)$ are Carathéodory functions. We first recall the notion of a weak solution to (1.1) referring to functions which are weakly differentiable and which solve the system in a weak integral form.

Definition 1.1. *A function $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is called a weak solution of* (1.1) *if*

$$\int_\Omega a(x, u, Du) \cdot D\varphi \, dx + \int_\Omega a_0(x, u, Du) \cdot \varphi \, dx = 0 \qquad \textit{for all } \varphi \in C_0^\infty(\Omega, \mathbb{R}^N)\,.$$

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By approximation, this identity continues to hold for all functions $\varphi \in W_0^{1,p}(\Omega,\mathbb{R}^N)\cap L^\infty(\Omega,\mathbb{R}^N)$. For the investigation of existence and regularity of weak solutions (in the sense of continuity), we may restrict ourselves to the case $p \in (1,n]$ since the Sobolev embedding immediately implies Hölder continuity with exponent $1-n/p$ for all function in $W^{1,p}(\Omega,\mathbb{R}^N)$ with $p > n$. We further require corresponding coercivity and ellipticity/monotonicity conditions on the vector field a, see the assumptions (2.1)–(2.5) for $p = n$.

Existence of weak solutions. Concerning the existence of weak solutions to such systems, there are several classical approaches, such as Leray-Schauder theorems or the theory of monotone operators. However, as a consequence of the critical growth assumption on the inhomogeneity, we do not necessarily have that $a_0(x, v, Dv)$ belongs to the dual space $W^{-1,p'}(\Omega,\mathbb{R}^N)$ if $v \in W^{1,p}(\Omega,\mathbb{R}^N)$. Thus, none of these approaches is directly applicable. Instead, under particular circumstances existence of weak solutions can be established via an approximation process, which shall be discussed in more detail further below. The reason for the interest in systems under the *natural* growth condition (1.3) is motivated for example by the minimization of variational integrals of the form

$$w \to \int_\Omega F(x, w, Dw)\,dx \tag{1.4}$$

among functions $w \in W^{1,p}(\Omega,\mathbb{R}^N)$, with an integrand F satisfying a standard p-growth condition. If F is sufficiently regular, each minimizer is a weak solution to the associated Euler-Lagrange system (1.1) with

$$a(x, u, z) := D_zF(x, u, z) \qquad \text{and} \qquad a_0(x, u, z) := D_uF(x, u, z)\,.$$

Classical minimization problems with integrand of splitting type, such as $F(x, u, z) = F_0(x, u)|z|^p$, naturally lead to Euler-Lagrange systems with inhomogeneities satisfying the natural growth condition (1.3). Here, the existence of minimizers follows from the direct method of the calculus of variation, combined with lower semi-continuity properties of the functional (implied by a suitable convexity assumption on F).

Regularity of weak solutions. We are further interested in the regularity properties of weak solutions. In the vectorial case $N > 1$ with $p < n$, there are many examples of variational and non-variational systems with irregular solutions, see e. g. the example of Giusti and Miranda [16] and its modification in [1]. In particular, for $p = 2$ and $n \geq 3$ Nečas [32, 18] gave examples where F depends only on the gradient variable (hence $a_0 = 0$) and where the solution is only Lipschitz, but not of class C^1. Moreover, the paper of Šverák and Yan [36] contains, among many other cases, an integrand $F(z)$ for $p = 2$, $n \geq 5$, where the solution is even unbounded. Thus, we can expect only partial regularity for weak solutions, even for homogeneous systems with $a_0 = 0$. An elaborate overview on partial regularity theory for such systems can be found in Mingione's survey article [30]. In order to obtain full regularity results, we are forced to make some further restriction on the systems under consideration. One idea is to stay "close" to a critical setting. This might refer to the structure of the vector field a, in the sense that it is of a very particular structure (which allows for example to use techniques known from the scalar case), see e. g. [35, 22], or that the growth of the vector field is coupled to the space dimension ($p = n$), in the sense that Sobolev's embedding almost gives the desired result (note $W^{1,n} \subset \mathrm{BMO}$, and we have the embedding $W^{1,n+\varepsilon} \subset C^\alpha$ for some $\alpha > 0$ iff $\varepsilon > 0$). We now provide a general heuristic of both approaches (always restricting ourselves to systems with inhomogeneity of natural growth), explain how several different settings are related and which results are expected. We here neither aim at a complete description of the results known in literature, nor do we go into too much detail.

Diagonal structure condition. We start by discussing one research direction related to a special structure condition, namely the case where the principal part is diagonal, i. e. $a_i^\alpha(x, u, z) = A_{ik}(x)z_k^\alpha$ (in particular, we here deal with $p = 2$). For such systems, the coupling between the different vectors of the solution takes place only via the inhomogeneity. However, we need further assumptions on the inhomogeneity, as we can easily see by the following well-known example.

Example 1.2. *Let $n = N \geq 3$. The function $u(x) = x/|x| \in W^{1,2}(B_1^n(0),\mathbb{R}^n) \cap L^\infty(B_1^n(0),\mathbb{R}^n)$ is a weak solution to the system $-\Delta u - u|Du|^2 = 0$ in $B_1^n(0)$.*

This elliptic system is solved by any critical point for the minimization of the Dirichlet energy among all maps from the unit ball $B_1^n(0)$ into the unit sphere $\partial B_1^n(0)$. Note that, although the vector field a and the boundary values are smooth, at least one solution develops a discontinuity since the inhomogeneity in some sense dominates the Laplacean. A possible condition how to prevent such irregularities is to limit the growth of the inhomogeneity a_0 in terms of the ellipticity of the principle part a

$$a(x,u,z)\cdot z \geq \nu|z|^p\,, \tag{1.5}$$

and possibly of the L^∞-norm of the solution itself. In this regard there are several possibilities to manage such inhomogeneous systems.

(I_1) smallness condition on the solution u: $K_0\|u\|_{L^\infty(\Omega,\mathbb{R}^N)} < \nu$;

(I_2) angle condition: there exists $\vartheta \in [0,\frac{\pi}{2})$ such that $a_0(x,u,z)\cdot u \geq |u||a_0(x,u,z)|\cos\vartheta$;

(I_3) one-sided condition: $a_0(x,u,z)\cdot u \geq -\nu_0|z|^p$ for some $\nu_0 < \nu$;

(I_4) two-sided condition: $a_0(x,u,z)\cdot u \geq -\nu_0|z|^p$ for some $\nu_0 \geq 0$ such that $\nu_0 + K_0\|u\|_{L^\infty(\Omega,\mathbb{R}^N)} < 2\nu$.

We first observe that the angle condition I_2 roughly means that the perturbation $a_0(x,u,Du)$ and the solution u point in the same direction. This is for example satisfied for systems of the form $a_0(x,u,z) = b(x,u,z)u$ with some non-negative, scalar $b(x,u,z)$. Furthermore, I_2 implies the one-sided condition I_3, and I_3 will be one of the main interests for the present paper. Also I_1 usually implies I_3 (taking into account the fact that I_3 is employed for u equal to the weak solution only). As we will discuss below, I_1 has an huge advantage for the proof of regularity properties of such solutions. However, since it is a condition on the solution itself, in practice it is in general not possible to show existence of such solutions.

The investigation of the regularity properties for weak solutions to diagonal systems has a long history. First results can be found in the book of Ladyzhenskaya and Uralt'seva [24]. Then it was shown by Hildebrandt, Widman and Wiegner [19, 39, 38] that $n \geq 3$ and I_1 imply Hölder continuity of a given weak solution (and in the scalar case actually any bounded solution is regular). A generalization under condition I_4 was then given in [20], and a proof based on a geometric approach in [6]. Also the condition I_3 seemed promising since some tools from the scalar theory are available. However, only in the two-dimensional case $n = 2$, Hölder continuity of all bounded weak solutions is obtained [41] (and smallness of boundary values gives a priori estimate of the $C^\alpha(\Omega,\mathbb{R}^N)$-norm), and an example of Struwe [33] – again with the Laplace operator as principal part – shows that this is not the case for $n \geq 3$. Moreover, these results are optimal in the sense that we cannot allow $K_0\|u\|_{L^\infty(\Omega,\mathbb{R}^N)} = \nu$ or $\nu_0 = \nu$, which is obvious from the above Example 1.2 for $n \geq 3$ and a modification of the counterexample [7] for $n = 2$, see also [19, 28].

In this context it is also worth mentioning the connection between interior regularity results for bounded solutions and Liouville-type theorems, cf. [9, 29], i.e. generalizations of Liouville's copacetic theorem stating that a bounded harmonic function in $\mathbb{R}^n$ is actually constant. Accordingly, we say that the system (1.1) has the Liouville property if any bounded function which is a weak solution on every bounded domain is constant. As above, the conditions I_1 or I_4 imply the Liouville property – for $N = 1$ without the smallness assumption on the L^∞-norm of the solution – and are sharp, see [21]. The condition I_3 instead implies the Liouville property only for $n = 2$, but cannot be expected for $n \geq 3$ and $N > 1$ (not even with $\nu_0 = 0$), see [28].

Systems of variational structure. We next discuss briefly another particular structure condition, namely that the system (not necessarily diagonal) arises as Euler-Lagrange system of the variational integral (1.4). Weak solutions then allow the interpretation as critical points. As explained above, the direct method of the calculus of variations leads to the existence of minimizers, for which Morrey [31, Chapter 4.3] proved Hölder continuity, provided that $p = n$. Hence, existence of regular weak solutions is guaranteed. Moreover, for quadratic-type functionals with diagonal coefficients of the form $F(x,u,z) = A_{ik}(x,u)z_k^\alpha z_i^\alpha$ Giaquinta and Giusti [14] proved that every bounded local minimum is Hölder continuous, for arbitrary $n \geq 2$, provided that a corresponding one-sided condition is fulfilled. For more general variational systems (also for inhomogeneities which differ from $D_uF(x,u,z)$ in a controllable way) the existence of a regular solution still holds true for $n \geq p = 2$ if in addition to the one-sided condition a structure condition on the principle part is assumed, such as a generalized splitting condition, see [4, 5]. It is an interesting, open problem whether *all* solutions (such as non-extremal solutions of the Euler equation) with smooth data are Hölder continuous, in particular for the two-dimensional case $n = p = 2$.

Non-diagonal systems and $\mathbf{p = n}$. Let us turn our attention to general non-diagonal systems. As explained above, we consider the critical integrability exponent $p = n$. In this situation some results are available in the literature, requiring additional conditions. For example, under a more restrictive smallness condition than I_1, continuity of the solution is obtained, see [17]. Furthermore, under the one-sided condition I_3, which is of central interest here, the existence of one regular solution was obtained by the second author [9] for the two-dimensional case $n = 2$.

Theorem 1.3 ([9]). *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain and assume that the structure assumptions* (2.1)–(2.5) *and the one-sided condition* (2.6) *below are fulfilled. Then the elliptic system* (1.1) *has a weak solution $u \in C^\alpha(\Omega, \mathbb{R}^N) \cap W_0^{1,2}(\Omega, \mathbb{R}^N)$ for some $\alpha > 0$.*

The one-sided condition I_3 does not prevent non-uniqueness and there may exist further bounded solutions which are discontinuous, see [29]. It is obvious from the positive results above that the non-diagonal principal part is indispensable for this dichotomy in the two-dimensional case, and for general $n > 2$ non-diagonality might at least allow an easier construction of counterexamples.

Our motivation of the present paper is twofold. Our first aim is to provide several examples for the existence of irregular solutions – with different scales for the lack of regularity (bounded and discontinuous or even unbounded) – to different types of elliptic systems. These examples shall be described in the next paragraph concerning the optimality of the regularity results presented here. The second aim is to provide a simplified proof of Theorem 1.3, which is achieved by the use of some modern tools. At the same time, we generalize the previous result to arbitrary dimensions $n = p \geq 2$, that is, we demonstrate the existence of a regular solution under the one-sided condition I_3.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and assume that the structure assumptions* (2.1)–(2.5) *and the one-sided condition* (2.6) *are fulfilled. Then the elliptic system* (1.1) *has a weak solution $u \in C^\alpha_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap W_0^{1,n}(\Omega, \mathbb{R}^N)$ for some $\alpha > 0$.*

We highlight that both previous theorems include an existence result for such inhomogeneous systems under the natural growth condition (1.3) with $p = n$. As explained at the beginning of this introduction, we do not have an a priori existence result, which then allows us to work directly on such a solution and then prove its regularity a posteriori. Instead, existence and regularity are proved simultaneously. The strategy of proof is to employ a double approximation procedure via variational inequalities and to derive for each of them first local and then uniform Morrey-estimates (with respect to the sequence under consideration), which then are preserved in the limit. Thus, the existence of a Hölder continuous solution follows from an embedding theorem. This line of argument is classical and was for example accomplished in [38, 9, 41]. However, the uniform estimates are here established by taking advantage of uniform smallness of the n-energy as a consequence of the Trudinger-Moser inequality (the specific choice $p = n$ is crucial). In Section 7 we finally show, how this approximation procedure can be applied in the setting of some particular Bellman-type systems which model discount control problems.

For sake of completeness we note that a similar strategy via a suitable approximation scheme can be accomplished if the angle condition I_2 is supposed. This applies in particular for all $p \in (1, n]$ and yields (under further assumptions) the existence of a weak solution, but no further regularity result, see [40, 25, 26].

Also here in the non-diagonal setting some Liouville-type results are available, in any dimension $n \geq 2$. In the scalar case $N = 1$ the Liouville property holds for all systems, whereas in the vectorial case $N > 1$, this property follows from each of the conditions I_1 and I_3, see [9, 28, 12].

Sharpness of the regularity results. We next investigate the optimality of the positive results above, pursuing the second aim of the present paper of gaining a better understanding of the regularity results stated above. To this end we discuss several examples presented in Section 3.

We first construct an elliptic system with inhomogeneity of critical growth (satisfying the one-sided condition I_3 or even the angle-condition I_2), which admits a discontinuous solution. This shows that the result from Theorem 1.4 is optimal, in the sense that non-uniqueness and irregular solutions may actually occur.

Theorem 1.5. *For every $n \geq 2$ there exist non-diagonal elliptic systems* (1.1) *which satisfy the structure assumptions* (2.1)–(2.5)*, the one-sided condition* (2.6)*, and which admit a bounded, irregular solution in some bounded, regular domain $\Omega \subset \mathbb{R}^n$ with regular boundary values.*

This examples still leaves open the possibility that – even though not all solutions are continuous – all solutions might still be bounded. This possibility is ruled out by another examples which shows for a similar elliptic system the existence of an unbounded solution.

Theorem 1.6. *For every $n \geq 2$ there exist non-diagonal elliptic systems* (1.1) *which satisfy the structure assumptions* (2.1)–(2.5)*, the one-sided condition* (2.6)*, and which admit an unbounded solution in some bounded, regular domain $\Omega \subset \mathbb{R}^n$ with regular boundary values.*

Concerning systems of variational structure we first note that for the scalar case $N = 1$ examples of integrands F were constructed which are analytic for $n \geq 3$ and such that the associated Euler-Lagrange equation has an unbounded $W^{1,p}$-solution, see [8]. For the vectorial case, an examples of an integrand F is known which is discontinuous with respect to the variable x and such that a bounded discontinuous solution to the Euler system exists for $p = n = 2$, see [10]. These examples violate the one-sided condition (so the scalar example is optimal since a one-sided condition would imply L^∞-bounds for the solution which in turn would lead to C^α-regularity via the theory of Ladyzhenskaya-Uralt'seva), and moreover, the system in [10] is of non-diagonal structure. In the present paper we give an example of a *diagonal* system (in particular, the splitting type condition of the result from [5] is satisfied) arising as Euler-Lagrange system of a variational integral (1.4) with a regular integrand such that an unbounded extremal exists.

Theorem 1.7. *Let $n \geq 2$. There exists a diagonal, elliptic Euler system arising from a variational integral with integrand of the form $F_0(x, u)|z|^2$ with the following properties: the structure assumptions* (2.1)–(2.5) *are satisfied (with exponent $p = 2$ instead of n), F_0 is continuous in x, analytic in u, and the system admits an unbounded weak solution in some bounded, regular domain $\Omega \subset \mathbb{R}^n$ with regular boundary values.*

Summary. Summarizing the results for diagonal and non-diagonal systems (with $p = 2$ and $p = n$, respectively) under the assumptions I_1 or I_3 in the vectorial case, we obtain the following table for the vectorial setting:

setting (with $N > 1$) / result	diagonal ($p = 2$)		non-diagonal ($p = n$)	
	I_1	I_3	I_1	I_3
existence of a regular solution	?	Yes ($n = 2$)	?	Yes
regularity of bounded solutions	Yes	Yes iff $n = 2$	?	No
Liouville property	Yes	Yes iff $n = 2$	Yes	

2 Structure assumptions

We here collect the assumptions on the vector field a and the inhomogeneity a_0 (already sketched in the introduction). We first assume the following growth assumptions

$$|a_i(x, u, z)| \leq K(1 + |u|^q + |z|^{n-1}), \qquad i = 1, \dots n, \tag{2.1}$$

$$|a_0(x, u, z)| \leq K(1 + |u|^q + |z|^n) \tag{2.2}$$

for some $q \geq n$ and for all $x \in \Omega$, $u \in \mathbb{R}^N$, and $z \in \mathbb{R}^{Nn}$. Furthermore, we need some coercivity of the problem, which is one of the essential ingredients in order to prove existence. For ease of notation we introduce

$$\langle Tu, \varphi \rangle := \langle A(u), \varphi \rangle + \langle B(u), \varphi \rangle := \int_\Omega a(x, u, Du) \cdot D\varphi \, dx + \int_\Omega a_0(x, u, Du) \cdot \varphi \, dx$$

for all $u \in W^{1,n}(\Omega, \mathbb{R}^N)$ and $\varphi \in L^\infty(\Omega, \mathbb{R}^N) \cap W_0^{1,n}(\Omega, \mathbb{R}^N)$, with the obvious abbreviations. Clearly, the above identity vanishes for all such φ, whenever u is a weak solution to (1.1). Our coercivity condition is then expressed in terms of the operator T, with

$$\langle Tu, u \rangle > 0 \quad \text{as } \|u\|_{W^{1,n}} \to \infty,\ u \in L^\infty(\Omega, \mathbb{R}^N) \cap W_0^{1,n}(\Omega, \mathbb{R}^N)\,. \tag{2.3}$$

We also assume the following pseudo-monotonicity condition:

$$\begin{cases} \text{if for a sequence } (u_m)_{m\in\mathbb{N}} \in L^\infty(\Omega, \mathbb{R}^N) \cap W_0^{1,n}(\Omega, \mathbb{R}^N) \text{ there hold} \\ u_m \rightharpoonup u \text{ weakly in } L^\infty(\Omega, \mathbb{R}^N) \cap W_0^{1,n}(\Omega, \mathbb{R}^N) \text{ and} \\ \limsup \int_\Omega \big(a(x, u_m, Du_m) - a(x, u_m, Du)\big) \cdot (Du_m - Du)\, dx \le 0\,, \\ \text{then } u_m \to u \text{ strongly in } W_0^{1,n}(\Omega, \mathbb{R}^N)\,. \end{cases} \tag{2.4}$$

This is a natural condition and for example satisfied if the usual monotonicity of the vector field a (pointwise or in an integral sense) holds. The last assumption on the principal part a is ellipticity (in integral form)

$$\int_\Omega a(x, u, Dw) \cdot Dw \, dx \ge \nu \int_\Omega |Dw|^n \, dx - K \int_{\mathrm{spt} w} |u|^q \, dx - K |\mathrm{spt} u \cap \mathrm{spt} w| \tag{2.5}$$

for all $u, w \in W_0^{1,n}(\Omega, \mathbb{R}^N)$. Concerning the inhomogeneity, we finally require an additional one-sided condition of the form

$$a_0(x, u, z) \cdot u \ge -\nu_0 |z|^n - K(|u|^q + 1) \tag{2.6}$$

for some $\nu_0 < \nu$ and all $x \in \Omega$, $u \in \mathbb{R}^N$ and $z \in \mathbb{R}^{Nn}$.

3 Examples

We start by giving several examples, which demonstrate the sharpness of some positive results concerning the existence of regular solutions. The construction of the various solutions and the associated system are in parts similar. However, in order to have a clearer exposition, we prefer to state the various examples in separate subsections (though not providing full details for the calculations for all of them). In Section 3.1 we give an example of a system admitting a bounded, but discontinuous solution. This example is then modified in Section 3.2, and a discontinuous solution to a system arising from discount control problems is obtained. Section 3.3 is devoted to the construction of unbounded weak solutions, which is again based on the construction of the first example, but with an additional parameter which for suitable choices is responsible for the emergence of a singularity of the solution. In Section 3.4 we finally sketch briefly the construction of an elliptic system of variational structure (i. e. as Euler-Lagrange system of a convex variational integral) which admits bounded and unbounded irregular critical points.

3.1 Examples with a bounded, irregular solution

Given $n \ge 2$, we here provide a family of weak solutions to inhomogeneous n-dimensional quasilinear systems of the form (1.1), which satisfy all assumptions of Section 2, in particular the inhomogeneity satisfies the one-sided condition (2.6). The construction of these examples is motivated from [7]. We study the system

$$\begin{cases} -(\Delta_n u)^1 + \lambda(\Delta_n u)^2 = 2|Du|^n (1 + |u|^2)^{-1} \big((1 + \alpha^{-1}\lambda) u^1 + (\alpha^{-1} - \lambda) u^2\big) \\ -(\Delta_n u)^2 - \lambda(\Delta_n u)^1 = 2|Du|^n (1 + |u|^2)^{-1} \big((1 + \alpha^{-1}\lambda) u^2 + (-\alpha^{-1} + \lambda) u^1\big) \end{cases} \tag{3.1}$$

in an n-dimensional, bounded domain Ω, with parameters $\lambda, \alpha \in \mathbb{R}$, $\alpha \neq 0$. We recall that the n-Laplace operator is defined as a

$$\Delta_n u = \sum_{i=1}^{n} D_i\big(|Du|^{n-2} D_i u\big)$$

(and in the two-dimensional case this reduces to the definition of the Laplace operator), with $(\Delta_n u)^i$ denoting the component functions. In the notation of system (1.1), we thus consider coefficients and inhomogeneity given by

$$\begin{aligned}
a_i(x,u,z) \equiv a_i(z) &:= |z|^{n-2}(z_i^1 - \lambda z_i^2, z_i^2 + \lambda z_i^1)^t\,, \qquad i = 1,\dots n\,,\\
a_0(x,u,z) \equiv a_0(u,z) &:= 2|z|^n(1+|u|^2)^{-1}\\
&\qquad \times \big((-1-\alpha^{-1}\lambda)u^1 + (\lambda - \alpha^{-1})u^2, (\alpha^{-1}-\lambda)u^1 + (-1-\alpha^{-1}\lambda)u^2\big)^t\,.
\end{aligned}$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$ and $z \in \mathbb{R}^{Nn}$.

Lemma 3.1. *The assumptions* (2.1)–(2.6) *from Section 2 are satisfied if* $\alpha \in \mathbb{R} \setminus \{0\}$ *and* $\lambda \in \mathbb{R}$ *verify the inequalities* $\alpha^{-1}\lambda < -1/2$ *and* $(n-2)|\lambda| < 1$.

Proof. The growth conditions (2.1) and (2.2) are clear by definition. It is further easy to calculate

$$a(Du)\cdot Du = |Du|^n \quad \text{and} \quad a_0(u,Du)\cdot u = -(1+\alpha^{-1}\lambda)\frac{2|u|^2}{1+|u|^2}|Du|^n\,.$$

Hence, (2.5) follows for $\nu = 1$, and if $\alpha^{-1}\lambda < -1/2$ we also get (2.3) and (2.6). Finally, we have the pointwise monotonicity condition

$$\big(a(z) - a(\widetilde{z})\big)\cdot(z-\widetilde{z}) \geq c(n)(1-(n-2)|\lambda|)|z-\widetilde{z}|^n\,,$$

which implies (2.4) if $(n-2)|\lambda| < 1$. □

Remark 3.2. *It is easy to check that the Wiegner-Landes' angle condition is satisfied for* $\alpha^{-1}\lambda < -1$.

We next take a function $f \in W^{1,1}(\Omega, \mathbb{R}_+)$ which satisfies $|Df|^n/f^n \in L^1(\Omega)$ and which is n-harmonic in $\Omega \setminus A$ for some set A of $W^{1,n}$-capacity zero (note that for $n > 1$ single points in $\mathbb{R}^n$ have vanishing $W^{1,n}$-capacity). Next we define the function $u \in W^{1,n}(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2)$ via

$$u^1(x) = \sin\big((n-1)\alpha \ln f(x)\big)\,, \qquad u^2(x) = \cos\big((n-1)\alpha \ln f(x)\big)\,.$$

Proposition 3.3. *For every* $\alpha, \lambda \in \mathbb{R}$, $\alpha \neq 0$, *and every* f *given as above the function* u *is a weak solution to* (3.1) *in* Ω.

Proof. We obtain this proposition by elementary calculations, using in particular the following easy consequences of the definition of the function f:

$$\begin{aligned}
D_i u^1 &= (n-1)\alpha\frac{D_i f}{f}u^2\,, \quad D_i u^2 = -(n-1)\alpha\frac{D_i f}{f}u^1 \qquad i = 1,\dots n\,,\\
|Du|^2 &= (n-1)^2\alpha^2\frac{|Df|^2}{f^2}\\
(\Delta_n u)^1 &= -(u^1 + \alpha^{-1}u^2)(n-1)^n|\alpha|^n\frac{|Df|^n}{f^n}\\
(\Delta_n u)^2 &= -(u^2 - \alpha^{-1}u^1)(n-1)^n|\alpha|^n\frac{|Df|^n}{f^n}
\end{aligned}$$

in $\Omega \setminus A$. This yields immediately that (3.1) is satisfied in $\Omega \setminus A$. Hence, since $a_i(Du) \in L^{n/(n-1)}(\Omega, \mathbb{R}^2)$ for $i \in \{1,2\}$ by construction and since A is of vanishing $W^{1,n}$-capacity by assumption, we find that u is indeed a weak solution (3.1) for all possible choices of α, λ, f as given in the statement, and so the assertion of the proposition is proved. □

Remark 3.4. *Let $\Omega = B_{1/e^2}(0) \subset \mathbb{R}^2$ and consider the function $f = \ln |x|^{-1}$ (harmonic outside of the origin). Then, with $\alpha = 1$ and $\lambda = 0$ we recover the example [7], and with $\alpha = 1$ and $\lambda = -1$ Meier's modification [29] of the previous example.*

As a consequence of the above construction, we can now prove Theorem 1.5.

Proof of Theorem 1.5. According to Lemma 3.1 the system (3.1) satisfies the assumptions (2.1)–(2.6) for every $\alpha, \lambda \in \mathbb{R}$ such that $\alpha^{-1}\lambda < 0$ and $(n-2)|\lambda| < 1$. Then the existence of an irregular solution follows by applying the previous proposition with $\Omega = B_{1/e}(0) \subset \mathbb{R}^n$ and $f = \ln |x|^{-1}$ (which is n-harmonic outside and discontinuous at the origin). □

Remark 3.5. *In the statement of Theorem 1.5 we can obtain zero boundary values on $\partial\Omega$, just by modifying the above construction. Multiplying u with a smooth cut-off function η (with $\eta \equiv 1$ in a neighbourhood of the singularity of f) we can easily calculate that the function $u\eta$ is an irregular solution to an elliptic system with the original principal part and a new inhomogeneity which differs from the original one only by a smooth function.*

Remark 3.6. *From the previous example we can construct further systems of type* (3.1) *which are solved by the same function u given above. Indeed, using the identity*

$$g^i(u, Du) := 2(1 + |u|^2)^{-1}(u^1 D_i u^2 - u^2 D_i u^1) = -\alpha(n-1)\frac{D_i f}{f}$$

for $i \in \{1, \dots, n\}$, we can replace the non-diagonal part in the vector field a. Setting for simplicity $\lambda = -\alpha$ — implying that the one-sided condition is satisfied with $\nu_0 = 0$ —, it is then easy to check that u also solves

$$\begin{cases} -(\Delta_n u)^1 - 2\alpha \sum_{i=1}^n D_i \big[|Du|^{n-2}(1 + |u|^2)^{-1} u^1 g^i(u, Du)\big] = 2|Du|^n (1 + |u|^2)^{-1}(\alpha^{-1} + \alpha) u^2 \,, \\ -(\Delta_n u)^2 - 2\alpha \sum_{i=1}^n D_i \big[|Du|^{n-2}(1 + |u|^2)^{-1} u^2 g^i(u, Du)\big] = -2|Du|^n (1 + |u|^2)^{-1}(\alpha^{-1} + \alpha) u^1 \,. \end{cases}$$

Clearly, this system is more involved than the original one, due to the explicit u-dependence in the principal part. However, it might give incitation for the construction of further counterexamples in this context.

Remark 3.7. *In the two-dimensional case $n = 2$ we further give the (straightforward) complex reformulation. The function $w\colon \mathbb{R}^2 \to \mathbb{C}$ defined as $w = u^1 + \mathrm{i}u^2$ solves*

$$(-1 - \lambda \mathrm{i})\Delta w = 2|Dw|^2 (1 + |w|^2)^{-1}\big[(1 + \alpha^{-1}\lambda) w + (-\alpha^{-1} + \lambda)\mathrm{i} w\big] \,.$$

3.2 A counterexample for discount control

The theory of stochastic differential games with infinite horizon leads to diagonal elliptic systems – so called Bellman systems – satisfied by the value function of the players. A standard system including discount control is

$$-\Delta u^i + \gamma u^i = h^i(u)|Du^i|^2 + G(u, Du)Du^i - F(u, Du)u^i + f^i \tag{3.2}$$

for $i = \{1, \dots, N\}$, with $\gamma > 0$, $0 \le F(u, z) \le K|z|^2 + K$, $|G(u, z)| \le K|z| + K$, and $f, h \in L^\infty$, cf. [2, 3]. The term "discount control" here refers to the fact that control of the discount factor in the cost functional of the players is admitted. An important open question is the existence of regular solutions (or even of weak solutions). While Wiegner's technique from [40] works also in the presence of the term $G(u, Du)Du^i$, the term $h_i(u)|Du^i|^2$ leads to difficulties. From this point of view it is important to know that indeed irregular solutions may occur in the case of the one-sided condition (2.6). Nevertheless, the existence of at least one regular solution was worked out for $n = 2$, see [3], with methods partially similar as in the present paper, and for $n = p \ge 3$ and corresponding p-growth assumptions, it is sketched at the end of this paper. For $n \ge 3$ and $p = 2$, the regularity theory for the above system is an interesting open problem.

We now discuss the above mentioned examples of a system related to discount control which admits an irregular solution, but with the difference that the term $F(u, Du)u^i$ is replaced by a pair (T_1, T_2)

satisfying the one-sided condition, but not the angle condition (in fact, we will have $T_1 u^1 + T_2 u^2 = 0$). Our construction is based on the old example [7]

$$\begin{cases} -\Delta u^1 = (u^1 + u^2 - 4)|Du|^2 \\ -\Delta u^2 = (u^2 - u^1)|Du|^2 \end{cases} \tag{3.3}$$

in $B_{1/e^2}(0) \subset \mathbb{R}^2$, which admits the discontinuous, positive solution

$$u^1(x) = \sin\big(\ln\ln|x|^{-1}\big) + 2\,, \qquad u^2(x) = \cos\big(\ln\ln|x|^{-1}\big) + 2\,.$$

The following lemma states that the right-hand side of (3.3) can be written in the form of (3.2), with $F(u, Du)u^i$ replaced by (T_1, T_2) satisfying $T(v, Dv) \cdot v = 0$.

Lemma 3.8. *Let $v = (v^1, v^2), g = (g^1, g^2) \in \mathbb{R}^2$, with $v^i > 0$ for $i \in \{1, 2\}$, and $z = (z_1, z_2)^t \in \mathbb{R}^{2\times 2}$, with $z_i = (z_i^1, z_i^2) \in \mathbb{R}^2$. Then the following identity holds*

$$\big(g^1|z|^2, g^2|z|^2\big) = \big(\widetilde{g}^1|z^1|^2, \widetilde{g}^2|z^2|^2\big) + \big(T_1^1, T_1^2\big) + \big(T_2^1, T_2^2\big)$$

with

$$\widetilde{g}^1 = g^1 + g^2\frac{v^2}{v^1} \qquad \text{and} \qquad \widetilde{g}^2 = g^2 + g^1\frac{v^1}{v^2}$$

and T_i^j defined in (3.4), (3.5) such that $T_i^1 v^1 + T_i^2 v^2 = 0$ for $i \in \{1, 2\}$.

Proof. We write for the pair of functions

$$\big(g^1|z|^2, g^2|z|^2\big) = \big(g^1|z^1|^2, g^2|z^2|^2\big) + \big(g^1|z^2|^2, 0\big) + \big(0, g^2|z^1|^2\big)\,.$$

Next, we observe

$$\big(g^1|z^2|^2, 0\big) = \big(T_1^1, T_1^2\big) + \big(0, g^1|z^2|^2\frac{v^1}{v^2}\big)\,,$$

with

$$T_1^1 = g^1|z^2|^2 \qquad \text{and} \qquad T_1^2 = -g^1|z^2|^2\frac{v^1}{v^2}\,. \tag{3.4}$$

Obviously, we have $T_1^1 v^1 + T_1^2 v^2 = 0$. Similarly, the third term in the decomposition above is treated, and we find

$$\big(0, g^2|z^1|^2\big) = \big(T_2^1, T_2^2\big) + \big(g^2|z^1|^2\frac{v^2}{v^1}, 0\big)$$

with

$$T_2^1 = -g^2|z^1|^2\frac{v^2}{v^1} \qquad \text{and} \qquad T_2^2 = g^2|z^1|^2\,, \tag{3.5}$$

and accordingly, we find $T_2^1 v^1 + T_2^2 v^2 = 0$. Hence, we end up with the desired representation. □

Remark 3.9. *The previous Lemma 3.8 can easily be generalized to n dimensions. The terms arising in the decomposition given at the beginning of the proof are then vectors $e_k g^k(|z|^2 - |z^k|)$ (with e_k denoting the k-th coordinate vector), which then are rewritten analogously to above as*

$$e_k g^k(|z|^2 - |z^k|) = (T_k^1, \ldots, T_k^n) + \sum_{i=1, i\neq k}^{n} e_i g^k|z^i|^2\frac{v^k}{v^i}$$

with

$$T_k^k = g^k(|z|^2 - |z^k|^2) \qquad \text{and} \qquad T_k^i = -g^k|z^i|^2\frac{v^k}{v^i} \text{ for } i \neq k\,.$$

Obviously, this choice implies $\sum_{j=1}^n T_k^j v^j = 0$ and gives immediately the statement also for the general n-dimensional case.

Corollary 3.10. *The system* (3.3) *can be written in the form*

$$-\Delta u^i + \gamma u^i = h^i(u)|Du^i|^2 - T^i(u, Du) + f^i\,,$$

where $\gamma > 0$, $f, h \in L^\infty(B_{1/e^2}(0), \mathbb{R}^2)$, *and where* T_i *satisfies the natural growth assumption* (2.2) *and the condition* $T(u,z)\cdot u = 0$ *for all* $u \in \mathbb{R}^2$ *and* $z \in \mathbb{R}^{Nn}$.

Proof. We first note that the right-hand side of (3.3) can be multiplied by a smooth function $\ell(u^1, u^2)$ with $\ell(s,t) = 1$ for $s, t \in [1,3]$ and $\ell(s,t) = 0$ for s or $t \notin [1/2, 4]$, and the function u defined above remains a weak solution, since it has values only in $[1,3]^2$. Consequently, with the abbreviation of right-hand side of (3.3) as $-a_0(u,z) = \ell(u)|z|^2(u^1 + u^2 - 4, u^2 - u^1)^t$ for $u \in \mathbb{R}^2$ and $z \in \mathbb{R}^{2n}$, it is sufficient to rewrite $a_0(u,z)$ only for u with values in $[1/2, 4]^2$. The application of the previous Lemma 3.8 with $g^1 = u^1 + u^2 - 4$, $g^2 = u^2 - u^1$ yields in this case

$$-a_0(u,z)^t = (h^1(u)|z^1|^2, h^2(u)|z^2|^2) + (T^1(u,z), T^2(u,z))$$

with

$$\begin{aligned}
h^1(u) &= u^1 - 4 + \frac{(u^2)^2}{u^1}\,, \qquad h^2(u) = u^2 + \frac{(u^1)^2}{u^2} - 4\frac{u^1}{u^2} \\
T^1(u,z) &= \Big(-\frac{(u^2)^2}{u^1} + u^2\Big)|z^1|^2 + (u^1 + u^2 - 4)|z^2|^2\,, \\
T^2(u,z) &= (u^2 - u^1)|z^1|^2 + \Big(-\frac{(u^1)^2}{u^2} - u^1 + 4\frac{u^1}{u^2}\Big)|z^2|^2\,,
\end{aligned}$$

and $T(u,z)\cdot u = 0$ is clearly satisfied. It remains to mention that the function u defined above is bounded, hence we can define $f = \gamma u \in L^\infty(B_{1/e^2}(0), \mathbb{R}^2)$ for an arbitrary number $\gamma > 0$, and the proof of the corollary is complete. □

3.3 Examples with an unbounded solution

We next provide families of *unbounded* weak solutions to inhomogeneous quasilinear systems. The first example in this subsection concerns $W^{1,n}$-solutions in n dimensions. The second example is about $W^{1,2}$-solutions in n dimension (that is, not in the critical dimension) to linear systems with inhomogeneity of subquadratic growth. Such a non-critical (or controllable) growth assumption usually allows to prove better regularity properties. For this reason we believe that this example of linear growth is interesting on its own even if the principle part satisfies the main assumptions of the present paper with $n > p = 2$.

Principle part of (n−1)-growth and critical n-growth of the inhomogeneity

We start by showing that a construction similar to that from Section 3.1 can be used to provide examples of systems under a one-sided condition (2.6), which admit an unbounded solution. This shows that not every weak solution in this critical situation is necessarily of class L^∞. The system under consideration is essentially the one from (3.1), with some slight variation caused by the new parameter θ. We now study the system

$$\left\{\begin{aligned}
-(\Delta_n u)^1 + \lambda(\Delta_n u)^2 &= |Du|^n \max\{|u|^2, 1\}^{-1}\big((b_1 + \lambda b_2)u^1 + (b_2 - \lambda b_1)u^2\big) \\
-(\Delta_n u)^2 - \lambda(\Delta_n u)^1 &= |Du|^n \max\{|u|^2, 1\}^{-1}\big((-b_2 + \lambda b_1)u^1 + (b_1 + \lambda b_2)u^2\big)
\end{aligned}\right. \tag{3.6}$$

in an n-dimensional, bounded domains Ω, for $\alpha, \theta, \lambda \in \mathbb{R}$ with $\alpha \neq 0$, and coefficients $b_1 = b_1(n,\theta,\alpha)$ and $b_2 = b_2(n,\theta,\alpha)$ defined by

$$b_1(n,\theta,\alpha) := \frac{-\theta(\theta-1)(n-1) + \alpha^2}{\alpha^2 + \theta^2} \qquad \text{and} \qquad b_2(n,\theta,\alpha) := \frac{-n\theta\alpha + (n-1)\alpha}{\alpha^2 + \theta^2}\,.$$

Obviously, this system can again be rewritten in the notation as in (1.1). Exactly as before for Lemma 3.1 we check that this system satisfies all assumptions from Section 2.

Lemma 3.11. *The assumptions* (2.1)–(2.6) *are satisfied if* $b_1 + \lambda b_2 < 1$ *and* $(n-2)|\lambda| < 1$.

Remark 3.12. *Replacing* $\max\{|u|^2, 1\}$ *in the inhomogeneity by a smooth function* $m(u)$ *with* $m(u) = |u|^{-2}$ *for* $|u| \geq 1$ *and* $m(u) \in [1/2, 1]$ *otherwise, one easily obtains an inhomogeneity which is smooth with respect to the variables* u, z *(and this modified system still satisfies* (2.1)–(2.6) *under a similar condition on* b_1 *and* b_2*).*

Similarly as in Section 3.1 we take a function $f \in W^{1,1}(\Omega, [1, \infty))$ which satisfies $|Df|^n f^{n(\theta-1)} \in L^1(\Omega)$ and which is n-harmonic in $\Omega \setminus A$ for some set A of $W^{1,n}$-capacity zero. Then we define the function $u \in W^{1,n}(\Omega, \mathbb{R}^2)$ via

$$u^1(x) = f(x)^\theta \sin\big(\alpha \ln f(x)\big)\,, \qquad u^2(x) = f(x)^\theta \cos\big(\alpha \ln f(x)\big)\,.$$

Proposition 3.13. *For every* $\alpha, \theta, \lambda \in \mathbb{R}$, $\alpha \neq 0$, *and every* f *given as above the function* u *is a weak solution to* (3.6) *in* Ω.

Proof. We first calculate in $\Omega \setminus A$ the partial derivatives of u:

$$\begin{aligned} D_i u^1(x) &= f(x)^{\theta-1} D_i f(x)\big(\theta \sin(\alpha \ln f(x)) + \alpha \cos(\alpha \ln f(x))\big)\,, \\ D_i u^2(x) &= f(x)^{\theta-1} D_i f(x)\big(\theta \cos(\alpha \ln f(x)) - \alpha \sin(\alpha \ln f(x))\big) \end{aligned}$$

for $i \in \{1, \ldots, n\}$. This allows to deduce $|Du(x)|^2 = f(x)^{2(\theta-1)}|Df(x)|^2(\alpha^2 + \theta^2)$, and we further find

$$\begin{aligned} (\Delta_n u)^1(x) &= |Du(x)|^n(\alpha^2+\theta^2)^{-1} f(x)^{-2\theta}\big((\theta(\theta-1)(n-1) - \alpha^2)u^1(x) + (n\theta\alpha - (n-1)\alpha)u^2(x)\big)\,, \\ (\Delta_n u)^2(x) &= |Du(x)|^n(\alpha^2+\theta^2)^{-1} f(x)^{-2\theta}\big((\theta(\theta-1)(n-1) - \alpha^2)u^2(x) - (n\theta\alpha - (n-1)\alpha)u^1(x)\big)\,. \end{aligned}$$

Employing the definitions of b_1 and of b_2 we immediately check that (3.6) is satisfied pointwise in $\Omega \setminus A$, which in turn shows (taking into account that $f \geq 1$ holds by assumption) that u is a weak solution to the system (3.6) as asserted. □

Proof of Theorem 1.6. According to Lemma 3.11 the system (3.6) satisfies the assumptions (2.1)–(2.6) for all $\alpha, \theta, \lambda \in \mathbb{R}$ with $\alpha \neq 0$ such that $b_1 + \lambda b_2 < 1$ and $(n-2)|\lambda| < 1$. We further note that such choices are always possible. For this purpose it is sufficient to observe that equality $b_1 + \lambda b_2 = 1$ holds for $\theta = -\lambda\alpha$ or $\theta = (n-1)/n$, and hence, the strict inequality can always be achieved for a given θ by a suitable choice of $\alpha\lambda$. Now the existence of an unbounded solution follows by applying the previous proposition with $\Omega = B_{1/e}(0) \subset \mathbb{R}^n$, $f = \ln|x|^{-1}$ and $\theta \in (0, (n-1)/n)$. Indeed, with this choice we observe that

$$|Df|^n f^{n(\theta-1)} = |x|^{-n}(\log|x|^{-1})^{n(\theta-1)} \in L^1(\Omega)\,,$$

and due to $|u(x)| = (\ln|x|^{-1})^\theta$ we have $u \notin L^\infty(\Omega, \mathbb{R}^2)$. □

Principle part of linear growth and subquadratic growth of the inhomogeneity

Given $n \geq 3$, we next construct a family of unbounded weak solutions to inhomogeneous quasilinear systems, where the principle part satisfies a linear growth condition and where the inhomogeneity is of subquadratic growth still obeying a one-sided condition (2.6). The motivation for this counterexamples comes from the theory of ergodic control problems where Bellman systems of the form (3.2) introduced above are studied, with the goal to analyze the passage to the limit $\gamma \searrow 0$. Under certain conditions, uniform estimates for $\|Du\|_{L^2}$ and the oscillations of u are available which is necessary for finding regularity estimates. Unfortunately, in this particular situation of systems (3.2) it is very difficult to obtain the uniform estimate for the oscillations. For this reason one is willing to pay the prize to restrict the inhomogeneity to subquadratic growth (which is indeed satisfied by appropriate models). However, our example shows that even under a subquadratic growth condition unbounded solutions may exist.

More precisely, we now consider the system

$$\left\{ \begin{aligned} -\Delta u^1 + \lambda \Delta u^2 &= |Du|^{\frac{2}{1+\gamma}}\big((\hat{b}_1 + \lambda \hat{b}_2)u^1 + (\hat{b}_2 - \lambda \hat{b}_1)u^2\big) \\ -\Delta u^2 - \lambda \Delta u^1 &= |Du|^{\frac{2}{1+\gamma}}\big((-\hat{b}_2 + \lambda \hat{b}_1)u^1 + (\hat{b}_1 + \lambda \hat{b}_2)u^2\big) \end{aligned} \right. \tag{3.7}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ containing the origin, for $\alpha, \gamma, \lambda \in \mathbb{R}$ with $\alpha \neq 0$, and coefficients $\hat{b}_1 = \hat{b}_1(n, \gamma, \alpha)$ and $\hat{b}_2 = \hat{b}_2(n, \gamma, \alpha)$ defined by

$$\hat{b}_1(n, \gamma, \alpha) := \frac{\gamma(n-\gamma-2)+\alpha^2}{(\alpha^2+\gamma^2)^{1/(1+\gamma)}} \qquad \text{and} \qquad \hat{b}_2(n, \gamma, \alpha) := \frac{-\alpha n + 2\alpha(\gamma+1)}{(\alpha^2+\gamma^2)^{1/(1+\gamma)}} .$$

Again, following the notation of this paper, we can define the vector field a and inhomogeneity a_0 (in dependence of n, α, γ and λ) as functions of z and of u, z, respectively. Analogously to the system from Section 3.1 it is easily verified that the system (3.7) is elliptic and that the inhomogeneity satisfies the one-sided condition, provided that further assumptions on the various parameters are satisfied.

Lemma 3.14. *The assumptions* (2.1)–(2.6) *are satisfied (with exponent $p = 2$ instead of n), for all choices of $\alpha, \lambda, \gamma \in \mathbb{R}$ with $\alpha \neq 0$ such that $2\gamma > n-2$ or such that $(n+2)\gamma \geq n-2$ and $\hat{b}_1 + \lambda \hat{b}_2 < 0$.*

Proof. The assumptions (2.1)–(2.5) follow exactly as in the proof of Lemma 3.1, and in particular, we note that the growth of the inhomogeneity a_0 is *subcritical* in the gradient variable, i. e. the growth with respect to the gradient variable is less than $|z|^2$. Therefore, under the assumption on γ the inhomogeneity satisfies a critical growth condition in the sense of $a_0(w, Dw) \in L^1(\Omega)$ for every $w \in W^{1,2}(\Omega, \mathbb{R}^2)$. Concerning the one-sided condition (2.6), we observe by Young's inequality

$$a_0(u,z) \cdot u = -|z|^{\frac{2}{1+\gamma}} (\hat{b}_1 + \lambda \hat{b}_2)|u|^2 \geq -\nu_0 |z|^2 - c(n, \gamma, \alpha, \nu_0)|u|^{2\frac{1+\gamma}{\gamma}}$$

holds true. The first assumption of the lemma implies $2(1+\gamma)/\gamma < 2n/(n-2)$ whereas in the second case the left-hand side is always positive. Consequently, the one-sided condition (2.6) is always satisfied, and the proof of the lemma is concluded. □

We next restrict ourselves to the systems with parameter $\gamma < n/2 - 1$, and we define the function $u \in W^{1,2}(\Omega, \mathbb{R}^2)$ via

$$u^1(x) = |x|^{-\gamma} \sin(\alpha \ln |x|) , \qquad u^2(x) = |x|^{-\gamma} \cos(\alpha \ln |x|) .$$

Proposition 3.15. *For every $\alpha, \lambda, \gamma \in \mathbb{R}$, with $\alpha \neq 0$ and $\gamma < n/2 - 1$, the function u is a weak solution to the system* (3.7) *in Ω.*

Proof. We first calculate in $\Omega \setminus \{0\}$ the partial derivatives of u:

$$\begin{aligned} D_i u^1(x) &= |x|^{-\gamma-2} x_i \big(-\gamma \sin(\alpha \ln |x|) + \alpha \cos(\alpha \ln |x|) \big) , \\ D_i u^2(x) &= |x|^{-\gamma-2} x_i \big(-\gamma \cos(\alpha \ln |x|) - \alpha \sin(\alpha \ln |x|) \big) \end{aligned}$$

for $i \in \{1, \ldots, n\}$. This allows to deduce $|Du(x)|^2 = |x|^{-2\gamma-2}(\alpha^2+\gamma^2)$, and we further find

$$\begin{aligned} -\Delta u^1(x) &= \big(\gamma(n-\gamma-2)+\alpha^2\big)|x|^{-2} u^1(x) + \big(-\alpha n + 2\alpha(\gamma+1)\big)|x|^{-2} u^2(x) , \\ -\Delta u^2(x) &= \big(\gamma(n-\gamma-2)+\alpha^2\big)|x|^{-2} u^2(x) + \big(\alpha n - 2\alpha(\gamma+1)\big)|x|^{-2} u^1(x) . \end{aligned}$$

Employing the definitions of $\hat{b}_1$ and of $\hat{b}_2$ we immediately check that (3.7) is satisfied pointwise in $\Omega \setminus \{0\}$, which in turn shows that u is a weak solution to the system (3.7) as asserted. □

As a consequence of the above construction, we now deduce the existence of unbounded solutions:

Theorem 3.16. *For every $n \geq 3$ there exist non-diagonal elliptic systems* (1.1) *which satisfy the assumptions* (2.1)–(2.6) *for $p = 2$ and which admit a discontinuous or even an unbounded weak solution in some bounded, regular domain $\Omega \subset \mathbb{R}^n$.*

Proof. According to Lemma 3.14 the system (3.7) satisfies the assumptions (2.1)–(2.6) for every all possible choices of $\alpha, \lambda, \gamma \in \mathbb{R}$, with $\alpha \neq 0$ such that $(n+2)\gamma \geq n-2$ and $\hat{b}_1 + \lambda \hat{b}_2 < 0$ (or such that $2\gamma > n-2$). It is easy to check that the condition $\hat{b}_1 + \lambda \hat{b}_2 < 0$ can be verified. For $\gamma \in [0, n/2-1)$, the function $u \in W^{1,2}(B, \mathbb{R}^2)$ defined above is a weak solution to this system (3.7) (and $(n+2)\gamma \geq n-2$ is trivially satisfied), and it is discontinuous in the origin. Moreover, for every $\gamma \in (0, n/2-1)$ it is even unbounded. □

3.4 A variational example

We lastly apply a similar strategy as in the examples given above to provide a discontinuous, unbounded solution to an Euler-Lagrange system, with diagonal principle part (but necessarily violating the one-sided condition). This proves that the corresponding one-sided assumption in the positive theory is indeed mandatory in order to obtain continuity of all solutions. For sake of simplicity we only give a simple example. We take $\Omega \subset \mathbb{R}^n$ a bounded, regular domain and we define a function $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ via

$$F(x,u,z) := |z|^2\big(1 + \hat{h}\big(f^{-2\theta}(x)|u|^2\big) + h\big(f^{-\theta}(x)(u^1\cos(\alpha\ln f(x)) - u^2\sin(\alpha\ln f(x)))\big)\big)\,.$$

Here, we take $f \in W^{1,1}(\Omega,\mathbb{R}^+)$ which satisfies $f^{2\theta}, f^{2\theta-2}|Df|^2 \in L^1(\Omega)$ for some $\theta \in \mathbb{R}$ and which is harmonic in $\Omega \setminus A$ for some set A of $W^{1,2}$-capacity zero. Furthermore, $h, \hat{h}\colon \mathbb{R} \to \mathbb{R}$ denote bounded, monotone C^∞-function with the property $h(0) = 0$. Defining the variational integral

$$\mathcal{F}[w] := \int_\Omega F(x,w,Dw)\,dx$$

with $w \in W^{1,2}(\Omega,\mathbb{R}^N)$, we may then investigate critical points of $\mathcal{F}$, that is weak solutions to the system

$$\operatorname{div} D_zF(x,u,Du) = D_uF(x,u,Du)\,.$$

From the definition of F it is clear that this system is elliptic whenever $\sup|h| + \sup|\hat{h}| < 1$, and the inhomogeneity is of quadratic, natural growth with respect to the gradient variable. Moreover, defining $u \in W^{1,2}(\Omega,\mathbb{R}^2)$ via

$$u^1(x) = f(x)^\theta \sin\big(\alpha\ln f(x)\big)\,, \qquad u^2(x) = f(x)^\theta\cos\big(\alpha\ln f(x)\big)\,,$$

it is easily derived the following condition on the parameters for u to be a critical point of $\mathcal{F}$.

Proposition 3.17. *The function $u \in W^{1,2}(\Omega,\mathbb{R}^2)$ is a critical point of $\mathcal{F}$ if the equations*

$$\begin{aligned}(1+\hat{h}(1))(\theta(\theta-1)-\alpha^2) &= (\alpha^2+\theta^2)\hat{h}'(1)\\ 2(1+\hat{h}(1))(2\theta-1)\alpha &= (\alpha^2+\theta^2)h'(0)\end{aligned}$$

are satisfied.

Proof. The computations are straightforward. After observing that the arguments of $\hat{h}$ and h are constant equal to 1 and to 0, respectively, for our choice of u, the system equations reduce to

$$\begin{cases} 2\Delta u^1(x)(1+\hat{h}(1)) = |Du(x)|^2\big(2f^{-2\theta}(x)u^1(x)\hat{h}'(1) + f^{-\theta}(x)\cos\big(\alpha\ln f(x)\big)h'(0)\big)\,,\\ 2\Delta u^2(x)(1+\hat{h}(1)) = |Du(x)|^2\big(2f^{-2\theta}(x)u^2(x)\hat{h}'(1) - f^{-\theta}(x)\sin\big(\alpha\ln f(x)\big)h'(0)\big)\,.\end{cases}$$

At this stage one essentially has to use the formulas for the derivatives of u derived in Proposition 3.13, and the claim is proved. □

Via the direct method in the calculus of variations, a minimizer in a given Dirichlet class in $W^{1,2}$ exists, which in the two-dimensional case is also known to be regular (due to by now classical results pioneered by Morrey [31]). Theorem 1.7 states that not necessarily all weak solutions of such systems – though obtained from a regular, variational integral – are continuous.

Proof of Theorem 1.7. We choose the system given above of variational structure for the special choices of $\Omega = B_{1/e}(0) \subset \mathbb{R}^n$, $f(x) = \ln|x|^{-1}$, $\theta \in (0,1/2)$ arbitrary, and the functions $\hat{h}(t) = -\hat{\lambda}\arctan(t)$ and $h(t) = (1-\hat{\lambda})\arctan(\lambda t)/2$ for free parameters $\hat{\lambda} \in (0,1)$, $\lambda \in \mathbb{R}$. By construction, the system is elliptic and satisfies (2.1)–(2.5). Furthermore, we need to satisfy the equations stated in Proposition 3.17. The first equation reduces to a linear equation in $\hat{\lambda}$, and for the choice $\alpha^2 = \theta$, we have $\hat{\lambda} \in (0,1)$. Finally, we observe that also the second equation is linear in λ, and thus can be satisfied for a suitable choice $\lambda \in \mathbb{R}$. Consequently, the function u defined above is the desired unbounded weak solution. □

Remark 3.18. *We highlight that the system given in the proof is of diagonal structure, continuous with respect to the x-variable, and smooth with respect to u and z. The variational system constructed by the second author in [10] instead is non-diagonal and only measurable in the x-variable (and it admits a discontinuous, bounded weak solution). It would be interesting to know whether a discontinuous solution to a variational system may also exist if the integrand is also smooth in x.*

Remark 3.19. *A similar construction is possible with $|z|^2$ replaced by $|z|^p$, ending up with a variational system with principle part containing the p-Laplace operator. However, since the positive theory available puts the emphasis on quadratic-type functionals, we decided to state the result only for $p = 2$. However, we note that the preceding result might be generalized in particular to the existence of regular, elliptic systems of variational structure of $(n-1)$-growth which admit an unbounded, discontinuous solution (and obviously also minimizers and therefore continuous weak solutions exist simultaneously).*

4 Preliminaries

We now present some auxiliary tools for the second part of the paper which deals with the existence of Hölder continuous solutions. We start with a basic lemma which ensures for an arbitrary L^1-function the existence of a suitable annulus $B_{2r}(x_0) \setminus B_r(x_0)$ (with radius depending on the given middle point x_0) on which the L^1-norm decays as $(|\ln r| \ln |\ln r|)^{-1}$ with respect to the radius r of the annulus.

Lemma 4.1 ([11], Lemma 3.4)**.** *Let $\Omega \subset \mathbb{R}^n$ and $g \in L^1(\Omega)$. For every $x_0 \in \Omega$ and every $R \leq 1/4$ there exists*

$$r = r(x_0) \in [r_m, R] \quad \text{with } r_m = r_m(R) = 2^{-(\ln R/\ln 2)^e}$$

such that

$$\int_{\Omega \cap B_{2r}(x_0) \setminus B_r(x_0)} |g| \, dx \leq \frac{1}{|\ln r| \ln |\ln r|} \int_{\Omega \cap B_{2R}(x_0)} |g| \, dx \,.$$

This tool will later be applied to control the growth of the n-energy.

The second lemma concerns mean values of functions which satisfy a certain exponential integrability. It states that the mean values might blow up in terms of the size of the domain of integration only in a logarithmic way.

Lemma 4.2 ([11], Lemma 3.5)**.** *Let $u \colon \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ be a measurable function such that*

$$\int_\Omega e^{\alpha |u|^q} \, dx \leq M$$

for some positive parameters α and q. Then there exists a constant c depending only on α, q and M such that for all measurable sets $A \subset \Omega$ with $|A| \leq \frac{1}{2}$ we have

$$\int_A |u| \, dx \leq c |A| |\ln |A||^{\frac{1}{q}}$$

This lemma can in particular be applied to functions in $W^{1,n}(\Omega, \mathbb{R}^N)$ with a bounded domain $\Omega \subset R^n$. In fact, in this critical case the embedding in L^∞ is not available, but with Trudinger-Moser inequality [34] the prerequisite of the lemma is satisfied for some positive α and $q = \frac{n}{n-1}$. Hence, it is possible to control the growth of the mean values for $W^{1,n}$-functions.

We will further work with Morrey spaces $L^{p,\sigma}(\Omega, \mathbb{R}^N)$, with $1 \leq p < \infty$, $\sigma > 0$, which are defined as the linear space of all functions $u \in L^p(\Omega, \mathbb{R}^N)$ such that

$$\|u\|^p_{L^{p,\sigma}(\Omega, \mathbb{R}^N)} := \sup_{y \in \Omega, 0 < \rho \leq \operatorname{diam} \Omega} \rho^{-\sigma} \int_{B_\rho(y) \cap \Omega} |u|^p \, dx < \infty \,.$$

This condition depends only on the behavior of u for radii $\rho \searrow 0$. The space $L^{p,\sigma}(\Omega, \mathbb{R}^N)$ is complete with respect to the norm $\| \cdot \|_{L^{p,\sigma}(\Omega, \mathbb{R}^N)}$. For details and some fundamental properties of the Morrey spaces we refer to the monographs of Giusti [15, Chapter 2.3] or of Giaquinta [13, Chapter 3]. In the sequel we shall use the following isomorphy (only for the case $p = n$).

Theorem 4.3 ([23], Theorem 2.2)**.** *Let $B_r \subset \mathbb{R}^n$ be a ball, $p \in (1,n]$, and $\lambda \in (0,1]$. If $u \in W^{1,p}(B_r,\mathbb{R}^N)$ and $Du \in L^{p,n-p+p\lambda}(B_r,\mathbb{R}^{Nn})$, then $u \in C^{0,\lambda}(\overline{B}_r,\mathbb{R}^N)$. Moreover, there exists a constant c depending only on n and p (but independent of the radius r) such that*

$$[u]_{C^{0,\lambda}(\overline{B_r},\mathbb{R}^N)} \leq c\|Du\|_{L^{p,n-p+p\lambda}(B_r,\mathbb{R}^N)}\,.$$

The same result holds true if B_r is replaced by a bounded Lipschitz domain Ω. In this case, the constant c also depends on the Lipschitz constant of $\partial\Omega$.

5 Approximation via a variational inequality

5.1 The variational inequality

Let $L \in \mathbb{R}^+$. We consider the auxiliary variational inequality.

$$\begin{cases} \text{Find } u_{\delta,L} \in W_0^{1,n}(\Omega,\mathbb{R}^N),\ \|u\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq L, \text{ such that} \\ \langle\, A(u_{\delta,L}), u_{\delta,L}-v\,\rangle + \langle\, B_\delta(u_{\delta,L}), u_{\delta,L}-v\,\rangle \leq 0 \quad \text{for all } v \in W_0^{1,n}(\Omega,\mathbb{R}^N),\ \|v\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq L\,. \end{cases} \tag{5.1}$$

Here we have introduced for $\delta \geq 0$

$$\langle\, B_\delta(u),\varphi\,\rangle := \int_\Omega \frac{a_0(x,u,Du)}{1+\delta|a_0(x,u,Du)|}\cdot\varphi\,dx$$

(and obviously B_0 reduces to the original operator B). We note that the growth condition (2.1) implies that A is a continuous mapping from $W_0^{1,n}(\Omega,\mathbb{R}^N)$ into its dual, and the same holds for B_δ for fixed $\delta > 0$ since $(1+\delta|a_0(x,u,z)|)^{-1}a_0(x,u,z)$ is bounded by δ^{-1}. This allows us to apply the theory of monotone operators which then leads to existence of a solution to (5.1).

Proposition 5.1. *The variational inequality* (5.1) *has a solution $u_{\delta,L}$ for fixed $\delta, L > 0$, and there holds $\|u_{\delta,L}\|_{W_0^{1,n}(\Omega,\mathbb{R}^N)} \leq c(L,\nu,\nu_0,K,|\Omega|)$ for all $\delta > 0$.*

Proof. We verify the conditions of Leray-Lions-Vishik from the theory of monotone operators, see [27, Chapitre 2.8]. In fact, we first observe that due to (2.5) and (2.6) we have that the coercivity

$$\langle\, A(u)+B_\delta(u),u\,\rangle \geq (\nu-\nu_0)\int_\Omega |Du|^n\,dx - 2K_L|\Omega| \tag{5.2}$$

holds true for all $u \in W_0^{1,n}(\Omega,\mathbb{R}^N)$ with $\|u\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq L$, independently for all δ. Next we discuss the pseudo-monotonicity of the operator $A(\cdot)+B_\delta(\cdot)$. For this purpose we take a sequence $(u_m)_{m\in\mathbb{N}} \subset W_0^{1,n}(\Omega,\mathbb{R}^N)$ converging weakly to a function u, with $\|u_m\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq L$ for all $m \in \mathbb{N}$, and we further suppose

$$\limsup_{m\to\infty}\langle\, A(u_m)+B_\delta(u_m),u_m-u\,\rangle \leq 0\,.$$

In view of the boundedness of $B_\delta(u_m)$ and Rellich's theorem, we find $\langle\, B_\delta(u_m),u_m-u\,\rangle \to 0$ as $m\to\infty$, and analogously we have

$$\int_\Omega a(x,u_m,Du)\cdot(Du_m-Du)\,dx = 0\,.$$

Hence, the monotonicity assumption (2.4) on the vector field a indeed guarantees strong convergence of $u_m \to u$ in $W_0^{1,n}(\Omega,\mathbb{R}^N)$. But then, the growth assumption (2.1) and boundedness of $B_\delta(\cdot)$ yield also strong convergence of $A(u_m) \to A(u)$ and $B_\delta(u_m) \to B_\delta(u)$ in the dual of $W_0^{1,n}(\Omega,\mathbb{R}^N)$, for all $\delta > 0$. In particular, this implies pseudo-monotonicity in the sense of Lions, and we thus obtain existence of a solution $u_{\delta,L}$ to the variational inequality (5.1) for all $\delta > 0$ and every $L > 0$, see [27, Théorème 8.2].

The second statement about uniform boundedness of $(u_{\delta,L})_{\delta>0}$ immediately follows from the variational inequality (5.1), applied with $v=0$, and the coercivity of the approximation (5.2). □

5.2 Caccioppoli's inequality

We next establish two types of the Caccioppoli inequality, for solutions $u_{\delta,L} \in W_0^{1,n}(\Omega,\mathbb{R}^N)$ – in both cases $\delta > 0$ and $\delta = 0$ – to the variational inequality (5.1), provided that $\|u_{\delta,L}\|_{L^\infty(\Omega,\mathbb{R}^N)} \le L$.

Lemma 5.2. *Let $L \in \mathbb{R}^+$, $\delta \ge 0$, and assume that $u_{\delta,L} \in W_0^{1,n}(\Omega,\mathbb{R}^N)$ is a solution of (5.1). Then there exist a number $\theta(K,\nu,n) \in (0,1)$, an exponent $\gamma(q)$ and constants $c_1(K,\nu,\nu_0,n,\|u_{\delta,L}\|_{W^{1,n}(\Omega,\mathbb{R}^N)})$, $c_2(K,\nu,\nu_0,n,q,\Omega,\|u_{\delta,L}\|_{W^{1,n}(\Omega,\mathbb{R}^N)})$ (all independent of the parameters δ and L) such that for all $x_0 \in \bar\Omega$ and $r \in (0,1)$ there hold*

$$\int_{\Omega\cap B_r(x_0)} |Du_{\delta,L}|^n\,dx \le c_1 \int_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)} \big(|Du_{\delta,L}|^n + r^{-1}|u_{\delta,L}||Du_{\delta,L}|^{n-1}\big)\,dx + c_1 r^\gamma$$

and

$$\int_{\Omega\cap B_r(x_0)} |Du_{\delta,L}|^n\,dx \le \theta \int_{\Omega\cap B_{2r}(x_0)} |Du_{\delta,L}|^n\,dx + c_2 r^\gamma + c_2 \operatorname{osc}(u_{\delta,L},\Omega\cap B_{2r}(x_0))\Big(r^\gamma + \int_{\Omega\cap B_{2r}(x_0)} \min\big\{\delta^{-1},|Du_{\delta,L}|^n\big\}\,dx\Big)\,.$$

Proof. For $x_0 \in \bar\Omega$, $r \in (0,1)$ we fix a smooth cut-off function $\tau \in C_0^\infty(B_{3r/2}(x_0),[0,1])$ satisfying $\tau \equiv 1$ in $B_r(x_0)$ and $\|D\tau\|_{L^\infty(\Omega,\mathbb{R}^n)} \le cr^{-1}$. To establish the desired inequalities, we now use two functions $v_1 = u_{\delta,L} - u_{\delta,L}\tau^2$ and $v_2 = u_{\delta,L} - (u_{\delta,L}-\xi)\tau^2$ for testing (5.1), where $\xi \in \mathbb{R}^N$ is an arbitrary constant with $|\xi| \le L$ (implying $\|v\|_{L^\infty(\Omega,\mathbb{R}^N)} \le L$) to be fixed later if $\operatorname{dist}(x_0,\partial\Omega) > 3r/2$, and $\xi = 0$ otherwise. This choice of ξ distinguishes the interior and the boundary situation and guarantees zero-boundary values for all possible choices of x_0, r. Note that v_1 and v_2 differ only by $\xi\tau^2$. For $i \in \{1,2\}$ we then find

$$\begin{aligned}\int_\Omega a\big(x,u_{\delta,L},D((u_{\delta,L}-\xi)\tau^2)\big)\cdot D((u_{\delta,L}-\xi)\tau^2)\,dx \\ + \int_\Omega \big[a\big(x,u_{\delta,L},Du_{\delta,L}\big) - a\big(x,u_{\delta,L},D((u_{\delta,L}-\xi)\tau^2)\big)\big]\cdot D((u_{\delta,L}-\xi)\tau^2)\,dx \\ + \int_\Omega a\big(x,u_{\delta,L},Du_{\delta,L}\big)\cdot D(v_2-v_i)\,dx \\ + \int_\Omega \frac{a_0(x,u_{\delta,L},Du_{\delta,L})}{1+\delta|a(x,u_{\delta,L},Du_{\delta,L})|}\cdot(u_{\delta,L}\tau^2 + v_1 - v_i)\,dx \le 0\,.\end{aligned}$$

Note here that the first integral was simply added to the weak formulation of the system equation and is fully compensated by the second term in the second integral. We then estimate the different integrals appearing in the previous inequality. First, by ellipticity (2.5) of the principal part, we find

$$\begin{aligned}\int_\Omega a\big(x,u_{\delta,L},D((u_{\delta,L}-\xi)\tau^2)\big)\cdot D((u_{\delta,L}-\xi)\tau^2)\,dx \\ \ge \nu\int_\Omega |D((u_{\delta,L}-\xi)\tau^2)|^n\,dx - K\int_{\Omega\cap B_{2r}(x_0)} (1+|u_{\delta,L}|^q)\,dx\,.\end{aligned}$$

Next, by the growth condition (2.1) and noting $\tau \equiv 1$ in $B_r(x_0)$, we find

$$\begin{aligned}-\int_\Omega \big[a\big(x,u_{\delta,L},Du_{\delta,L}\big) - a\big(x,u_{\delta,L},D((u_{\delta,L}-\xi)\tau^2)\big)\big]\cdot D((u_{\delta,L}-\xi)\tau^2)\,dx \\ \le c(K)\int_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)} \big(1+|u_{\delta,L}|^{q\frac{n}{n-1}} + |Du_{\delta,L}|^n + r^{-n}|u_{\delta,L}-\xi|^n\big)\,dx\,.\end{aligned}$$

The third integral only occurs for $i = 1$ and again gives a contribution only on the annulus. In fact, with (2.1) we get

$$-\int_\Omega a\big(x,u_{\delta,L},Du_{\delta,L}\big)\cdot D(\xi\tau^2)\,dx \le c(K)|\xi|r^{-1}\int_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)} \big(1+|u_{\delta,L}|^q + |Du_{\delta,L}|^{n-1}\big)\,dx\,.$$

It remains to estimate the last integral involving a_0. For $i = 1$ we employ the one-sided condition (2.6). This yields

$$-\int_\Omega \frac{a_0(x,u_{\delta,L},Du_{\delta,L})}{1+\delta|a_0(x,u_{\delta,L},Du_{\delta,L})|}\cdot u_{\delta,L}\tau^2\,dx \le \nu_0\int_\Omega |Du_{\delta,L}|^n\tau^2\,dx + K\int_{\Omega\cap B_{2r}(x_0)}\big(1+|u_{\delta,L}|^q\big)\,dx\,.$$

Alternatively, if $i = 2$, we use the growth condition (2.2) to find

$$\begin{aligned}
-\int_\Omega &\frac{a_0(x,u_{\delta,L},Du_{\delta,L})}{1+\delta|a_0(x,u_{\delta,L},Du_{\delta,L})|}\cdot (u_{\delta,L}-\xi)\tau^2\,dx\\
&\le c(K)\|u_{\delta,L}-\xi\|_{L^\infty(\Omega\cap B_{2r}(x_0),\mathbb{R}^N)}\int_{\Omega\cap B_{2r}(x_0)}\min\big\{\delta^{-1},|Du_{\delta,L}|^n\big\}\,dx\\
&\quad + c(K)\|u_{\delta,L}-\xi\|_{L^\infty(\Omega\cap B_{2r}(x_0),\mathbb{R}^N)}\int_{\Omega\cap B_{2r}(x_0)}(1+|u_{\delta,L}|^q)\,dx\,.
\end{aligned}$$

Combining the estimates (recalling $\tau\equiv 1$ in $B_r(x_0)$), we then find with the triangle inequality and Young the preliminary Caccioppoli-type inequalities

$$\begin{aligned}
\int_{\Omega\cap B_r(x_0)}&|Du_{\delta,L})|^n\,dx\\
&\le c(K,\nu,\nu_0)\int_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)}\big(|Du_{\delta,L}|^n + r^{-1}|u_{\delta,L}||Du_{\delta,L}|^{n-1} + r^{-n}|u_{\delta,L}-\xi|^n\big)\,dx\\
&\quad + c(K,\nu,\nu_0)\int_{\Omega\cap B_{2r}(x_0)}\big(1+|u_{\delta,L}|^{q\frac{n}{n-1}}\big)\,dx
\end{aligned}$$

or alternatively

$$\begin{aligned}
\int_{\Omega\cap B_r(x_0)}&|Du_{\delta,L})|^n\,dx\\
&\le c(K,\nu)\int_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)}\big(|Du_{\delta,L}|^n + r^{-n}|u_{\delta,L}-\xi|^n\big)\,dx\\
&\quad + c(K,\nu)(\|u_{\delta,L}-\xi\|_{L^\infty(\Omega\cap B_{2r}(x_0),\mathbb{R}^N)}+1)\int_{\Omega\cap B_{2r}(x_0)}(1+|u_{\delta,L}|^{q\frac{n}{n-1}})\,dx\\
&\quad + c(K,\nu)\|u_{\delta,L}-\xi\|_{L^\infty(\Omega\cap B_{2r}(x_0),\mathbb{R}^N)}\int_{\Omega\cap B_{2r}(x_0)}\min\big\{\delta^{-1},|Du_{\delta,L}|^n\big\}\,dx\,.
\end{aligned}$$

Now, in the interior situation when $\operatorname{dist}(x_0,\partial\Omega) > 3r/2$, we set ξ as the mean value over the annulus $B_{2r}(x_0)\setminus B_r(x_0)$ in Ω, i. e.

$$\xi = (u_{\delta,L})_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)} = |\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)|^{-1}\int_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)} u_{\delta,L}\,dx$$

(thus $|\xi|\le L$ is guaranteed), and we recall that we have chosen $\xi = 0$ otherwise. These choices allow us, for both in the interior and close to the boundary, to apply the Poincaré inequality to the function $u_{\delta,L}-\xi$ on $\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)$. Note here that whenever $\operatorname{dist}(x_0,\partial\Omega)\le 3r/2$, then $u_{\delta,L}$ vanishes on a quantified proportion of the boundary of $\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)$, since Ω is assumed to be a Lipschitz domain. Furthermore, by Hölder and then the Sobolev-Poincaré inequality we estimate

$$\|u_{\delta,L}\|_{L^p(\Omega\cap B_{2r}(x_0),\mathbb{R}^N)} \le c(n,p,\Omega)r^\gamma\|u_{\delta,L}\|_{W^{1,n}(\Omega,\mathbb{R}^N)}$$

for any $p<\infty$ and some exponent $\gamma(p)$. Hence, we obtain the inequalities

$$\int_{\Omega\cap B_r(x_0)}|Du_{\delta,L}|^n\,dx \le c_1\int_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)}\big(|Du_{\delta,L}|^n + r^{-1}|u_{\delta,L}||Du_{\delta,L}|^{n-1}\big)\,dx + c_1 r^\gamma$$

and

$$\int_{\Omega\cap B_r(x_0)} |Du_{\delta,L}|^n\,dx \le c_0(K,\nu,n)\int_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)} |Du_{\delta,L}|^n\,dx + c_2 r^\gamma + c_2\|u_{\delta,L}-\xi\|_{L^\infty(\Omega\cap B_{2r}(x_0),\mathbb{R}^N)}\Big(r^\gamma + \int_{\Omega\cap B_{2r}(x_0)} \min\big\{\delta^{-1}, |Du_{\delta,L}|^n\big\}\,dx\Big)\,,$$

with two constants $c_1 = c_1(K,\nu,\nu_0,n,\|u_{\delta,L}\|_{W^{1,n}(\Omega,\mathbb{R}^N)})$ and $c_2 = c_2(K,n,q,\nu,\|u_{\delta,L}\|_{W^{1,n}(\Omega,\mathbb{R}^N)})$, which do not explicitly depend on the parameters L and δ (only via the dependence on $\|u_{\delta,L}\|_{W^{1,n}(\Omega,\mathbb{R}^N)}$). This yields the first Caccioppoli-type inequality stated in the lemma, and the second one follows from filling the hole and the fact that $\|u_{\delta,L}-\xi\|_{L^\infty(\Omega\cap B_{2r}(x_0),\mathbb{R}^N)}$ is controlled by the oscillation of $u_{\delta,L}$ in $\Omega\cap B_{2r}(x_0)$, as a direct consequence of the definition of ξ. □

5.3 Uniform smallness of the n-energy and Morrey estimates

We continue to study sequences of functions in $W^{1,n}$ and we are now interested in some consequences of Caccioppoli-type inequalities of the form of those derived in the previous section. The first consequence concerns uniform smallness of the n-energy and is obtained following the arguments from [11, Section 3.4]. Secondly, we provide estimates on the Morrey-norm for the gradient of solutions to the variational inequality (5.1).

Lemma 5.3. *Let $(v_k)_{k\in\mathbb{N}}$ be a sequence of functions in $W^{1,n}(\Omega,\mathbb{R}^N)$ which is uniformly bounded, i. e. $\|v_k\|_{W^{1,n}(\Omega,\mathbb{R}^N)} \le C_0$ for all $k\in\mathbb{N}$, and which satisfies for all $x_0\in\bar\Omega$ and $r\in(0,1)$*

$$\int_{\Omega\cap B_r(x_0)} |Dv_k|^n\,dx \le C_1\int_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)} \big(|Dv_k|^n + r^{-1}|v_k||Dv_k|^{n-1}\big)\,dx + C_2\omega(r)$$

with $\omega(\cdot)$ a modulus of continuity. Then there exists a critical radius $r_c(\varepsilon,\omega(\cdot),C_0,C_1,C_2,\Omega)>0$ such that

$$\int_{\Omega\cap B_{r_c}(x_0)} |Dv_k|^n\,dx < \varepsilon^n$$

uniformly in $x_0\in\bar\Omega$ and $k\in\mathbb{N}$.

Proof. We essentially have to estimate the mixed term, which is done by taking advantage of Lemma 4.1 and Lemma 4.2. We start with a fixed $x_0\in\bar\Omega$, $R\in(0,1/4)$ and determine, according to Lemma 4.1, a radius $r=r(x_0,k,R)<1$ such that

$$\int_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)} |Dv_k|^n\,dx \le \frac{1}{|\ln r|\ln|\ln r|}\int_{\Omega\cap B_{2R}(x_0)} |Dv_k|^n\,dx\,.$$

We then observe that by the Trudinger-Moser inequality [34] we find positive constants M,α (depending only on the uniform $W^{1,n}$-norm of the sequence v_k) such that

$$\int_\Omega e^{\alpha|v_k|^{n/(n-1)}}\,dx \le M \qquad \text{for all } k\in\mathbb{N}\,.$$

Hence, by Lemma 4.2 we find

$$|(v_k)_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)}| \le c|\ln r|^{\frac{n-1}{n}}\,.$$

Now, by Hölder's and Poincaré's inequality (applied to the function $v_k-(v_k)_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)}$ on the annulus

$B_{2r}(x_0) \setminus B_r(x_0)$ in Ω for the radius r determined above), the Caccioppoli-type inequality becomes

$$\begin{aligned}\int_{\Omega\cap B_r(x_0)} |Dv_k|^n\,dx &\le c\int_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)} |Dv_k|^n\,dx \\ &\quad + c|\ln r|^{\frac{n-1}{n}}\Big(\int_{\Omega\cap B_{2r}(x_0)\setminus B_r(x_0)} |Dv_k|^n\,dx\Big)^{\frac{n-1}{n}} + C_2\omega(r) \\ &\le c(|\ln r|\ln|\ln r|)^{-1}\int_{\Omega\cap B_{2R}(x_0)} |Dv_k|^n\,dx \\ &\quad + c(\ln|\ln r|)^{\frac{1-n}{n}}\Big(\int_{\Omega\cap B_{2R}(x_0)} |Dv_k|^n\,dx\Big)^{\frac{n-1}{n}} + C_2\omega(r)\,.\end{aligned}$$

At this stage we conclude uniform smallness of the n-energy, in the sense that to a given $\varepsilon > 0$ we can choose R sufficiently small such that the right-hand side of the previous inequality is smaller than ε^n. Since r is bounded from below according to Lemma 4.1 (note that this bound is independent of x_0 and k), we hence obtain the statement. □

We next state the announced Morrey-estimates for sequences of functions, which – under further prerequisites – allow to pass from non-uniform to uniform estimates.

Lemma 5.4. *Let $(v_k)_{k\in\mathbb{N}}$ be a sequence of functions in $W^{1,n}(\Omega,\mathbb{R}^N)$ with the following properties:*

(i) non-uniform Morrey-estimate: there exists some $\alpha > 0$ (independent of k) such that there holds $[Dv_k]_{L^{n,n\alpha}(\Omega,\mathbb{R}^{Nn})} \le C(k)$ for all $k \in \mathbb{N}$;

(ii) uniform smallness condition: for every $\varepsilon > 0$ there exists a radius $r_c > 0$ such that there holds $\|Dv_k\|_{L^n(\Omega\cap B_{r_c}(x_0),\mathbb{R}^{Nn})} \le \varepsilon$;

(iii) Caccioppoli-type inequality: for all $x_0 \in \bar\Omega$, every $r > 0$ and some $\theta < 1$ there holds

$$\begin{aligned}\int_{\Omega\cap B_r(x_0)} |Dv_k|^n\,dx &\le \theta\int_{\Omega\cap B_{2r}(x_0)} |Dv_k|^n\,dx + C_1 r^{n\alpha} \\ &\quad + C_1\operatorname{osc}(v_k,\Omega\cap B_{2r}(x_0))\Big(r^{n\alpha} + \int_{\Omega\cap B_{2r}(x_0)} |Dv_k|^n\,dx\Big)\,.\end{aligned}$$

Then there exists $\beta \in (0,\alpha]$ (independent of k) such that

$$\|Dv_k\|_{L^{n,n\beta}(\Omega,\mathbb{R}^{Nn})} \le C\big(C_1, r_c(n,C_1,\Omega,\theta), \|Dv_k\|_{L^n(\Omega,\mathbb{R}^{Nn})}\big)\,.$$

Proof. We start by fixing a number $\gamma \in (\theta,1)$ and choose $\beta < \alpha$ such that $2^{n\beta}\theta \le \gamma$. Then we divide the Caccioppoli-type inequality in (iii) by $r^{n\beta}$. Estimating $(2r)^{-\beta}\|Dv_k\|_{L^n(\Omega\cap B_{2r})}$ by $\|Dv_k\|_{L^{n,n\beta}(\Omega)}$ and using Morrey's Theorem 4.3, we then find

$$\begin{aligned}r^{-n\beta}\int_{\Omega\cap B_r(x_0)} |Dv_k|^n\,dx &\le \gamma\|Dv_k\|^n_{L^{n,n\beta}(\Omega,\mathbb{R}^{Nn})} + C_1 \\ &\quad + c(n,C_1,\Omega)\big(\|Dv_k\|_{L^n(\Omega\cap B_{2r}(x_0),\mathbb{R}^{Nn})} + r^{n\alpha}\big)\|Dv_k\|^n_{L^{n,n\beta}(\Omega,\mathbb{R}^{Nn})}\,.\end{aligned}$$

Hence, for $\varepsilon = \varepsilon(n,C_1,\Omega,\gamma)$ sufficiently small, we deduce from (ii) that

$$r^{-n\beta}\int_{\Omega\cap B_r(x_0)} |Dv_k|^n\,dx \le \frac{1+\gamma}{2}\|Dv_k\|^n_{L^{n,n\beta}(\Omega,\mathbb{R}^{Nn})} + C_1$$

provided that $r \le r_c(\varepsilon) = r_c(n,C_1,\Omega,\theta)$. However, for larger radii, the left-hand side is trivially estimated by $r_c(n,C_1,\Omega,\theta)^{-n\beta}\|Dv_k\|^n_{L^n(\Omega,\mathbb{R}^{Nn})}$, and we hence end up with

$$r^{-n\beta}\int_{\Omega\cap B_r(x_0)} |Dv_k|^n\,dx \le \frac{1+\gamma}{2}\|Dv_k\|^n_{L^{n,n\beta}(\Omega,\mathbb{R}^{Nn})} + C_1 + r_c(n,C_1,\Omega,\theta)^{-n\beta}\|Dv_k\|^n_{L^n(\Omega,\mathbb{R}^{Nn})}$$

for all $x_0 \in \bar\Omega$, $r > 0$. Passing to the supremum over all $x_0 \in \bar\Omega$, $r > 0$, we can replace the left-hand side of the last estimate by $\|Dv_k\|^n_{L^{n,n\beta}(\Omega,\mathbb{R}^{Nn})}$, and the assertion then follows from $\gamma < 1$. □

6 Proof of Theorem 1.4

Proof of Theorem 1.4. The proof is divided into a series of steps. We now study solutions $u_{\delta,L}$ to the variational inequality (5.1) for $\delta, L > 0$, and subsequently we extract information about the two families $(u_{\delta,L})_{\delta\in(0,1)}$, with $L > 0$ fixed, and $(u_L)_{L\in\mathbb{N}} = (u_{0,L})_{L\in\mathbb{N}}$, using the results provided in Section 5.

Step 1: The sequence $(u_{\delta,L})_{\delta>0}$ for $L > 0$ fixed. In this step we shall sometimes use the fact that $|u_{\delta,L}| \le L$ for all $\delta \in (0,1)$. Sequentially we now derive the following properties.

a) Existence and uniform bound on the $W_0^{1,n}$-norm. This is a consequence of Proposition 5.1, which in particular shows $\|u_{\delta,L}\|_{W_0^{1,n}(\Omega,\mathbb{R}^N)} \le c(K, L, \nu, \nu_0, |\Omega|)$ for all $\delta > 0$.

b) Morrey estimate and Hölder continuity, via a classical hole-filling technique introduced by Widman [37]. Indeed, the Caccioppoli-type inequality in Lemma 5.2 shows the existence of a number $\alpha(\theta, n) = \alpha(K, \nu, n) > 0$, independent of δ, L, such that

$$\|Du_{\delta,L}\|_{L^{n,n\alpha}(\Omega,\mathbb{R}^{Nn})} \le c(\delta, K, L, \nu, \nu_0, n, q, \Omega, \|Du_{\delta,L}\|_{L_0^n(\Omega,\mathbb{R}^{Nn})}) .$$

Moreover, we obtain Hölder continuity of $u_{\delta,L}$ with exponent α via Theorem 4.3, with Hölder constant depending in particular on δ, L.

c) Uniform smallness of the n-energy. From the first Caccioppoli-type inequality in Lemma 5.2, combined with the uniform bound from Step 1a), and Lemma 5.3 we immediately find for every $\varepsilon > 0$ a radius $r_c(\varepsilon, K, L, \nu, \nu_0, n, q, \Omega) > 0$ such that $\|Du_{\delta,L}\|_{L^n(\Omega\cap B_{r_c}(x_0))} \le \varepsilon$ holds, uniformly in $x_0 \in \bar{\Omega}$ and for all $\delta \in (0,1)$.

d) Uniform Morrey-estimate in $\delta \in (0,1)$. With the (non-uniform) Hölder regularity derived in Step 1b), the uniform smallness from Step 1c), and the second Caccioppoli-type inequality in Lemma 5.2 we are in the position to apply Lemma 5.4. This ensures the existence of some $\beta \in (0,\alpha]$ such that

$$\|Du_{\delta,L}\|_{L^{n,n\beta}(\Omega,\mathbb{R}^{Nn})} \le C(n, K, L, \nu, \nu_0, q, \Omega, \|Du_{\delta,L}\|_{L^n(\Omega,\mathbb{R}^{Nn})}) .$$

Step 2: The passage to the limit $\delta \searrow 0$. As already noted in Step 1a), the sequence $(u_{\delta,L})_{\delta>0}$ is bounded uniformly in $W_0^{1,n}(\Omega, \mathbb{R}^N)$, with $L > 0$ fixed. Hence, we can extract a subsequence which converges weakly to a function $u_L \in W_0^{1,n}(\Omega, \mathbb{R}^N)$. Moreover, u_L is a solution to the variational inequality (5.1) for $\delta = 0$ (thus, we establish in particular existence of a solution to (5.1) for $\delta = 0$). This is seen as follows. First, since for all $L > 0$ we also have convergence $u_{\delta,L} \to u_L$ in some Hölder space, we get

$$\langle B_\delta(u_{\delta,L}), u_{\delta,L} - u_L \rangle \to 0 \quad \text{as } \delta \searrow 0 .$$

Hence, as in the proof of Proposition 5.1, we find

$$\limsup_{\delta\searrow 0} \int_\Omega \big(a(x, u_{\delta,L}, Du_{\delta,L}) - a(x, u_{\delta,L}, Du_L)\big) \cdot (Du_{\delta,L} - Du_L)\, dx \le 0 .$$

The monotonicity condition (2.4) then implies strong convergence $u_{\delta,L} \to u_L$ in $W_0^{1,n}(\Omega, \mathbb{R}^N)$, which in turn yields the desired variational inequality, due to the convergence

$$\langle A(u_{\delta,L}), u_{\delta,L} - v \rangle + \langle B_\delta(u_{\delta,L}), u_{\delta,L} - v \rangle \to \langle A(u_L), u_L - v \rangle + \langle B(u_L), u_L - v \rangle$$

for all $v \in W_0^{1,n}(\Omega, \mathbb{R}^N)$ with $\|v\|_{L^\infty(\Omega,\mathbb{R}^N)} \le L$.

Step 3: The sequence $(u_L)_L$. The regularity properties of u_L derived in Step 1 may still depend on L. In order to remove this L-dependence, we next proceed analogously to above and show the corresponding properties for the sequence $(u_L)_L$ (instead of $(u_{\delta,L})_\delta$), uniformly in L.

a) Existence and uniform bound on the $W_0^{1,n}$-norm. Existence was proved in Step 2, and the uniform bound is an immediate consequence of the coercivity (2.3) of the operator T.

b) Morrey estimate and Hölder continuity. A non-uniform bound (depending on L) for Du_L in the Morrey norm $L^{n,n\beta}(\Omega,\mathbb{R}^{Nn})$ follows by construction via the passage to the limit in Step 1b).

c) Uniform smallness of the n-energy. Again, the first Caccioppoli-type inequality in Lemma 5.2 (now with $\delta = 0$) together with Lemma 5.3 implies that, for every $\varepsilon > 0$, there exists a radius $r_c(\varepsilon, K, \nu, \nu_0, n, q, \Omega, \sup_L \|Du_L\|_{L^n(\Omega,\mathbb{R}^{Nn})}) > 0$ such that $\|Du_{\delta,L}\|_{L^n(B_{r_c}(x_0),\mathbb{R}^{Nn})} \leq \varepsilon$, uniformly in $x_0 \in \bar{\Omega}$ and $L > 0$.

d) Uniform Morrey-estimate in $L > 0$. With Step 3b), Step 3c), and the second Caccioppoli-type inequality in Lemma 5.2, we may apply Lemma 5.4 as above in Step 1d). This yields $\widetilde{\beta} \in (0, \beta]$ such that

$$\|Du_L\|_{L^{n,n\widetilde{\beta}}(\Omega,\mathbb{R}^{Nn})} \leq C(n, K, \nu, \nu_0, q, \Omega, \sup_L \|Du_L\|_{L^n(\Omega,\mathbb{R}^{Nn})}) \,.$$

Step 4: The passage to the limit $L \to \infty$. Exactly as in Step 2, we obtain a subsequence of $(u_L)_L$ which converges (weakly by compactness, strongly via the monotonicity condition) to some $u \in W_0^{1,n}(\Omega,\mathbb{R}^N) \cap C^{0,\widetilde{\beta}}(\Omega,\mathbb{R}^N)$. The variational inequality

$$\langle\, A(u), u - v\,\rangle + \langle\, B(u), u - v\,\rangle \leq 0 \quad \text{for all } v \in W_0^{1,n}(\Omega,\mathbb{R}^N)$$

then immediately gives equality, and hence, u is a solution to the system (1.1). This completes the proof of the theorem. □

7 Application to discount control problems

The theory provided above also applies to obtain the existence of a regular solution for other problems having an inhomogeneity of critical growth, at least in the critical dimension. For illustration, we state the corresponding result exemplary for equations related to discount control. The proof differs in some parts, but the general strategy via approximation and the validity of uniform Morrey-type estimates remains the same. For simplicity, we here restrict ourselves to the following system of equations (cf. Section 3.2)

$$\begin{aligned} -\sum_{i=1}^{n} D_i\big[(1+|Du|^2)^{\frac{p-2}{2}} D_i u^\alpha\big] + \gamma u^\alpha &= H_0^\alpha(\cdot,u,Du) + G(\cdot,u,Du)Du^\alpha - F(\cdot,u,Du)u^\alpha + f^\alpha \\ &=: H^\alpha(x,u,Du) + f^\alpha \end{aligned} \tag{7.1}$$

for all $\alpha = 1,\dots,N$ in a regular domain $\Omega \subset \mathbb{R}^n$, some $\gamma > 0$, and with functions $H_0\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn} \to \mathbb{R}^N$, $G\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^n$, $F\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ and $f\colon \Omega \to \mathbb{R}^N$ on the right-hand side. We further assume that this inhomogeneity satisfies the Carathéodory condition and that the following growth assumptions hold true:

$$\begin{cases} |H_0^\alpha(x,u,z)| \leq K|z|^{p-2}|z^\alpha|^2 + K \quad \text{for } \alpha = 1,\dots,N \\ |G(x,u,z)| \leq K|z|^{p-1} + K \\ 0 \leq F(x,u,z) \leq K|z|^p + K \\ f \in L^\infty(\Omega,\mathbb{R}^N) \end{cases} \tag{7.2}$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$, and $z \in \mathbb{R}^{Nn}$, cf. [2, 3]. We emphasize that the function H_0^α only depends on z^β for $\beta \neq \alpha$ with $(p-2)$-growth. This system can be considered as a natural extension of discount control problems ($p = 2$). We further note that we here consider only principal parts which are of p-Laplace structure. However, one might easily adapt the theory in order to cover more general systems of monotone structure. With an approximation technique which is similar to that applied before we find the existence of a regular solution to (7.1).

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and assume that the aforementioned growth assumptions* (7.2) *are fulfilled for $p = n$. Then the elliptic system* (7.1) *has a weak solution $u \in C^\alpha_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \cap W_0^{1,n}(\Omega,\mathbb{R}^N)$ for some $\alpha > 0$.*

Sketch of proof. Since the arguments are very similar to those in [3] (and partially also to those applied before), we only describe briefly the intermediate steps of the proof and leave the details to the reader. We start by approximating the right-hand side $H(x,u,z)$ in (7.1) via a bounded inhomogeneity $H_\delta(x,u,z) = H(x,u,z)/(1+\delta|z|^n)$. We then introduce the approximating system via

$$-\sum_{i=1}^{n} D_i\big[(1+|Du|^2)^{\frac{p-2}{2}} D_i u_\delta\big] + \gamma u_\delta = H_\delta(x,u,Du) + f \qquad \text{in } \Omega\,. \tag{7.3}$$

Observation 1: Boundedness of (u_δ)*.* The approximate elliptic systems have continuous solutions $u_\delta \in W_0^{1,n}(\Omega,\mathbb{R}^N)$ with uniform L^∞-bound in terms of only the data (that is, K, $\|f\|_{L^\infty(\Omega,\mathbb{R}^N)}$ and γ). The proof is based on the theory of monotone operators, combined with a weak maximum principles involving truncations, and follows the arguments in [3, Section 8]. With this bound at hand, we need only the approximation with $\delta \searrow 0$ (and not the additional approximation with $L \nearrow \infty$ for the bounds on the L^∞-norms of the solutions to the approximating system).

Observation 2: A Caccioppoli-type inequality I. We test the approximating system (7.3) with iterated exponential functions of the form

$$\begin{aligned}\varphi^\alpha &= \tau\big(\exp(\lambda u_\delta^\alpha) - \exp(-\lambda u_\delta^\alpha)\big)\exp\Big(\lambda\sum_{\beta=1}^{N}(\exp(\lambda u_\delta^\beta) + \exp(-\lambda u_\delta^\beta))\Big)\\ &:= \tau\big(\exp(\lambda u_\delta^\alpha) - \exp(-\lambda u_\delta^\alpha)\big)\exp(h(u_\delta))\end{aligned}$$

with τ either a localization function or $\tau \equiv 1$ (both is possible due to the zero-boundary condition on the family $(u_\delta)_{\delta\in(0,1)}$), and $\lambda > 0$ a parameter to be chosen later. Similarly as in [3, Section 6], we now study the effects of this test function on the principle part of (7.3). We find

$$\begin{aligned}&\int_\Omega (1+|Du_\delta|^2)^{\frac{p-2}{2}} Du_\delta \cdot D\varphi\,dx\\ &\quad = \lambda\sum_{\alpha=1}^{N}\int_\Omega (1+|Du_\delta|^2)^{\frac{p-2}{2}}|Du_\delta^\alpha|^2\big(\exp(\lambda u_\delta^\alpha) + \exp(-\lambda u_\delta^\alpha)\big)\exp(h(u_\delta))\tau\,dx\\ &\qquad + \int_\Omega (1+|Du_\delta|^2)^{\frac{p-2}{2}}\Big|\sum_{\alpha=1}^{N}\nabla(\exp(\lambda u_\delta^\alpha) + \exp(-\lambda u_\delta^\alpha))\Big|^2\exp(h(u_\delta))\tau\,dx\\ &\qquad + \sum_{\alpha=1}^{N}\int_\Omega (1+|Du_\delta|^2)^{\frac{p-2}{2}} Du_\delta^\alpha\big(\exp(\lambda u_\delta^\alpha) - \exp(-\lambda u_\delta^\alpha)\big)\exp(h(u_\delta))\cdot\nabla\tau\,dx\,.\end{aligned}$$

Concerning the right-hand side of the approximating system 7.3, we can estimate the first three terms as follows (noting $(1+\delta|z|^p) \geq 1$ in the regularization of the inhomogeneity):

$$\int_\Omega |H_0(\cdot,u_\delta,Du_\delta)\cdot\varphi|\,dx \leq K\int_\Omega (1+|Du_\delta|^{p-2}|Du_\delta^\alpha|^2)\big(\exp(\lambda u_\delta^\alpha) + \exp(-\lambda u_\delta^\alpha)\big)\exp(h(u_\delta))\tau\,dx$$

$$\begin{aligned}\int_\Omega |G(\cdot,u_\delta,Du_\delta)Du_\delta\cdot\varphi|\,dx &\leq \varepsilon\int_\Omega |Du_\delta|^p\exp(h(u_\delta))\tau\,dx\\ &\quad + \lambda^{-2}c(\varepsilon,K)\int_\Omega (1+|Du_\delta|^2)^{\frac{p-2}{2}}\\ &\qquad\times\Big|\sum_{\alpha=1}^{N}\nabla(\exp(\lambda u_\delta^\alpha) + \exp(-\lambda u_\delta^\alpha))\Big|^2\exp(h(u_\delta))\tau\,dx\end{aligned}$$

$$-F(x,u_\delta(x),Du_\delta(x))u_\delta(x)\cdot\varphi(x) \leq 0 \qquad \text{for } x\in\Omega\,.$$

For the last inequality we have used the facts that $F \geq 0$ and that $t(\exp(\lambda t) - \exp(-\lambda t))$ for all possible choices of $t \in \mathbb{R}$ and $\lambda > 0$. With $\lambda = \lambda(K)$ chosen in a suitable way, we then obtain the desired

Caccioppoli-type inequality

$$\sum_{\alpha=1}^{N}\int_{\Omega}\Big[(1+|Du_\delta|^2)^{\frac{p-2}{2}}|Du_\delta^\alpha|^2\big(\exp(\lambda u_\delta^\alpha)+\exp(-\lambda u_\delta^\alpha)\big)$$
$$+(\gamma u_\delta^\alpha-f^\alpha)\big(\exp(\lambda u_\delta^\alpha)-\exp(-\lambda u_\delta^\alpha)\big)\Big]\exp(h(u_\delta))\tau\,dx$$
$$\le-\sum_{\alpha=1}^{N}\int_{\Omega}(1+|Du_\delta|^2)^{\frac{p-2}{2}}Du_\delta^\alpha\big(\exp(\lambda u_\delta^\alpha)-\exp(-\lambda u_\delta^\alpha)\big)\exp(h(u_\delta))\cdot\nabla\tau\,dx\,.$$

Observation 3: A Caccioppoli-type inequality II. Testing the approximating system (7.3) with the function $(u_\delta-\xi)\tau^2$ for suitable choices of $\xi\in\mathbb{R}^N$ and with $\tau\in C_0^\infty(B_{2r}(x_0),[0,1])$ a standard cut-off function with $\tau\equiv 1$ in $B_r(x_0)$ and $\|D\tau\|_{L^\infty(\Omega,\mathbb{R}^n)}\le cr^{-1}$ and using a hole-filling argument one arrives at a classical Caccioppoli inequality, in a very similar way as in Lemma 5.2. In particular, this establishes a non-uniform Morrey-type estimate for Du_δ, which in particular guarantees that assumption (i) of Lemma 5.4 is satisfied for the sequence $(u_{1/k})_{k\in\mathbb{N}}$.

Consequence 1: Uniform $W^{1,p}$-estimate. Using the previous Caccioppoli-type inequality I for the choice $\tau\equiv 1$, we immediately obtain a uniform bound on the $W^{1,n}$-norm of the family $(u_\delta)_{\delta\in(0,1)}$, by taking advantage of the inequalities

$$\exp(\lambda t)+\exp(-\lambda t)\ge 2\quad\text{and}\quad t\big(\exp(\lambda t)-\exp(-\lambda t)\big)\ge 0$$

for all $t\in\mathbb{R}$ and $\lambda>0$. This bound is needed later for the passage to the limit $\delta\searrow 0$.

Consequence 2: Logarithmic Morrey estimate. In the Caccioppoli-type inequality I we next choose $\tau=|\ln(\operatorname{diam}(\Omega)|x-x_0|/2)|^a$ for $a\in(0,1)$. Using Young's inequality, this gives

$$\int_{\Omega}(1+|Du_\delta|^2)^{\frac{p-2}{2}}|Du_\delta|^2|\ln(\operatorname{diam}(\Omega)|x-x_0|/2)|^a\,dx$$
$$\le c\int_{\Omega}|\ln(\operatorname{diam}(\Omega)|x-x_0|/2)|^a\,dx$$
$$+c\int_{\Omega}(1+|Du_\delta|^2)^{\frac{p-2}{2}}|Du_\delta||\ln(\operatorname{diam}(\Omega)|x-x_0|/2)|^{a-1}|x-x_0|^{-1}\,dx$$
$$\le\frac{1}{2}\int_{\Omega}(1+|Du_\delta|^2)^{\frac{p-2}{2}}|Du_\delta|^2|\ln(\operatorname{diam}(\Omega)|x-x_0|/2)|^a\,dx+c\big(a,K,\|f\|_{L^\infty(\Omega,\mathbb{R}^N)},\operatorname{diam}(\Omega)\big)\,.$$

Hence, after absorbing the first integral on the right-hand side, we end up with

$$\int_{B_r(x_0)\cap\Omega}|Du_\delta|^p|\ln(|x-x_0|)|^a\,dx\le c$$

for all $x_0\in\Omega$ and every $r\le 1/2$. In particular, this provides the uniform smallness condition of the L^n-norm of the sequence $(Du_\delta)_{\delta\in(0,1)}$ as required in Lemma 5.4 (ii).

Conclusion and passage to the limit. Proceeding exactly as in the proof of Theorem 1.4, one now employs Lemma 5.4 in order to obtain a uniform Morrey estimate for the sequence $(u_{1/k})_{k\in\mathbb{N}}$. Passing to the limit $k\to\infty$ it turns out that the sequence $(u_{1/k})_{k\in\mathbb{N}}\subset W_0^{1,n}(\Omega,\mathbb{R}^N)$ of solutions to the approximating system (7.3) (with $1/k$ instead of δ) form an approximating sequence of a regular solution $u\in C^{0,\alpha}(\Omega,\mathbb{R}^N)\cap W_0^{1,n}(\Omega,\mathbb{R}^N)$ of the original system (7.1). This finishes the proof of the theorem. □

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